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Commutative Schur rings over symmetric groups II: The case $n = 6$

Research Article

Amanda E. Francis, Stephen P. Humphries

Abstract: We determine the commutative Schur rings over S_6 that contain the sum of all the transpositions in S_6. There are eight such types (up to conjugacy), of which four have the set of all the transpositions as a principal set of the Schur ring.

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1. Introduction

Given a finite group G it is sometimes possible to characterize the Schur rings over G in some way. This has been accomplished for the family of cyclic groups [3, 6–9]. In [4] we characterized some commutative Schur rings over symmetric groups. In this paper we use those results to characterize certain commutative Schur rings over S_6.

For a finite group G and $X = \{x_1, \ldots, x_k\} \subseteq G, |X| = k$, we let $\overline{X} = x_1 + \cdots + x_k \in \mathbb{C}G$. We also let $X^{-1} = \{x^{-1} : x \in X\}$.

Let $\mathcal{C}_2$ denote the class of transpositions in the symmetric group S_n. As a consequence of the main result of [4] we have:

Corollary 1.1. *If $\mathfrak{S}$ is a commutative Schur ring over S_6 containing $\overline{\mathcal{C}_2}$, then $\mathfrak{S}$ determines (up to conjugacy) one of the following partitions of $\mathcal{C}_2$:*

(i) $\mathcal{C}_2 = \mathcal{C}_2$;

(ii) $\mathcal{C}_2 = C_1 \cup C_2 \cup C_3$ *where*

$$C_1 = \{(1,2)\}, \quad C_2 = \{(i,j) : 3 \leq i < j \leq 6\} \text{ and } C_3 = \mathcal{C}_2 \setminus (C_1 \cup C_2);$$

Amanda E. Francis (Corresponding Author), Stephen P. Humphries; Department of Mathematics, Brigham Young University, Provo, UT 84602, U.S.A. (email: amanda@mathematics.byu.edu, steve@mathematics.byu.edu).

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(iii) $\mathcal{C}_2 = C_1 \cup C_2$ *where*

$$C_1 = \{(1,2),(3,4),(5,6)\} \text{ and } C_2 = \mathcal{C}_2 \setminus C_1;$$

(iv) $\mathcal{C}_2 = C_1 \cup C_2$ *where*

$$C_1 = \{(1,2),(1,3),(2,3),(4,5),(4,6),(5,6)\} \text{ and } C_2 = \mathcal{C}_2 \setminus C_1;$$

(v) $\mathcal{C}_2 = C_1 \cup C_2$ *where*

$$C_1 = \{(i,j) : 1 \leq i < j \leq 5\} \text{ and } C_2 = \mathcal{C}_2 \setminus C_1.$$

If $H \leq G$, then the set of orbits of elements of G under the action of conjugation by elements of H gives a Schur-ring that we denote $\mathfrak{S}(G, H)$.

In [4] we found all commutative Schur rings over $S_n, n \leq 5$, that contain $\overline{\mathcal{C}_2}$. In this paper we do the same for S_6, this being our main result:

Theorem 1.2. *The only commutative Schur rings over* S_6 *containing* $\overline{\mathcal{C}_2}$ *are (up to conjugacy):*

(1) $Z(\mathbb{C}S_6)$;

(2) $\mathfrak{S}(S_6, H_{120})$, *where* $H_{120} = \langle (1,4)(3,5),(1,4,6,2,5,3)\rangle \cong S_5$ *is a* 3*-transitive subgroup;*

(3) $\mathfrak{S}(S_6, H)$, *where* $H = A_6$;

(4) $\mathfrak{S}_{36}$ *(to be constructed in* §4*).*

(5) $\mathfrak{S}(S_6, H)$, *where* $H = S_2 \times S_4 \leq S_6$;

(6) $\mathfrak{S}(S_6, H)$, *where* $H = S_3 \wr S_2 \leq S_6$;

(7) $\mathfrak{S}(S_6, H)$, *where* $H = S_2 \wr S_3 \leq S_6$;

(8) $\mathfrak{S}(S_6, H)$, *where* $H = S_5$.

Of these, (1)-(4) are those where $\mathcal{C}_2$ *is a principal set of the Schur ring.*

These Schur rings have dimensions $11, 19, 12, 12, 34, 26, 34, 19$, respectively

We have shown in [5] that the dimension of a commutative Schur-ring over G is bounded by $s_G := \sum_{i=1}^{r} d_i$, where the $d_i, i \leq r$, are the irreducible character degrees of G. In [5] we showed that some (specific) groups realize this bound, while others do not. For example, we showed that S_3, S_4, S_5 each have a commutative Schur-ring of this maximal dimension. We note that $s_{S_6} = 76$. Then Theorem 1.2 allows us to prove

Corollary 1.3. *There is no commutative Schur-ring over* S_6 *of dimension* $s_{S_6} = 76$.

Another consequence of our main result is that there are non-Schurian Schur rings over S_6 (see (4) above).

There is of necessity a certain amount of computation involved in the proof of Theorem 1.2, however we have tried to restrict such computations to: the Schur-ring algorithm that is described in §2; computations of Gröbner bases for certain ideals of polynomial algebras; a small number of small enumerations of possibilities.

All computer computations involved in the preparation of this paper were accomplished using Magma [1].

2. Schur-rings

We now define the concept of a Schur ring [2, 9–11]:

A *Schur-ring* (or *S-ring*) over a finite group G is a subring $\mathfrak{S}$ of $\mathbb{C}G$ that is constructed from a partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ of the elements of G: $G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, with $\Gamma_1 = \{id\}$, satisfying:

(1) if $1 \le i \le m$, then there is some $j \ge 1$ such that $\Gamma_i^{-1} = \Gamma_j$;

(2) if $1 \le i, j \le m$, then $\overline{\Gamma}_i \overline{\Gamma}_j = \sum_{k=1}^{m} \lambda_{ijk} \overline{\Gamma}_k$, where $\lambda_{ijk} \in \mathbb{Z}^{\ge 0}$ for all i, j, k.

The Γ_i are called the *principal sets* of the S-ring.

An S-ring naturally gives rise to a subalgebra of $\mathbb{C}G$ by extending coefficients. We will usually think of S-rings as $\mathbb{C}$-algebras in this way.

We refer to the survey [9] for an account of recent developments and applications of the theory of Schur rings.

We recall the following fact (called the *Schur-Wielandt principle*, see Proposition 22.3 of [11]):

Lemma 2.1. *Let $\mathfrak{S}$ be an S-ring over a group G. If $C \subseteq G$ satisfies $\overline{C} \in \mathfrak{S}$ and $\sum_{g \in G} \lambda_g g \in \mathfrak{S}$, then for all $\lambda \in \mathbb{R}$ the element $\sum_{g \in C} \delta_{\lambda_g, \lambda} g$ is in $\mathfrak{S}$; here δ is the Kronecker delta function i.e. $\delta_{x,y} = 0$ if $x \ne y$ and is 1 otherwise.* $\square$

S-ring Algorithm: Suppose that we have a subalgebra H of $\mathbb{C}G$ ($|G| < \infty$) and we wish to find the smallest S-ring that contains H. Suppose that we start with a (ring) generating set $c_1, \ldots, c_r$ for the subalgebra $H \subset \mathbb{C}G$. For each c_i partition the elements of G according to their coefficients in c_i. For each such subset C of this partition add $\overline{C}$ and $\overline{C^{-1}}$ to your set of generators; do this for each i and consider this new set of generators, that we will denote by $d_1, \ldots, d_t$. We simplify the set $d_1, \ldots, d_t$ by eliminating any $\mathbb{C}$-linear dependences. Now the d_i determine and are determined by subsets of G, so that $t \le |G|$. Now consider the products $d_i d_j, 1 \le i, j \le t$. Again we partition the terms of $d_i d_j$ according to their coefficients, and add in $\overline{C}, \overline{C^{-1}}$, for all sets C in the partition. Simplify this generating set using any linear dependences. This describes the basic step. If, after this basic step, one has a $\mathbb{C}$-basis for an S-ring, then we are done; otherwise we repeat the basic step. The process is guaranteed to terminate since $|G| < \infty$. It is easy to write a program to implement this algorithm (in, say, Magma [1]).

3. Covers of complete graphs

Let K_n be the complete graph on n vertices. Then for $\lambda \in \mathbb{N}$ and a graph P, a *cover of λK_n by P's* is a set $\mathfrak{T}$ of subgraphs $P_1, \ldots, P_m$ of K_n, each of which is isomorphic to P, and such that every edge of K_n occurs λ times in $P_1, \ldots, P_m$. We will need the following result in §4.

Lemma 3.1. *Any cover $\mathfrak{T}$ of λK_6 by r distinct triangles must have $5|r$. Further, the cases $r = 5, 15$ do not happen, and the case $r = 10$ is unique up to a permutation by an element of S_6:*

$$\{\{1,2,3\},\{1,2,4\},\{2,4,5\},\{1,3,5\},\{1,5,6\},\{3,4,5\},\{3,4,6\},\{2,5,6\},\{2,3,6\},\{1,4,6\}\}.$$

The situation $r = 20$ is where we have all the triangles in K_6.

Proof. Let $\mathfrak{T}$ be a set of r triangles giving the cover of λK_6. Then counting edges we see that $3r = 15\lambda$, so that $5|r$. Further, the number of triangles in K_6 is 20. If $r = 15$, then taking the complement of $\mathfrak{T}$ in the set of all triangles gives the case $r = 5$. Thus we consider the cases $r \in \{5, 10\}$, since if $r = 20$, then we have all the triangles.

Case A: $r = 5$. Here $\lambda = 1$. Then (by permuting if necessary) we can assume that $\{1,2,3\}, \{1,4,5\} \in \mathfrak{T}$. Considering the edge $\{2,4\}$ one is forced (since $\lambda = 1$) to have $\{2,4,6\} \in \mathfrak{T}$. But now one checks that the edge $\{2,5\}$ cannot be in any triangle. Thus this case does not arise.

Case B: $r = 10$. Here $\lambda = 2$. We consider two cases:

Case B1: there is some $K_4 \subset K_6$, all of whose triangles are in $\mathfrak{T}$. Here we can assume that the vertices of the K_4 are $1, 2, 3, 4$. Then one sees that the only possibilities for extra triangles in $\mathfrak{T}$ are

$\{i,5,6\}, i = 1,2,3,4$ (otherwise we have edges with multiplicities greater than 2). But this makes it impossible to cover all edges and have the multiplicity of the edge $\{5,6\}$ be 2.

Now assume that Case B1 does not happen.

Case B2: there is some $K_4 \subset K_6$, all but one of whose triangles are in $\mathfrak{T}$. So assume that $\{1,2,3\},\{1,2,4\},\{1,3,4\} \in \mathfrak{T}$. Then, considering the edge $\{2,4\}$ we see that, up to permuting $5,6$, this forces $\{2,4,5\} \in \mathfrak{T}$. Considering the edge $\{2,5\}$ forces either (a) $\{2,5,6\} \in \mathfrak{T}$ or (b) $\{2,3,5\} \in \mathfrak{T}$.

Assume that we have (a) $\{2,5,6\} \in \mathfrak{T}$. Considering the edge $\{1,6\}$ forces $\{2,3,6\} \in \mathfrak{T}$. Considering the edge $\{2,6\}$ forces $\{1,5,6\} \in \mathfrak{T}$, and that this is the only triangle that can be in $\mathfrak{T}$ that contains $\{1,6\}$. Thus this case cannot happen.

If we have (b) $\{2,3,5\} \in \mathfrak{T}$, then considering the edge $\{2,6\}$ gives a contradiction.

Now assume that Case B1 and Case B2 do not happen. Then we can assume that $\{1,2,3\}$, $\{1,2,4\} \in \mathfrak{T}$, $\{1,3,4\},\{2,3,4\} \notin \mathfrak{T}$. Considering the edge $\{2,4\}$ forces $\{2,4,5\} \in \mathfrak{T}$ (up to permuting $5,6$). Considering the edge $\{1,5\}$ (and using the fact that Case B2 does not occur) we must have $\{1,3,5\},\{1,5,6\} \in \mathfrak{T}$. Considering the edge $\{3,4\}$ (and using the fact that Case B2 does not occur) we must have $\{3,4,5\},\{3,4,6\} \in \mathfrak{T}$. Similarly, considering the edge $\{5,6\}$ we must have $\{2,5,6\} \in \mathfrak{T}$. Considering the edge $\{2,6\}$ we must have $\{2,3,6\} \in \mathfrak{T}$. This leaves $\{1,4,6\}$ as the remaining triangle. □

Let $\mathcal{C}_3$ denote the class of 3-cycles in S_n. In general for $\mu \vdash n$ we let $\mathcal{C}_\mu$ denote the class of elements of S_n of cycle type μ.

To each element $(i,j,k) \in \mathcal{C}_3$ there is associated the triangle $\{i,j,k\}$ in K_6. We let the following subset of $\mathcal{C}_3$ represent the set of triangles in the above Lemma:

$$\mathfrak{C}_3 = \{(1,2,3),(1,2,4),(2,4,5),(1,3,5),(1,5,6),(3,4,5),(3,4,6),(2,5,6),(2,3,6),(1,4,6)\}.$$

The set of triangles corresponding to the elements of $\mathfrak{C}_3$ can be represented as the set of triangles of the hemi-icosahedron $\mathfrak{H}$ (a polyhedral decomposition of the projective plane). See Figure 1, where we have drawn the ten triangles (2-simplices) of $\mathfrak{H}$, and the outside edges of these 2-simplices are identified in pairs as usual.

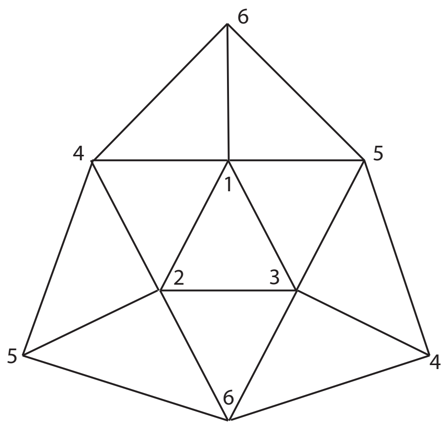

Figure 1. The hemi-icosahedron corresponding to $\mathfrak{C}_3$

The automorphism group of this 2-complex is $A_6 \leq S_6$, and it acts transitively on the ten triangles corresponding to the elements of $\mathfrak{C}_3$.

4. The case where $\mathcal{C}_2$ is a principal set

In this section we assume that $\mathcal{C}_2$ is a principal set of a commutative S-ring $\mathfrak{S}$. This is the most difficult case. Since $\overline{\mathcal{C}_2} \in \mathfrak{S}$ it follows from Lemma 4.1 of [5] that $Z(\mathbb{C}G)$ is a subring of $\mathfrak{S}$.

It follows from this and the Schur-Wielandt principle that if C is a principal set of $\mathfrak{S}$, then C is contained in some conjugacy class, namely the class of one of its elements.

The class $\mathcal{C}_3$.

We first consider the principal sets $C \subseteq \mathcal{C}_3$ of $\mathfrak{S}$, with the goal of showing $C = \mathcal{C}_3$.

For $\mu \vdash n$ and $\alpha = \sum_{g\in G} \lambda_g g$ we let

$$\alpha_\mu = \sum_{g\in\mathcal{C}_\mu} \lambda_g g.$$

Lemma 4.1. *If $C \subseteq \mathcal{C}_3 \subset S_n$ is a principal set of $\mathfrak{S}$ and $\left(\overline{C}\cdot\overline{\mathcal{C}_2}\right)_{(2,1^{n-2})} = \lambda\overline{\mathcal{C}_2}$, then C determines a cover of λK_n by triangles. Moreover, we have*

$$3\cdot|C| = \lambda\cdot|\mathcal{C}_2|.$$

Proof. We have

$$(i,j,k)\cdot((i,j)+(j,k)+(i,k)) = (i,j)+(j,k)+(i,k).$$

Thus each 3-cycle $(i,j,k) \in C$ contributes $(i,j)+(j,k)+(i,k)$ to the product $\overline{C}\cdot\mathcal{C}_2$.

Further, for each $\alpha = (i,j,k) \in \mathcal{C}_3$, there are only three $\beta \in \mathcal{C}_2$ (namely $\beta \in \{(i,j),(j,k),(k,i)\}$) with $\alpha\beta \in \mathcal{C}_2$. Since each $(i,j) \in \mathcal{C}_2$ occurs λ times in $\overline{C}\cdot\mathcal{C}_2$ we have a cover of λK_n by triangles, and the first part of the result follows. By counting edges we see that $3\cdot|C| = \lambda\cdot|\mathcal{C}_2|$. □

If $n = 6$, then Lemma 3.1 together with the fact that either $C = C^{-1}$ or $C \cap C^{-1} = \emptyset$, shows that (up to conjugacy) we have one of:

(I) $C = \mathcal{C}_3$;

(II) $|C| = 20$ where $\overline{C} = \sum_{\alpha\in\mathfrak{C}_3} \alpha + \alpha^{-1}$;

(III) $|C| = 10$ where $\overline{C} = \sum_{\alpha\in\mathfrak{C}_3} \alpha^{\varepsilon(\alpha)}$ with $\varepsilon(\alpha) \in \{1,-1\}$;

(IV) $\overline{C} = \sum_{1\le i<j<k\le 6} (i,j,k)^{\varepsilon(i,j,k)}$, for some $\varepsilon(i,j,k) \in \{1,-1\}$.

We want to show that (I) is the only possibility. If we have (II), then applying the S-ring algorithm to the algebra generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ gives an S-ring that is not commutative. Thus (II) does not happen.

Suppose that we have (III), so that $\overline{C} = \sum_{\alpha\in\mathfrak{C}_3} \alpha^{\varepsilon(\alpha)}$ with $\varepsilon(\alpha) \in \{1,-1\}$.

We will use Figure 1 to determine a *standard orientation* (counter clockwise) for each of the triangles of $\mathfrak{C}_3$, so that, for example the order $1,2,3$ determines a positive orientation for the triangle $\{1,2,3\}$ in $\mathfrak{C}_3$.

Since we have (III), every $(i,j,k) \in C$ has the form $\alpha^{\varepsilon(\alpha)}$, for some $\alpha \in \mathfrak{C}_3$. Each such $\alpha^{\varepsilon(\alpha)} = (i,j,k) \in C$ determines a triangle $\mathfrak{t}(\alpha^{\varepsilon(\alpha)}) = \{i,j,k\}$ of $\mathfrak{H}$. In fact $\alpha^{\varepsilon(\alpha)} = (i,j,k) \in C$ determines an *orientation* for $\mathfrak{t}(\alpha^{\varepsilon(\alpha)})$, this being $i \mapsto j \mapsto k \mapsto i$. We denote the set of oriented triangles of $\mathfrak{H}$ determined by C as $\mathfrak{H}(C)$. Further, each oriented triangle determines an orientation on each of its edges.

An edge of $\mathfrak{H}$ will be said to be *oppositely-oriented in* $\mathfrak{H}(C)$ if that edge has different orientations in the two triangles of $\mathfrak{H}(C)$ in which it is contained. Let $o(C)$ denote the number of oppositely-oriented edges in $\mathfrak{H}(C)$. We note that $o(C) < 15$, since $\mathfrak{H}$ has 15 edges.

Lemma 4.2. *(a) If $(i,j,k),(r,s,t) \in \mathcal{C}_3$, then $(i,j,k)(r,s,t) \in \mathcal{C}_3$ if and only if either*

(i) $(i,j,k) = (r,s,t)$; or

(ii) $\{i,j,k\} \cap \{r,s,t\} = \{u,v\}$ has size 2 and the edge u,v in (i,j,k) is oriented differently from that in (r,s,t).

(b) Two (oriented) triangles of $\mathfrak{H}(C)$ that share an edge will contribute two elements of $\mathcal{C}_3$ to $\overline{C}^2$ if and only if their common edge is oriented differently in the two triangles, if and only if the two triangles are oriented the same. In any other case they contribute no elements of $\mathcal{C}_3$.

(c) $\left(\overline{C}^2 - \overline{C^{-1}}\right)_{(3,1^3)}$ is a sum of $2o(C)$ elements of $\mathcal{C}_3$.

Proof. (a) We may assume that $i,j,k,r,s,t \le 6$, and one checks a small number of cases.

(b) Now applying Lemma 4.2 (a) to $\overline{C}^2|_{(3,1^3)}$ we see that there is the contribution that comes from $\sum_{\alpha \in C} \alpha^2$. We note that none of these elements α^2 of $\mathcal{C}_3$ are in C, since $C \cap C^{-1} = \emptyset$. Any other product comes from a pair $\alpha_1, \alpha_2 \in C$ that satisfy Lemma 4.2 (a) (ii). Thus α_1, α_2 determine an orientation of two of the triangles of $\mathfrak{H}$ that have opposite orientations on the common edge of these two triangles.

So considering $\mathfrak{H}(C)$ we see that two triangles of $\mathfrak{H}(C)$ that share an edge will contribute two elements of $\mathcal{C}_3$ to $\overline{C}^2$ if and only if their common edge is oriented differently in the two triangles, if and only if the two triangles are oriented the same.

(c) This follows from (b) and its proof. □

Let $\Omega(C)$ be the subgraph of the 1-skeleton $\mathfrak{H}(C)^{(1)}$ of $\mathfrak{H}(C)$ consisting of edges that are oppositely-oriented. Let f_+, f_- be the number of triangles of $\mathfrak{C}_3$ that are oriented positively or negatively (respectively) in $\mathfrak{H}(C)$. Then $f_+ + f_- = 10$ and we may assume that $f_+ \ge f_-$ (else replace C by C^{-1}).

Lemma 4.3. *(a) The graph $\Omega(C)$ contains all the vertices of $\mathfrak{H}(C)^{(1)}$.*

(b) $o(C) \in \{5, 10\}$.

(c) $f_+ \equiv f_- \equiv o(C) \mod 2$. In particular,

if $f_- = 0$, then $o(C) = 12$;

if $f_- = 1$, then $o(C) \in \{9, 11\}$;

if $f_- = 2$, then $o(C) \in \{6, 8, 10\}$;

if $f_- = 3$, then $o(C) \in \{5, 7, 9\}$;

if $f_- = 4$, then $o(C) \in \{6, 8, 10\}$;

if $f_- = 5$, then $o(C) \in \{3, 7\}$;

if $f_- = 6$, then $o(C) = 6$.

(d) If $o(C) = 5$, then in Figure 2 we show the two possibilities for C (up to a permutation of $1, \ldots, 6$) where we have shaded the positively (say) oriented triangles, and where the oppositely oriented edges are drawn dashed.

Proof. (a) This follows, since every edge of $\mathfrak{H}(C)^{(1)}$ has degree 5, so that for every vertex v of $\mathfrak{H}(C)$, at least two adjacent triangles that share the vertex v must have the same orientation in $\mathfrak{H}(C)$.

(b) We know that every principal set contained in $\mathcal{C}_3$ has size a multiple of 5. Lemma 4.2 (c) shows that $10 | o(C)$. But $o(C) < 15$ then gives the result.

(c) Changing the orientation of any triangle of $\mathfrak{H}$ changes the orientation of the three edges of that triangle. It follows that $f_+ \equiv f_- \equiv o(C) \mod 2$. The rest follows by looking at each particular case.

(d) We use (c) to check this. □

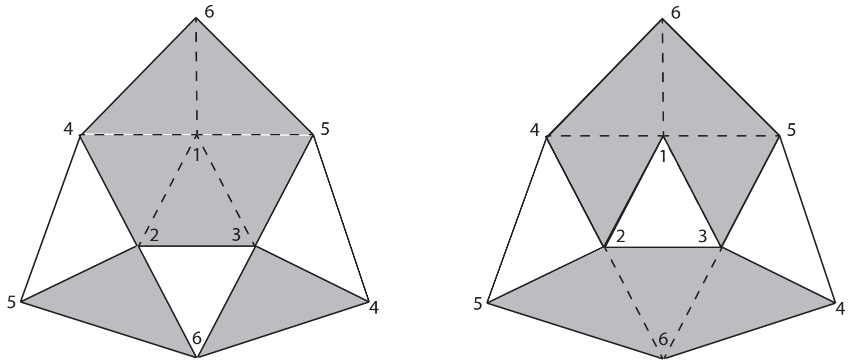

Figure 2. Orientations of $\mathfrak{H}(C)$ giving $o(C) = 5$.

Thus Case (III) has been reduced to showing that the two S-rings determined by Lemma 4.3 (d) do not give commutative S-rings. One checks this latter fact using the S-ring algorithm. This concludes consideration of (III).

Now assume that we have (IV), so that any subset of $\mathcal{C}_3$ that is a principal set has size 20. Thus $C^{-1} \neq C$ is also a principal set. Thus we have

$$\overline{C} = \sum_{1 \le i < j < k \le n} (i,j,k)^{\varepsilon(i,j,k)} \text{ where } \varepsilon(i,j,k) \in \{1,-1\}.$$

Lemma 4.4. *Let $a, b, c \in \mathcal{C}_3$. Then at most one of the eight products $a^{\pm 1}b^{\pm 1}, b^{\pm 1}a^{\pm 1}$ can be equal to c.*

Proof. (a) We may assume that $c = (1,2,3)$, and then one only has to check the cases where $a, b \in \mathcal{C}_3 \cap S_4$. □

For $c = (i,j,k) \in C$ we now wish to determine the coefficient of c in $\overline{C}^2$ as a polynomial function of the $\varepsilon(r,s,t)$.

First assume that $\varepsilon(i,j,k) = 1$. Suppose $a = (r,s,t)^{\delta_1}$, $b = (u,v,w)^{\delta_2}$, are distinct where $\delta_1 = \delta_{1,i,j,k,r,s,t,u,v,w}$, $\delta_2 = \delta_{2,i,j,k,r,s,t,u,v,w} \in \{1,-1\}$, $r < s < t, u < v < w$, and c is one of ab, ba. Then by Lemma 4.4 (a) the coefficient of c in $(a+b)^2$ is 1. Thus if $a, b \in C$, then they will contribute 1 to the coefficient of c in $\overline{C}^2$. So when $a = (r,s,t)^{\varepsilon(r,s,t)}, b = (u,v,w)^{\varepsilon(u,v,w)} \in C$ are distinct, then they will contribute 1 to the coefficient of c in $\overline{C}^2$ if and only if $\varepsilon(r,s,t) = \delta_1, \varepsilon(u,v,w) = \delta_2$.

Thus we have proved (i) of the following result:

Lemma 4.5. *(i) When $c = (i,j,k) \in C$, the coefficient of $c = (i,j,k)$ in $\overline{C}^2$ coming from the pair $(r,s,t)^{\varepsilon(r,s,t)}, (u,v,w)^{\varepsilon(u,v,w)}$ is*

$$\frac{(1 + \delta_1 \varepsilon(r,s,t))}{2} \cdot \frac{(1 + \delta_2 \varepsilon(u,v,w))}{2}.$$

(ii) When $c = (i,j,k) \notin C$, the coefficient of c in $\overline{C}^2$ coming from the pair $(r,s,t)^{\varepsilon(r,s,t)}, (u,v,w)^{\varepsilon(u,v,w}$ is

$$\frac{(1 - \delta_1 \varepsilon(r,s,t))}{2} \cdot \frac{(1 - \delta_2 \varepsilon(u,v,w))}{2}.$$

(iii) For any $c = (i,j,k)^{\varepsilon(i,j,k)}, 1 \le i < j < k \le 6$, the coefficient of c in $\overline{C}^2$ coming from the pair $a = (r,s,t)^{\varepsilon(r,s,t)}, b = (u,v,w)^{\varepsilon(u,v,w}$ is

$$\frac{(1+\delta_1\varepsilon(r,s,t)\varepsilon(i,j,k))}{2} \cdot \frac{(1+\delta_2\varepsilon(u,v,w)\varepsilon(i,j,k))}{2}.$$

Proof. The proof of (ii) is similar to what we did above for (i), and (iii) is a restatement of (i) and (ii). □

Now let P be a polynomial ring over $\mathbb{Q}$ with generators $X(i,j,k)$ where $1 \le i < j < k \le 6$ and define, for $c = (i,j,k)^{\varepsilon(i,j,k)}, a = (r,s,t)^{\varepsilon(r,s,t)}, b = (u,v,w)^{\varepsilon(u,v,w)}, a \neq b$, where c is one of ab, ba. We define the polynomial

$$S(a,b,c) = \frac{(1+\delta_1 X(r,s,t)X(i,j,k))}{2} \cdot \frac{(1+\delta_2 X(u,v,w)X(i,j,k))}{2}.$$

Here $\delta_1 = \delta_{1,i,j,k,r,s,t,u,v,w}, \delta_2 = \delta_{2,i,j,k,r,s,t,u,v,w} \in \{1,-1\}$, $r < s < t, u < v < w$, as in the above. Thus the $X(i,j,k)$ correspond to the $\varepsilon(i,j,k)$. We let $E_\varepsilon : P \to \mathbb{Q}$ be the evaluation map $X(i,j,k) \mapsto \varepsilon_{i,j,k}$.

For each $c = (i,j,k)^{\varepsilon(i,j,k)}, i < j < k$, we let $S(c)$ be the sum of all $S(a,b,c)$ where $a = (r,s,t)^{\varepsilon(r,s,t)}, b = (u,v,w)^{\varepsilon(u,v,w)} \in C$ are distinct and one of ab, ba is c. Thus $S(c)$ is a sum of nine of the $S(a,b,c)$s. It is clear that if $c = (i,j,k)^{\varepsilon(i,j,k)}, i < j < k$, and $c' = (i',j',k')^{\varepsilon(i',j',k')}, i' < j' < k'$, then $E_\varepsilon(S(c)) = E_\varepsilon(S(c'))$, since C is a principal set.

We now define the ideal I whose generators are all $S(c)-S(c')$, for c, c' as above, and all $X(i,j,k)^2-1$ for $1 \le i < j < k \le 6$. Since we can always conjugate C so that $\varepsilon(1,2,3) = 1 = \varepsilon(4,5,6)$ we also let $X(1,2,3)-1, X(4,5,6)-1$ be in I.

Now if there is such a subset C, then the ideal I will not be equal to P. However a Gröbner basis calculation ([1]) shows that $I = P$, and so (IV) is not possible. Thus we now have:

Proposition 4.6. *Any commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set, also contains $\mathcal{C}_3$ as a principal set.* □

The class $\mathcal{C}_{(2^2,1^2)}$.

We let $\mathcal{C}_{2,2}$ denote $\mathcal{C}_{(2^2,1^2)}$. Here we have:

Lemma 4.7. *If $C \subset \mathcal{C}_{2,2}$ satisfies $\overline{C} \in \mathfrak{S}$, where $\mathfrak{S}$ is a commutative S-ring over S_n containing $\mathcal{C}_2$ as a principal set, then C determines a cover of λK_n by pairs of disjoint edges.*

Proof. This follows from

$$\begin{aligned}[(i,j)(k,m)\cdot\overline{\mathcal{C}_2}]_{(2,1^{n-2})} &= [(i,j)(k,m)\cdot[(i,j)+(k,m)+\cdots]]_{(2,1^{n-2})}\\ &= (i,j)+(k,m),\end{aligned}$$

together with the fact that $\mathcal{C}_2$ is a principal set, so that $[\overline{C}\cdot\overline{\mathcal{C}_2}]_{(2,1^{n-2})} = \lambda\overline{\mathcal{C}_2}$. □

In the above situation we have: $2|C| = \lambda|\mathcal{C}_2|$.

For each of $n = 4, \lambda = 1; n = 5, \lambda = 1$, and $n = 5, \lambda = 2$, one can check that there is a unique such cover. For $n = 6$ we have $2|C| = 15\lambda$, so that λ must be even, and 15 divides $|C|$. We note that $|\mathcal{C}_{(2^2,1^2)}| = 45$, so that we can assume $|C| < 45$. If $|C| = 30$, then we replace C by $\mathcal{C}_{(2^2,1^2)} \setminus C$, so that we can assume $|C| = 15$. We now show

Lemma 4.8. *For $C \subset \mathcal{C}_{(2^2,1^2)}, |C| = 15$, as above, the only possibilities for $\overline{C}$ are those conjugate to the following element (or its complement in $\mathcal{C}_{(2^2,1^2)}$)*

$$\begin{aligned}&(1,2)(3,4) + (1,2)(5,6) + (3,4)(5,6)\\ &+ (1,3)(2,5) + (1,3)(4,6) + (2,5)(4,6)\\ &+ (1,5)(2,4) + (1,5)(3,6) + (2,4)(3,6)\\ &+ (1,6)(2,3) + (1,6)(4,5) + (2,3)(4,5)\\ &+ (1,4)(2,6) + (1,4)(3,5) + (2,6)(3,5).\end{aligned}$$

The stabilizer of this element (under conjugation by elements of S_6) is

$$H_{120} = \langle (1,4)(3,5), (1,4,6,2,5,3)\rangle,$$

so that $[S_6 : H_{120}] = 6$, and there are 6 conjugates of this element.

Further, the S-ring generated by $Z(\mathbb{C}G)$ and the above element is $\mathfrak{S}(S_6, H_{120})$.

Proof. The last two paragraphs are simple computations once we have the first part.

For the first part, let K_n^e denote the graph with vertices the edges $\{i,j\}$ of K_n, and where we have an edge of K_n^e whenever the corresponding vertices (edges of K_n) are disjoint. Thus each edge $\{i,j\}-\{k,m\}$ of K_n^e corresponds to an element $(i,j)(k,m) \in \mathcal{C}_{(2^2,1^{n-4})}$, so that we can refer to an edge of K_n^e by that element.

Now let $C \in \mathcal{C}_{(2^2,1^2)}$ determine a cover of $2K_6$ by subgraphs isomorphic to $K_2 \cup K_2$. Then every vertex of K_6^e is in exactly two edges. Thus C determines a set of disjoint cycles $\Gamma_1, \ldots, \Gamma_r$ in K_6^e. Each cycle $\Gamma_i, i \le r$, has length at least three. We will think of the Γ_i as subsets of C. We have:

Lemma 4.9. *Let $n = 6$. (i) If $\{i,j\} - \{k,m\} - \{r,s\}$ represent three consecutive vertices in a cycle Γ_u, then (up to a permutation) we have one of*

(a) $\{1,2\} - \{3,4\} - \{5,6\}$;

(b) $\{1,2\} - \{3,4\} - \{1,5\}$.

(ii) If $\{i,j\} - \{k,m\} - \{r,s\} - \{u,v\}$ represent four consecutive vertices in a cycle Γ_u, then (up to a permutation) we have one of

(a) $\{1,2\} - \{3,4\} - \{1,5\} - \{2,6\}$;

(b) $\{1,2\} - \{3,4\} - \{1,5\} - \{2,4\}$;

(c) $\{1,2\} - \{3,4\} - \{1,5\} - \{3,4\}$;

(d) $\{1,2\} - \{3,4\} - \{5,6\} - \{3,6\}$.

Further, in a path Γ_u of length at least 4 with distinct non-consecutive edges $\{i,j\} - \{k,m\}$ and $\{i',j'\} - \{k',m'\}$, the product $(i,j)(k,m) \cdot (i',j')(k',m')$ is not in $\mathcal{C}_{(2^2,1^2)}$.

(iii) If Γ_i, Γ_j are distinct cycles, then no term of $\overline{\Gamma_i} \cdot \overline{\Gamma_j}$ is in $\mathcal{C}_{(2^2,1^2)}$.

(iv) If Γ_i has length greater than 4, then the number of elements of the form $\alpha\beta, (\alpha, \beta \in \Gamma_i)$, that are in $\mathcal{C}_{(2^2,1^2)}$ is exactly $|\Gamma_i|$, and none of these elements are in C. If Γ_i has length 4, then the number of elements of the form $\alpha\beta, (\alpha, \beta \in \Gamma_i)$, that are in $\mathcal{C}_{(2^2,1^2)}$ is 2, and none of these elements are in C.

Proof. (i) In such a path $\{i,j\} - \{k,m\} - \{r,s\}$ we have $|\{r,s\} \cap \{i,j\}| = 0, 1$, which give the two cases.

(ii) The first part is easy to check using (i). If all the vertices of $\{i,j\} - \{k,m\}, \{i',j'\} - \{k',m'\}$ are distinct, then one sees that $(i,j)(k,m) \cdot (i',j')(k',m')$ is not in $\mathcal{C}_{(2^2,1^2)}$.

(iii) If Γ_i, Γ_j are distinct cycles with $\alpha = \alpha_1\alpha_2 \in \Gamma_1, \beta = \beta_1\beta_2 \in \Gamma_j, \alpha_k, \beta_k \in \mathcal{C}_2, k = 1,2$, then $\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\} = \emptyset$, since Γ_i, Γ_j do not share a vertex. Thus no element of $\overline{\Gamma_i} \cdot \overline{\Gamma_j}$ is in $\mathcal{C}_{(2^2,1^2)}$.

(iv) This now follows from (ii), which shows that only products of the form $(i,j)(r,s)\cdot(r,s)(u,v)$ will produce elements of $\mathcal{C}_{2,2}$. □

If we have a cycle Γ_i of size greater than three, then Lemma 4.9 (iv) shows that all cycles Γ_j have size greater than 3. The same result shows that no cycle has length four (since then we would have a principal set in $\mathcal{C}_{(2^2,1^2)}$ of size less than 15). Thus $(\overline{C}^2)_{(2^2,1^2)} = 0\overline{C} + \overline{D}$, where $|D| = 15$, $C \cap D = \emptyset$. We then similarly have $(\overline{D}^2)_{(2^2,1^2)} = 0\overline{D} + \overline{C}$. Thus $\mathcal{C}_{(2^2,1^2)} \setminus (C \cup D)$ has size 15, and all of its cycles must have size 3. Thus we may assume that all cycles Γ_i of C have size 3.

In this case it is easy to see that the components Γ_i of C are as shown in Figure 3 (up to conjugacy):

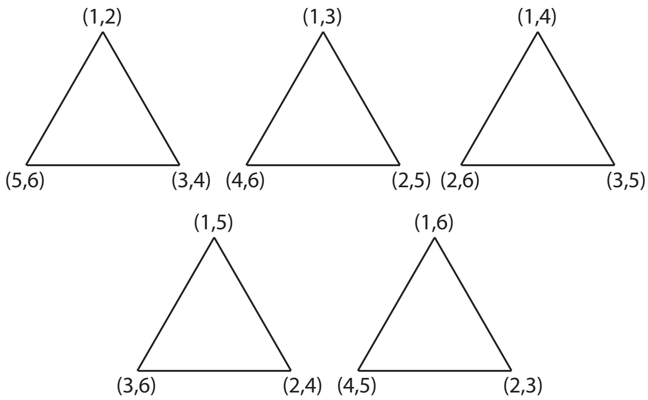

Figure 3.

One now finds that the S-ring determined by $\overline{C}$ and $Z(\mathbb{C}S_6)$ is $\mathfrak{S}(S_6, H_{120})$, and that the complement of C in $\mathcal{C}_{(2^2,1^2)}$ is also a principal set (of size 30). Thus we cannot have any principal sets C in $\mathfrak{S}$ with $|C| = 15$ and with cycles Γ_i of length greater than three.

This completes the proof of Lemma 4.8 and our discussion of the case $\lambda = 2$. □

For $n = 6$ the element $\overline{\mathcal{C}_{(2^2,1^2)}}$ gives a $\lambda = 6$ cover, thus if we have a cover C with $\lambda = 4$ then $\overline{\mathcal{C}_{(2^2,1^2)}} - \overline{C}$ has $\lambda = 2$, and the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is the same as the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{\mathcal{C}_{(2^2,1^2)}} - \overline{C}$. Thus we have:

Proposition 4.10. *Any commutative S-ring $\mathfrak{S}$ over S_6 that contains $\mathcal{C}_2$ as a principal set, also contains $\mathcal{C}_3$ as a principal set, and either $\mathcal{C}_{2,2}$ is a principal set or $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$.*

Lastly, any principal set of $\mathfrak{S}$ that is properly contained in $\mathcal{C}_{2,2}$ is an orbit of one of its elements under the action of a conjugate of H_{120}.

The $\mathcal{C}_{(2^3)}$ case

Lemma 4.11. *The only non-empty and proper subsets C of $\mathcal{C}_{(2^3)} \subset S_6$ that satisfy*

$$\left(\overline{C}\cdot\overline{\mathcal{C}_{2,2}}\right)_{(2,1^4)} = \lambda\overline{\mathcal{C}_2}$$

for some $\lambda \geq 1$ are conjugates of

(*i*) $C_1 = \{(1,2)(3,4)(5,6), (1,6)(2,3)(4,5), (1,5)(2,4)(3,6), (1,4)(2,6)(3,5), (1,3)(2,5)(4,6)\}$; *or*

(*ii*) $C_2 = \mathcal{C}_{(2^3)} \setminus C_1$.

Further, any conjugate of (i) or (ii) generates (together with $Z(\mathbb{C}S_6)$) an S-ring conjugate to $\mathfrak{S}(S_6, H_{120})$.

Proof Since $|\mathcal{C}_{(2^3)}| = 15$ we may assume that $|C| \leq 7$ and that $(1,2)(3,4)(5,6) \in C$. For $\alpha \in \mathcal{C}_{(2^3)}$ one sees that $[\alpha \cdot \overline{\mathcal{C}_{(2^2,1^2)}}]_{(2,1^4)}$ is a sum of three elements. Thus $3|C| = 15\lambda$ i.e. $|C| = 5\lambda$. Since $|C| \leq 7$ we see that $\lambda = 1$ and $|C| = 5$. It is now easy to see that up to a permutation we can assume that $(1,6)(2,3)(4,5) \in C$. Then there are two cases: (a) $(1,4)(2,6)(3,5) \in C$; and (b) $(1,4)(2,5)(3,6) \in C$. Either case quickly gives a unique C of size 5, these Cs being conjugate. One checks that C_1 and C_2 are principal sets of $\mathfrak{S}(S_6, H_{120})$. This gives the result. $\square$

This gives:

Proposition 4.12. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set. Then either*

(i) $\mathcal{C}_3, \mathcal{C}_{(2^3)}$ and $\mathcal{C}_{2,2}$ are principal sets of $\mathfrak{S}$, or

(ii) $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$.

Lastly, any principal set of $\mathfrak{S}$ that is properly contained in $\mathcal{C}_{(2^3)}$ is an orbit of one of its elements under the action of a conjugate of H_{120}. $\square$

The $\mathcal{C}_{(3,2,1)}$ case

Let $\mathfrak{S}$ be a commutative S-ring over $S_n, n \geq 5$, that contains $\mathcal{C}_2, \mathcal{C}_3$ as principal sets. For a principal set $C \subseteq \mathcal{C}_{(3,2,1^{n-5})} \subset S_n$ we define sets $D_{i,j} \subset \mathcal{C}_3 \subset S_n$ by

$$\overline{C} = \sum_{1 \leq i < j \leq n} (i,j)\overline{D_{i,j}}.$$

Since $[(i,j)(r,s,t) \cdot \overline{\mathcal{C}_3}]_{(2,1^{n-2})} = (i,j)$, and $\mathcal{C}_2$ is a principal set of $\mathfrak{S}$ we see that there is some $\lambda \in \mathbb{N}$ such that

$$\left(\overline{C} \cdot \overline{\mathcal{C}_3}\right)_{(2,1^{n-2})} = \lambda \overline{\mathcal{C}_2}.$$

Further, from the definition of the $D_{i,j}$ we see that $\lambda = |D_{i,j}|$ for all $1 \leq i < j \leq n$.

Since $[(i,j)(r,s,t) \cdot \overline{\mathcal{C}_2}]_{(3,1^{n-3})} = (r,s,t)$ and $\mathcal{C}_3$ is a principal set of $\mathfrak{S}$ we also have $\mu \in \mathbb{N}$ such that

$$\left(\overline{C} \cdot \overline{\mathcal{C}_2}\right)_{(3,1^{n-3})} = \mu \overline{\mathcal{C}_3}.$$

But we also have

$$\left(\overline{C} \cdot \overline{\mathcal{C}_2}\right)_{(3,1^{n-3})} = \sum_{1 \leq i < j \leq n} \overline{D_{i,j}}.$$

Thus from $\mu \overline{\mathcal{C}_3} = \sum_{1 \leq i < j \leq n} \overline{D_{i,j}}$ we see that for all $1 \leq i < j \leq n$ we have

$$\binom{n}{2} |D_{i,j}| = \binom{n}{2} \lambda = \mu |\mathcal{C}_3| = \mu \frac{n(n-1)(n-2)}{3}.$$

From this we obtain

Lemma 4.13. *$\lambda = |D_{i,j}|$ and*

$$3 \cdot |D_{i,j}| = 2\mu(n-2). \quad \square$$

Since any $(r,s,t) \in D_{i,j}$ must satisfy $i, j \notin \{r,s,t\}$ we see that

$$|D_{i,j}| \leq \frac{(n-2)(n-3)(n-4)}{3}.$$

When $n = 6$ this gives $|D_{i,j}| \le 8$. But if $n = 6$, then Lemma 4.13 shows that 3 divides μ so that

$$|D_{i,j}| = 2 \cdot \frac{\mu}{3} \cdot 4 \ge 8.$$

It follows from these two equations that $|D_{i,j}| = 8$ for all $1 \le i < j \le n = 6$, so that $D_{i,j}$ is maximal, i.e. $D_{i,j} = \{(r,s,t)^{\pm 1} : 1 \le r < s < t \le 6, i, j \notin \{r,s,t\}\}$, and $C = \mathcal{C}_{(3,2,1)}$. Thus we now have shown

Proposition 4.14. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set. Then $\mathcal{C}_{(3,2,1)}$ is a principal set of $\mathfrak{S}$, and either*

(i) $\mathcal{C}_3, \mathcal{C}_{(2^3)}, \mathcal{C}_{(3,2,1)}$ and $\mathcal{C}_{2,2}$ are principal sets of $\mathfrak{S}$, or

(ii) $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$. □

We note that in (ii) the only non-trivial conjugacy classes that do not split are $\mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_{(3,2,1)}$.

The $\mathcal{C}_{(4,1^2)}$ case

Let $C \subseteq \mathcal{C}_{(4,1^2)}$ be a principal set. Then the fact that

$$\left((i,j,k,m) \cdot \overline{\mathcal{C}_3}\right)_{(2,1^4)} = (i,j) + (j,k) + (k,m) + (m,i),$$

shows that C determines a cover of λK_n by 4-cycles. Thus, if $[\overline{C} \cdot \overline{\mathcal{C}_3}]_{(2,1^4)} = \lambda \overline{\mathcal{C}_2}$, then

$$4 \cdot |C| = 15\lambda.$$

Thus 15 divides $|C|$. Now there are twelve $\alpha \in \mathcal{C}_{(4,1^2)}$ such that $(1,2,3,4)\alpha$ is a 3-cycle. Thus considering $\left(\overline{C} \cdot \overline{\mathcal{C}_4}\right)_{(3,1^3)}$ we see that (since $|\mathcal{C}_3| = 40$) there is some $\mu \ge 1$ such that:

$$12 \cdot |C| = 40\mu,$$

(since $\mathcal{C}_3$ is a principal set) which shows that 10 divides $|C|$.

Thus $|C|$ is divisible by 30. Note that $|\mathcal{C}_{(4,1^2)}| = 90$. If $|C| = 60$, then we replace C by $\mathcal{C}_{(4,1^2)} \setminus C$, so as to have $|C| = 30$.

Now if $C^{-1} \ne C$, then $\mathcal{C}_{(4,1^2)} \setminus (C \cup C^{-1})$ is a principal set (since it has size 30) that is its own inverse. Thus we may assume that the principal set C satisfies $|C| = 30$ and $C = C^{-1}$.

Write $\overline{C} = \alpha_1 + \alpha_1^{-1} + \cdots + \alpha_{15} + \alpha_{15}^{-1}$, so that $(\overline{C}^2)_{(2^2,1^2)} = 2\alpha_1^2 + 2\alpha_2^2 + \cdots + 2\alpha_{15}^2$, which shows that $\{\alpha_1^2, \alpha_2^2, \ldots, \alpha_{15}^2\} \subset \mathcal{C}_{(2^2,1^2)}$ is a principal set of $\mathfrak{S}$ of size 15. Thus Lemma 4.8 shows that such a set is conjugate to the set of 15 elements given in that lemma. Call this set of 15 elements W_{15}.

We note that the map

$$\iota^2 : \mathcal{C}_{(4,1^2)} \to \mathcal{C}_{(2^2,1^2)}, \quad \alpha \mapsto \alpha^2,$$

is a 2 to 1 surjection, where $(\iota^2)^{-1}((i,k)(j,m)) = \{(i,j,k,m), (i,m,k,j)\}$, is an inverse pair. Since $C^{-1} = C$ we see that C is completely determined by $\iota^2(C)$, which is conjugate to W_{15}. Thus C must be a conjugate of

$$\begin{aligned}\{&(1,2,5,4),(1,4,5,2),(1,4,3,6),(2,6,5,4),(1,3,4,5),(1,4,6,5),(2,5,3,4),\\ &(2,3,4,6),(3,5,4,6),(1,4,2,3),(1,2,4,6),(1,5,4,3),(3,6,4,5),(2,3,6,5),\\ &(1,6,2,5),(2,4,5,6),(1,3,5,6),(2,4,3,5),(1,3,6,2),(1,2,3,5),(1,6,3,4),\\ &(2,5,6,3),(1,5,3,2),(1,5,6,4),(1,2,6,3),(1,3,2,4),(2,6,4,3),(1,6,4,2),\\ &(1,5,2,6),(1,6,5,3)\}.\end{aligned} \tag{4.1}$$

Now one finds that the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is $\mathfrak{S}(S_6, H_{120})$. Using the fact that any conjugate of $\overline{W_{15}}$ that is not equal to $\overline{W_{15}}$ does not commute with $\overline{W_{15}}$, this now easily gives:

Proposition 4.15. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set. Then either*

(i) $\mathcal{C}_3, \mathcal{C}_{(2^3)}, \mathcal{C}_{(3,2,1)}, \mathcal{C}_{(4,1^2)}$ and $\mathcal{C}_{2,2}$ are principal sets of $\mathfrak{S}$, or

(ii) $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$.

Lastly, any principal set of $\mathfrak{S}$ that is properly contained in $\mathcal{C}_{(4^2,1^2)}$ is an orbit of one of its elements under the action of a conjugate of H_{120}, and is either a conjugate of (4.1) or of its complement in $\mathcal{C}_{(4,1^2)}$. □

The $\mathcal{C}_{(4,2)}$ case

There is a natural bijection $\mathcal{C}_{4,2} \to \mathcal{C}_4, (i,j,k,m)(r,s) \mapsto (i,j,k,m)$. So that the image of the set of elements (4.1) of $\mathcal{C}_{(4,1^2)}$ is

$$\begin{aligned}\{&(1,2,5,4)(3,6),(1,4,5,2)(3,6),(1,4,3,6)(2,5),(2,6,5,4)(1,3),(1,3,4,5)(2,6),\\ &(1,4,6,5)(2,3),(2,5,3,4)(1,6),(2,3,4,6)(1,5),(3,5,4,6)(1,2),(1,4,2,3)(5,6),\\ &(1,2,4,6)(3,5),(1,5,4,3)(2,6),(3,6,4,5)(1,2),(2,3,6,5)(1,4),(1,6,2,5)(3,4),\\ &(2,4,5,6)(1,3),(1,3,5,6)(2,4),(2,4,3,5)(1,6),(1,3,6,2)(4,5),(1,2,3,5)(4,6),\\ &(1,6,3,4)(2,5),(2,5,6,3)(1,4),(1,5,3,2)(4,6),(1,5,6,4)(2,3),(1,2,6,3)(4,5),\\ &(1,3,2,4)(5,6),(2,6,4,3)(1,5),(1,6,4,2)(3,5),(1,5,2,6)(3,4),(1,6,5,3)(2,4)\}.\end{aligned} \tag{4.2}$$

This case then follows by following the proof of the $\mathcal{C}_4$ case just completed. This then gives:

Proposition 4.16. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set. Then either*

(i) $\mathcal{C}_3, \mathcal{C}_{(2^3)}, \mathcal{C}_{(3,2,1)}, \mathcal{C}_{(4,1^2)}, \mathcal{C}_{(4,2)}$ and $\mathcal{C}_{2,2}$ are principal sets of $\mathfrak{S}$, or

(ii) $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$.

Lastly, any principal set of $\mathfrak{S}$ that is properly contained in $\mathcal{C}_{(4^2,2)}$ is an orbit of one of its elements under the action of a conjugate of H_{120}, and is either a conjugate of (4.2) or of its complement in $\mathcal{C}_{(4,2)}$. □

The $\mathcal{C}_{(3^2)}$ case

Consider a principal set $C \subseteq \mathcal{C}_{(3^2)}$. Note that if $\alpha = ab \in \mathcal{C}_{(3^2)}$ where $a, b \in \mathcal{C}_3$, and if $c \in \mathcal{C}_3$ with $\alpha \cdot c \in \mathcal{C}_3$, then $c \in \{a^{-1}, b^{-1}\}$.

Let $\overline{C} = \sum_{i=1}^{r} \alpha_i, \alpha_i = a_i b_i \in \mathcal{C}_{(3^2)}, a_i, b_i \in \mathcal{C}_3$. Then

$$\left(\overline{C} \cdot \overline{\mathcal{C}_3}\right)_{(3,1^3)} = \sum_{i=1}^{r} (a_i + b_i).$$

Since $\mathcal{C}_3$ is a principal set we see that

$$\sum_{i=1}^{r} (a_i + b_i) = \lambda \overline{\mathcal{C}_3}.$$

Since $|\mathcal{C}_3| = 40$ this gives $20\lambda = r$. Now $r \leq |\mathcal{C}_{(3^2)}| = 40$, so that we have $\lambda \leq 2$. If $\lambda = 2$, then $r = 40$ and $C = \mathcal{C}_{(3^2)}$.

So assume that $\lambda = 1, r = |C| = 20$. We may also assume (by conjugating) that $(1,2,3)(4,5,6) \in C$. We have $[\overline{C} \cdot \overline{\mathcal{C}_3}]_{(3,1^3)} = 1\overline{\mathcal{C}_3}$, thus for each $\alpha \in \mathcal{C}_3$ there is a unique $\alpha^* \in \mathcal{C}_3$ such that $\alpha\alpha^* \in C$. Since $(1,2,3)(4,5,6) \in C$ we cannot have $(1,3,2)(4,5,6) \in C$, and so we must have $(1,3,2)(4,6,5) \in C$. Thus $C = C^{-1}$.

For $\alpha = ab \in \mathcal{C}_{(3^2)}, a, b \in \mathcal{C}_3$, we let $S(\alpha) = \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\}$. Then the $S(\alpha), \alpha \in \mathcal{C}_3$, partition $\mathcal{C}_{(3^2)}$ into ten subsets $S_1, \ldots, S_{10}$ of size 4. Then any set $C \subset \mathcal{C}_{(3^2)}$ as above is a union of an inverse pair $\{\alpha_i, \alpha_i^{-1}\} \subset S_i$ from each S_i. There are thus 2^{10} possible such C's, but only 2^9 if one assumes that (say) $(1,2,3)(4,5,6) \in C$

Of these 2^9 choices for C one finds that only twelve satisfy $\left(\overline{C}^2\right)_{(3,3)} = \lambda\overline{C} + \mu\overline{\mathcal{C}_{(3^2)} \setminus C}$ for some $\lambda, \mu \geq 0$. These twelve elements are in three conjugacy classes. Of the representatives from these three conjugacy classes one checks that the subalgebra of $\mathbb{C}S_6$ generated by such a $\overline{C}$ and $Z(\mathbb{C}S_6)$ is only an S-ring if (up to conjugacy) we have one of the following two cases.

$$\begin{aligned}\overline{C}_1 = {} & (1,3,5)(2,4,6) + (1,5,3)(2,6,4) + (1,6,4)(2,5,3) + (1,4,5)(2,3,6) \\ & + (1,2,6)(3,5,4) + (1,5,2)(3,4,6) + (1,3,4)(2,5,6) + (1,6,5)(2,4,3) \\ & + (1,6,3)(2,4,5) + (1,3,6)(2,5,4) + (1,3,2)(4,6,5) + (1,5,6)(2,3,4) \\ & + (1,4,6)(2,3,5) + (1,5,4)(2,6,3) + (1,2,5)(3,6,4) + (1,4,2)(3,6,5) \\ & + (1,2,3)(4,5,6) + (1,2,4)(3,5,6) + (1,4,3)(2,6,5) + (1,6,2)(3,4,5); \text{ or} \\ \overline{C}_2 = {} & (1,4,5)(2,3,6) + (1,2,6)(3,5,4) + (1,5,2)(3,4,6) + (1,6,4)(2,3,5) \\ & + (1,6,5)(2,3,4) + (1,6,3)(2,4,5) + (1,4,3)(2,5,6) + (1,3,6)(2,5,4) \\ & + (1,5,6)(2,4,3) + (1,3,2)(4,6,5) + (1,5,4)(2,6,3) + (1,5,3)(2,4,6) \\ & + (1,4,6)(2,5,3) + (1,2,5)(3,6,4) + (1,3,4)(2,6,5) + (1,4,2)(3,6,5) \\ & + (1,2,3)(4,5,6) + (1,2,4)(3,5,6) + (1,3,5)(2,6,4) + (1,6,2)(3,4,5).\end{aligned} \tag{4.3}$$

However in each of these two cases the S-ring that one obtains is just a conjugate of $\mathfrak{S}(S_6, H_{120})$ again.

Thus we now have:

Proposition 4.17. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set. Then either*

(i) $\mathcal{C}_3, \mathcal{C}_{(2^3)}, \mathcal{C}_{(3,2,1)}, \mathcal{C}_{(4,1^2)}, \mathcal{C}_{(4,2)}$, $\mathcal{C}_{2,2}$ and $\mathcal{C}_{(3^2)}$ are principal sets of $\mathfrak{S}$, or

(ii) $\mathfrak{S}$ contains a conjugate of $\mathfrak{S}(S_6, H_{120})$.

Lastly, any principal set of $\mathfrak{S}$ that is properly contained in $\mathcal{C}_{(3^2)}$ is an orbit of one of its elements under the action of a conjugate of H_{120}, and is either a conjugate of one of the elements shown in (4.3) or of its complement in $\mathcal{C}_{(3^2)}$. □

The $\mathcal{C}_{(5,1)}$ case

Consider a principal set $C \subseteq \mathcal{C}_5 = \mathcal{C}_{(5,1)}$. Now there are five elements $\alpha \in \mathcal{C}_2$ such that $(1,2,3,4,5)\alpha \in \mathcal{C}_{(3,2,1)}$; since $\mathcal{C}_{(3,2,1)}$ is a principal set we see that $5 \cdot |C| = \lambda \cdot |\mathcal{C}_{(3,2,1)}| = 120\lambda$. This gives $|C| = 24\lambda$.

Also, there are five $\alpha \in \mathcal{C}_3$ such that $(1,2,3,4,5)\alpha \in \mathcal{C}_{(3,1^3)}$. Since $\mathcal{C}_{(3,1^3)}$ is a principal set we see that $5 \cdot |C| = \mu \cdot |\mathcal{C}_{(3,1^3)}| = 40\mu$.

Case 1: $|C| = 24$. Here $\lambda = 1, \mu = 3$, and so we have (i) $[\overline{C} \cdot \overline{\mathcal{C}_2}]_{(3,2,1)} = \overline{\mathcal{C}_{(3,2,1)}}$ and (ii) $[\overline{C} \cdot \overline{\mathcal{C}_3}]_{(3,1^3)} = 3\overline{\mathcal{C}_{(3,1^3)}}$.

Let $\alpha_1, \ldots, \alpha_{72} \in \mathcal{C}_{(5,1)}$ represent the inverse pairs in $\mathcal{C}_{(5,1)}$, so that $\mathcal{C}_{(5,1)} = \cup_{i=1}^{72}\{\alpha_i, \alpha_i^{-1}\}$.

Assume first that $C^{-1} = C$. Then we can write $\overline{C} = \sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1})$. Here $x_i = 0, 1$ satisfy

(a) $x_i^2 - x_i = 0$ for $1 \leq i \leq 72$;

(b) $\sum_{i=1}^{72} x_i = 12$;

(c) $\left(\sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1}) \cdot \overline{\mathcal{C}_2}\right)_{(3,2,1)} = \overline{\mathcal{C}_{(3,2,1)}}$;

(d) $\left(\sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1}) \cdot \overline{\mathcal{C}_3}\right)_{(3,1^3)} = 3\overline{\mathcal{C}_3}$.

If we now think of the x_i as indeterminates in the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_{72}]$, then each of (a)–(d) gives relations satisfied by the x_i. Let I be the ideal of R generated by these relations. Finding a Gröbner basis for I ([1]) we see that all but five of the x_i are determined (in fact by a conjugacy we can assume that one of these five is equal to zero). This leaves a small number of possibilities for C, and one checks that they are all equal to a conjugate of the H_{120} orbit of some element of $\mathcal{C}_{(5,1)}$. Further each such element, together with the class sums, generates an S-ring that is just some conjugate of $\mathfrak{S}(S_6, H_{120})$. This does the case $|C| = 24, C = C^{-1}$.

So now assume $|C| = 24, C^{-1} \neq C$. Then we write $\overline{C} = \sum_{i=1}^{72} x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1}$. Here $x_i = 0, 1$ satisfy

(a) $x_i^2 - x_i = 0$ for $1 \leq i \leq 144$, and $x_{2i-1}x_{2i} = 0$ for $1 \leq i \leq 144$;

(b) $\sum_{i=1}^{144} x_i = 24$;

(c) $\left(\sum_{i=1}^{72}(x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1}) \cdot \overline{\mathcal{C}_2}\right)_{(3,2,1)} = \overline{\mathcal{C}_{(3,2,1)}}$;

(d) $\left(\sum_{i=1}^{72}(x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1}) \cdot \overline{\mathcal{C}_3}\right)_{(3,1^3)} = 3\overline{\mathcal{C}_3}$.

Finding the ideal of $R = \mathbb{Q}[x_1, \ldots, x_{144}]$ determined by these equations gives no solutions. The above shows that if there is a $C \subset \mathcal{C}_{(5,1)}, |C| = 24$, then the S-ring contains a conjugate of $\mathfrak{S}(S_6, H_{120})$, and a principal set of size 24 in $\mathcal{C}_{(5,1)}$ is an orbit under the same conjugate of H_{120}.

Case 2: $|C| = 48$. Here $\lambda = 2, \mu = 6$, and we aim to show that no such set exists. First note that if $C \neq C^{-1}$, then $D = \mathcal{C}_{(5,1)} \setminus (C \cup C^{-1})$ satisfies $D = D^{-1}, |D| = 48$, and $\overline{D}$ is in the S-ring. If D is a union of disjoint sets of size 24, then these sets are conjugates of the sets of size 24 determined in Case 1. One checks, however, that no two of these sets (there are six of them) commute, and so we see that D must also be a principal set. Thus we can assume that $C = C^{-1}$. Then we can write $\overline{C} = \sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1})$. Here, as in Case 1, the $x_i = 0, 1$ satisfy

(a) $x_i^2 - x_i = 0$ for $1 \leq i \leq 72$;

(b) $\sum_{i=1}^{72} x_i = 24$;

(c) $\left(\sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1}) \cdot \overline{\mathcal{C}_2}\right)_{(3,2,1)} = 2\overline{\mathcal{C}_{(3,2,1)}}$;

(d) $\left(\sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1}) \cdot \overline{\mathcal{C}_3}\right)_{(3,1^3)} = 6\overline{\mathcal{C}_3}$.

Let I be the ideal of $R = \mathbb{Q}[x_1, \ldots, x_{72}]$ generated by the polynomials determined by these relations. Finding a Gröbner basis for I ([1]) we see that all but twelve of the x_i are determined (again by a conjugacy we can assume that one of these twelve is equal to zero). Choosing values in $\{0, 1\}$ for these eleven variables, one checks the 2^{11} cases, and one finds that there are only 30 (non-trivial) elements C (any other case gives $I = R$), and for each of these 30 cases one finds those such that $\left(\overline{C}^2\right)_{(3,1^3)}$ has the form $m\overline{\mathcal{C}_3}, m \geq 0$. There are 10 of these, and one finds that they are all conjugate. Thus, if C is one of these, then we check that the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is commutative and has dimension 34, but does not contain $\mathcal{C}_2$ as a principal set. (Note: each of these ten conjugate Cs does give a commutative S-ring containing $\overline{\mathcal{C}_2}$, however $\mathcal{C}_2$ partitions as $\{(1, 2), (3, 4), (5, 6)\}$ and the complement; see Corollary 1.1 (iii).) Thus there are no principal sets of size 48 in $\mathcal{C}_{(5,1)}$ if $\mathcal{C}_2$ is a principal set.

Case 3: $|C| = 72$. Here $\lambda = 3, \mu = 9$.

Assume first that $C = C^{-1}$. Then we can write $\overline{C} = \sum_{i=1}^{72} x_i(\alpha_i + \alpha_i^{-1})$. Here, as in the above, $x_i = 0, 1$, satisfy

(a) $x_i^2 - x_i = 0$ for $1 \leq i \leq 72$;

(b) $\sum_{i=1}^{72} x_i = 36$;

(c) $\left(\sum_{i=1}^{72} x_i(\alpha_i+\alpha_i^{-1})\cdot\overline{\mathcal{C}_2}\right)_{(3,2,1)} = 3\overline{\mathcal{C}_{(3,2,1)}}$;

(d) $\left(\sum_{i=1}^{72} x_i(\alpha_i+\alpha_i^{-1})\cdot\overline{\mathcal{C}_3}\right)_{(3,1^3)} = 9\overline{\mathcal{C}_3}$;

(e) $\left(\sum_{i=1}^{72} x_i(\alpha_i+\alpha_i^{-1})\cdot\overline{\mathcal{C}_4}\right)_{(2,1^4)} = 24\overline{\mathcal{C}_2}$;

(f) Let $\left(\overline{C}^2\right)_{(3,1^3)} = \sum_{\alpha\in\mathcal{C}_3} x_\alpha\alpha$, where $x_\alpha \in R = \mathbb{Q}[x_1,\ldots,x_{72}]$. Then we have $x_\alpha = x_\beta$ for all $\alpha,\beta\in\mathcal{C}_3$, since $\mathcal{C}_3$ is a principal set.

Here (e) is obtained in the same way as (c) and (d). Also, (f) is a consequence of the fact that $\mathcal{C}_3$ is a principal set of $\mathfrak{S}$. Constructing the ideal I (including all $x_\alpha - x_\beta$ as in (f)) again one finds that there are 8 variables that determine the rest (7 if one uses conjugacy to set one of them to be zero). Choosing values in $\{0,1\}$ for these variables gives 2^7 cases to consider, of which all but eleven give $I = R$. These eleven Cs are in two conjugacy classes of sizes 1 and 10. The element C_1, representing the single conjugacy class, is the orbit of some element of $\mathcal{C}_{(5,1)}$ under the action of A_6. Let C_2 be one of the ten conjugate elements. Then there is a subgroup H_{36} of size 36 such that the orbits of H_{36} on $\mathcal{C}_{(5,1)}$ are O_1,O_2,O_3,O_4 where $|O_i| = 36, i = 1,2,3,4$. Further, these sets O_i can be numbered so that

$$C_1 = O_1\cup O_2, \quad C_2 = O_1\cup O_3.$$

A representative for H_{36} up to conjugacy is

$$H_{36} = \langle(1,6)(2,3,4,5),(1,2,5,4)(3,6)\rangle.$$

We note that if one uses $O_1\cup O_4$, then one obtains a commutative S-ring of dimension 26 that contains $\{(1,3),(3,5),(1,5),(2,6),(4,6),(2,4)\}$ and its complement in $\mathcal{C}_2$, as principal sets - see Corollary 1.1 (iii).

One finds that each of the S-rings $\langle Z(\mathbb{C}S_6),\overline{C_1}\rangle, \langle Z(\mathbb{C}S_6),\overline{C_2}\rangle$ has dimension 12 (recall that $Z(\mathbb{C}S_6)$ has dimension 11) where $C_i, \mathcal{C}_{(5,1)}\setminus C_i, i = 1,2$, are principal sets of these S-rings. This concludes the case where $|C| = 72, C = C^{-1}$. We will let C_{36} denote the element C_2 in what follows, and we will denote the S-ring $\langle Z(\mathbb{C}S_6),\overline{C_{36}}\rangle$ by $\mathfrak{S}_{36}$.

Now if $C \neq C^{-1}$, then we can write $\overline{C} = \sum_{i=1}^{72} x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1}$. Here $x_i = 0,1$ satisfy

(a) $x_i^2 - x_i = 0$ for $1\le i\le 144$ and $x_{2i-1}x_{2i} = 0$ for $1\le i\le 72$;

(b) $\sum_{i=1}^{72} x_i = 72$;

(c) $\left(\sum_{i=1}^{72}(x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1})\cdot\overline{\mathcal{C}_2}\right)_{(3,2,1)} = 3\overline{\mathcal{C}_{(3,2,1)}}$;

(d) $\left(\sum_{i=1}^{72}(x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1})\cdot\overline{\mathcal{C}_3}\right)_{(3,1^3)} = 9\overline{\mathcal{C}_3}$;

(e) $\left(\sum_{i=1}^{72}(x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1})\cdot\overline{\mathcal{C}_4}\right)_{(2,1^4)} = 24\overline{\mathcal{C}_2}$;

(f) Let $\left(\overline{C}^2\right)_{(3,1^3)} = \sum_{\alpha\in\mathcal{C}_3} x_\alpha\alpha$, where $x_\alpha \in R = \mathbb{Q}[x_1,\ldots,x_{72}]$. Then we have $x_\alpha = x_\beta$ for all $\alpha,\beta\in\mathcal{C}_3$, since $\mathcal{C}_3$ is a principal set.

One then finds that the ideal determined by (a)–(f) is the whole ring, so that there are no solutions in this situation.

Now if $C \subsetneq \mathcal{C}_{(5,1)}$ is a principal set of $\mathfrak{S}$ with $|C| > 72$, then there are certainly principal sets D of $\mathfrak{S}$ with $|D| < 72$. However in the above considerations of Cases 1,2 we have shown that the S-ring generated by D and $Z(\mathbb{C}S_6)$ does not have a principal set of size greater than 72. Thus the situation $|C| > 72$ does not occur, and we have now proved:

Proposition 4.18. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 containing $\overline{\mathcal{C}_2}$. Then any principal set $C\subset\mathcal{C}_{(5,1)}$ of $\mathfrak{S}$ is either a conjugate of C_{36}, or is the orbit of one of its elements under the action of A_6 or of a conjugate of H_{120}.*

The $\mathcal{C}_{(6)}$ case

Consider a principal set $C \subsetneq \mathcal{C}_{(6)}$. Now there are six elements $\alpha \in \mathcal{C}_{(4,1^2)}$ such that $(1,2,3,4,5,6)\alpha \in \mathcal{C}_{(3,1^3)}$; since $\mathcal{C}_{(3,1^3)}$ is a principal set we see that $6 \cdot |C| = \lambda_1 \cdot |\mathcal{C}_{(3,1^3)}| = 40\lambda_1$. This gives (i) $3|C| = 20\lambda_1$.

Similarly, by considering

(ii) $\left(\overline{C} \cdot \overline{\mathcal{C}_{(5,1)}}\right)_{(2,1^4)} = \lambda_2 \overline{\mathcal{C}_2}$, we see that $2|C| = 5\lambda_2$;

(iii) $\left(\overline{C} \cdot \overline{\mathcal{C}_{(3,2,1)}}\right)_{(3,1^3)} = \lambda_3 \overline{\mathcal{C}_3}$, we see that $3|C| = 10\lambda_3$;

(iv) $\left(\overline{C} \cdot \overline{\mathcal{C}_{2,2}}\right)_{(3,2,1)} = \lambda_4 \overline{\mathcal{C}_{(3,2,1)}}$, we see that $3|C| = 20\lambda_4$.

One sees that $|C|$ is divisible by 20. Also $|\mathcal{C}_{(6)}| = 120$. Let $\alpha_i, i = 1, \ldots, 60$ be representatives for the inverse pair sets in $\mathcal{C}_{(6)}$.

Case 1: $|C| = 20$. Here $\lambda_1 = 3, \lambda_2 = 8, \lambda_3 = 6, \lambda_4 = 3$.

First assume that $C = C^{-1}$. Then we can write $\overline{C} = \sum_{i=1}^{60} x_i(\alpha_i + \alpha_i^{-1})$. Here, as in the above, $x_i = 0, 1$ satisfy

(a) $x_i^2 - x_i = 0$ for $1 \leq i \leq 60$;

(b) $\sum_{i=1}^{60} x_i = 10$;

(c) relations for each of (i)–(iv) above;

(d) let $\left(\overline{C}^2\right)_{(3,1^3)} = \sum_{\alpha \in \mathcal{C}_3} x_\alpha \alpha$, where $x_\alpha \in R = \mathbb{Q}[x_1, \ldots, x_{10}]$. Then we have $x_\alpha = x_\beta$ for all $\alpha, \beta \in \mathcal{C}_3$, since $\mathcal{C}_3$ is a principal set.

(e) Let $\left(\overline{C}^2 \cdot \overline{\mathcal{C}_{2,2}}\right)_{(3,1^3)} = \sum_{\alpha \in \mathcal{C}_3} x_\alpha \alpha$, where $x_\alpha \in R = \mathbb{Q}[x_1, \ldots, x_{10}]$. Then we have $x_\alpha = x_\beta$ for all $\alpha, \beta \in \mathcal{C}_3$, since $\mathcal{C}_3$ is a principal set.

Constructing the ideal I (including all $x_\alpha - x_\beta$ as in (d),(e)) again one finds that there are 5 variables that determine the rest (4 if one uses conjugacy to set one of them to be zero). Looking at the 2^4 cases one finds that there are 5 possibilities for C. These are all in a single conjugacy class, so we need only consider one of them, C say. One finds that the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$. This does the case where $C = C^{-1}$.

Now if $C \neq C^{-1}$, then we can write $\overline{C} = \sum_{i=1}^{60} x_{2i-1}\alpha_i + x_{2i}\alpha_i^{-1}$. Using the same tests as in the $C = C^{-1}$ case above one creates an ideal that has nine variables that determine the rest. Enumerating the various possibilities for C gives ten non-conjugate elements. One checks that each of these generates $\mathbb{C}S_6$ as an S-ring. This shows that this case cannot occur. We further note

Lemma 4.19. *If $C_1, C_2 \subseteq \mathcal{C}_6$ are distinct principal sets of a commutative S-ring over S_6 that contains $\mathcal{C}_2$ as a principal set, then at most one of C_1, C_2 can have size* 20.

Proof. If $|C_1| = 20$, then the above shows that C_2 is a conjugate of C_1; but there are only six such conjugates and it is easy to check that no two such distinct commute. □

Case 2: $|C| = 40$. Here $\lambda_1 = 6, \lambda_2 = 16, \lambda_3 = 12, \lambda_4 = 6$.

First assume that $C = C^{-1}$. Then we can write $\overline{C} = \sum_{i=1}^{60} x_i(\alpha_i + \alpha_i^{-1})$. Applying the same ideal calculation as in the $|C| = 20$ case one obtains an ideal having 4 variables that determine the rest. Enumerating the various possibilities for C gives four elements, all of them conjugate to each other. Considering one of them, C say, one finds that the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$. This does the case where $C = C^{-1}$.

Repeating the above for the situation where $|C| = 40, C \neq C^{-1}$, one finds that there are no solutions.

Case 3: $|C| = 60$. Here $\lambda_1 = 9, \lambda_2 = 24, \lambda_3 = 18, \lambda_4 = 9$.

First assume that $C = C^{-1}$. Then we can write $\overline{C} = \sum_{i=1}^{60} x_i(\alpha_i + \alpha_i^{-1})$. Applying the same ideal calculation as in the $|C| = 20$ case one obtains an ideal having 5 variables that determine the rest. Enumerating the various possibilities for C gives six possibilities for C, and there are two conjugacy classes of such elements. If C say, represents either of these classes, one finds that the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$. This does the case where $C = C^{-1}$.

Repeating the above for the situation where $|C| = 60, C \neq C^{-1}$, one finds that there are no solutions.

The cases $|C| > 60$ are dealt with as in the situation $|C| > 72$ of Case 2.

This concludes consideration of all conjugacy classes of S_6 where $\mathfrak{S}$ has $\mathcal{C}_2$ as a principal set.

The following is mostly a summary of what we have done:

Proposition 4.20. *Let $\mathfrak{S}$ be a commutative S-ring over S_6 containing $\mathcal{C}_2$ as a principal set. Then*

(i) $\mathcal{C}_{(3,1^3)}$ and $\mathcal{C}_{(3,2,1)}$ are principal sets of $\mathfrak{S}$.

(ii) If $C \subsetneq \mathcal{C}_{2,2}$ is a principal set of $\mathfrak{S}$, then $|C| = 15$ or $|C| = 30$, and $\overline{C}$ is either a conjugate of the element shown in Lemma 4.8, or is $\overline{\mathcal{C}_{2,2}} - \overline{C}$ for such a set of size 15. There are six conjugates of each such set C. No two distinct such conjugates commute. The S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is $\mathfrak{S}(S_6, H_{120})$.

(iii) If $C \subsetneq \mathcal{C}_{(2^3)}$ is a principal set of $\mathfrak{S}$, then $|C| = 5$ or $|C| = 10$. If $|C| = 5$, then C is conjugate to the element shown in Lemma 4.11 (i). There are six conjugates of each such set C. No two distinct such conjugates commute. The S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$.

(iv) If $C \subsetneq \mathcal{C}_{(4,1^2)}$ is a principal set of $\mathfrak{S}$, then $|C| = 30$ or $|C| = 60$. If $|C| = 30$, then C is a conjugate of the element shown in (4.1), otherwise it is the complement in $\mathcal{C}_{(4,1^2)}$ of such an element. There are six conjugates of each such set C. No two distinct such conjugates commute. The S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$.

(v) If $C \subsetneq \mathcal{C}_{(4,2)}$ is a principal set of $\mathfrak{S}$, then $|C| = 30$ or $|C| = 60$. If $|C| = 30$, then C is a conjugate of the element shown in (4.2), otherwise it is the complement in $\mathcal{C}_{(4,2)}$ of such an element. There are six conjugates of each such set C. No two distinct such conjugates commute. The S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$.

(vi) If $C \subsetneq \mathcal{C}_{(3^2)}$ is a principal set of $\mathfrak{S}$, then $|C| = 20$ and $\overline{C}$ is one of the two elements $\overline{C_1}$, $\overline{C_2}$ shown in (4.3). Now $\overline{C_1}$ and $\overline{C_2}$ commute. Each of C_1, C_2 has six conjugates; no two distinct conjugates of each $\overline{C_i}, i = 1, 2$, commute, and each $\overline{C_1}^g$ only commutes with a single conjugate of $\overline{C_2}$. For $i = 1, 2$, the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C_i}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$.

(vii) If $C \subsetneq \mathcal{C}_{(5,1)}$ is a principal set of $\mathfrak{S}$, then either

(viia) $|C| = 24$ and C is an orbit of an element of $\mathcal{C}_{(5,1)}$ under the action of a conjugate of H_{120}; or

(viib) $|C| = 120$ and C is an orbit of an element of $\mathcal{C}_{(5,1)}$ under the action of a conjugate of H_{120}; or

(viic) $|C| = 72$ and C is an A_6-orbit of an element of $\mathcal{C}_{(5,1)}$.

(viid) $|C| = 72$ and C is a conjugate of C_{36}.

Further, no two distinct conjugates of C, for C of type (viia) commute; no two distinct conjugates of type (viib) commute; no conjugate of type (viia) commutes with a conjugate of type (viib) unless their sum is $\overline{\mathcal{C}_{(5,1)}}$. Any conjugate of type (viic) commutes with any conjugate of type (viia) or (viib).

Any element of type (viia) or (viib) generates (with $Z(\mathbb{C}S_6)$) an S-ring which is a conjugate of $\mathfrak{S}(S_6, H_{120})$. Any element of type (viic) generates (with $Z(\mathbb{C}S_6)$) an S-ring which is a conjugate of $\mathfrak{S}(S_6, A_6)$.

(viii) If $C \subsetneq \mathcal{C}_{(6)}$ is a principal set of $\mathfrak{S}$, then $|C| \in \{20, 40, 60\}$, and in each case the S-ring generated by $Z(\mathbb{C}S_6)$ and $\overline{C}$ is a conjugate of $\mathfrak{S}(S_6, H_{120})$. Let O_{20}, O_{40}, O_{60} denote the sets of conjugates (of the sums

$\overline{C}$) for each case, so that each O_i has size six. Then for $i \in \{20, 40, 60\}$ no two distinct elements of O_i commute, and for distinct $i, j \in \{20, 40, 60\}$ elements of O_i, O_j commute if and only if the corresponding sets are disjoint.

Further, the only principal elements of $\mathfrak{S}(S_6, H_{120})$ that $\overline{C_{36}}$ commutes with are $\overline{\mathcal{C}_2}, \overline{\mathcal{C}_3}$ and $\overline{\mathcal{C}_{(3,2,1)}}$.

Proof. Here (i) follows from Proposition 4.6 and Proposition 4.14. Part (ii) follows from Lemma 4.8, the fact that $[S_6 : H_{120}] = 6$, and a calculation to show that distinct conjugates don't commute. (iii) follows from Lemma 4.11, the fact that $[S_6 : H_{120}] = 6$, and a calculation to show that distinct conjugates don't commute.

Part (iv) follows from the proof of Proposition 4.15, and a calculation to show that distinct conjugates don't commute. Part (v) follows from the discussion of the $\mathcal{C}_{(4,2)}$ case, and a calculation to show that distinct conjugates don't commute. Part (vi) follows from the discussion of the $\mathcal{C}_{(3^2)}$ class that resulted in the two cases shown in (4.3) (up to conjugacy), and a calculation to show that distinct conjugates don't commute. Parts (vii) and (viii) follow from the discussion of the $\mathcal{C}_{(5,1)}$ and $\mathcal{C}_{(6)}$ classes, and a calculation to show that certain conjugates of these elements don't commute. ☐

We use this to prove:

Theorem 4.21. *The only commutative Schur rings over S_6 containing $\mathcal{C}_2$ as a principal set are:*

(1) $\mathfrak{S}(S_6, S_6)$*;*

(2) $\mathfrak{S}(S_6, H_{120})$*;*

(3) $\mathfrak{S}(S_6, A_6)$*;*

(4) $\mathfrak{S}_{36}$*.*

Proof. The last statement of Proposition 4.20 shows that such an S-ring cannot have split S_6 classes that are orbits under a conjugate of H_{120}, and also have C_{36} (or its complement) in it. One checks that the S-ring generated by any conjugate of $\mathfrak{S}_{36}$ and $\mathfrak{S}(S_6, A_6)$ is not commutative. Given the fact that, according to Proposition 4.20, not very many principal elements of conjugates of $\mathfrak{S}(S_6, H_{120})$ commute, it is now easy to prove the theorem. ☐

5. When $\mathcal{C}_2$ is not a principal set

We consider each case as enumerated in Corollary 1.1.

Case (ii): $\mathcal{C}_2 = C_1 \cup C_2 \cup C_3$, where $C_1 = \{(1, 2)\}; C_2 = \{(i, j) : 3 \leq i < j \leq 6\}$, and $C_2 = \mathcal{C}_2 \setminus (C_1 \cup C_2)$.

One finds that the S-ring generated by $Z(\mathbb{C}S_6)$, $\overline{C_1}, \overline{C_2}$ and $\overline{C_3}$ is $\mathfrak{S}(S_6, S_2 \times S_4)$, where $S_2 \times S_4$ is the subgroup $\langle (1, 2), (3, 4), (3, 4, 5, 6)\rangle \leq S_6$. There are 34 principal sets $O_1, \ldots, O_{34}$ of sizes $1, 3, 6, 8, 12, 16, 24, 48$. The idea is to show that no proper, non-empty subset of each O_i can be a principal set of a commutative S-ring containing $\mathfrak{S}(S_6, S_2 \times S_4)$.

One can check this directly (using [1]) for each O_i with $|O_i| \leq 8$. For the rest let $O_i = \{o_1, \ldots, o_m\}$. Let $R = \mathbb{Q}[x_1, \ldots, x_m]$ be a polynomial ring, and let $e = \sum_{j=1}^{m} x_j o_j$. One considers the ideal I of R generated by all the coefficients of the elements $e\overline{O_k} - \overline{O_k}e$, these being linear polynomials in the x_j. One finds that a Gröbner basis for I has only elements of the form $P_h - P_{u_k}$, where the P_{u_k} are free variables, and there are at most six of the P_{u_k}. One chooses a subset of the P_{u_k}, and puts these equal to 1, while one puts the rest equal to 0. This produces an ideal, that determines an element in $\mathbb{C}S_6$. One then sees whether this element (together with $Z(\mathbb{C}S_6)$) generates a commutative S-ring. One finds that $\mathfrak{S}(S_6, S_2 \times S_4)$ is the only possible case.

Case (iii): $\mathcal{C}_2 = C_1 \cup C_2$ where $C_1 = \{(1, 2), (3, 4), (5, 6)\}$ and $C_2 = \mathcal{C}_2 \setminus C_1$.

One finds that the S-ring generated by $Z(\mathbb{C}S_6)$, $\overline{C_1}$ and $\overline{C_2}$ is $\mathfrak{S}(S_6, S_2 \wr S_3)$, where $S_2 \wr S_3$ is the (wreath product) subgroup $\langle (1,2),(3,4),(5,6),(1,3)(2,4),(3,5)(4,6)\rangle \leq S_6$. There are 34 principal sets $O_1,\dots,O_{34}$ of sizes $1,3,6,8,12,16,24,48$.

We note that S_6 has an outer automorphism α such that $\alpha(\mathfrak{S}(S_6, S_2 \times S_4)) = \mathfrak{S}(S_6, S_2 \wr S_3)$. Thus this case follows from Case (ii).

Case (iv): $\mathcal{C}_2 = C_1 \cup C_2$ where $C_1 = \{(1,2),(1,3),(2,3),(4,5),(4,6),(5,6)\}$ and $C_2 = \mathcal{C}_2 \setminus C_1$.

One finds that the S-ring generated by $Z(\mathbb{C}S_6)$, $\overline{C_1}$ and $\overline{C_2}$ is $\mathfrak{S}(S_6, S_3 \wr S_2)$, where $S_3 \wr S_2$ is the (wreath product) subgroup $\langle (1,2),(2,3),(5,6),(1,4)(2,5)(3,6)\rangle \leq S_6$. There are 26 principal sets $O_1,\dots,O_{26}$ of sizes $1,4,6,9,12,18,36,72$. The idea is to again show that no proper, non-empty subset of each O_i can be a principal set of a commutative S-ring containing $\mathfrak{S}(S_6, S_3 \wr S_2)$.

Case (v): $\mathcal{C}_2 = C_1 \cup C_2$ where $C_1 = \{(1,6),(2,6),(3,6),(4,6),(5,6)\}$ and $C_2 = \mathcal{C}_2 \setminus C_1$.

One finds that the S-ring generated by $Z(\mathbb{C}S_6)$, $\overline{C_1}$ and $\overline{C_2}$ is $\mathfrak{S}(S_6, S_5)$. There are 19 principal sets $O_1,\dots,O_{19}$ of sizes $1,5,10,15,20,24,30,40,60,120$. We note that S_6 has an outer automorphism, and that $\mathfrak{S}(S_6, H_{120})$ and $\mathfrak{S}(S_6, S_5)$ are related by this automorphism. Thus this case follows from (iv).

This concludes the proof of Theorem 1.2. $\square$

Proof of Corollary 1.3 In [5] it is shown that any commutative S-ring $\mathfrak{S}$, of maximal dimension s_G, over a group G contains $Z(\mathbb{C}G)$. Thus the principal sets of $\mathfrak{S}$ give a partition of G that is a refinement of the partition of G by conjugacy classes. In particular, for $G = S_6$, we must have $\overline{\mathcal{C}_2} \in \mathfrak{S}$. Thus any such S-ring must be in the list given in Theorem 1.2. However none of the S-rings listed in Theorem 1.2 has dimension $76 = s_{S_6}$. This proves Corollary 1.3. $\square$

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