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The rainbow vertex-index of complementary graphs*

Research Article

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Abstract: A vertex-colored graph G is *rainbow vertex-connected* if two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. If for every pair u, v of distinct vertices, G contains a vertex-rainbow $u-v$ geodesic, then G is *strongly rainbow vertex-connected*. The minimum k for which there exists a k-coloring of G that results in a strongly rainbow-vertex-connected graph is called the *strong rainbow vertex number* $srvc(G)$ of G. Thus $rvc(G) \leq srvc(G)$ for every nontrivial connected graph G. A tree T in G is called a *rainbow vertex tree* if the internal vertices of T receive different colors. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. For $S \subseteq V(G)$ and $|S| \geq 2$, an S-Steiner tree T is said to be a *rainbow vertex S-tree* if the internal vertices of T receive distinct colors. The minimum number of colors that are needed in a vertex-coloring of G such that there is a rainbow vertex S-tree for every k-set S of $V(G)$ is called the *k-rainbow vertex-index* of G, denoted by $rvx_k(G)$. In this paper, we first investigate the strong rainbow vertex-connection of complementary graphs. The k-rainbow vertex-index of complementary graphs are also studied.

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1. Introduction

The graphs considered in this paper are finite undirected and simple graphs. We follow the notation of Bondy and Murty [1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $n(G)$, $m(G)$, and $\overline{G}$, respectively, be the set of vertices, the set of edges, the order, the size, and the complement graph of G.

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Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-coloring graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected, whereas any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, in [4] L. Chen, X. Li, H. Lian defined the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. They showed that $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G. For more results on the rainbow connection, we refer to the survey paper [2],[3],[4] and [12], and a new book [10] of Li and Sun.

In [8], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. A vertex-colored graph G is *rainbow vertex-connected* if two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. For more results on the rainbow vertex-connection, we refer to the survey paper [5] and [9]. An easy observation is that if G is of order n, then $rvc(G) \leq n - 2$ and $rvc(G) = 0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2.

If for every pair u, v of distinct vertices, G contains a vertex-rainbow $u - v$ geodesic, then G is *strong rainbow vertex-connected*. The definition of strongly rainbow vertex-connected was defined by Li et al. in [11]. The minimum k for which there exists a k-coloring of G that results in a strongly rainbow vertex-connected graph is called *the strong rainbow vertex-connection number* $srvc(G)$ of G. Thus $rc(G) \leq srvc(G)$ for every nontrivial connected graph G.

If G is a nontrivial connected graph of order n whose diameter is $diam(G)$, then

$$diam(G) - 1 \leq rvc(G) \leq srvc(G) \leq n - s, \tag{1}$$

where s denote the number of pendent vertices in G.

Proposition 1.1. *Let G be a nontrivial connected graph of order n. Then*

(*a*) *$srvc(G) = 0$ if and only if G is a complete graph;*

(*b*) *$srvc(G) = 1$ if and only if $diam(G) = 2$ if and only if $rvc(G) = 1$.*

A tree T in G is called a *rainbow vertex tree* if the internal vertices of T receive different colors. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. For more problems on S-Steiner tree, we refer to [6] and [7].

For $S \subseteq V(G)$ and $|S| \geq 2$, an S-Steiner tree T is said to be a *rainbow vertex S-tree* if the internal vertices of T receive distinct colors. The minimum number of colors that are needed in an vertex-coloring of G such that there is a rainbow vertex S-tree for every k-set S of $V(G)$ is called the *k-rainbow vertex-index* of G, denoted by $rvx_k(G)$. The *vertex-rainbow index* of a graph was first defined by Yaping Mao in [13].

2. The strong rainbow vertex-connection of complementary graphs

In this section, we investigate the rainbow vertex-connection number of a graph G according to some constraints to its complement $\overline{G}$. We give some conditions to guarantee that $srvc(G)$ is bounded by a constant.

We investigate the rainbow vertex-connection number of connected complement graphs of graphs with diameter at least 3.

Theorem 2.1. *If G is a connected graph with $diam(G) \geq 3$, then*

$$srvc(\overline{G}) = \begin{cases} 1, & if \quad diam(G) \geq 4; \\ 2, & if \quad diam(G) = 3. \end{cases}$$

Proof. We choose a vertex x with $ecc_G(x) = diam(G) = d \geq 3$. Let $N_G^i(x) = \{v : d_G(x,v) = i\}$ where $0 \leq i \leq d$. So $N_G^0(x) = \{x\}, N_G^1(x) = N_G(x)$ as usual. Then $\bigcup_{0 \leq i \leq d} N_G^i(x)$ is a vertex partition of $V(G)$ with $|N_G^i(x)| = n_i$. Let $A = \bigcup_{i \ is \ even} N_G^i(x)$, $B = \bigcup_{i \ is \ odd} N_G^i(x)$. For example, see Figure 1, a graph with $diam(G) = 5$.

So, if $d = 2k (k \geq 2)$, then $A = \bigcup_{0 \leq i \leq d \ is \ even} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d-1 \ is \ odd} N_G^i(x)$; if $d = 2k+1 (k \geq 2)$ then $A = \bigcup_{0 \leq i \leq d-1 \ is \ even} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d \ is \ odd} N_G^i(x)$. Then by the definition of complement graphs, we know that $\overline{G}[A]$ ($\overline{G}[B]$) contains a spanning complete k_1-partite subgraph(complete k_2-partite subgraph) where $k_1 = \lceil \frac{d+1}{2} \rceil$ ($k_2 = \lceil \frac{d}{2} \rceil$). For example, see Figure 1, $\overline{G}[A]$ contains a spanning complete tripartite subgraph K_{n_0,n_2,n_4}, $\overline{G}[B]$ contains a spanning complete tripartite subgraph K_{n_1,n_3,n_5}.

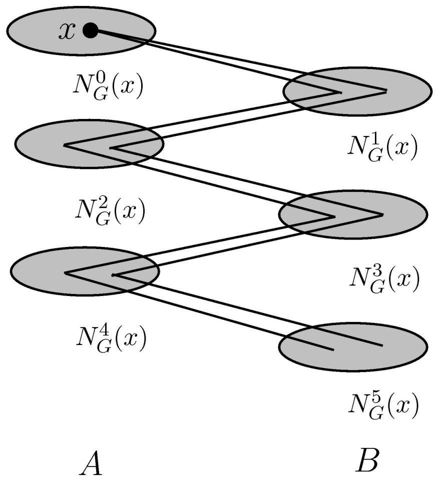

Figure 1. Graphs for the proof of Theorem 2.

First of all, we see that $\overline{G}$ must be connected, since otherwise, $diam(G) \leq 2$, contradicting the condition $diam(G) \geq 3$.

Case 1. $d \geq 5$.

In this case, $k_1, k_2 \geq 3$. We will show that $diam(\bar{G}) \leq 2$ in this case. For $u, v \in V(\overline{G})$, we consider the following cases:

Subcase 1.1. $u, v \in A$ or $u, v \in B$.

If $u, v \in A$, then u, v is contained in the spanning complete k_1-partite subgraph of $\overline{G}[A]$. Thus $d_{\overline{G}}(u,v) \leq 2$. The same is true for $u, v \in B$.

Subcase 1.2. $u \in A$ and $v \in B$.

If $u = x$, $v \in B$, then u is adjacent to all vertices in $\overline{G}[B] \setminus N_G^1(x)$. So $d_{\overline{G}}(u,v) = 1$ for $v \in \overline{G}[B] \setminus N_G^1(x)$. For $v \in N_G^1(x)$, let $P = ux_3v$, where $x_3 \in N_G^3(x)$. Clearly, $d_{\overline{G}}(u,v) = 2$.

If $u \neq x$, without loss of generality, we assume that $u \in N_G^2(x)$ and $v \in N_G^1(x)$. Let $Q = ux_5v$, where $x_5 \in N_G^5(x)$. Thus $d_{\overline{G}}(u,v) = 2$.

From the above, we conclude that $diam(\overline{G}) \leq 2$. So, by Proposition 1($b$), we have $srvc(\overline{G}) = 1$.

Case 2. $d = 4$.

It is obvious that $A = N_G^0(x) \cup N_G^2(x) \cup N_G^4(x)$, $B = N_G^1(x) \cup N_G^3(x)$. So $\overline{G}[A](\overline{G}[B])$ contains a spanning complete 3-partite subgraph K_{n_0,n_2,n_4} (complete bipartite subgraph K_{n_1,n_3}). So, we will show that $diam(G) \leq 2$.

Subcase 2.1. $u, v \in A$ or $u, v \in B$.

If $u, v \in A$, then u, v is contained in the spanning complete k_1-partite subgraph of $\overline{G}[A]$. Thus $d_{\overline{G}}(u, v) \leq 2$. If $u, v \in B$, then u, v is contained in the spanning complete bipartite subgraph of $\overline{G}[B]$. Also we have $d_{\overline{G}}(u, v) \leq 2$.

Subcase 2.2. $u \in A$ and $v \in B$.

If $u = x, v \in B$, then u is adjacent to all vertices in $N_G^3(x)$. For $v \in N_G^1(x)$, let $P = ux_3v$, where $x_3 \in N_G^3(x)$. Clearly, $d_{\overline{G}}(u, v) = 2$. So $d_{\overline{G}}(u, v) \leq 2$.

If $u \neq x$, then we assume that $u \in N_G^2(x)$ and $v \in N_G^1(x)$. Let $Q = ux_4v$, where $x_4 \in N_G^4(x)$. Thus $d_{\overline{G}}(u, v) = 2$. Suppose $u \in N_G^4(x)$ and $v \in N_G^3(x)$. Let $R = ux_1v$, where $x_1 \in N_G^1(x)$. Thus $d_{\overline{G}}(u, v) = 2$. If $u \in N_G^2(x)$ and $v \in N_G^3(x)$, then $S = uxv$ is a path of length 2. Then $diam(G) \leq 2$. So, by Proposition 1, we have $srvc(\overline{G}) = 1$.

Case 3. $d = 3$.

In this case, $A = N_G^0(x) \cup N_G^2(x)$, $B = N_G^1(x) \cup N_G^3(x)$. So $\overline{G}[A]$ contains a spanning complete bipartite subgraph K_{n_0,n_2}. So, we give $\overline{G}$ a vertex-coloring as follows: color vertex x with 1 and color all vertices of $N_G^3(x)$ with 2. It is easy to see that for any $u \in N_G^2(x)$, $v \in N_G^1(x)$, there is a rainbow $\{1, 2\}$ path connecting them in $\overline{G}$. So $srvc(\overline{G}) = 2$ in this case.

For the case of $diam(G) = 2$, $srvc(\overline{G})$ can be very large since $diam(\overline{G})$ may be very large. For example, let $G = K_n \setminus E(C_n)$, where C_n is a cycle of length n in K_n. Then $\overline{G} = C_n$ and $srvc(\overline{G}) \geq diam(\overline{G}) - 1 = \lceil \frac{n}{2} \rceil - 1$ by (1). $\square$

3. The k-rainbow vertex-index of complete multipartite graphs

Theorem 3.1. *Let $K_{n_1,n_2,\cdots n_l}$ be a complete multipartite graph. If $k < 2\ell$, then $rvx_k = 1$; If $k \geq 2\ell$, then $rvx_k = 2$. Where $S = \{v_1, v_2, \cdots v_k\}$ (that is the rainbow S-tree we choose) and $V_{n_i}, (1 \leq i \leq l)$ are the vertices of the partition of $K_{n_1,n_2,\cdots n_\ell}$.*

Proof. If $k < 2\ell$, then we can find a partition $V_{n_i}, (1 \leq i \leq l)$ of $K_{n_1,n_2,\cdots n_l}$ with $V_{n_i} \cap S \leq 1$. If $V_{n_i} \cap S = \emptyset$, then we can choose a vertex $v \in V_{n_i}$ as the root vertex of the rainbow S tree and all the other vertices are leaves. So $rvx_k(K_{n_1,n_2,\cdots n_\ell}) = 1$. If $V_{n_i} \cap S = 1$, then we choose the vertex $v \in V_{n_i}$ as the root vertex of the rainbow S tree, and all the other vertices are leaves. So $rvx_k(K_{n_1,n_2,\cdots n_\ell}) = 1$.

If $k \geq 2\ell$ and there exists V_{n_i} such that $|S \cap V_{n_i}| \leq 1$, then we can choose the vertex v in V_{n_i} as the root of the rainbow tree and all the other vertices are the leaves the same as when $k < 2\ell$. So $rvx_k(K_{n_1,n_2,\cdots n_l}) = 1$.

Suppose $k \geq 2\ell$ and $|S \cap V_{n_i}| \geq 2$ for any V_{n_i}. Now we give a rainbow vertex-coloring as follows.

$$c(V_{n_i}) = \begin{cases} 1, & if \quad 1 \leq i \leq \ell - 1; \\ 2, & if \quad i = \ell. \end{cases}$$

Next we prove it is a k-rainbow vertex-coloring. Choose one vertex v in V_{n_ℓ} as the root vertex of the rainbow tree. Obviously v is adjacent to all the vertices in $V_{n_1} \cap S, V_{n_2} \cap S, \ldots V_{n_{\ell-1}} \cap S$. Then choose a vertex in $v' \in V_{n_1}$. Since v' is adjacent to all the remaining vertices in $V_{n_\ell} \cap S$, one can prove that the tree is rainbow S-tree. $\square$

4. The k-rainbow vertex-index of complementary graphs

Theorem 4.1. *If G is a connected graph with $diam(G) \geq 3$, then $rvx_k(\bar{G}) \leq 2$ and the bound is tight.*

Proof. We choose a vertex x with $ecc_G(x) = diam(G) = d \geq 3$ as Figure 1. Then $\overline{G}[A](\overline{G}[B])$ contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph). If the rainbow S-tree contains in $\overline{G}[A](\overline{G}[B])$, then $rvx_k(\bar{G}) \leq 2$ by Theorem 3.1. Now we consider the rainbow S-tree does not contain in $\overline{G}[A]$ or $\overline{G}[B]$. If $S \cap N_G^1(G) = \emptyset$, then we choose x as the root vertex, and all the other vertices are the leaves. So one can prove that there is a rainbow S-tree. Suppose $S \cap N_G^1(G) \neq \emptyset$. Now we give a rainbow vertex-coloring as follows.

$$\begin{cases} c(x) = 1, \\ c(v) = 2, \;\; v \in V(G) \backslash x. \end{cases}$$

We choose the vertex x as the root of the rainbow tree. We know x is adjacent to all the vertices in $N_G^j(x) \cap S, (j \in \{2, 3, 4, \cdots\})$, and there must be a $v \in N_G^j(x), (j \in \{2m+1 \; and \; m \geq 1\})$ such that v is adjacent to $N_G^1(x) \cap S$. one can prove that the tree is rainbow S-tree.

Let G is a connected graph of $diam(G) = 3$. We have $rvx_k(\bar{G}) = 2$, so the bound is tight. □

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