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Some properties of topological pressure on cellular automata*

Research Article

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Abstract: This paper investigates the ergodicity and the power rule of the topological pressure of a cellular automaton. If a cellular automaton is either leftmost or rightmost permutive (due to the terminology given by Hedlund [Math. Syst. Theor. 3, 320-375, 1969]), then it is ergodic with respect to the uniform Bernoulli measure. More than that, the relation of topological pressure between the original cellular automaton and its power rule is expressed in a closed form. As an application, the topological pressure of a linear cellular automaton can be computed explicitly.

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1. Introduction

Let $\mathcal{A} = \{0, 1, \dots, \upsilon - 1\}$ for some $\upsilon \geq 2$, and let $X = \mathcal{A}^{\mathbb{Z}}$ be the space of infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$. A continuous map $F : X \to X$ satisfying $F \circ \sigma = \sigma \circ F$ is called a cellular automaton (CA), where σ is the shift map defined by $\sigma(x)_i = x_{i+1}$, $i \in \mathbb{Z}$. CA is a particular class of dynamical systems introduced by S. Ulam [21] and J. von Neumann [23] as a model for self-production. It is widely studied in a variety of contexts in physics, biology and computer science [6, 7, 10, 12, 16, 20, 22, 26, 27].

Greenberg and Hastings [13] transfer a reaction-diffusion equation into a CA and demonstrate that such a simplified model can still generate spatial-temporal structures similar to the original system [11–13]. In 1976, Hardy et al. [14] proposed the so-called HPP model for the study of hydrodynamics. Notably, this model is a CA.

Since CA is widely applied in physics, it is also interesting to consider the role thermodynamics plays in CA. The second law of thermodynamics says that the entropy of a system and its surroundings (universe) do not decrease. However, this encounters with a paradox to Poincaré Recurrence Theorem.

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To overcome this difficulty, L. Boltzmann introduces “Ergodic Hypothesis". Reader may refer to [17]. This leads to the development of ergodic theory.

This elucidation studies two properties of CA. The first part investigates the ergodic property of CA. Then the second part gives an application for ergodic CA by demonstrating the power rule of topological pressure of CA.

One-dimensional CA consists of infinite lattice with finite states and an associated mapping, say local rule. A systematic study of CA from purely mathematical point of view is discussed by Hedlund [15]. He treats CA as a particular class of symbolic dynamics and discovers that a permutive CA (defined later) implements many phenomena such as surjection. From the viewpoint of ergodic theory, it is interesting to investigate whether a CA is surjective or not since a surjective CA preserves the uniform Bernoulli measure [9]. Shereshevsky [18] demonstrates more ergodic properties for permutive CA, such as mixing, Bernoulli automorphism, Kolmogorov automorphism, and so on. He also defines the power rule of a CA and figures out many properties are preserved. Read is referred to Akın’s works [1, 2] for more results.

Shereshevsky and Rogers [19] conjecture that every surjective CA, except those of the form $Fx_i = kx_i$, where $g.c.d.(k, \upsilon) = 1$, is ergodic with respect to the uniform Bernoulli measure. This conjecture does not hold in general. A counterexample is discussed in Section 3.

The first result of this paper is, if a CA is permutive, then it is ergodic. Besides, the permutivity of CA is necessary for the ergodicity for a particular class of CA. Using this property, the power rule of topological pressure of CA can be expressed in a closed form.

For $T : \Omega \to \Omega$ a continuous transformation defined on a compact metric space, it is well-known that the power rule of topological entropy of T satisfying $h_{top}(T^n) = nh_{top}(T)$ for all $n \in \mathbb{N}$. If, moreover, $(\Omega, \mathcal{B}, \mu)$ is a probability space and $T : \Omega \to \Omega$ is also a measure-preserving transformation. Then $h_\mu(T^n) = nh_\mu(T)$ for all $n \in \mathbb{N}$, where $h_\mu(T)$ is the measure-theoretic entropy of T. Also, for the topological pressure of T, $P(T, \cdot) : C(\Omega, \mathbb{R}) \to \mathbb{R}$, there is a similar expression

$$P(T^n, S_n g) = nP(T, g), \quad \text{where } S_n g = \sum_{i=0}^{n-1} g \circ T^i.$$

In [5], the authors investigated the topological pressure of permutive CA. Theorem 4.4 indicates that, if F is a permutive CA, then the topological pressure of F and its nth power rule can be presented in a closed form:

$$P(F^n, g) = nP(F, g) - (n-1)\int g \, d\mu, \tag{1}$$

for g continuous function, $n \in \mathbb{N}$. Notably, this expression does not hold in general. A counterexample is studied in Section 4. As an application, the topological pressure of linear CA can be formulated rigorously (cf. [3]), which extends our previous result [4].

The materials of this elucidation are organized as follows. Section 2 states some definitions and notations. Section 3 investigates the sufficient condition for the ergodic CA, and figures out a necessary condition for the ergodicity for a particular class of CA. Section 4 studies the power rule of topological pressure of a CA. A closed expression is given via a strong generator.

2. Preliminaries

Let $\mathcal{A} = \{0, 1, \ldots, \upsilon - 1\}$ be a finite alphabet and let $X = \mathcal{A}^{\mathbb{Z}}$ be the space of infinite sequence $x = (x_n)_{-\infty}^{\infty}$. Whenever a local rule $f : \mathcal{A}^{2k+1} \to \mathcal{A}$ is given, there associates unique CA, say $F : X \to X$, defined by

$$F(x)_i = f(x_{i-k}, \ldots, x_{i+k}),$$

where

$$\mathcal{A}_n = \{a_1 a_2 \cdots a_n : a_i \in \mathcal{A}, 1 \le i \le n\}, \quad n \in \mathbb{N}.$$

For any $m \ge 1$, f can be extended to the mapping $f_m : \mathcal{A}^{2k+m} \to \mathcal{A}^m$ by

$$f_m(x_{-k}, \ldots, x_{k+m-1}) = (f(x_{-k}, \ldots, x_k), \ldots, f(x_{-k+m-1}, \ldots, x_{k+m-1})),$$

where $f_1 = f$. In addition, the nth power rule of f, denote by $f^n : \mathcal{A}^{2nk+1} \to \mathcal{A}$, is defined as

$$f^n(x_{-nk}, \ldots, x_{nk}) = f(f^{n-1}(x_{-nk}, \ldots, x_{(n-1)k}), \ldots, f^{n-1}(x_{-(n-2)k}, \ldots, x_{nk})),$$

for $n \ge 1$.

Hedlund [15] shows a necessary and sufficient condition for the surjection of a CA.

Proposition 2.1 ([15])**.** *Consider F a CA with local rule $f : \mathcal{A}^{2k+1} \to \mathcal{A}$. Then F is surjective if and only if card $f_m^{-1}(y_1, \ldots, y_m) = (\text{card } \mathcal{A})^{2k}$ for all $m \in \mathbb{N}$ and all $(y_1, \ldots, y_m) \in \mathcal{A}^m$, where card S denotes the cardinality of S.*

The study of the local rule of a CA is essential for the understanding of this system. A particular class of local rules, say permutive, is initially introduced by Hedlund [15].

Definition 2.2. *The local rule $f : \mathcal{A}^{2k+1} \to \mathcal{A}$ for a given CA is said to be leftmost (respectively rightmost) permutive in x_i if there exists an integer i, $-k \le i \le -1$ (respectively $1 \le i \le k$), such that*

(i) for $(y_1, \ldots, y_{2k}) \in \mathcal{A}^{2k}$, $g(x_i) \equiv f(y_1, \ldots, y_{2k}; x_i) : \mathcal{A} \to \mathcal{A}$ is a permutation;

(ii) f does not depend on x_j for $j < i$ (respectively $j > i$),

where $x = (x_j)_{j=-k}^{k} \in \mathcal{A}^{2k+1}$.

Definition 2.3. *The local rule f is called bipermutive if it is not only rightmost permutive but also leftmost permutive. f is called permutive if f is one of these three cases, i.e., f is either leftmost or rightmost or bipermutive.*

Hedlund demonstrates a permutive CA is also surjective.

Proposition 2.4 ([15])**.** *If f is permutive, then F is surjective.*

It is easily verified that the permutivity is preserved by power rule.

Lemma 2.5. *If f is permutive, then so is f^n for all $n \in \mathbb{N}$. More precisely, f^n preserves the type of permutivity of f for all $n \in \mathbb{N}$.*

Proof. This follows immediately from the definition of f^n. □

Example 2.6. Consider the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$ and local rule $f : \mathcal{A}^3 \to \mathcal{A}$ defined by

$$f(x_{-1}, x_0, x_1) = 2x_{-1} + x_1 \mod 4.$$

Then f is rightmost permutive.

Observe that the second and the third power rule of f are

$$f^2 : \mathcal{A}^5 \to \mathcal{A}, \quad f^2(x_{-2}, \ldots, x_2) = x_2,$$

and

$$f^3 : \mathcal{A}^7 \to \mathcal{A}, \quad f^3(x_{-3}, \ldots, x_3) = 2x_1 + x_3,$$

respectively. It comes by induction that

$$f^n : \mathcal{A}^{2n+1} \to \mathcal{A}, \quad f^n(x_{-n}, \ldots, x_n) = \begin{cases} x_n, & n \text{ is even;} \\ 2x_{n-2} + x_n, & n \text{ is odd.} \end{cases} \tag{2}$$

Hence f^n is rightmost permutive for all $n \in \mathbb{N}$.

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving transformation. T is called ergodic if the only elements $A \in \mathcal{B}$ with $T^{-1}A = A$ satisfy $\mu(A) = 0$ or $\mu(A) = 1$. The following theorem gives some equivalent conditions for the ergodicity of T.

Theorem 2.7 ([24])**.** *If $T : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$, then the following statements are equivalent:*

(i) *T is ergodic.*

(ii) *If g is measurable and $g \circ T = g$ a.e., then g is constant a.e.*

(iii) *For every $A, B \in \mathcal{B}$ with $\mu(A) > 0, \mu(B) > 0$, there exists $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap B) > 0$.*

Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{v^{|i|}}, \quad x, y \in X. \tag{3}$$

It is easy to verify that d is a metric and (X, d) is a compact metric space.

Consider μ an invariant probability measure on (X, F). Let α and β be two finite measurable partitions of X, define

$$\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}.$$

It is easily seen that $\alpha \vee \beta$ is a refinement of α and β. We define the entropy of the partition α by

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A).$$

The entropy of the measure preserving transformation F with respect to the partition α is given by

$$h_\mu(F, \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1} F^{-i}\alpha\Big), \tag{4}$$

reader may refer to [24] for the existence of the limit. And the measure-theoretic entropy of F is defined by

$$h_\mu(F) = \sup h_\mu(F, \alpha), \tag{5}$$

where the supremum is taken over all finite measurable partitions of X.

Let (X, F) be an endomorphism and let $\mathcal{P}$ be an open cover of X. Set

$$H(\mathcal{P}) = \inf\{\log card \hat{\mathcal{P}}\},$$

where the infimum is taken over the set of finite subcovers $\hat{\mathcal{P}}$ of $\mathcal{P}$ and $cardA$ is the cardinality of A. The topological entropy of the measure preserving transformation F with respect to the open cover $\mathcal{P}$ is defined by

$$h(F, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} F^{-i}\mathcal{P}\Big). \tag{6}$$

For the existence of the limit, we refer reader to [24]. The topological entropy of F is defined by

$$h_{top}(F) = \sup h(F, \mathcal{P}), \tag{7}$$

where the supremum is taken over all finite open covers of X.

In addition, for α an open cover of X and $\phi \in C(X, \mathbb{R})$ a continuous function from X to $\mathbb{R}$, denote by

$$p_n(F, \phi, \alpha) = \inf\{\sum_{B \in \beta} \sup_{x \in B} e^{(S_n\phi)(x)} : \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} F^{-i}\alpha\},$$

where $n \in \mathbb{N}$ and $S_n\phi = \sum_{i=0}^{n-1} \phi \circ F^i$. Then $\lim_{n\to\infty} \frac{1}{n} \log p_n(F, \phi, \alpha)$ exists [24]. For each $\delta > 0$, define

$$P(F, \phi, \delta) = \sup\{\lim_{n\to\infty} \frac{1}{n} \log p_n(F, \phi, \alpha) : diam(\alpha) \leq \delta\}, \tag{8}$$

and

$$P(F, \phi) = \lim_{\delta \to 0} P(F, \phi, \delta). \tag{9}$$

The map $P(F, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is called the topological pressure of F. It comes immediately that $P(F, 0) = h_{top}(F)$.

3. Ergodicity on Permutive Cellular Automata

This section studies the ergodicity of permutive CA. For simplicity, a local rule $f : \mathcal{A}^{2k+1} \to \mathcal{A}$ is presumed that f only depends on x_i, $r \leq i \leq s$, for some $r \leq s$, $r, s \in \mathbb{Z}$.

In the rest of this elucidation, we refer μ to the uniform Bernoulli measure unless otherwise stated.

Theorem 3.1. *If f is permutive, then F is ergodic.*

Proof. It suffices to show that F is ergodic if f is rightmost permutive. The case that f is leftmost permutive can be demonstrated via the same argument, thus is omitted.

Denote by $C(i, j) = {}_i[a_i, \ldots, a_j]_j$ a cylinder of X, i.e., for all $x = (x_n) \in X$, $x_n = a_n$, where $i \leq n \leq j$ and $i \leq j$. First we show that, for any two cylinder $C(i, j) = {}_i[c_i, \ldots, c_j]_j, D(m, l) = {}_m[d_m, \ldots, d_l]_l$, there exists $N \in \mathbb{N}$ such that $\mu(F^{-N}C(i, j) \cap D(m, l)) > 0$, where $i \leq j, m \leq l$.

Without loss of generality, we may assume that $r = 0$. The case that $r \neq 0$ can be done analogously.

We construct a scheme to demonstrate the ergodicity of F.

Since f is rightmost permutive, it comes immediately that $F^{-n}C(i, j)$ is a collection of cylinders $E(i, j + ns)$ with cardinality $card\ F^{-n}C(i, j) = \nu^{ns}$, for $n \in \mathbb{N}$. Therefore, if $l < i$ or $j < m - s$, then the permutivity of f demonstrates that $\mu(F^{-1}C(i, j) \cap D(m, l)) > 0$.

If $m \leq i \leq l \leq j$. Let $\alpha \in \mathbb{Z}^+$ be the smallest number such that $l + \alpha - i$ is a multiple of s, say $l + \alpha - i = \tau s$ for some $\tau \in \mathbb{N}$. Pick $D'(m, l + \alpha) = {}_m[d_m, \ldots, d_l, 0, \ldots, 0]_{l+\alpha} \equiv [d_m, \ldots, d_{l+\alpha}]$ a subcylinder of $D(m, l)$. Define $\kappa_{1,1}, \ldots, \kappa_{1,s}, \kappa_{2,1}, \ldots$ by

$$\begin{aligned}
\kappa_{1,\ell} &= f^{\tau-1}(d_{i+\ell-1}, \ldots, d_{i+\ell+(\tau-1)s-1}), \quad 1 \leq \ell \leq s, \\
\kappa_{2,\ell} &= f^{\tau-2}(d_{i+\ell-1}, \ldots, d_{i+\ell+(\tau-2)s-1}), \quad 1 \leq \ell \leq 2s, \\
&\vdots \\
\kappa_{\tau-1,\ell} &= f(d_{i+\ell-1}, \ldots, d_{i+\ell+s-1}), \quad 1 \leq \ell \leq (\tau - 1)s.
\end{aligned}$$

Since f is rightmost permutive, there exist unique $\beta_{1,1}, \ldots, \beta_{1,j-i+1}$ such that $F([\kappa_{1,1}, \ldots, \kappa_{1,s}, \beta_{1,1}, \ldots, \beta_{1,j-i+1}]) = C(i, j)$, i.e.,

$$E_1 \equiv {}_i[\kappa_{1,1}, \ldots, \kappa_{1,s}, \beta_{1,1}, \ldots, \beta_{1,j-i+1}]_{j+s} \subset F^{-1}C(i, j).$$

Similarly, there exist unique $\beta_{2,1}, \ldots, \beta_{2,j-i+1}$ such that

$$F([\kappa_{2,1}, \ldots, \kappa_{2,2s}, \beta_{2,1}, \ldots, \beta_{2,j-i+1}]) = E_1,$$

i.e.,

$$E_2 \equiv {}_i[\kappa_{2,1}, \ldots, \kappa_{2,2s}, \beta_{2,1}, \ldots, \beta_{2,j-i+1}]_{j+2s} \subset F^{-1}E_1 \subset F^{-2}C(i,j).$$

Inductively we can conclude that $D'(m, l+\alpha) \subset F^{-\tau}C(i,j)$. This implies $\mu(F^{-\tau}C(i,j) \cap D(m,l)) > 0$.

The other cases, such as $m \leq i \leq j \leq l$, $i \leq m \leq j \leq l$, and so on, can be done analogously. Hence, for any two cylinders $C(i,j)$ and $D(m,l)$, there exists $N \in \mathbb{N}$ such that $\mu(F^{-1}C(i,j) \cap D(m,l)) > 0$.

For any two measurable sets A, B with positive measure and $\epsilon > 0$, there exist two collections of pairwise disjoint cylinders $A_1, \ldots, A_m$, $B_1, \ldots, B_l$ such that

(i) $\cup A_i \subseteq A$ and $\cup B_j \subseteq B$;

(ii) $\mu(A \setminus \cup A_i) < \epsilon$ and $\mu(B \setminus \cup B_j) < \epsilon$.

It can be easily verified that the permutivity of F asserts $F^{-1}A_i \cap F^{-1}A_j = \varnothing$ and $F^{-1}B_i \cap F^{-1}B_j = \varnothing$ for all $i \neq j$. Moreover,

$$F^{-n}A \cap B \supseteq (F^{-n}(\cup A_i)) \cap (\cup B_j) = \cup_{i,j}(F^{-n}A_i \cap B_j),$$

for all $n \in \mathbb{N}$. Therefore, there exists $N \in \mathbb{N}$ such that $\mu(F^{-N}A \cap B) > 0$.

The proof is completed. □

Example 3.2. For $\mathcal{A} = \{0, 1, \ldots, \nu-1\}$, $\nu \geq 2$. Let $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the shift map. Then σ is rightmost permutive.

It is easy to see that σ is ergodic. For any two distinct cylinders $C(i,j)$ and $D(m,l)$, where $i \leq j, m \leq l$. There exists $n \in \mathbb{N}$ such that $i + n > l$. Thus $\sigma^{-n}C(i,j) \cap D(m,l) = C(i+n, j+n) \cap D(m,l)$ is of positive measure. That is, σ is ergodic.

Example 3.3 (Continued)**.** In Example 2.6, we consider the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$ and local rule $f : \mathcal{A}^3 \to \mathcal{A}$ defined by

$$f(x_{-1}, x_0, x_1) = 2x_{-1} + x_1 \mod 4.$$

Moreover,

$$f^n : \mathcal{A}^{2n+1} \to \mathcal{A}, \quad f^n(x_{-n}, \ldots, x_n) = \begin{cases} x_n, & n \text{ is even;} \\ 2x_{n-2} + x_n, & n \text{ is odd.} \end{cases} \tag{10}$$

It is easily seen that if F^n is ergodic for some $n \in \mathbb{N}$, then so is F. For all $k \in \mathbb{N}$, $f^{2k}(x_{-2k}, \ldots, x_{2k}) = x_{2k}$. This indicates F^{2k} is kind of a "multiple-shift map", thus is ergodic. Therefore, F is ergodic.

If the cardinality of states $\mathcal{A}$ is a prime, then every additive CA is permutive except for those local rules of the form $f : \mathcal{A} \to \mathcal{A}, f(x) = \alpha x$ for some $0 \leq \alpha \leq p-1$, where *card* $\mathcal{A} = p$.

There is an example that, f is permutive in x_0 but not ergodic since f is not rightmost permutive.

Proposition 3.4. *Consider* $\mathcal{A} = \{0, 1\}$ *and local rule* $f : \mathcal{A}^3 \to \mathcal{A}$ *defined by*

$$f(x_0, x_1, x_2) = x_0 + x_1(x_2 + 1) \mod 2.$$

Then F *is not ergodic.*

Proof. It comes from [8] that F is not topologically transitive. Therefore, F is not ergodic. □

However, the assumption of permutivity is the necessary condition for the ergodicity in a particular class of CA.

Proposition 3.5. *Consider $\mathcal{A} = \{0, 1, \ldots, \nu - 1\}$ and F a CA satisfying the following conditions:*

(i) *f is additive, i.e., $f : \mathcal{A}^{2k+1} \to \mathcal{A}$ can be represented as $f(x_{-k}, \ldots, x_k) = \sum_{i=-k}^{k} a_i x_i$.*

(ii) *There exists unique $\ell, -k \leq \ell \leq k$, such that $a_\ell = 1$.*

(iii) *If f only depends on x_i, $r \leq i \leq s$, for some $-k \leq r \leq s \leq k$, then g.c.d.$(a_m, \nu) > 1$ for $r < m < s, m \neq \ell$.*

Then F is ergodic if and only if f is either leftmost or rightmost permutive.

Proof. It suffices to show that, if f is neither leftmost nor rightmost permutive, then F is not ergodic.

Write $f(x_r, \ldots, x_s) = \sum_{i=r}^{s} a_i x_i$. There exists $\kappa \in \mathbb{N}, \kappa \neq 0 \mod \nu$, such that $\kappa a_i = 0 \mod \nu$ for all $i \neq \ell$. Then

$$\kappa F(x)_j = \kappa \sum_{i=r}^{s} a_i x_{i+j} = \kappa x_{j+\ell}.$$

Define

$$g : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R} \quad \text{by } g(x) = \sum \kappa x_i,$$

it is easily seen that g is measurable and $g \circ F = g$ on $\mathcal{A}^{\mathbb{Z}}$. The fact that g is not constant demonstrates F is not ergodic.

This completes the proof. □

Example 3.6. Consider $\mathcal{A} = \{0, 1, 2, 3\}$ and local rule $f : \mathcal{A}^3 \to \mathcal{A}$ defined by

$$f(x_{-1}, x_0, x_1) = 2x_{-1} + x_0 + 2x_1 \mod 4.$$

Then f is neither rightmost nor leftmost permutive.

Consider $g : X \to \mathbb{R}$ defined by $g(x) = 2x_0$. Then g is measurable and nonconstant, however, $g \circ F = g$ for all $x \in X$. This demonstrates F is not ergodic.

4. The Topological Pressure of Power Rule of CA

This section investigates the topological pressure via a strong generator.

Consider (X, F) an endomorphism and ξ a finite cover of X, we call ξ a strong generator if, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\| \bigvee_{i=0}^{m-1} F^{-i}\xi \| < \delta$ for all $m \geq N$, where $\| A \|$ denotes the diameter of A. In other words, ξ is a strong generator if and only if

$$\| \bigvee_{i=0}^{m-1} F^{-i}\xi \| \to 0, \quad \text{as } n \to \infty.$$

For $a, b \in \mathbb{Z}$, $a \leq b$, denote by ${}_a[s_a, \ldots, s_b]_b = \{x \in X : x_a = s_a, \ldots, x_b = s_b\}$ a cylinder in X and $\xi(a, b) = \{{}_a[x_a, \ldots, x_b]_b : x_a, \ldots, x_b \in S\}$ a partition of X.

Lemma 4.1. *Let f be permutive and let $\xi \equiv \xi(a,b)$ be a partition of X for $a \leq b$, $a,b \in \mathbb{Z}$. Then $\xi(a,b)$ is a strong generator.*

Proof. We show the case that f is rightmost permutive, the other cases can be done similarly.

Assume that f is rightmost permutive in x_s and does not depend on x_j for all $j < r$, where $r < s$. Denote by $C(i,j) = {}_i[a_i, \ldots, a_j]_j$ a cylinder of X, i.e., for all $x = (x_m) \in X$, $x_m = a_m$, where $i \leq m \leq j$ and $i \leq j$. It is easy to verify that $F^{-1}A \cap F^{-1}B = \varnothing$ since f is rightmost permutive. Moreover, for any cylinder $C(i,j)$, $F^{-1}C(i,j) \subset \xi(i+r, j+s)$ with cardinality v^{s-r}. In other words, $F^{-1}\xi(a,b) = \xi(a+r, b+s)$.

Denote $\xi \equiv \xi(a,b)$ for simplicity. It follows from the discussion above that

$$\xi \vee F^{-1}\xi = \begin{cases} \xi(a, b+s), & 0 \leq r; \\ \xi(a+r, b+s), & 0 \geq r. \end{cases} \tag{11}$$

Inductively,

$$\xi \vee F^{-1}\xi \vee \cdots \vee F^{-(m-1)}\xi = \begin{cases} \xi(a, b+(m-1)s), & 0 \leq r; \\ \xi(a+(m-1)r, b+(m-1)s), & 0 \geq r. \end{cases} \tag{12}$$

This elucidates that

$$\| \xi \vee F^{-1}\xi \vee \cdots \vee F^{-(m-1)}\xi \| \to 0, \quad \text{as } m \to \infty,$$

i.e., ξ is a strong generator.

The proof is completed. □

Example 4.2 (Continued)**.** Consider $\mathcal{A}$ and f same as given in Example 2.6. Pick $\xi(0) = \{[0]_0, [1]_0, [2]_0, [3]_0\}$ the standard partition of $X = \mathcal{A}^{\mathbb{Z}}$. Then

$$\begin{aligned}
F^{-1}[0]_0 &= \{{}_{-1}[0,\cdot,0]_1, {}_{-1}[1,\cdot,2]_1, {}_{-1}[2,\cdot,0]_1, {}_{-1}[3,\cdot,2]_1\}, \\
F^{-1}[1]_0 &= \{{}_{-1}[0,\cdot,1]_1, {}_{-1}[1,\cdot,3]_1, {}_{-1}[2,\cdot,1]_1, {}_{-1}[3,\cdot,3]_1\}, \\
F^{-1}[2]_0 &= \{{}_{-1}[0,\cdot,2]_1, {}_{-1}[1,\cdot,0]_1, {}_{-1}[2,\cdot,2]_1, {}_{-1}[3,\cdot,0]_1\}, \\
F^{-1}[3]_0 &= \{{}_{-1}[0,\cdot,3]_1, {}_{-1}[1,\cdot,1]_1, {}_{-1}[2,\cdot,3]_1, {}_{-1}[3,\cdot,1]_1\}.
\end{aligned}$$

It follows that

(i) $F^{-1}[i]_0 \cap F^{-1}[j]_0 = \varnothing$ for $i \neq j$;

(ii) for $0 \leq i \leq 3$, $A \cap B = \varnothing$ for $A \neq B$, $A, B \in F^{-1}[i]_0$;

(iii) $F^{-1}\xi(0) = \xi(-1,1)$.

It is easily verified that

$$\bigvee_{i=0}^{n-1} F^{-i}\xi(0) = \xi(-(n-1), n-1), \quad \text{and } \| \bigvee_{i=0}^{n-1} F^{-i}\xi(0) \| \to 0,$$

as $n \to \infty$. In other words, $\xi(0)$ is a strong generator. In addition, it can be checked without difficulty that $\xi(a,b)$ is a strong generator for $a \leq b$.

It is well-known that $h_{top}(F^n) = nh_{top}(F)$ and $P(F^n, S_n g) = nP(F,g)$ for all $n \in \mathbb{N}$, where $S_n g = \sum_{i=0}^{n-1} g \circ F^i$ and $g \in C(X, \mathbb{R})$ [24]. There is an intuitive connection between F^n and the nth power rule of local rule f.

Lemma 4.3. *Consider a CA F with local rule f. Then the nth power rule of F, F^n, is a CA with local rule f^n for $n \in \mathbb{N}$.*

Proof. This comes from the definition of F^n and f^n, thus is omitted. ☐

The main theorem of this section then follows.

Theorem 4.4. *Consider F a CA with permutive local rule f. The topological pressure of F^n can be expressed as*

$$P(F^n, g) = nP(F, g) - (n-1) \int_X g \; d\mu, \tag{13}$$

for all $n \in \mathbb{N}$, $g \in C(X, \mathbb{R})$.

Theorem 3.1 shows that, for a CA F with the local rule f, if f is permutive, then F is ergodic. Therefore, there is an immediate consequence comes from Birkhoff Ergodic Theorem.

Lemma 4.5. *Consider $g \in C(X, \mathbb{R})$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $\ell \geq N$,*

$$|\sup_{x \in A} S_\ell g(x) - \sup_{x \in B} S_\ell g(x)| < \ell\epsilon,$$

for any two disjoint measurable set A, B, where $S_\ell g = \sum_{i=0}^{\ell-1} g \circ F^{ni}$.

Proof. The permutivity of f implies f^n is permutive, thus F^n is ergodic. Birkhoff Ergodic Theorem demonstrates that $\frac{1}{\ell} S_\ell g$ converges a.e. to a function $\bar{g}$ and $\bar{g} = \int_X g \; d\mu$ is constant a.e.

The proof is completed. ☐

Proof of Theorem 4.4. ξ is a strong generator implies that

$$P(F^n, g) = \lim_{m \to \infty} \frac{1}{m} \log p_m(F^n, g, \xi),$$

for $g \in C(X, \mathbb{R})$. Same discussion as in the proof of Lemma 4.1 indicates that

$$\bigvee_{i=0}^{m-1} F^{-ni} \xi = \begin{cases} \xi(a, b + (m-1)ns), & 0 \leq r; \\ \xi(a + (m-1)nr, b + (m-1)ns), & 0 \geq r. \end{cases}$$

and

$$card \bigvee_{i=0}^{m-1} F^{-ni} \xi = \begin{cases} \upsilon^{b-a+1+(m-1)ns}, & 0 \leq r; \\ \upsilon^{b-a+1+(m-1)n(s-r)}, & 0 \geq r. \end{cases}$$

Hence

$$\begin{aligned} P(F^n, g) &= \lim_{m \to \infty} \frac{1}{m} \log p_m(F^n, \phi, \xi) \\ &= \lim_{m \to \infty} \frac{1}{m} \log \inf \{ \sum_B \sup_{x \in B} e^{(S_m g)(x)} : B \in \bigvee_{i=0}^{m-1} F^{-ni} \xi \} \\ &= \lim_{m \to \infty} \frac{1}{m} \log \begin{cases} \upsilon^{b-a+1+(m-1)ns} e^{(S_m g)}, & 0 \leq r; \\ \upsilon^{b-a+1+(m-1)n(s-r)} e^{(S_m g)}, & 0 \geq r. \end{cases} \\ &= \begin{cases} ns \log \upsilon + \int_X g \; d\mu, & 0 \leq r; \\ n(s-r) \log \upsilon + \int_X g \; d\mu, & 0 \geq r. \end{cases} \end{aligned}$$

by Lemma 4.5.

It is easily seen that $P(F^n, g) = nP(F, g) - (n-1) \int_X g \; d\mu$.

This completes the proof. ☐

Example 4.6 (Continued)**.** Let $\mathcal{A}$ and f same as in Example 2.6. Replace n by 1 in the proof above, we have

$$P(F, g) = 2 \log 4 + \int_X g \, d\mu,$$

and

$$P(F^n, g) = 2n \log 4 + \int_X g \, d\mu.$$

Hence $P(F^n, g) = nP(F, g) - (n-1) \int_X g \, d\mu$ for all $n \in \mathbb{N}$.

Furthermore, Ward [25] shows that $h_{top}(F) = 2 \log 4$. This implies

$$P(F^n, g) = nh_{top}(F) + \int_X g \, d\mu = h_{top}(F^n) + \int_X g \, d\mu.$$

Notably, Theorem 4.4 fails if the permutivity of F is omitted.

Proposition 4.7. *Consider F a CA with local rule*

$$f : \mathcal{A}^{2k+1} \to \mathcal{A}, \quad \text{defined by } f(x_{-k}, \dots, f_k) = \kappa x_0,$$

where $0 \le \kappa \le \upsilon - 1$. Then $P(F^n, g) = 0$ for all $n \in \mathbb{N}, g \in C(X, \mathbb{R})$.

Proof. It can be easily verified that F is not permutive.

If $\kappa = 0$, it comes immediately that $P(F^n, g) = 0$ for all $n \in \mathbb{N}, g \in C(X, \mathbb{R})$. It remains to show the cases $\kappa \neq 0$.

For $n \in \mathbb{N}$ and $\xi(a, b)$ a partition of X, where $a \le b$. Since $f(x_{-k}, \dots, x_k) = \kappa x_0$, we have $\bigvee_{i=0}^{m-1} F^{-ni}\xi(a, b) = \xi(a, b)$. This implies $P(F^n, g) = 0$ for any $g \in C(X, \mathbb{R})$.

The proof is completed. □

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