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On the spectral characterization of kite graphs*

Research Article

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Abstract: The *Kite graph*, denoted by $Kite_{p,q}$ is obtained by appending a complete graph K_p to a pendant vertex of a path P_q. In this paper, firstly we show that no two non-isomorphic kite graphs are cospectral w.r.t the adjacency matrix. Let G be a graph which is cospectral with $Kite_{p,q}$ and let $w(G)$ be the clique number of G. Then, it is shown that $w(G) \geq p - 2q + 1$. Also, we prove that $Kite_{p,2}$ graphs are determined by their adjacency spectrum.

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1. Introduction

All of the graphs considered here are simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For a given graph F, if G does not contain F as an induced subgraph, then G is called $F - free$. A complete subgraph of G is a *clique of G. The clique number of* G is the number of the vertices in the largest clique of G and it is denoted by $w(G)$. Let $A(G)$ be the *(0,1)-adjacency matrix of* G and d_k denotes the degree of the vertex v_k. The polynomial $P_{A(G)}(\lambda) = det(\lambda I - A(G))$ is the *adjacency characteristic polynomial of* G, where I is the identity matrix. Eigenvalues of the matrix $A(G)$ are *adjacency eigenvalues.* Since $A(G)$ is real and symmetric matrix, adjacency eigenvalues are all real numbers and could be ordered as $\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \ldots \geq \lambda_n(A(G))$. *Adjacency spectrum of the graph* G consists of the adjacency eigenvalues with their multiplicities. The largest absolute value of the adjacency eigenvalues of a graph is known as its *adjacency spectral radius.* Two graphs G and H are said to be *cospectral* if they have same spectrum (i.e., same characteristic polynomial). A graph G is *determined by its adjacency spectrum*, shortly *DAS*, if every graph cospectral with G w.r.t the adjacency matrix, is isomorphic to G. It is conjectured in [5] that almost all graphs are determined by their spectrum, *DS* for short. But it is difficult to show that a given

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graph is *DS*. Up to now, some graphs are proved to be *DS* [1, 2, 4–11, 13, 15]. Recently, some papers have appeared in the literature that researchers focus on some special graphs (oftenly under some conditions) and prove that these special graphs are *DS* or *non-DS* [1, 2, 6–11, 13, 15]. For a recent survey, one can see [5].

The *Kite graph*, denoted by $Kite_{p,q}$, is obtained by appending a complete graph with p vertices K_p to a pendant vertex of a path graph with q vertices P_q. If $q = 1$, it is called *short kite graph*.

In this paper, firstly we obtain the characteristic polynomial of kite graphs and show that no two non-isomorphic kite graphs are cospectral w.r.t the adjacency matrix. Then for a given graph G which is cospectral with $Kite_{p,q}$, the clique number of G is $w(G) \geq p - 2q + 1$. Also we prove that $Kite_{p,2}$ graphs are *DAS* for all p.

2. Preliminaries

First, we give some lemmas that will be used in the next sections of this paper.

Lemma 2.1. *[8] Let x_1 be a pendant vertex of a graph G and x_2 be the vertex which is adjacent to x_1. Let G_1 be the induced subgraph obtained from G by deleting the vertex x_1. If x_1 and x_2 are deleted, the induced subgraph G_2 is obtained. Then,*

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) - P_{A(G_2)}(\lambda)$$

Lemma 2.2. *[4] For nxn matrices A and B, followings are equivalent :*

- ***(i)*** *A and B are cospectral*
- ***(ii)*** *A and B have the same characteristic polynomial*
- ***(iii)*** *$tr(A^i) = tr(B^i)$ for $i = 1, 2, ..., n$*

Lemma 2.3. *[4] For the adjacency matrix of a graph G, following parameters can be deduced from the spectrum;*

- ***(i)*** *the number of vertices*
- ***(ii)*** *the number of edges*
- ***(iii)*** *the number of closed walks of any fixed length.*

Theorem 2.4. *[14] If a given connected graph G has the same order, same clique number and same spectral radius with $Kite_{p,q}$ then G is isomorphic to $Kite_{p,q}$.*

In the rest of the paper, we denote the number of subgraphs of a graph G which are isomorphic to complete graph K_3 by $t(G)$.

Theorem 2.5. *[14] For any integers $p \geq 3$ and $q \geq 1$, if we denote the spectral radius of $A(Kite_{p,q})$ with $\rho(Kite_{p,q})$ then*

$$p - 1 + \frac{1}{p^2} + \frac{1}{p^3} < \rho(Kite_{p,q}) < p - 1 + \frac{1}{4p} + \frac{1}{p^2 - 2p}$$

Theorem 2.6. *[12] Let G be a graph with n vertices, m edges and spectral radius μ. If G is $K_{r+1} - free$, then*

$$\mu \leq \sqrt{2m(\frac{r-1}{r})}$$

Lemma 2.7. *[3]**(Interlacing Lemma)** If G is a graph on n vertices with eigenvalues $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ and H is an induced subgraph on m vertices with eigenvalues $\lambda_1(H) \geq \ldots \geq \lambda_m(H)$, then for $i = 1, \ldots, m$*

$$\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$$

Lemma 2.8. *[3] A connected graph with the largest adjacency eigenvalue less than 2 are precisely induced subgraphs of the Smith graphs shown in Figure 1.*

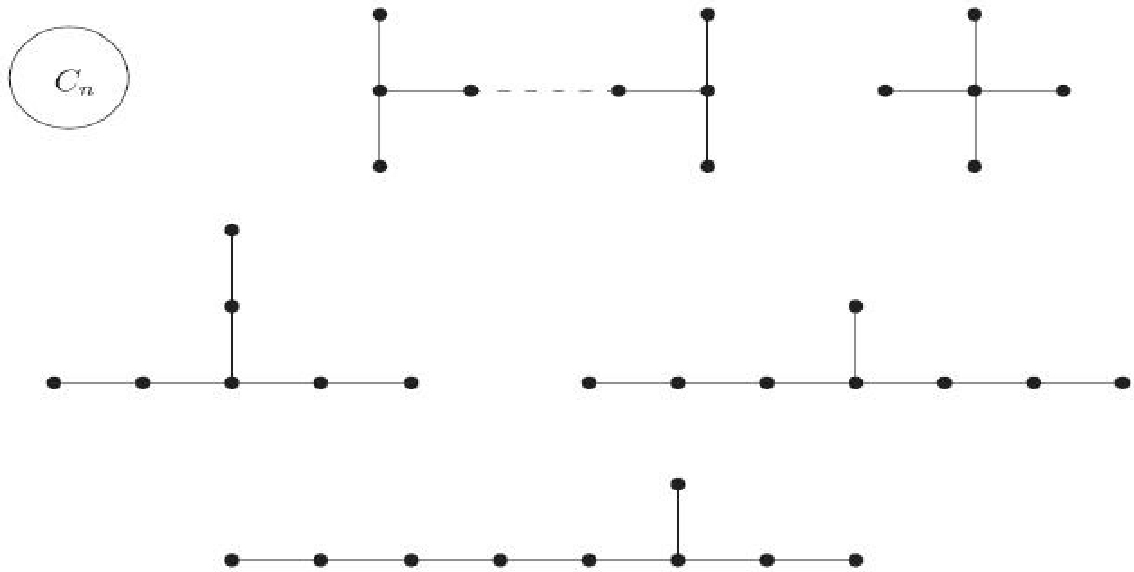

Figure 1. Smith graphs

3. Characteristic polynomial of kite graphs

We use the method similar to that given in [8] to obtain the general form of characteristic polynomials of $Kite_{p,q}$ graphs. Obviously, if we delete the vertex with one degree from short kite graph, the induced subgraph will be the complete graph K_p. Then, by deleting the vertex with one degree and its adjacent vertex, we obtain the complete graph K_{p-1} with $p-1$ vertices. From Lemma 2.1, we get

$$\begin{aligned} P_{A(Kite_{p,1})}(\lambda) &= \lambda P_{A(K_p)}(\lambda) - P_{A(K_{p-1})}(\lambda) \\ &= \lambda(\lambda - p + 1)(\lambda+1)^{p-1} - [(\lambda - p + 2)(\lambda+1)^{p-2}] \\ &= (\lambda+1)^{p-2}[(\lambda^2 - \lambda p + \lambda)(\lambda+1) - \lambda + p - 2] \\ &= (\lambda+1)^{p-2}[\lambda^3 - (p-2)\lambda^2 - \lambda p + p - 2]. \end{aligned}$$

Similarly for $Kite_{p,2}$, induced subgraphs will be $Kite_{p,1}$ and K_p respectively. By Lemma 2.1, we get

$$\begin{aligned} P_{A(Kite_{p,2})}(\lambda) &= \lambda P_{A(Kite_{p,1})}(\lambda) - P_{A(K_p)})(\lambda) \\ &= \lambda(\lambda P_{A(K_p)}(\lambda) - P_{A(K_{p-1})}(\lambda)) - P_{A(K_p)})(\lambda) \\ &= (\lambda^2 - 1)P_{A(K_p)}(\lambda) - \lambda P_{A(K_{p-1})}(\lambda). \end{aligned}$$

By using these polynomials, we calculate the characteristic polynomial of $Kite_{p,q}$ where $n = p + q$. Again, by Lemma 2.1 we have

$$P_{A(Kite_{p,1})}(\lambda) = \lambda P_{A(K_p)}(\lambda) - P_{A(K_{p-1})}(\lambda).$$

Coefficients of above equation are $b_1 = -1$, $a_1 = \lambda$. Simultaneously, we get

$$P_{A(Kite_{p,2})}(\lambda) = (\lambda^2 - 1)P_{A(K_p)}(\lambda) - \lambda P_{A(K_{p-1})}(\lambda).$$

and coefficients of above equation are $b_2 = -a_1 = -\lambda$, $a_2 = \lambda a_1 - 1 = \lambda^2 - 1$. Then for $Kite_{p,3}$, we have

$$\begin{aligned} P_{A(Kite_{p,3})}(\lambda) &= \lambda P_{A(Kite_{p,2})}(\lambda) - P_{A(Kite_{p,1})})(\lambda) \\ &= (\lambda(\lambda^2 - 1) - \lambda)P_{A(K_p)}(\lambda) - ((\lambda^2 - 1)P_{A(K_{p-1})}(\lambda)) \end{aligned}$$

and coefficients of above equation are $b_3 = -a_2 = -(\lambda^2 - 1), a_3 = \lambda a_2 - a_1 = \lambda(\lambda^2 - 1) - \lambda$. In the following steps, for $n \geq 3$, $a_n = \lambda a_{n-1} - a_{n-2}$. From this difference equation, we get

$$a_n = \sum_{k=0}^{n} (\frac{\lambda + \sqrt{\lambda^2 - 4}}{2})^k (\frac{\lambda - \sqrt{\lambda^2 - 4}}{2})^{n-k}$$

Now, let $\lambda = 2cos\theta$ and $u = e^{i\theta}$. Then, we have

$$a_n = \sum_{k=0}^{n} u^{2k-n} = \frac{u^{-n}(1 - u^{2n+2})}{1 - u^2}$$

and by calculation the characteristic polynomial of any kite graph $Kite_{p,q}$ where $n = p + q$ is

$$\begin{aligned} P_{A(Kite_{p,q})}(u + u^{-1}) &= a_{n-p}P_{A(K_p)}(u + u^{-1}) - a_{n-p-1}P_{A(K_{p-1})}(u + u^{-1}) \\ &= \frac{u^{-n+p}(1 - u^{2n-2p+2})}{1 - u^2}.((u + u^{-1} - p + 1).(u + u^{-1} + 1)^{p-1}) \\ &\quad - \frac{u^{-n+p+1}(1 - u^{2n-2p+4})}{1 - u^2}.((u + u^{-1} - p + 2).(u + u^{-1} + 1)^{p-2}) \\ &= \frac{u^{-n+p}(1 + u - u^{-1})^{p-2}}{1 - u^2}[(2 - p).(1 + u^{-1} - u^{2n-2p+2} - u^{2n-2p+3}) \\ &\quad +(u^{-2} - u^{2n-2p+4})] \\ &= \frac{u^{-q}(1 + u - u^{-1})^{p-2}}{1 - u^2}[(2 - p).(1 + u^{-1} - u^{2q+2} - u^{2q+3}) \\ &\quad +(u^{-2} - u^{2q+4})]. \end{aligned}$$

Theorem 3.1. *No two non-isomorphic kite graphs have the same adjacency spectrum.*

Proof. Assume that there are two cospectral kite graphs with number of vertices respectively, $p_1 + q_1$ and $p_2 + q_2$. Since they are cospectral, they must have same number of vertices and same characteristic polynomials. Hence, $n = p_1 + q_1 = p_2 + q_2$ and we get

$$P_{A(Kite_{p_1,q_1})}(u + u^{-1}) = P_{A(Kite_{p_2,q_2})}(u + u^{-1})$$

i.e.,

$$\begin{aligned} &\frac{u^{-n+p_1}(1 + u - u^{-1})^{p_1-2}}{1 - u^2}[(2 - p_1).(1 + u^{-1} - u^{2n-2p_1+2} - u^{2n-2p_1+3}) \\ &+(u^{-2} - u^{2n-2p_1+4})] \\ = &\frac{u^{-n+p_2}(1 + u - u^{-1})^{p_2-2}}{1 - u^2}[(2 - p_2).(1 + u^{-1} - u^{2n-2p_2+2} - u^{2n-2p_2+3}) \\ &+(u^{-2} - u^{2n-2p_2+4}]) \end{aligned}$$

i.e.,

$$\begin{aligned} & u^{p_1}.(1+u-u^{-1})^{p_1}.[(2-p_1).(1+u^{-1}-u^{2n-2p_1+2}-u^{2n-2p_1+3}) \\ & +(u^{-2}-u^{2n-2p_1+4})] \\ = \; & u^{p_2}.(1+u-u^{-1})^{p_2}.[(2-p_2).(1+u^{-1}-u^{2n-2p_2+2}-u^{2n-2p_2+3}) \\ & +(u^{-2}-u^{2n-2p_2+4})] \end{aligned}$$

Let $p_1 > p_2$. It follows that $n - p_2 > n - p_1$. Then, we have

$$\begin{aligned} & u^{p_1-p_2}.(1+u-u^{-1})^{p_1-p_2}\{[(2-p_1).(1+u^{-1}-u^{2n-2p_1+2}-u^{2n-2p_1+3}) \\ & +(u^{-2}-u^{2n-2p_1+4})]-[(2-p_2).(1+u^{-1}-u^{2n-2p_2+2}-u^{2n-2p_2+3}) \\ & +(u^{-2}-u^{2n-2p_2+4})]\}=0 \end{aligned}$$

By using the fact that $u \neq 0$ and $1 + u + u^{-1} \neq 0$, we get

$$\begin{aligned} f(u) &= [(2-p_1).(1+u^{-1}-u^{2n-2p_1+2}-u^{2n-2p_1+3})+(u^{-2}-u^{2n-2p_1+4})] \\ & \quad -[(2-p_2).(1+u^{-1}-u^{2n-2p_2+2}-u^{2n-2p_2+3})+(u^{-2}-u^{2n-2p_2+4})] \\ &= 0 \end{aligned}$$

Since $f(u) = 0$, the derivation of $(2n - 2p_2 + 5)$th of f equals to zero again. Thus, we have

$$[(p_1-2)(2n-2p_2+4)!(u^{-2n+2p_2-6})]-[(p_2-2).(2n-2p_2+4)!(u^{-2n+2p_2-6})]=0$$

i.e.,

$$[(p_1-2)-(p_2-2)].(u^{-2n+2p_2-6})=0$$

i.e.,

$$p_1 = p_2$$

since $u \neq 0$. This is a contradiction with our assumption that is $p_1 > p_2$. For $p_2 > p_1$, we get the similar contradiction. So p_1 must be equal to p_2. Hence $q_1 = q_2$ and these graphs are isomorphic. □

4. Spectral characterization of Kite$_{p,2}$ graphs

Lemma 4.1. *Let G be a graph which is cospectral with $Kite_{p,q}$. Then we get*

$$w(G) \geq p - 2q + 1.$$

Proof. Since G is cospectral with $Kite_{p,q}$, from Lemma 2.3, G has the same number of vertices, same number of edges and same spectrum with $Kite_{p,q}$. So, if G has n vertices and m edges, $n = p + q$ and $m = \binom{p}{2} + q = \frac{p^2-p+2q}{2}$. Also, $\rho(G) = \rho(Kite_{p,q})$. From Theorem 2.6, we say that if $\mu > \sqrt{2m(\frac{r-1}{r})}$ then G isn't $K_{r+1} - free$. It means that, G contains K_{r+1} as an induced subgraph. Now, we claim that for $r < p - 2q$, $\sqrt{2m(\frac{r-1}{r})} < \rho(G)$. By Theorem 2.5, we've already known that $p - 1 + \frac{1}{p^2} + \frac{1}{p^3} < \rho(G)$. Hence, we need to show that $\sqrt{2m(\frac{r-1}{r})} < p - 1 + \frac{1}{p^2} + \frac{1}{p^3}$, when $r < p - 2q$. Indeed,

$$\begin{aligned}(\sqrt{2m(\frac{r-1}{r})})^2-(p-1+\frac{1}{p^2}+\frac{1}{p^3})^2 &= (p^2-p+2q)(r-1)-r(p-1+\frac{1}{p^2}+\frac{1}{p^3})^2\\ &= (p^2-p+2q)(r-1)-\\ &\quad (\frac{r(p^2+p^3)}{p^5})(2(p-1)+\frac{(p^2+p^3)}{p^5})\\ &= (pr-p^2+p+(2q-1)r-2q)-\\ &\quad (\frac{r(p^2+p^3)}{p^5})(2(p-1)+\frac{(p^2+p^3)}{p^5})\end{aligned}$$

By the help of *Mathematica*, for $r<p-2q$ we can see

$$(pr-p^2+p+(2q-1)r-2q)-(\frac{r(p^2+p^3)}{p^5})(2(p-1)+\frac{(p^2+p^3)}{p^5})<0$$

i.e.,

$$(\sqrt{2m(\frac{r-1}{r})})^2-(p-1+\frac{1}{p^2}+\frac{1}{p^3})^2<0$$

i.e.,

$$(\sqrt{2m(\frac{r-1}{r})})^2<(p-1+\frac{1}{p^2}+\frac{1}{p^3})^2$$

Since $\sqrt{2m(\frac{r-1}{r})}>0$ and $p-1+\frac{1}{p^2}+\frac{1}{p^3}>0$, we get

$$\sqrt{2m(\frac{r-1}{r})}<p-1+\frac{1}{p^2}+\frac{1}{p^3}<\rho(G).$$

Thus, we proved our claim and so G contains K_{r+1} as an induced subgraph such that $r<p-2q$. Consequently, $w(G)\geq p-2q+1$. □

Theorem 4.2. *$Kite_{p,2}$ graphs are determined by their adjacency spectrum for all p.*

Proof. If $p=1$ or $p=2$, $Kite_{p,2}$ graphs are actually the path graphs P_3 or P_4. Also if $p=3$, then we obtain the lollipop graph $H_{5,3}$. As is known, these graphs are already DAS [8]. Hence we will continue our proof for $p\geq 4$. Adjacency characteristic polynomial of $Kite_{p,2}$ is as below,

$$P_{A(Kite_{p,2})}(\lambda)=(\lambda+1)^{p-2}[\lambda^4+(2-p)\lambda^3-(p+1)\lambda^2+(2p-4)\lambda+p-1]$$

By calculation, for the adjacency eigenvalues of $Kite_{p,2}$, we obtain the following facts;
$p-1<\lambda_1(A(Kite_{p,2}))<p$, $0<\lambda_2(A(Kite_{p,2}))<2$, $\lambda_3(A(Kite_{p,2}))<0$, $\lambda_4(A(Kite_{p,2}))=\ldots=\lambda_{p+1}(A(Kite_{p,2}))=-1$ and $\lambda_{p-1}(A(Kite_{p,2}))<-1$.

For a given graph G with n vertices and m edges, assume that G is cospectral with $Kite_{p,2}$. Then by Lemma 2.3, $n=p+2$, $m=\binom{p}{2}+2=\frac{p^2-p+4}{2}$ and $t(G)=t(Kite_{p,2})=\binom{p}{3}=\frac{p^3-3p^2+2p}{6}$. From

Lemma 4.1, $w(G) \geq p-2q+1$. When $q=2$, $w(G) \geq p-3 = n-5$. It's well-known that complete graph K_n is DS. So $w(G) \neq n$. If $w(G) = n-1 = p+1$, then G contains at least one clique with size $p-1$. It means that the edge number of G is greater than or equal to $\binom{p+1}{2}$. But it is a contradiction since $\binom{p+1}{2} > \binom{p}{2} + 2 = m$. Hence, $w(G) \neq n-1$. Because of these facts, we get $p-3 \leq w(G) \leq p$. From interlacing lemma, G can not contain the graphs in the following figure as an induced subgraph because $\lambda_3(G_1) = \lambda_3(G_2) = 0$.

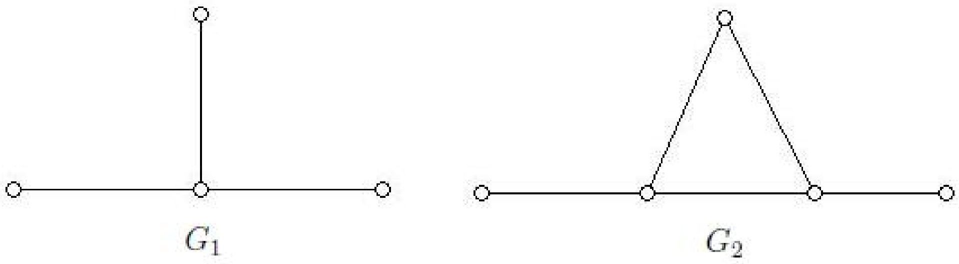

Figure 2. Graphs G_1 and G_2

If G is disconnected, from Lemma 2.8, components of G except one of them must be induced subgraphs of Smith graphs. Clearly, this is impossible because G_1 is forbidden and any path graph (since they have symmetric eigenvalues) can not be a component of G. Hence G must be a connected graph. If $w(G) = p$, then by Theorem 2.4., $G \cong Kite_{p,2}$. So we continue for $w(G) < p$. Since $w(G) \geq p-3$, G contains at least one clique with size at least $p-3$. This clique is denoted by $K_{w(G)}$. There may be at most five vertices out of the clique $K_{w(G)}$. Let us label these five vertices respectively with $1,2,3,4,5$ and call the set of these five vertices with A. So, we get $|A| \leq 5$. Moreover, $\forall i,j \in A$ we get $i \sim j$ since G_1, G_2 are not induced subgraphs of G and there is no isolated vertex in G. Then, we can say that $p \geq 6$ since $w(G) \geq p-3$.

For $i \in A$, x_i denotes the number of adjacent vertices of i in $K_{w(G)}$. By the fact that $p-1 \geq w(G) \geq p-3$, for all $i \in A$ we say

$$x_i \leq w(G) - |A| + 1 \tag{1}$$

Also, $x_{i\wedge j}$ denotes the number of common adjacent vertices in $K_{w(G)}$ of i and j such that $i,j \in A$ and $i<j$. Similarly, if $i \sim j$ then

$$x_{i\wedge j} \leq w(G) - |A| \tag{2}$$

Let d denotes the number of edges between the vertices of A and $K_{w(G)}$, also α denotes the number of cliques with size 3 which are not contained by A or $K_{w(G)}$. Then, we get

$$m = \binom{p}{2} + 2 = \binom{w(G)}{2} + \binom{|A|}{2} + d. \tag{3}$$

Similarly, we get

$$t(G) = \binom{p}{3} = \binom{w(G)}{3} + \binom{|A|}{3} + \alpha. \tag{4}$$

On the other hand for α and d, we have

$$d = \sum_{i=1}^{|A|} x_i \tag{5}$$

and

$$\alpha = \sum_{i=1}^{|A|} \binom{x_i}{2} + \sum_{i \sim j} x_{i \wedge j}. \tag{6}$$

If $w(G) = p - 3$ then $|A| = 5$ and so $p \geq 8$. Thus we have

$$d = 3p - 14 \tag{7}$$

and

$$\alpha = \binom{p}{3} - \binom{p-3}{3} - 10 = \frac{3p^2}{2} - \frac{15p}{2}. \tag{8}$$

From (1),(2),(5),(6) and (7) we have

$$\begin{aligned} \alpha = \sum_{i=1}^{5} \binom{x_i}{2} + \sum_{i \sim j} x_{i \wedge j} &\leq 3\binom{p-7}{2} + \binom{7}{2} + 2\sum_{i=1}^{5} x_i \\ &= 3\binom{p-7}{2} + \binom{7}{2} + 6p - 28 \\ &= \frac{3p^2 - 33p}{2} + 77. \end{aligned}$$

But obviously for $p = 8$ this result gives contradiction. Also for $p > 8$,

$$\frac{3p^2 - 33p}{2} + 77 < \frac{3p^2 - 15p}{2} = \alpha.$$

So this is again a contradiction.

If $w(G) = p - 2$ then $|A| = 4$ and so $p \geq 7$. Thus we have

$$d = 2p - 7$$

and

$$\alpha = \binom{p}{3} - \binom{p-2}{3} - 4 = p^2 - 4p.$$

On the other hand we have

$$\begin{aligned} \alpha = \sum_{i=1}^{4} \binom{x_i}{2} + \sum_{i \sim j} x_{i \wedge j} &\leq 2\binom{p-5}{2} + \binom{3}{2} + 2\sum_{i=1}^{4} x_i \\ &= p^2 - 7p + 19. \end{aligned}$$

Clearly for $p \geq 7$,

$$p^2 - 7p + 19 < p^2 - 4p = \alpha.$$

So this is a contradiction.

Similarly, if $w(G) = p - 1$ then $|A| = 3$ and so $p \geq 6$. Hence we have

$$d = p - 2$$

and

$$\alpha = \frac{p^2 - 3p}{2}.$$

Also we have

$$\alpha = \sum_{i=1}^{3} \binom{x_i}{2} + \sum_{i\sim j} x_{i\wedge j} \leq \binom{p-3}{2} + p - 2$$
$$= \frac{p^2-5p}{2} + 4.$$

Clearly for $p \geq 6$,

$$\frac{p^2-5p}{2} + 4 < \frac{p^2-3p}{2} = \alpha.$$

Again we obtain a contradiction.

By all of these facts, we can conclude that our assumption is actually false, then $w(G) \not< p$. Hence $w(G) = p$ and so that by Theorem 2.4., $G \cong Kite_{p,2}$. □

In the final of the paper, we give a conjecture below.

Conjecture 4.3. *For $q > 2$, $Kite_{p,q}$ graphs are DAS.*

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