# Skew cyclic codes over $\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q}{ }^{*}$ 

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Abstract: In this paper, we study skew cyclic codes over the ring $R=\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q}$, where $u^{2}=u, v^{2}=$ $v, u v=v u, q=p^{m}$ and $p$ is an odd prime. We investigate the structural properties of skew cyclic codes over $R$ through a decomposition theorem. Furthermore, we give a formula for the number of skew cyclic codes of length $n$ over $R$.

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## 1. Introduction

Cyclic codes form an important subclass of linear block codes, studied from the fifties onward. Their clear algebraic structures as ideals of a quotient ring of a polynomial ring makes for an easy encoding. A landmark paper [11] has shown that some important binary nonlinear codes with excellent error-correcting capabilities can be identified as images of linear codes over $\mathbb{Z}_{4}$ under the Gray map.

Recently, in [3], D. Boucher et al. gave skew cyclic codes defined by using the skew polynomial ring with an automorphism $\theta$ over the finite field with $q$ elements. The definition generalizes the concept of cyclic codes over non-commutative polynomial rings. Soon afterwards, D. Boucher et al. studied skew constacyclic codes in [5]. Later, in [4], some important results on the duals of the skew cyclic codes over $\mathbb{F}_{q}[x ; \theta]$ are given. In [12], I. Siap et al. presented the structure of skew cyclic codes of arbitrary length. Further, S. Jitman et al. in [10] defined skew constacyclic codes over the skew polynomial ring with coefficients from finite rings. In [1], T. Abualrub and P. Seneviratne studied skew cyclic codes over ring

[^0]$\mathbb{F}_{2}+v \mathbb{F}_{2}$ with $v^{2}=v$. Moreover, J. Gao [6] and F. Gursoy et al. [8] presented skew cyclic codes over $\mathbb{F}_{p}+v \mathbb{F}_{p}$ and $\mathbb{F}_{q}+v \mathbb{F}_{q}$ with different automorphisms, respectively. In [7], J. Gao et al. also studied skew generalized quasi-cyclic codes over finite fields.

In this article, we mainly study skew cyclic codes over ring $R=\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q}$, where $u^{2}=u, v^{2}=v, u v=v u$ and $q=p^{m}$.

In our work, the automorphism $\theta$ on the $\operatorname{ring} R$ is defined to be

$$
\theta\left(b_{0}+b_{1} u+b_{2} v+b_{3} u v\right)=b_{0}^{p}+b_{2}^{p} u+b_{1}^{p} v+b_{3}^{p} u v,
$$

for all $b_{0}+b_{1} u+b_{2} v+b_{3} u v \in R$, where $b_{i} \in \mathbb{F}_{q}$, and $i=0,1,2,3$. In fact, for any $a_{1} \eta_{1}+a_{2} \eta_{2}+a_{3} \eta_{3}+a_{4} \eta_{4} \in$ $R$, we have

$$
\theta\left(a_{1} \eta_{1}+a_{2} \eta_{2}+a_{3} \eta_{3}+a_{4} \eta_{4}\right)=\theta\left(a_{1}\right) \eta_{1}+\theta\left(a_{2}\right) \eta_{2}+\theta\left(a_{4}\right) \eta_{3}+\theta\left(a_{3}\right) \eta_{4}
$$

Note that if $m$ is even, the order of the ring automorphism $|\langle\theta\rangle|$ is $m$, otherwise, $2 m$.
The material is organized as follows. In Section 2, we show the basics of codes over ring $R$ that we need for further reference. Section 3 derives the structure of linear codes over $R$. In Section 4, we introduce skew cyclic codes over ring $R$ and give the structural properties of skew cyclic codes over $R$ through a decomposition theorem. Section 5, we give a example to illustrate the discussed results.

## 2. Preliminary

Let $\mathbb{F}_{q}$ be a finite field with $q$ elements, where $q=p^{m}, p$ is an odd prime. Throughout, we let $R$ denote the commutative ring $\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q}$, where $u^{2}=u, v^{2}=v$, and $u v=v u$. Let $\eta_{1}=1-u-v+u v$, $\eta_{2}=u v, \eta_{3}=u-u v, \eta_{4}=v-u v$. It is easy to verify that $\eta_{i}^{2}=\eta_{i}, \eta_{i} \eta_{j}=0$, and $\sum_{k=1}^{4} \eta_{k}=1$, where $i, j=1,2,3,4$, and $i \neq j$. According to [2], we have $R=\eta_{1} R \oplus \eta_{2} R \oplus \eta_{3} R \oplus \eta_{4} R$. By calculating, we can easily obtain that $\eta_{i} R \cong \mathbb{F}_{q}, i=1,2,3,4$. Therefore, for any $r \in R, r$ can be expressed uniquely as $r=\sum_{i=1}^{4} \eta_{i} a_{i}$, where $a_{i} \in \mathbb{F}_{q}$ for $i=1,2,3,4$.

We recall the definition of the Gray map over $R$ in [13]

$$
\begin{aligned}
\Phi: R=\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q} & \rightarrow \mathbb{F}_{q}^{4} \\
\eta_{1} a+\eta_{2} b+\eta_{3} c+\eta_{4} d & \rightarrow(a, a+b, a+c, a+b+c+d) .
\end{aligned}
$$

Equivalently, if $r=a^{\prime}+b^{\prime} u+c^{\prime} v+d^{\prime} u v \in R$, then

$$
\Phi(r)=\left(a^{\prime}, 2 a^{\prime}+b^{\prime}+c^{\prime}+d^{\prime}, 2 a^{\prime}+b^{\prime}, 4 a^{\prime}+2 b^{\prime}+2 c^{\prime}+d^{\prime}\right) .
$$

This map can be naturally extended to the case over $R^{n}$.
For any element $r=a+b u+c v+d u v \in R$, we define the Lee weight of $r$ as $w_{L}(r)=w_{H}(a, a+b, a+$ $c, a+b+c+d)$, where $w_{H}$ denotes the ordinary Hamming weight for $q$-ary codes. The Lee distance of $r \in R$ can be similarly defined.

From the definition of the Gray map $\Phi$, we can easily check that $\Phi$ is $\mathbb{F}_{q}$-linear and it is also a distance-reserving isometry from $\left(R^{n}, d_{L}\right)$ to $\left(\mathbb{F}_{q}^{4 n}, d_{H}\right)$, where $d_{L}$ and $d_{H}$ denote the Lee and Hamming distance in $R^{n}$ and $\mathbb{F}_{q}^{4 n}$, respectively.

## 3. Linear codes over $R$

In this section, we mainly show some familiar structural properties of $R$. The proofs of the following theorems can be found in [13], so we omit them here.

If $A_{i}(i=1,2,3,4)$ are codes over $R$, we denote their direct sum by

$$
A_{1} \oplus A_{2} \oplus A_{3} \oplus A_{4}=\left\{a_{1}+a_{2}+a_{3}+a_{4} \mid a_{i} \in A_{i}, i=1,2,3,4\right\}
$$

Definition 3.1. Let $C$ be a linear code of length $n$ over $R$, we define that

$$
\begin{aligned}
& C_{1}=\left\{\boldsymbol{a} \in \mathbb{F}_{q}^{n}\left|\exists \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{F}_{q}^{n}\right| \eta_{1} \boldsymbol{a}+\eta_{2} \boldsymbol{b}+\eta_{3} \boldsymbol{c}+\eta_{4} \boldsymbol{d} \in C\right\}, \\
& C_{2}=\left\{\boldsymbol{b} \in \mathbb{F}_{q}^{n}\left|\exists \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{F}_{q}^{n}\right| \eta_{1} \boldsymbol{a}+\eta_{2} \boldsymbol{b}+\eta_{3} \boldsymbol{c}+\eta_{4} \boldsymbol{d} \in C\right\}, \\
& C_{3}=\left\{\boldsymbol{c} \in \mathbb{F}_{q}^{n}\left|\exists \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d} \in \mathbb{F}_{q}^{n}\right| \eta_{1} \boldsymbol{a}+\eta_{2} \boldsymbol{b}+\eta_{3} \boldsymbol{c}+\eta_{4} \boldsymbol{d} \in C\right\}, \\
& C_{4}=\left\{\boldsymbol{d} \in \mathbb{F}_{q}^{n}\left|\exists \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}_{q}^{n}\right| \eta_{1} \boldsymbol{a}+\eta_{2} \boldsymbol{b}+\eta_{3} \boldsymbol{c}+\eta_{4} \boldsymbol{d} \in C\right\} .
\end{aligned}
$$

It is clear that $C_{i}(i=1,2,3,4)$ are linear codes over $\mathbb{F}_{q}^{n}$. Furthermore, $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$, and $|C|=\left|C_{1}\right| \cdot\left|C_{2}\right| \cdot\left|C_{3}\right| \cdot\left|C_{4}\right|$. Throughout the paper $C_{i}(i=1,2,3,4)$ will be reserved symbols referring to these special subcodes.

According to Definition 3.1 and [13], we have the following theorem.
Theorem 3.2. Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a linear code of length $n$ over $R$. Then $C^{\perp}=$ $\eta_{1} C_{1}^{\perp} \oplus \eta_{2} C_{2}^{\perp} \oplus \eta_{3} C_{3}^{\perp} \oplus \eta_{4} C_{4}^{\perp}$.

According to the definition of the Gray map $\Phi$, we can easily obtain the following theorem.
Theorem 3.3. Let $C$ be a linear code of length $n$ over $R,|C|=q^{k}$ and $d_{L}(C)=d$. Then $\Phi(C)$ is a $q$-ary linear code with parameter $[4 n, k, d]$.

Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a linear code of length $n$ over $R$. Since $C$ is a $\mathbb{F}_{q}$-module, then we have the following lemma.

Lemma 3.4. If $G_{i}$ are generator matrices of $q$-ary linear codes $C_{i}(i=1,2,3,4)$, respectively, then the generator matrix of $C$ is

$$
G=\left(\begin{array}{l}
\eta_{1} G_{1} \\
\eta_{2} G_{2} \\
\eta_{3} G_{3} \\
\eta_{4} G_{4}
\end{array}\right)
$$

Moreover, if $G_{1}=G_{2}=G_{3}=G_{4}$, then $G=G_{1}$.
In the light of the definition of Gray map $\Phi$, we can easily obtain the following proposition.
Proposition 3.5. If $C$ is a linear code of length $n$ over $R$ with generator matrix $G$, then we have

$$
\Phi(G)=\left(\begin{array}{l}
\Phi\left(\eta_{1} G_{1}\right) \\
\Phi\left(\eta_{2} G_{2}\right) \\
\Phi\left(\eta_{3} G_{3}\right) \\
\Phi\left(\eta_{4} G_{4}\right)
\end{array}\right)=\left(\begin{array}{cccc}
G_{1} & G_{1} & G_{1} & G_{1} \\
\boldsymbol{O} & G_{2} & \boldsymbol{O} & G_{2} \\
\boldsymbol{O} & \boldsymbol{O} & G_{3} & G_{3} \\
\boldsymbol{0} & \boldsymbol{O} & \boldsymbol{O} & G_{4}
\end{array}\right)
$$

## 4. Skew cyclic codes over $R$

In this section, we assume $C_{3}$ and $C_{4}$ are equal. Before studying skew cyclic codes over $R$, we define a skew polynomial ring $R[X ; \theta]$ and skew cyclic codes over $R$. Next, we determine the structural properties of skew cyclic codes over $R$ through a decomposition theorem.

Definition 4.1. We define the skew polynomial ring as $R[x ; \theta]=\left\{a_{0}+a_{1} x+\cdots+a_{n} x^{n} \mid a_{i} \in R, i=\right.$ $0,1, \cdots, n\}$, where the coefficients are written on the left of the variable $x$. The multiplication is defined by the basic rule $\left(a x^{i}\right)\left(b x^{j}\right)=a \theta^{i}(b) x^{i+j}$, and the addition is defined to be the usual addition rule of polynomials.

It is easily checked that the ring $R[x ; \theta]$ is not commutative unless $\theta$ is the identity automorphism on $R$.

Definition 4.2. A nonempty subset $C$ of $R^{n}$ is called a skew cyclic code of length $n$ if $C$ satisfies the following conditions: (1) $C$ is a submodule of $R^{n}$; (2) if $r=\left(r_{0}, r_{1}, \cdots, r_{n-1}\right) \in C$, then skew cyclic shift $\rho(r)=\left(\theta\left(r_{n-1}\right), \theta\left(r_{0}\right), \cdots, \theta\left(r_{n-2}\right)\right) \in C$.

Theorem 4.3. Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a linear code of length $n$ over $R$, where $C_{i}(i=$ $1,2,3,4)$ are codes over $\mathbb{F}_{q}$ of length $n$. Then $C$ is a skew cyclic code with respect to the automorphism $\theta$ if and only if $C_{i}$ are skew cyclic codes over $\mathbb{F}_{q}$ with respect to the automorphism $\theta$.

Proof. For any $r=\left(r_{0}, r_{1}, \cdots, r_{n-1}\right) \in C$, let $r_{i}=\eta_{1} a_{i}+\eta_{2} b_{i}+\eta_{3} c_{i}+\eta_{4} d_{i}$ for $0 \leq i \leq n-1$, where $a=\left(a_{0}, a_{1}, \cdots, a_{n-1}\right) \in C_{1}, b=\left(b_{0}, b_{1}, \cdots, b_{n-1}\right) \in C_{2}, c=\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in C_{3}$ and $d=$ $\left(d_{0}, d_{1}, \cdots, d_{n-1}\right) \in C_{4}$. If $C_{i}$ are skew cyclic codes, then $\rho(r)=\rho\left(\eta_{1} a+\eta_{2} b+\eta_{3} c+\eta_{4} d\right)=\eta_{1} \rho(a)+$ $\eta_{2} \rho(b)+\eta_{3} \rho(d)+\eta_{4} \rho(c)=\eta_{1} \rho(a)+\eta_{2} \rho(b)+\eta_{3} \rho(c)+\eta_{4} \rho(d) \in C$. This implies that $C$ is a skew cyclic code over $R$.

On the other hand, if $C$ is a skew cyclic code over $R$, we have $\rho(r)=\left(\theta\left(r_{n-1}\right), \theta\left(r_{0}\right), \cdots, \theta\left(r_{n-2}\right)\right)=$ $\eta_{1} \rho(a)+\eta_{2} \rho(b)+\eta_{3} \rho(c)+\eta_{4} \rho(d) \in C$, which implies $\rho(a) \in C_{1}, \rho(b) \in C_{2}, \rho(c) \in C_{3}, \rho(d) \in C_{4}$. Thus $C_{i}$ are skew cyclic codes over $\mathbb{F}_{q}$.

According to ([4], Corollary 18), we know that the dual code of every skew cyclic code over $\mathbb{F}_{q}$ is also skew cyclic. By using this connection and Theorem 4.3, we get the following corollary.

Corollary 4.4. If $C$ is a skew cyclic code over $R$, then the dual code $C^{\perp}$ is also skew cyclic.
The following theorem determines the generator polynomials of a skew cyclic code of length $n$ over $R$.

Theorem 4.5. Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a skew cyclic code of length $n$ over $R$ and suppose that $g_{i}(x)$ are generator polynomials of $C_{i}(i=1,2,3,4)$ respectively. Then $C=$ $\left\langle\eta_{1} g_{1}(x), \eta_{2} g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle$ and $|C|=q^{4 n-\sum_{i=1}^{4} \operatorname{deg}\left(g_{i}(x)\right)}$.

Proof. Since $C_{i}=\left\langle g_{i}(x)\right\rangle$, for $i=1,2,3,4$, and $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$, then

$$
C=\left\{c(x)=\sum_{i=1}^{4} \eta_{i} r_{i}(x) g_{i}(x) \mid r_{i}(x) \in \mathbb{F}_{q}[x ; \theta]\right\} .
$$

Hence $C \subseteq\left\langle\eta_{1} g_{1}(x), \eta_{2} g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle$. Conversely, for any $\sum_{i=1}^{4} \eta_{i} k_{i}(x) g_{i}(x) \in\left\langle\eta_{1} g_{1}(x), \eta_{2}\right.$. $\left.g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle$, where $k_{i}(x) \in R[x ; \theta] /\left(x^{n}-1\right)$, then there exist $r_{i} \in \mathbb{F}_{q}[x ; \theta]$ such that $\eta_{i} k_{i}(x)=\eta_{i} r_{i}(x), i=1,2,3,4$. Thus $\left\langle\eta_{1} g_{1}(x), \eta_{2} g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle \subseteq C$, which implies $C=$ $\left\langle\eta_{1} g_{1}(x), \eta_{2} g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle$.

Since $|C|=\left|C_{1}\right| \cdot\left|C_{2}\right| \cdot\left|C_{3}\right| \cdot\left|C_{4}\right|$, we obtain that $|C|=q^{4 n-\sum_{i=1}^{4} \operatorname{deg}\left(g_{i}(x)\right)}$.
Theorem 4.6. Let $C_{i}(i=1,2,3,4)$ be skew cyclic codes over $\mathbb{F}_{q}$ and $g_{i}(x)$ be the monic generator polynomials of these codes respectively, then there is a unique polynomial $g(x) \in R[x ; \theta]$ such that $C=$ $\langle g(x)\rangle$ and $g(x)$ is a right divisor of $x^{n}-1$, where $g(x)=\sum_{i=1}^{4} \eta_{i} g_{i}(x)$.

Proof. By Theorem 4.5, we know $C=\left\langle\eta_{1} g_{1}(x), \eta_{2} g_{2}(x), \eta_{3} g_{3}(x), \eta_{4} g_{4}(x)\right\rangle$. We take $g(x)=\eta_{1} g_{1}(x)+$ $\eta_{2} g_{2}(x)+\eta_{3} g_{3}(x)+\eta_{4} g_{4}(x)$, obviously, we have $\langle g(x)\rangle \subseteq C$. On the other hand, one can check that $\eta_{i} g_{i}(x)=\eta_{i} g(x)(i=1,2,3,4)$, which implies $C \subseteq\langle g(x)\rangle$. Hence $C=\langle g(x)\rangle$. Since $g_{i}(x)$ are monic right divisors of $x^{n}-1 \in \mathbb{F}_{q}[x ; \theta]$, then there exist $r_{i}(x) \in \mathbb{F}_{q}[x ; \theta]$ such that $x^{n}-1=r_{i}(x) g_{i}(x)$. Thus

$$
\begin{aligned}
{\left[\eta_{1} r_{1}(x)+\eta_{2} r_{2}(x)+\eta_{3} r_{3}(x)+\eta_{4} r_{4}(x)\right] g(x) } & =\sum_{i=1}^{4} \eta_{i} r_{i}(x) \cdot \sum_{i=1}^{4} \eta_{i} g_{i}(x) \\
& =\sum_{i=1}^{4} \eta_{i} r_{i}(x) g_{i}(x) \\
& =\sum_{i=1}^{4} \eta_{i}\left(x^{n}-1\right) \\
& =x^{n}-1 .
\end{aligned}
$$

This implies $g(x)$ is a right divisor of $x^{n}-1$.
Corollary 4.7. Every left submodule of $R[x ; \theta] /\left(x^{n}-1\right)$ is principally generated.
Let $g(x)=g_{0}+g_{1} x+\cdots+g_{t} x^{t}$ and $h(x)=h_{0}+h_{1} x+\cdots+h_{n-t} x^{n-t}$ be polynomials in $\mathbb{F}_{q}[x ; \theta]$ such that $x^{n}-1=h(x) g(x)$ and $C$ be the skew cyclic code generated by $g(x)$ in $\mathbb{F}_{q}[x ; \theta] /\left(x^{n}-1\right)$, according to Corollary 18 in [4], then the dual code of $C$ is a skew cyclic code generated by $\widetilde{h}(x)=$ $h_{n-t}+\theta\left(h_{n-t-1}\right) x+\cdots+\theta^{n-t}\left(h_{0}\right) x^{n-t}$. Therefore we have the following corollary.

Corollary 4.8. Let $C_{i}$ be skew cyclic codes over $\mathbb{F}_{q}$ and $g_{i}(x)$ be their generator polynomial such that $x^{n}-1=h_{i}(x) g_{i}(x)$ in $\mathbb{F}_{q}[x ; \theta]$. If $C$ is a skew cyclic code over $R$, then $C^{\perp}=\left\langle\sum_{i=1}^{4} \eta_{i} \widetilde{h}_{i}(x)\right\rangle$ and $\left|C^{\perp}\right|=q^{\sum_{i=1}^{4} \operatorname{deg}\left(g_{i}(x)\right)}$.

Let $t$ be the order of $\theta$. The following theorem can be obtain by applying similar steps of the Theorem 3.7 in [6].

Theorem 4.9. Let $(n, t)=1$ and $C$ be a skew cyclic code of length $n$, then $C$ is a cyclic code of length $n$ over $R$.

In [8], the factorization of $x^{n}-1$ in $\mathbb{F}_{q}\left[x ; \theta_{i}\right]$ is unique if $\left(n, t_{i}\right)=1$. Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a skew cyclic code of length $n$ over $R$ and suppose that $g_{i}(x)$ are generator polynomials of $C_{i}(i=$ $1,2,3,4)$ respectively. Then each $g_{i}(x)$ is a right divisor of $x^{n}-1$ in $\mathbb{F}_{q}[x ; \theta] . \theta$ acts on $\mathbb{F}_{q}$ as follows, $\theta(a)=a^{p}$ for all $a \in \mathbb{F}_{q}$. Thus the order of $\theta$ on $\mathbb{F}_{q}$ is $m$. Hence if $(n, m)=1$ then the factorization of $x^{n}-1$ in $\mathbb{F}_{q}[x ; \theta]$ is unique. Now we can determine the number of distinct skew cyclic codes of length $n$ over $R$, where $(n, m)=1$.

Corollary 4.10. Let $(n, m)=1$ and $x^{n}-1=\prod_{i=1}^{r} p_{i}^{s_{i}}(x)$, where $p_{i}(x) \in \mathbb{F}_{q}\left[x ; \theta_{i}\right]$ is irreducible, then the number of distinct skew cyclic codes of length $n$ over $R$ is equal to the number of ideals in $R[x] /\left(x^{n}-1\right)$, i.e. $\prod_{i=1}^{r}\left(s_{i}+1\right)^{3}$.

## 5. Application example

In this section, we will exhibit a example of skew cyclic codes and their Gray images over $G F(9)$. Before giving a example, we first give the definition of Plotkin Sum.

Let $C \oplus_{P} D$ denote the Plotkin sum of two linear codes $C$ and $D$, also called $(u \mid u+v)$ construction, where $u \in C, v \in D$. For more information on the Plotkin sum, one can see a good survey [9].

In the following, we assume $G_{i}$ are generator matrices of 9-ary linear codes $C_{i}$ for $i=1,2,3,4$, respectively. Let $C=\eta_{1} C_{1} \oplus \eta_{2} C_{2} \oplus \eta_{3} C_{3} \oplus \eta_{4} C_{4}$ be a linear code of length $n$ over $R$, then its Gray image $\Phi(C)$ is none other than

$$
\left(C_{1} \oplus_{P} C_{2}\right) \oplus_{P}\left(C_{3} \oplus_{P} C_{4}\right) .
$$

We construct skew cyclic codes over $G F(9)$ with some conditions. If $C_{1}$ is a $[20,1,20]$ code, $C_{2}$ is a $[20,9,4]$ code, $C_{3}$ is a $[20,10,2]$ code and $C_{4}$ is a [20,10,2] code, then the Gray image of $C$ has parameters [80, 30, 4] over $G F(9)$.

## 6. Conclusion

This paper is devoted to studying skew cyclic codes over $R=\mathbb{F}_{q}+u \mathbb{F}_{q}+v \mathbb{F}_{q}+u v \mathbb{F}_{q}$, where $u^{2}=$ $u, v^{2}=v, u v=v u, q=p^{m}$ and $p$ is an odd prime. First, we introduce the structure of linear codes over $R$ and show the structural properties of skew cyclic codes over $R$. Next, we give the enumeration of distinct skew cyclic codes over $R$ when $n$ is odd.

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