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A database of linear codes over $\mathbb{F}_{13}$ with minimum distance bounds and new quasi-twisted codes from a heuristic search algorithm

Research Article

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Abstract: Error control codes have been widely used in data communications and storage systems. One central problem in coding theory is to optimize the parameters of a linear code and construct codes with best possible parameters. There are tables of best-known linear codes over finite fields of sizes up to 9. Recently, there has been a growing interest in codes over $\mathbb{F}_{13}$ and other fields of size greater than 9. The main purpose of this work is to present a database of best-known linear codes over the field $\mathbb{F}_{13}$ together with upper bounds on the minimum distances. To find good linear codes to establish lower bounds on minimum distances, an iterative heuristic computer search algorithm is employed to construct quasi-twisted (QT) codes over the field $\mathbb{F}_{13}$ with high minimum distances. A large number of new linear codes have been found, improving previously best-known results. Tables of $[pm, m]$ QT codes over $\mathbb{F}_{13}$ with best-known minimum distances as well as a table of lower and upper bounds on the minimum distances for linear codes of length up to 150 and dimension up to 6 are presented.

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1. Introduction and motivation

Let $[n, k, d]_q$ denote a linear code of length n, dimension k and minimum distance (weight) d over the finite field $\mathbb{F}_q$. A central and fundamental problem in coding theory is to find the optimal values of the parameters of a linear code and construct codes with these parameters. The problem can be formulated in a few different ways. For example, we may wish to maximize the minimum distance d for the given block length n and dimension k; or minimize the block length n for the given dimension k and minimum

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distance. Let $d_q(n,k)$ denote the largest value of d for which there exists an $[n,k,d]$ code over $\mathbb{F}_q$, and $n_q(k,d)$ the smallest value of n for which there exists an $[n,k,d]$ code over $\mathbb{F}_q$. An $[n,k,d]$ code is called optimal (or length-optimal) if its block length n equals $n_q(k,d)$, or if its minimum distance d equals $d_q(n,k)$ (also called distance-optimal).

This optimization problem is very difficult. In general, it is only solved for the cases where either k or $n-k$ is small. Computers are often used in searching for codes with best parameters but there is an inherent difficulty: computing the minimum distance of a linear code is computationally intractable [19]. Since it is not possible to conduct exhaustive searches for linear codes if the dimension is large, researchers often focus on promising subclasses of linear codes with rich mathematical structures. As a generalization to cyclic and consta-cyclic codes, quasi-cyclic (QC) and quasi-twisted (QT) codes are known to have this characteristic. They have been shown to contain many good linear codes. With the help of modern computers, many record-breaking QC and QT codes have been constructed [2]-[13]. However, the problem still becomes intractable as the dimension and the block length of the code get large. The records of best-known linear codes are available. For example, the online database [21] is one that is commonly referred to. It contains records of best-known codes over $\mathbb{F}_q$ for $q \leq 9$ together with upper bounds on $d_q(n,k)$. The Magma software [20] also contains a similar database. The online database of QT codes contains best-known QC and QT codes [24]. These databases are updated as new codes are discovered.

There has been a growing interest in codes over $\mathbb{F}_{13}$ in recent years. Several papers in the literature deal with self-dual or maximum distance separable (MDS) codes over $\mathbb{F}_{13}$. For example, Betsumiya [25] et al studied MDS self-dual codes over $\mathbb{F}_{13}$ of lengths up to 24 and determined largest minimum weights of such codes for lengths up to 20. De Boer [16] constructed a self-dual [18, 9, 9] code and optimal codes with parameters [23, 3, 20] and [23, 17, 6] over $\mathbb{F}_{13}$. Newhart [26] studied the extended quadratic residue (QR) codes [18, 9, 9], [24, 12, 10] and [30, 15, 12] over $\mathbb{F}_{13}$. Grassl and Gulliver [28] showed non-existence of a self-dual MDS code over $\mathbb{F}_{13}$ with parameters [12,6,7]. In [29] the authors constructed a Euclidean self-dual near-MDS code over $\mathbb{F}_{13}$. Kotsireas et al. constructed many MDS and near-MDS self-dual codes over $\mathbb{F}_{13}$ [27].

Another reason for the interest in codes over $\mathbb{F}_{13}$ is the connection between linear codes and finite geometries. Codes of dimension 3 are closely related to arcs in a projective geometry, and a lot of research has been carried out on projective codes of dimension 3 over finite fields of size up to 19 [4].

Finally, Venkaiah and Gulliver [13] used the tabu search to construct quasi-cyclic codes over $\mathbb{F}_{13}$ of dimensions up to 6 and lengths less than 150. They constructed many QC codes of the form $[pk,k]$, for over $\mathbb{F}_{13}$, and presented their results in several tables (one for each value of k). These tables constitute the most comprehensive set of best-known linear codes over $\mathbb{F}_{13}$ to date.

In this paper, we present a database of linear codes over $\mathbb{F}_{13}$ for lengths $\leq$ 150 and dimensions $3 \leq k \leq 6$. We employed an iterative, heuristic algorithm [15] to conduct a computer search to produce new codes. With this algorithm, a large number of new QC and QT codes have been constructed many of which improve the previous results. We achieve improvements on the parameters of the codes presented in [13] in many cases. Combining the results presented in [13] with the new codes we have found, we create a comprehensive database of best-known linear codes over $\mathbb{F}_{13}$. To the best of our knowledge, this is the first time such a database appears in the literature.

The remainder of the paper is organized as follows. In Section 2, some basic definitions and facts on QT codes are presented. In Section 3, the iterative heuristic algorithm that is used to find good QT codes is described. Next, a database of linear codes over $\mathbb{F}_{13}$ with minimum distance bounds is presented. The paper contains several tables: tables of new, improved QC and QT codes, maximum known minimum distances for QT $[pm,m]$ codes, optimal QT codes, as well as a comprehensive table of lower and upper bounds on linear codes over $\mathbb{F}_{13}$ that covers the range $n \leq 150$ and $3 \leq k \leq 6$. With these concrete results, this work can serve as a foundation for future research on linear codes over $\mathbb{F}_{13}$ (e.g. a more comprehensive database).

2. Quasi-twisted codes

A linear q-ary $[n,k,d]$ code is said to be α-consta-cyclic if there is a non-zero element α of $\mathbb{F}_q$ such that for any codeword $(a_0,a_1,\ldots,a_{n-1})$, a consta-cyclic shift by one position, that is $(\alpha a_{n-1},a_0,\ldots,a_{n-2})$, is also a codeword [14]. Therefore, consta-cyclic codes are a generalization of cyclic codes, or a cyclic code is an α-consta-cyclic code with $\alpha=1$. A consta-cyclic code can be defined by a single generator polynomial. A code is said to be quasi-twisted (QT) if a consta-cyclic shift of any codeword by p positions is still a codeword. Thus a consta-cyclic code is a QT code with $p=1$, and a quasi-cyclic (QC) code is a QT code with $\alpha=1$. The length n of a QT code is a multiple of p, i.e., $n=pm$ for some positive integer m.

An α-consta-cyclic matrix of order n, also called a twistulant matrix, is defined as

$$C=\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ \alpha c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \alpha c_{n-2} & \alpha c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha c_1 & \alpha c_2 & \alpha c_3 & \cdots & c_0 \end{bmatrix} \tag{1}$$

Twistulant matrices are basic components in the generator matrix for a QT code. The algebra of $n\times n$ consta-cyclic matrices over $\mathbb{F}_q$ is isomorphic to the algebra of the quotient ring $\mathbb{F}_q[x]/(x^n-\alpha)$ if C is mapped onto the polynomial formed by the elements of its first row, $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}$, with the least significant coefficient on the left. The polynomial $c(x)$ is also called the defining polynomial of the matrix C. A twistulant matrix is called a circulant matrix if $\alpha=1$.

The generator matrix of a QT code can be transformed into rows of twistulant matrices by a suitable permutation of columns. Most research has been focused on 1-generator and 2-generator QT codes. The generator matrices for 1-generator and 2-generator QT codes consist of one row of twistulant matrices and two rows of twistulant matrices, respectively,

$$G=\begin{bmatrix} G_0 & G_1 & \cdots & G_{p-1} \end{bmatrix} \text{ and } G=\begin{bmatrix} G_{1,0} & G_{1,1} & \cdots & G_{1,p} \\ G_{1,0} & G_{2,1} & \ldots & G_{2,p} \end{bmatrix} \tag{2}$$

where G_j and G_{ij} are twistulant matrices, for $j=0,1,2,\ldots,p1$ and $i=1,2$. Let $g_{i,j}(x)$ and $g_{i,j}(x)$ be the defining polynomials for the corresponding twistulant matrices G_j and G_{ij}. Then, the defining polynomials $(g_0(x),g_1(x),g_2(x),\cdots,g_{p-1}(x))$ and $(g_{1,0}(x),g_{1,1}(x),g_{1,2}(x),\ldots,g_{1,p-1}(x);g_{2,0}(x),g_{2,1}(x),$ $g_{2,2}(x),\ldots,g_{2,p-1}(x))$ define a 1-generator QT $[pm,k,d]$ code and 2-generator QT $[pm,k,d]$ code, where k, the dimension of the code, is the rank of the generator matrix G. In Magma algebra system [20], the number of generators is called the height. The parameters of all the codes presented in this paper have been verified by Magma.

3. The search algorithm and new QT codes over $\mathbb{F}_{13}$

As a generalization to cyclic codes and consta-cyclic codes, quasi-cyclic (QC) codes and quasi-twisted (QT) codes have been known to contain many good codes. In fact, many record-breaking linear codes have been obtained from these classes [2]-[13].

Gulliver et al. [4, 5, 9, 13] have done much work on the computer searches for good QC and QT codes. By eliminating the equivalent generator polynomials, and eliminating all redundant information polynomials, an $r\times s$ weight matrix W is used in the constructions, as given below, where $c_k(x)$ is the kth generator polynomial, $i_j(x)$ is the jth information polynomial, w_{jk} is the Hamming weight of $i_j(x)c_k(x)$

$$W = \begin{array}{c|ccccccc|} & c_1(x) & c_2(x) & \cdots & c_k(x) & \cdots & c_s(x) \\ \hline i_1(x) & w_{11} & w_{12} & \cdots & w_{1k} & \cdots & w_{1s} \\ i_2(x) & w_{21} & w_{22} & \cdots & w_{2k} & \cdots & w_{2s} \\ \vdots & \vdots & \vdots & \cdots & \vdots & & \vdots \\ i_j(x) & w_{j1} & w_{j2} & \cdots & w_{jk} & \cdots & w_{js} \\ \vdots & \vdots & \vdots & \cdots & \vdots & & \vdots \\ i_r(x) & w_{r1} & w_{r2} & \cdots & w_{rk} & \cdots & w_{rs} \end{array}$$

mod $(x^m - \alpha)$, m is the size of the twistulant matrix and α is the shift constant. To construct a good QT $[pm, k]$ code, their algorithm selects a set of p columns among s columns such that the set of columns maximizes the smallest row sum of the corresponding p columns. When p and s are large, it is not possible to examine all (s, p) combinations. Gulliver's search is initialized with an arbitrary $[pm, k]$ code (usually a good one) with p columns (or generator polynomials). To improve the code, a new column is found to replace one presently in the code so that the minimum distance is increased. Later on, a stochastic optimization called tabu search has been used to construct good QC or QT codes by Gulliver and Östergård [9], and Daskalov et al. [10]. In a recent paper, Venkaiah and Gulliver [13] used the tabu search to find good QC codes over $\mathbb{F}_{13}$.

On the other hand, a method to obtain a weight matrix from a consta-cyclic simplex code of composite length was recently presented in [15]. The resulting weight matrix is cyclic, and therefore only one row is required to be in the memory during the search. A new iterative heuristic search is also presented, and many good QT codes have been constructed [15]. In this work, the algorithm from [15] is applied to both the weight matrix defined by Gulliver's method and the weight matrix derived from the consta-cyclic simplex code as given in [15]. As a result, many good QT codes have been obtained, allowing us to establish a database of linear codes over $\mathbb{F}_{13}$ with the range of parameters described above.

Given an $r \times s$ weight matrix $W = (w_{ij})$. The iterative algorithm tries to find a sequence of good QT $[im, k]_q$ codes, $i = 1, 2, \ldots, t$, where $t < s$. The basic idea of the algorithm is to extend a QT $[(i-1)m, k]_q$ code by one more column to obtain a good QT $[im, k]_q$ code, for $i = 2, 3, \ldots, t$. The algorithm is executed for a specified number of iterations. The algorithm records the best codes found so far, and stores them in files. When the algorithm stops, a summary of the codes found is presented. In the execution of the algorithm, the selection of columns is important as it determines if good codes can be found quickly. In order to avoid exhaustive search, we use a heuristic method to implement the selection. At each iteration, to obtain the best possible minimum distance for a QT $[im, k]_q$ code, we select a column that results in the largest minimum row sum (it is also the minimum distance of the constructed code). If there is more than one column that gives the same best minimum distance, we count how many such rows that result in the minimum row sum. We choose the column that will have the smallest number of such rows, since it is expected that such a selection will provide a better chance to get a good QT $[(i+1)m, k]_q$ code in the next extension. In this way, the algorithm is greedy and heuristic. If there is more than one choice, a column is selected at random among suitable choices. So the algorithm contains some randomization.

The effectiveness of this iterative heuristic search algorithm is evident from the fact that a large number of new QT codes over $\mathbb{F}_{13}$ for $k = 3, 4, 5,$ and 6 have been obtained as a result of the application of the algorithm. The new codes improve the previously known results.

Table 1 lists the new QT codes over $\mathbb{F}_{13}$ that have larger minimum distances than the corresponding codes given in [13]. The defining polynomials are listed with the lowest degree coefficient on the left, and the finite field $\mathbb{F}_{13}$ elements 10, 11, 12 are denoted by A, B and C (as commonly used in a hexa-decimal system). For example, $C024A9$ corresponds to the polynomial $12 + 2x^2 + 4x^3 + 10x^4 + 9x^5$.

Table 2 summarizes the maximum known minimum distances for QT $[pm, m]$ codes over $\mathbb{F}_{13}$ for p up to 25. The authors can provide all best known QT codes for n up to 255, upon request. Most entries in the table are from the results in [13], and the entries labeled with superscript "e" are new codes found with the algorithm in this paper. All codes with $k = 6$ are constructed from the weight matrix derived from the consta-cyclic simplex $[402234, 6, 371293]$ code. Since the weight matrix is cyclic, only one row of $402234/6 = 67039$ elements is required to be stored in memory. This makes it easier to search for good QT codes with $k = 6$ (otherwise, the weight matrix is too big to fit in the memory).

4. A database of linear codes over $\mathbb{F}_{13}$ with minimum distance bounds

4.1. Lower bounds on minimum distance

Since there are no good, general analytical lower bounds available for the parameters of a linear code, the lower bounds on minimum distances have been established by explicitly constructing the codes [1]. As commented earlier, constructing good linear codes is a difficult task because finding the minimum distance of a linear code is computationally expensive [19]. Therefore, researchers focus on certain promising classes of codes with rich mathematical structure. The class of QT codes has been an excellent source for producing best-known codes [2]-[13]. Constacyclic codes are a special case of QT codes. Following the approach given in [22], we have been able to compute all constacyclic codes exhaustively for most lengths since the dimension is restricted to $3 \leq k \leq 6$. Some of the best-known (or optimal) codes are constacyclic.

Another tool that can be used to obtain more new codes from existing codes in a computationally efficient way is to apply standard construction methods to derive codes from known codes, such as puncturing, shortening and extending [1]. With the codes constructed in [13], the new QT codes over $\mathbb{F}_{13}$ presented in the previous section, as well as the standard construction methods to derive new codes from existing codes, we are able to create a comprehensive table of lower bounds on the minimum distances for linear codes over $\mathbb{F}_{13}$ with dimensions between 3 and 6 and block length n up to 255. Table 3 includes the lower bounds for block lengths up to 150.

There is a connection between best-known linear codes and projective geometry. An (n, r)-arc in $PG(k-1, q)$ is a set of n points K with the property that every hyperplane is incident with at most r points of K and there is some hyperplane incident with exactly r points of K. It is known that there exists a projective $[n, 3, d]_q$ code if and only if there exists an $(n, n-d)$-arc in $PG(2, q)$ [13]. Ball [17] maintains an online table of bounds on the sizes of (n, r)-arcs in $PG(2, q)$ for $q \leq 19$. From that table, one can obtain lower bounds on the minimum distances of linear codes of dimension 3. Some of the entries in Table 3 for $k = 3$ can be derived from [17].

Table 4 lists the defining polynomials for the new codes found in this paper and that are used to establish the lower bounds in Table 3. There are 7 new 2-generator QT codes with $k = 6$ and $m = 3$ that are used to derive the lower bounds in Table 3.

5. Upper bounds on minimum distance

We also determined upper bounds on the minimum distances by applying the standard bounds (such as Griesmer, Elias, Sphere Packing etc.) [1] and taking the best result for each parameter set. In the range of parameters considered here, Griesmer bound turned out to be the best for most of the cases except that in some cases the Levenshtein bound performed better. When a code whose minimum distance equals to the upper bound, an optimal code is constructed and there is no room for improvement in the table. When there is a gap between the minimum distance of a best-known code and the upper bound on the minimum distance, this is indicated in the table by listing the both values. For example, for a [51,4]-code, the minimum distance of a best-known code is 43 whereas the theoretical upper bound is 45. It is worth noting that the theoretical upper bound may be unattainable. To save the space, only entries for the block length n up to 150 are given below (Table 3). Interested readers can obtain the full table from the authors.

5.1. Linear codes with dimension 3

Suppose $d \leq q^{k-1}$ and that C is an $[n, k, d]$ code over $\mathbb{F}_q$ which attains the Griesmer bound. Then C is projective [13]. Therefore, from the Ball's table, we conclude that there do not exist codes with the following parameters over $\mathbb{F}_{13}$: [15, 3, 13], [24, 3, 21], [25, 3, 22], [26, 3, 23], [27, 3, 24], [28, 3, 25], [29, 3, 26], [41, 3, 37], [42, 3, 38], [43, 3, 39], [54, 3, 49], [55, 3, 50], [56, 3, 51], [57, 3, 52], [70, 3, 64], [71, 3, 65], [80, 3, 73], [81, 3, 74], [82, 3, 75], [83, 3, 76], [84, 3, 77], [85, 3, 78], [93, 3, 85], [94, 3, 86], [95, 3, 87], [96, 3, 88], [97, 3, 89], [98, 3, 90], [99, 3, 91], [106, 3, 97], [107, 3, 98], [108, 3, 99], [109, 3, 100], [110, 3, 101], [111, 3, 102], [112, 3, 103], [113, 3, 104], [120, 3, 110], [121, 3, 111], [122, 3, 112], [123, 3, 113], [124, 3, 114], [125, 3, 115], [126, 3, 116], [127, 3, 117], [134, 3, 123], [135, 3, 124], [136, 3, 125], [137, 3, 126], [138,3 , 127], [139, 3, 128], [140, 3, 129], [141, 3, 130], [148, 3, 136], [149, 3, 137], and [150, 3, 138].

5.2. Some optimal codes over $\mathbb{F}_{13}$

Table 3 presents the lower and upper bounds on $d_{13}(n, k)$ for k up to 6. Many bounds are attained. It is possible that some of the current upper bounds may be improved and more codes may turn out to be optimal. In the rest of this section, we give more details on the optimal codes in Table 3.

With the algorithm given in the last section, many QC codes with $k = 3$ have been constructed whose minimum distances meet the Griesmer bounds, and thus are optimal. Table 5 lists those optimal QC $[pm, 3]$ codes that do not appear in [13]. It should be noted that codes with these parameters were not constructed in the QC form [17, 23]. Codes constructed in QC or QT form have advantages in practical implementation. Table 6 lists optimal QT $[pm, k]$ codes for $k = 4, 5$ and 6, over $\mathbb{F}_{13}$, and their defining polynomials. With the upper bounds given in Table 3, we now know that the QC [20, 4, 16] and [28, 4, 23] codes constructed in [13] are optimal, since they reach the upper bounds. The optimal [153, 4, 139] code is included here, since two other optimal codes are obtained from it by puncturing: [150, 4, 136] and [149, 4, 135] codes. The optimal [15, 6, 9] code given in the table is a 2-generator QT code with shift constant 6, and is constructed with the method given in [15]. With these codes, and results on (n, r)-arcs, the exact values on $d_{13}(n, k)$ in Table 3 are established.

6. Conclusion

In this paper, we present the construction of a large number of new QT codes over $\mathbb{F}_{13}$ obtained by an iterative heuristic search algorithm recently introduced. The results are presented in several tables. Combining the new results with earlier work on linear codes over $\mathbb{F}_{13}$, a database of linear codes over $\mathbb{F}_{13}$ with both lower and upper bounds on the minimum distances is presented for the first time. We hope that the results presented in this paper serve as a basis for future study on codes over $\mathbb{F}_{13}$.

Table 1 New QC and QT codes over F_{13}

Code	m	α	Defining polynomials
[63, 3, 57]	3	1	531, 51, 61, C11, B31, 21, A31, 321, 211, 341, 641, C31, B11, 611, 91, 921, C1, 261, 241, 311, 651
[40, 4, 34]	5	1	C1, 7B71, 7B611, 2911, A9511, 3B921, BC21, 69731
[48, 4, 41]	4	6	C55B, 529B, B301, 0AC5, A418, 4CA2, 1A21, 0995, 1625, 1C21, 93B1, 7A9C
[60, 4, 52]	4	6	C55B, 9578, 9997, 2586, A9C3, 4254, 6A96, A3A3, B0B4, B501, A61B, 45B3, 7255, 5C97, 2C3C
[68, 4, 60]	17	6	C9566572B03055915, 663CC4022720508C5, 8680977C590521B4A, 972A15A2473369C09
[68, 4, 59]	4	1	38A, 1B8, 191, B873, 6AA1, 6C4, 103, BA11, 417, 468B, 6521, 315, 6712, 7133, 6691, 422A, 9631
[72, 4, 63]	4	1	[68, 4,59] code, 6171
[76, 4, 67]	4	2	9012, BB74, 3631, 849, A98, 7C26, C8CA, 6C74, 7BA1, 661, C219, 4148, 1C37, BB21, 7A, 2489, 5797, C668, A751
[80, 4, 71]	4	2	[76, 4, 67] code, 28
[88, 4, 77]	4	1	681, 6B21, 2C21, 2711, A51, 3421, 4B41, A621, 2851, 6A71, 4A11, C431, 2B21, A91, 361, 451, 6211, 3B41, 51, C11, 6B1, 7121
[92, 4, 81]	4	6	C55B, 732A, 614, B965, 290C, BA84, 9113, 8251, 42C3, C71A, 7B64, C3A4, C867,2A73, C081, 1C88, AACB, 95A8, ABBA, A61B, 0549, 0837, 887B
[100, 4, 88]	4	1	691, 211, 581, 5281, A81, A11, 3231, 231, 8B31, 8531, 4941, 6531, 4621, A831, 4961, C411, 4A11, 2711, 5831, C111, 5721, 8321, 8911, C431, A51
[40, 5, 32]	5	1	8351, 6721, C1511, 83731, 5191, CA821, B3C31, 7A411
[75, 5, 63]	5	1	C841, 8611, A7521, 93211, AC81, 74B1, 2C411, 4B571, CA831, 48161, 9A721, B451, 4A131, 69A1, 38711
[85, 5, 72]	5	1	81C21, 41931, 47521, 98711, BAB41, 54721, 71611, A621, 471, C6A1, 69A21, B7C21, 7CC1, 3C81, A8111, BC821, 56131
[95, 5, 81]	5	1	B9261, 61811, 9751, C9C11, A3C21, 5811, C2641, 64C11, C251, 93C1, 89C1, C1A11, B3761, 61831, 1231, 601, 28B41, 8611, BCB21
[100, 5, 85]	5	1	B191, A291, B631, A8C1, 41611, 81711, 27B1, 7801, 3331, B361, CB521, 43261, BC921, 53641, CBB1, 69611, 32311, 35731, 65261, 3201
[105, 5, 90]	5	1	48911, A1211, 6A71, 24621, 17A1, 63921, CAC21, 6A651, 3B241, CA21, 37511, 46941, 1B91, 9C121, C2741, ABA1, B4821, 4481, 39A1, AC911, 89531
[110, 5, 94]	5	1	B191, A291, B631, A8C1, 41611, 81711, 27B1, 7801, 3331, B361, CB521, 43261, BC921, 53641, CBB1, 69611, 32311, 35731, 65261, 2411, 37C1, 1
[115, 5, 99]	5	1	53641, C4C21, 95511, 45861, 9401, A9511, BAB31, 5141, B3A1, 2211, 89641, 93B1, 66A1, 94321, 85C1, A161, 6A391, 7161, BB61, 3AB1, 58511, 64B21, 68111
[120, 5, 103]	5	1	B191, A291, B631, A8C1, 41611, 81711, 27B1, 7801, 3331, B361, CB521, 43261, BC921, 53641, CBB1, 69611, 32311, 35731, 65261, 2411, 37C1, 52411, 67A11, 6B1
[18, 6, 12]	6	6	C024C9, 16589B, AB836
[28, 6, 20]	28	1	83470747880B081737A7331
[36, 6, 27]	6	6	C024C9, 9064C3, A6666A, 980855, BCC956, 259089
[66, 6, 53]	6	6	C024C9, 422448, 5B6A6C, 918C06, 6016A2, 8111B4, 3C0676, 7C4A08, 1B18B, 32C246, B9C5A3
[72, 6, 58]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, 2C8C13
[84, 6, 69]	28	1	28BB602605A2731CB0B90B65031,9779C314425896634952A6B4541, 6AA2C9836262784120C570C3321
[90, 6, 74]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 01C3C
[96, 6, 79]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, BC1263
[102, 6, 84]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, BB1818
[108, 6, 90]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8
[112, 6, 94]	28	1	693580A3C5B4B6B4114264B25B1, 4B3388AC355242875B3105A841, 498A29A8A8489B2497587593661, 354B13A9088905C58328B301941
[114, 6, 95]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, B283
[120, 6, 100]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, 0AB0AB
[126, 6, 106]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, A67A0B,9C512A
[132, 6, 111]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, A67A0B, 115A6, 348297
[138, 6, 116]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, A67A0B, 115A6, C8833A, BB343
[140, 6, 119]	28	1	A351438B0147A3ABCB9A6BC5681, 15894745B677671461888533801, 2A2121C7A84423995189AB26401, 5396C6558B1B083BC216427981, CAB6C982774602546921BBB6241
[144, 6, 122]	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, A67A0B, 115A6, C8833A, 6A3704, AA727

Table 2 Maximum known minimum distances for QT [pk, k] codes over F_{13}

k\p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	4^o	7^o	10^o	12	15^o	18^o	20	23	26^o	29^o	32^o	34	37	40^o	43^o	45	48	51	54^o	57^{oe}	59	62	65^o	68^o
4	5^o	9^o	12	16^o	19	23	26	30	33	37	41^e	44	48	52^e	55	59^e	63^e	67^e	71^e	73	77^e	81^e	84	88^e
5	6	10	15^0	19	23	27	32^e	36	40	45	49	54	58	63^e	67	72^e	76	81^e	85^e	90^e	94^e	99^e	103^e	108^e
6	7	12	16	21	27^e	32	37	42	47	53^e	58^e	63	68	74^e	79^e	84^e	90^e	95^e	100^e	106^e	111^e	116^e	122^e	127^e

n^0 an optimal code

n^e new code found in this paper, and exceeds the best minimum distance in [13]

Table 3 Lower and upper bounds on minimum distances for linear codes over F_{13}

n	k = 3	4	5	6	n	k = 3	4	5	6	n	k = 3	4	5	6
1					51	45-46	43-45	41-44	39-43	101	92	89-91	86-90	83-89
2					52	46-47	44-46	42-45	40-44	102	93	90-92	87-91	84-90
3	1				53	47-48	45-47	43-46	41-45	103	94	91-93	88-92	85-91
4	2	1			54	48	46-48	44-47	42-46	104	95	92-94 [CA]	89-93	86-92
5	3	2	1		55	49	47-48	45-48	43-47	105	96 [BA]	92-95	90-93 [CA]	87-93
6	4	3	2	1	56	50	48-49	46-48 [CA]	44-48	106	96	93-96	90-94	88-93
7	5	4	3	2	57	51	49-50	46-49	45-48 [CA]	107	97	94-96	91-95	89-94
8	6	5	4	3	58	52	50-51	47-50	45-49	108	98	95-97	92-96	90-95
9	7	6	5	4	59	53	51-52	48-51	46-50	109	99	96-98	93-97	91-96
10	8	7	6	5	60	54	52-53	49-52	47-51	110	100	97-99	94 -98	92-97
11	9	8	7	6	61	55	53-54	50-53	48-52	111	101	98-100	95-99	93-98
12	10	9	8	7	62	56	54-55	51-54	49-53	112	102	99-101	96-100	94-99 [CA]
13	11	10	9	8	63	57	55-56	52-55	50-54	113	103	100-102	97-101	94-100
14	12 [Be]	11 [Be]	10 [Be]	9 [Be]	64	58 [BA]	56-57	53-56	51-55	114	104	101-103	98-102	95-101
15	12	11	10	9	65	58-59	57-58	54-57	52-56	115	105	102-104	99-103	96-102
16	13	12	11	10	66	59-60	58-59	55-58	53-57 [CA]	116	106	103-105	100-103	97-103
17	14	13	12	11	67	60	59	56-58	53-57	117	107	104-106	101-104	98-103 [CA]
18	15	14	13	12 [CA]	68	61	60 CA	57-59 [CA]	54-58	118	108 [BA]	105-107	102-105	98-104
19	16	15	14	12-13	69	62	60-61	57-60	55-59	119	108-109	106-108	103-106 [CA]	99-105
20	17	16 VG	15 VG	13-14	70	63	61-62	58-61	56-60	120	109	107-109	103-107	100-106
21	18	16-17	15-16	14-15 [CA]	71	64	62-63	59-62	57-61	121	110	108-109	104-108	101-107
22	19	17-18	16-17	14-16	72	65	63-64	60-63	58-62 [CA]	122	111	109-110	104-109	102-108
23	20 [DB]	18-19	17-18	15-17	73	66	64-65	61-64	58-63	123	112	110-111	106-110	103-109
24	20	19-20	18-19	16-18	74	67	65-66	62-65	59-64	124	113	111-112	107-111	104-110
25	21	20	19-20 [VG]	17-19	75	68	66-67	63-66 [CA]	60-65	125	114	112-113	108-112	105-111
26	22	21	19-20	18-20	76	69	67-68	63-67	61-66	126	115	113-114	109-113 [CA]	106-112 [C]
27	23	22	20-21	19-20	77	70	68-69	64-68	62-67	127	116	114-115	109-114	106-113

Table 3 Lower and upper bounds on minimum distances for linear codes over F_{13}

n	k = 3	4	5	6	n	k = 3	4	5	6	n	k = 3	4	5	6
28	24	23 VG	21-22	20-21 [CA]	78	71	69-70	65-69	63-68	128	117	115-116	110-114	107-114
29	25	23-24	22-23	20-22	79	72 [BA]	70-71	66-70	64-69	129	118	116-117	111-115	108-114
30	26	24-25	23-24	21-23	80	72	71-72 [CA]	67-71	65-70	130	119	117-118	112-116	109-115
31	27	25-26	24-25	22-24	81	73	71-72	68-72	66-71	131	120	118-119	113-117	110-116
32	28	26-27	25-26 [CA]	23-25	82	74	72-73	69-72	67-72	132	121 [BA]	119-120	114-118	111-117
33	29	27-28	25-27	24-26	83	75	73-74	70-73	68-72	133	121-122	120-121	115-119	112-118
34	30	28-29	26-28	25-27	84	76	74-75	71-74	69-73 [CA]	134	122	121-122	116-120	113-119
35	31	29-30	27-29	26-28	85	77	75-76 [CA]	72-75 [CA]	69-74	135	123	122-122	117-121	114-120
36	32	30-31	28-30	27-29 [CA]	86	78	75-77	72-76	70-75	136	124	123-123 [CA]	118-122 [CA]	115-121
37	33	31-32	29-31	27-30	87	79	76-78	73-77	71-76	137	125	123-124	118-123	116-122
38	34 [BA]	32-33	30-32	28-31	88	80	77-79	74-78	72-77	138	126	124-125	119-124	117-123
39	34-35	33-34	31-33	29-32	89	81	78-80	75-79	73-78	139	127	125-126	120-125	118-124
40	35-36	34-35 [CA]	32-34	30-33	90	82	79-81	76-80	74-79 [CA]	140	128	126-127	121-125	119-125 [C]
41	36	34-36	33-35	31-34	91	83	80-82	77-81	74-80	141	129	127-128	122-126	119-125
42	37	35-36	34-36 [CA]	32-35 [VG]	92	84 [BA]	81-83 [CA]	78-82	75-81	142	130	128-129	123-127	120-126
43	38	36-37	34-36	32-36	93	84	81-84	79-82	76-82	143	131	129-130	124-128	121-127
44	39	37-38	35-37	33-36	94	85	82-84	80-83	77-82	144	132	130-131	125-129	122-128
45	40	38-39	36-38	34-37	95	86	83-85	81-84	78-83	145	133 [BA]	131-132	126-130	123-129
46	41	39-40	37-39	35-38	96	87	84-86	82-85 [CA]	79-84	146	133-134	132-133	127-131	124-130
47	42	40-41	38-40	36-39	97	88	85-87	82-86	80-85	147	134-135	133-134	128-132 [CA]	125-131 [C]
48	43	41-42	39-41	37-40	98	89	86-88	83-87	81-86	148	135	134-135	128-133	125-132
49	44 [BA]	42-43	40-42 [CA]	38-41[CA]	99	90	87-89	84-88	82-97 [CA]	149	136	135	129-134	126-133
50	44-45	43-44	40-43	38-42	100	91	88-90 [CA]	85-89 [CA]	82-88	150	137	136	130-135	127-134 [C]

[BA] – Simeon Ball [17]　[VG] – quasi-cyclic code in [13]　[Be]—MDS code for n < 15 [14]

[DB] – de Boer code [16]　[CA] – new codes presented in this paper

Unmarked entries can be obtained by puncturing technique on longer codes or [23] if k = 3

Table 4 Other new QT codes that are found in this work and used to derive the lower bounds in Table 3

n	k	d	m	α	Defining polynomials
85	4	75	17	6	C9566572B03055915, 3B90BC822420AB4, 175010787B272AC12, A4586B7A4A1555AAA, 6824A1C67B8BA3B68
104	4	92	8	1	A9C77511, 5A613A31, C9695821, 9A4AC421, B8952121, 2AC2C01, 645B4621, A5B14521, 5661611, 74B99B1, 7809C21, B92B6B11, 43532621
136	4	123	34	6	C89658606957772CB509300552519B145A, 9A340568467B478A34AA1135A5253AAA9A, 33B2950BB9C98B2C26482101A4B841080A, 269882B42A512CA6B74B883B0A732B3638
153	4	139	17	6	C9566572B03055915, 3B90BC822420AB4, 175010787B272AC12, A4586B7A4A1555AAA, 6824A1C67B8BA3B68, 930647483A13A23A9, 298B252AB48307233, 8680977C590521B4A, 325B99BC68114818A
32	5	25	8	1	965C81, 117A5C1, 6B66521, 39612911
42	5	34	14	1	C364C197A1, C1406B7668B51, 273B691926081
49	5	40	7	1	5274A11, 8A21511, 2696211, 32991, CAC131, 373491, 9716131
56	5	46	7	1	6B65B1, C65C81, 4B4961, 569781, 4A16121, 84C1711, 9283C1, 87C661
68	5	57	17	1	26778C10B7B73731, 42C4C825639842A1, 196C27211272C691, 6A30BA1C25566381
96	5	82	8	1	3BA79211, C57B3311, 25B43731, C647121, A981611, 16715B1, 98C1C811, 2472C951, 8470C41, 29327121, 7A1A781, C85B6B1
119	5	103	17	1	158BCB3BCB851, 43227B2CB6976681, 1939C57875C9391, 550102645A546201, A463983A2316C151, 3839B779C2980661, 315895A570C81241
126	5	109	14	1	C364C197A1, 913937AB67CC1, 21472632237C1, 8986292BB3AB1, 3062CA237511, 76B23513759411, 68B307A1CC521, 35096BA095421, 963CA473580A1
136	5	118	8	1	5ABAB321, A7916721, 3462211, 286B9621, 979BAA1, B761AC1, 68C5971, 47681341, 79883511, A1A86A31, 95643721, B9643B51, B59B7321, 312B8C11, C7AC5A11, B78B7A1, 781B7B1
147	5	128	21	1	CBC6C37BC7B51A88C1111, C5843C3AA792418224C1, C46981C81A47C179B1C11, 2487C583226B53651B11, B1A625C6B3C6B5054C91, 64CB15B59406B66416A1, A34881B17C650445C121
150	5	130	5	1	B191, A291, B631, A8C1, 41611, 81711, 27B1, 7801, 3331, B361, CB521, 43261, BC921, 53641, CBB1, 69611, 32311, 35731, 65261, 2411, 37C1, 52411, 67A11, 4751, C4531, A6311, 67161, CBA31, 82911, 81
21	6	14	3	6	C2C, AA8, 9BB, 88C, A81, 71A, 042; 0C2, 3A9, 9C5, 38, 5C4, 3A4, 054
27	6	19	3	6	C2C, AA8, 9BB, 88C, A81, 71A, 5BB, A87, 608; 0C2, 3A9, 9C5, 38, 5C4, 3A4, 803, 939, A35
49	6	38	7	1	7B6A311, 3CC651, 5B39641, A9B171, C89CA1, 893C421, 953B911
57	6	45	3	6	C2C, 015, 1A, 95, 1A2, 10C, 435, 204, 169, 0A5, 676, 769, 82C, B8C, 524, 4BA, 52C, 963, 249; 0C2, 605, 916, CC4, 689, 6A4, 5AA, 361, 88C, 099, 80B, CB5, 902, B66, 7B6, 576, 9C3, B34, 49B
81	6	66	3	6	C2C, AA8, 9BB, 88C, A81, 71A, 5BB, A87, 288, 8B3, 391, CAB, 866, 371, 621, 453, 2A8, 9CB, 491, 345, 818, BA5, 26B, 248, 795, C6, 7B6; 0C2, 3A9, 9C5, 38, 5C4, 3A4, 803, 939, CA2, 479, 8A1, CCA, 205, 14B, 805, 181, 416, 7AA, 30B, AC, 489, 418, 482, 80A, C35, 511, 2C9
99	6	82	3	6	C2C, 043, 991, A4A, 73, 342, 103, B55, C51, 054, 962, 052, B18, 9C9, A35, 95A, 7CC, 042, 788, 3C6, 45B, CA3, 377, 43, 025, 9C3, 7B9, 049, A82, B75, 628, A19, 09C; 0C2, 11A, B96, 577, 743, 1B7, 867, 915, 68, 6B7, 4B5, 977, B09, 681, 7AC, A51, 91, 8BB, 879, 8C3, 839, 5AC, 414, 165, B7B, 304, 8BC, 36A, 003, 368, CB8, 313, 38
117	6	98	3	6	C2C, B51, 80B, B18, 525, 38B, B36, A76, 68B, C61, 24C, 865, 6A4, 82C, 331, 879, B6A, 04C, 391, A0C, 217, A7, 711, CCA, 35C, C84, A24, 0C1, 216, 993, C35, 8B8, A88, 747, A66, 3AB, 361, 044, 6A7; 0C2, C72, 4C3, 1A6, 901, 363, 4C8, 472, 5A6, C09, 2B9, B74, 729, B44, 2A4, 2B, 346, A04, C19, 304, 0C4, 6BC, 5CA, B6B, 6, 61C, C24, 7C8, 1B4, 282, 587, C87, 196, 03C, 68A, 346, 89B, C82, A98
135	6	114	3	6	C2C, 532, 556, CCB, 7B2, 59A, 26, BB4, C22, 53A, 968, 3AA, B3C, 37, 4C8, 905, 82B, 119, 271, 112, 565, A8, 9C5, B7B, 5AB, C22, 077, 216, 8B1, C14, 9CC, 06, 3C8, BAA, 745, 501, 295, 0CA, B9C, 404, 23, 16C, 5C2, 031, 1A4; 0C2, 81C, A1, 038 C86, 5C3, B7A, 31A, ABA, 54B, 591, BB4, 2A7, 096, 243, A64, 5B9, 37, 61B, C76, 1C9, 40A, 3A5, A42, C4B, 552, 252, 65, A68, 975, 96A, 989, 2B4, 383, 902, 94C, 626, 878, B45, 7A8, CCB, C48, 13B, 526, 374
147	6	125	21	1	15927A7C452B83136931, C8905A324208569C6611, 850175159485BB722991, 5796B3114C22C917C231, 32AA65B413508B2BA141, 46854B05322A8664661, B773147B873A94886041
150	6	127	6	6	C024C9, 015C, 73CA6A, 0073A6, 7742C9, 4C3651, 641374, 42BB6, 22133, 56723, 2CBA85, CB02A6, BC404B, 571AA2, 8227C, 834C31, 93857, 2130B8, 7C85C6, A67A0B, 115A6, C8833A, 6A3704, 25574C, BB1818

Table 5 Other new optimal QT [pm, 3] codes found in this work

n	p	m	d	α	Defining polynomials
8	2	4	6	1	11, 2321
14	2	7	12	1	18481, A8B8A11
20	5	4	17	1	11, 4961, 6831, 671, 231
32	8	4	28	1	11, 4961, 6831, 671, 4521, 3311, AB21, 8921
35	5	7	31	1	3C72B1, 19CC91, 6A6011, 3494311, B323B11
77	11	7	70	1	A8B8A11, 18481, 19CC91, 8535811, 3494311, 2437A1, 64BA931, 2959211, 75A821, B786451, C64971
78	26	3	71	1	211, 261, 531, 341, 1, B1, 6A1, 491, A31, 851, 21, 581, 241, 451, 811, 11, 351, 621, 31, B41, 671, 421, 911, 81, 321, B51
87	29	3	79	1	[78, 3, 71] code, 411, C71, 51
88	22	4	80	1	11, 4961, 6831, 671, 4521, 3311, AB21, 8921, 341, 5731, 561, 2321, 4851, 6611, BC1, 3531, 121, 8A31, 9A1, 2211, 4741, 6941
90	30	3	82	1	[87, 3, 79] code, 311
91	13	7	83	1	[77, 3, 70] code, 47BAC1, A39651
102	34	3	93	1	511, 321, C21, B51, A11, 821, 431, 231, 521, 241, 31, 611, 361, 911, 1, 541, 6A1, 91, 671, 581, B21, 921, 341, 471, 851, 311, C11, B11, 831, B41, 421, 11, 641, 491
105	35	3	96	1	421, 851, 411, C31, 21, 491, 11, 231, 61, 211, 511, 361, C91, 1, 341, 581, 921, 651, 6A1, 241, B1, A31, 51, 321, 471, 831, A1, 641, 531, 621, C21, 261, B51, 911, 541
116	29	4	106	1	11, 2321, 4961, 6941, 4521, 891, 2211, 341, 6831, 671, 9A1, 3A81, B361, 3531, C131, 4411, AB1, 8B41, 9A21, 8A31, 231, B141, 5621, 2651, 2431, 781, 5511, 4851, 6611
117	39	3	107	1	231, 51, 71, 431, 531, 261, B21, C91, 911, 811, C71, 1, 11, A11, 361, C31, 61, 541, 641, 671, 91, 611, 341, 621, C11, 471, 321, 21, 81, 491, B51, 831, 511, 451, 821, A31, 211, 921, 521
129	43	3	118	1	A31, 641, B31, C21, 921, 41, 321, 6A1, 531, A1, C11, 61, B51, 51, 511, 421, 11, 821, 541, B41, C71, 491, C31, 91, B1, 81, 471, 361, 241, 211, 711, 571, C1, 261, 341, 621, 71, 611, 431, 411, 581, 851, 671
132	44	3	121	1	341, C91, 261, 671, B1, 831, C1, B41, 921, 311, C31, 51, 821, 231, 91, 711, 811, 511, A1, 1, 431, 581, 471, 411, A31, 11, 491, 351, C21, 611, 6A1, 81, 641, 241, 31, 321, B31, A11, 911, 71, B11, 851, 451, C71
144	48	3	132	1	431, 581, 911, 711, 231, 471, 341, A11, 51, 321, 851, 361, 421, 821, B41, 261, C21, 81, B31, 511, 241, C91, 921, B51, 31, 211, 811, 611, A31, 411, 671, C11, C31, 521, 351, 21, 641, 451, 11, 491, 531, 311, 6A1, 651, 621, B1, C71, 1
159	53	3	146	1	211, 261, 531, 341, 1, B1, 6A1, 491, A31, 851, 21, 581, 241, 451, 811, 11, 351, 621, 31, B41, 671, 421, 911, 81, 321, B51, 411, C71, 311, 51, C31, 71, 361, 431, 651, 711, C21, B11, 511, A21, 571, C91, 611, 231, 921, 61, A11, C1, 471, 521, B31, C11, 41
160	40	4	147	1	11, 4961, 6831, 671, 4521, 3311, AB21, 8921, 341, 5731, 561, 2321, 4851, 6611, BC1, 3531, 121, 8A31, 9A1, 2211, 4741, 6941, 8B41, 231, 5621, 5511, 3A81, B361, C131, 781, 2651, 3421, AB1, B141, 2431, 4411, 3641, 6721, 451, 891
161	23	7	148	1	18481, A8B8A11, 19CC91, 8535811, 47BAC1, 75A821, B323B11, 2959211, CB6BC11, 985B41, 86B9321, 2437A1, 64BA931, 3494311, B786451, 7ACA711, 3548B21, 5747511, 6A6011, A39651, 522501, A96C521, 3C72B1
162	54	3	149	1	[159, 3, 146] + 821
164	41	4	151	1	11, 4961, 6831, 671, 4521, 3311, AB21, 8921, 341, 5731, 561, 2321, 4851, 6611, BC1, 3531, 121, 8A31, 9A1, 2211, 4741, 6941, 8B41, 231, 5621, 5511, 3A81, B361, C131, 781, 2651, 3421, AB1, B141, 2431, 4411, 3641, 6721, 451, C241, 891

Table 5 Other new optimal QT [pm, 3] codes found in this work

n	p	m	d	α	Defining polynomials
165	55	3	152	1	[162, 3, 149], 641
168	56	3	155	1	[165,3, 155], 91
171	57	3	157	1	C91, 361, A31, 921, B11, 1, 51, A11, 211, 851, 311, 241, C21, B1, 31, B51, 491, 621, 11, 451, 71, 651, 671, B41, 421, 511, 81, 541, C11, 531, 341, 611, 521, 431, C71, 21, 581, 471, 831, 811, 911, 61, A1, 641, 411, B31, A21, C31, 41, C1, 571, 231, 261, 6A1, 321, 351, 91
174	58	3	160	1	[171, 3, 157], 711
175	25	7	161	1	18481, A8B8A11, 19CC91, 8535811, A39651, 2437A1, 47BAC1, 75A821, CB6BC11, 6282611, B4A7621, 5747511, 6A6011, 7ACA711, 3548B21, B323B11, 3C72B1, 985B41, 522501, 2959211, 3494311, B786451, 64BA931, A96C521, C64971
177	59	3	163	1	[174, 3, 160] code, 821
180	60	3	166	1	[177, 3, 163] code, B21
182	26	7	168	1	[175, 3, 161] code, 86B9321
183	61	3	169	2	431, 521, 721, A1, 81, 651, 471, 511, A21, 811, 91, B51, 711, 121, A41, 11, 31, A11, 411, 531, 21, 451, 341, 41, 321, C91, 641 311, 831, 611, 71, 731, B71, B21, C1, 821, 621, 351, 891, B11, C11, C71, A81, 921, 1, B31, 941, 851, 61, 961, 911, 51, 111, B41, 951, 861, 671, A31, C31, C21, B1
186	62	3	170	1	C11, 811, 71, 471, 21, 521, 211, 491, C21, B11, 31, 11, 261, C91, 621, 921, 361, 51, 81, 641, A11, C71, B51, 611, B1, 571, A31, 671, 91, 411, 431, 41, 541, B21, A1, 341, 241, 581, 511, 421, 61, 6A1, 321, 711, 911, 531, 351, B41, 311, B31, B31, 651, C1, 451, 821, 1, C31, 851, 231, 831, 831, A21
188	47	4	172	1	11, 3A81, 2431, 231, AB1, 6941, 4521, 8B41, AB21, 9A21, B141, 3531, 2321, 4961, 2211, 6611, 3641, 8A31, 3421, 3311, 6721, 4851, 451, 341, C241, 9A1, 5511, BC1, 781, 121, 121, 561, C131, 891, 5621, 4411, 671, 4741, 5731, 2651, B361, 8921, 6831, C01, BC21, 6C71, AC31
192	48	4	176	1	C131, 5511, B361, 8921, AB1, 5621, BC1, 8921, 451, AB21, 671, 2211, 6941, 121, 6611, 6831, C01, 231, C241, 3531, 781, 5731, 2431, 341, 8A31, 3311, 4521, 9A21, 561, 3641, 6721, 11, 2651, 3421, 4741, B141, 891, 4851, 2321, 4961, 4411, 8B41, 8B41, 9A1, 3A81, CC11, 4C91, BC21
189	27	7	173	1	CB6BC11, 5747511, 522501, 18481, 2959211, 985B41, 3C72B1, 64BA931, B4A7621, C64971, 75A821, 86B9321, 8535811, 8535811, 6A6011, B323B11, B323B11, 19CC91, A8B8A11, 3494311, 2437A1, 47BAC1, 6282611, 7ACA711, A96C521, B786451, 3548B21
204	17	12	187	1	4BA782CA2831, 5A96A793501, A2085669411, 274A0532321, C4C133A1141, C4C133A1141, 5262343C8511, 55428CA6C1, 55428CA6C1, 6B299CBCA81, B36B4B85991, 246A8A98C621, 246A8A98C621, 91756140C61, 2C1662AB6711, 57B65C291421, B837A821C111

Table 6 Optimal QT [pm, 4] and [pm, 5] codes

n	k	p	m	d	α	Defining polynomials
10	4	2	5	7	1	C01, B2B11
14	4	2	7	11	1	C851, B636B11
15	4	3	5	11	1	C01, 96911, A111
16	4	4	4	12	1	4121, 7211, A431, 41
18	4	3	6	14	1	C71, 91A81, 9A6111
20	4	5	4	16 VG	1	116, 1B, 1186, 142, 134A
25	4	5	5	20	1	C01, 96911, 28111, 95B1, 5611
28	4	7	4	23 VG	1	14, 13, 1159, 163B, 1252, 112C, 1294
68	4	4	17	60	6	C9566572B03055915, 663CC4022720508C5, 8680977C590521B4A, 972A15A2473369C09
153	4	9	17	139	6	C9566572B03055915, 3B90BC822420AB4, 175010787B272AC12, A4586B7A4A1555AAA, 6824A1C67B8BA3B68, 930647483A13A23A9, 298B252AB48307233, 8680977C590521B4A, 325B99BC68114818A
10	5	2	5	6 VG	1	13A, 10AA
12	5	2	6	8	1	11, 512721
14	5	2	7	10	1	6B65B1, C65C81
15	5	3	5	10	1	B191, A291, 721
18	5	3	6	13	1	32B131, 8C4121, 51271
20	5	4	5	15 VG	1	18, 14AC4, 1C8B, 12B3C
15	6	5	3	9	6	C2C, AA8, 9BB, 88C, 2A2; 0C2, 3A9, 9C5, 38, A04
18	6	3	6	12	6	C024C9, 16589B, AB836

References

[1] F. J. MacWilliams F. J., N. J. A. Sloane, The theory of error-correcting codes, North-Holland, 1977.
[2] C. L. ACen and W. W. Peterson, *Some Results on Quasi-cyclic Codes*, Inf. Contr., 15, 407-423, 1969.
[3] H. C. A. van Tilborg, *On Quasi-cyclic Codes with rate 1/m*, IEEE Trans. Inform. Theory, 24, 628-629, 1978.
[4] T. A. Gulliver, V. K. Bhargava, *Some Best Rate 1/p and Rate (p-1)/p Systematic Quasi-cyclic Codes*, IEEE Trans. Inform. Theory, 37, 552-555, 1991.
[5] T. A. Gulliver, V. K. Bhargava, *Some Best Rate 1/p and Rate (p-1)/p Systematic Quasi-cyclic Codes over GF(3) and GF(4)*, IEEE Trans. Inform. Theory, 38, 1369-1374, 1992.
[6] P. P. Greenough, R. Hill, *Optimal Ternary Quasi-cyclic Codes*, Des. Codes Crypt.,2,81-91, 1992.
[7] E. Z. Chen, *Six New Binary Quasi-cyclic Codes*, IEEE Trans. Inform. Theory, 40, 1666-1667, 1994.
[8] R. N. Daskalov, T. A. Gulliver, E. Metodieva, *New Good Quasi-cyclic Ternary and Quaternary Linear Codes*, IEEE Trans. Inform. Theory, 43, 1647-1650, 1997.
[9] T. A. Gulliver, P. R. J. Östergård, *Improved Bounds for Ternary Linear Codes of Dimension 7*, IEEE Trans. Inform. Theory, 43, 1377-1388, 1997.
[10] R. N. Daskalov, T. A. Gulliver, E. Metodieva, *New Ternary Linear Codes*, IEEE Trans. Inform. Theory, 45, 1687-1688, 1999.
[11] N. Aydin, I. Siap, D. K. Ray-Caudhuri, *The Structure of 1-generator Quasi-twisted Codes and New Linear Codes*, Des. Codes Crypt.,24, 313-326, 2001.
[12] N. Aydin, I. Siap, *New Quasi-cyclic Codes over* $\mathbb{F}_5$, Appl. Math. Lett., 15, 833-836, 2002.
[13] V. C. Venkaiah, T.A. Gulliver, *Quasi-cyclic codes over* $\mathbb{F}_{13}$ *and enumeration of defining polynomials*, J. of Discrete Algorithms, 16, 249-257, 2012.
[14] E. R. Berlekamp, Algebraic Coding Theory, Aegean Park, (1984).
[15] E. Z. Chen, *A New Iterative Computer Search Algorithm for Good Quasi-twisted Codes*, accepted for publication in Des. Codes Crypt., in press.
[16] M. A. De Boer, *Almost MDS Codes*, Des. Codes Crypt., 9, 143-155, 1996.
[17] S. Ball, *Table of Bounds on Three Dimensional Linear Codes or* (n, r) *Arcs in* $PG(2, q)$, available at http://www-ma4.upc.es/∼ simeon/codebounds.html, Accessed July 22, 2014
[18] A. R. Caldebank, W. M. Kantor, *The geometry of two-weight codes*, Bull. London Math. Soc., 18, 97-122, 1986.
[19] A. Vardy, *The Intractability of Computing the Minimum Distance of a Code*, IEEE Trans. Inform. Theory, 43, 1757-1766, 1997.
[20] W. Bosma, L. Cannon, C. Playoust, *The Magma Algebra System I, The User Language*, J. Symbolic Comput., 24, 235-265, 1997.
[21] M. Grassl, *Bounds on the minimum distances of linear codes*, available at http://www.codetables.de, Accessed July 22, 2014.
[22] N. Aydin, J. Murphree, *New Linear Codes from Constacyclic Codes*, J. Franklin Inst., 351, 1691-1699, 2014.
[23] R. Hill, *Optimal Linear Codes*, in Cryptography and Coding II, edited by C. Mitchell, Oxford University Press, 74-104, 1992.
[24] E. Z. Chen, *Database of Quasi-twisted codes*, available at http://moodle.tec.hkr.se/∼chen/research /codes/searchqt.htm, Accessed July 22, 2014
[25] K. Betsumiya, S. Georgiou, T. A. Gulliver, M. Harada, C. Koukouvinos, *On Self-dual Codes over Some Prime Fields*, Discrete Math. 262, 37-58, 2003.
[26] D. W. Newhart, *On Minimum Weight Codewords in QR codes*, J. Combin. Theory Ser. A 48, 104-119, 1998.
[27] I. S. Kotsireas C. Koukouvinos, D. E. Simos, *MDS and near-MDS Self-dual Codes over Large Prime Fields*, Advances in Math. Commun.,3 , 349-361, 2009.
[28] M. Grassl, T. A. Gulliver, *On Self-dual MDS codes*, Proc. IEEE Int. Symp. Inform. Theory, 1954-1957, 2008.

[29] T. A. Gulliver, J-L. Kim, Y. Lee, *New MDS or near-MDS Self-dual Codes*, IEEE Trans. Inform. Theory, 54, 4354-4360, 2008.