Massive gravity with nonminimal coupling

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We propose new massive gravity theories with 5 dynamical degrees of freedom. We evade uniqueness theorems regarding the form of the kinetic and potential terms by adopting the "generalized massive gravity" framework, where a global translation invariance is broken. By exploiting the rotation symmetry in the field space, we determine two novel classes of theories. The first one is an extension of generalized massive gravity with a nonminimal coupling. On the other hand, the second theory produces a mass term that is different from de Rham, Gabadadze, Tolley construction and trivially has 5 degrees of freedom. Both theories allows for stable cosmological solutions without infinite strong coupling, which are free of ghost and gradient instabilities.

I. INTRODUCTION

Whether a consistent Lorentz-invariant massive gravity theory exists or not has been a long-standing issue for more than seven decades, starting with the pioneering work by Fierz and Pauli [1]. The Fierz-Pauli theory of a massive spin-2 field is constructed by choosing a specific combination of the mass terms to have 5 physical degrees of freedom, but a naive massless limit does not reduce to the massless theory, i.e., linearized general relativity [2,3]. This discontinuity was realized to be an artifact of the linear theory and can be solved by taking into account nonlinear completions of the Fierz-Pauli mass term [4]. Despite this elegant solution, an unwanted sixth ghost degree of freedom called the Boulware-Daser (BD) ghost appears at the nonlinear level [5], and thus nonlinear massive gravity was thought to be unstable [6]. Relatively recently, it was shown that the cure to this ghost problem is supplied by adding infinite nonlinear corrections to potential terms [7,8]. This ghost-free massive gravity called de Rham-Gabadadze-Tolley (dRGT) theory admits an open-FLRW solution, where the dRGT mass terms exactly behave as the cosmological constant [9], but the scalar and vector kinetic terms of perturbations around this background unfortunately vanishes, which is a signal of a strong coupling problem [10] and worse, a nonlinear ghost instability [11]. Thus, further extensions of the nonlinear massive gravity should be necessary to accommodate a stable cosmological solution.

Although one can simply extend the dRGT theory by introducing other fields such as the quasidilaton theory [12], mass-varying massive gravity [13], and Hassan-Rosen bigravity [14], it might be more interesting if the massive gravity can be further generalized without invoking additional degrees of freedom. So far, such extensions have been intensively investigated with the various approaches, but none of them are successful at this point [15–19]. All of the above investigations rely on theories invariant under the Poincaré symmetry in the internal field space consisted of the Stückelberg fields ϕ^a, which are responsible for restoring the general covariance [20]. However, once we abandon the translation invariance while keeping the global Lorentz invariance, a natural extension of the dRGT theory can be accessible [21]. In this theory called the generalized dRGT theory, the constant parameters in the graviton potential are promoted to be arbitrary functions of four auxiliary fields ϕ^a, and the total number of physical degrees of freedom remains the same as the dRGT theory; i.e., the BD ghost is absent. Furthermore, it has been recently shown that all perturbations around the open-FLRW background are free from any instabilities [22].

Now, it is interesting to know how far we can generalize (global) Lorentz-invariant massive gravity theories. In any massive gravity theory with 5 degrees of freedom, there exists the Hamiltonian constraint in unitary gauge, and this guarantees the absence of the BD ghost [23]. In the Stückelberg language, the degeneracy of the kinetic matrix of four scalar fields ϕ^a leads one of them to be nondynamical, implying that the would-be BD ghost is successfully eliminated [24]. Therefore, imposing the degeneracy of the four scalar fields ϕ^a, one should be able to explore new theories of massive gravity. In the present paper, we investigate the possibility of extending massive gravity theory which preserves the global Lorentz symmetry and derive two distinct classes of ghost-free theories.

To derive these novel theories, we make use of two specific properties of these field theories that become available once translation invariance is dropped. First, introducing the matter fields after performing conformal transformation of the physical metric generates a non-minimal coupling of the Stückelberg fields to the curvature. Second, we show that conformal and disformal deformations of the fiducial metric may evade the reappearance of the BD mode under certain conditions, which only leave the two unconnected theory classes that we present here. The first theory is a direct extension of generalized massive gravity with modified fiducial metric and nonminimal coupling. The derivation of this theory also reveals how one can generate the original generalized massive gravity action starting from constant parameter dRGT theory. The second theory is different from dRGT type constructions by directly projecting out 1 degree of freedom and is thus closely linked to Lorentz breaking theories.

This paper is organized as follows. In Sec. II, we briefly review the dRGT massive gravity and discuss the implications of breaking the translation invariance, which allows a conformal transformation of the physical metric. In Sec. III, we generalize the dRGT mass terms by using disformal transformation acted on the fiducial metric and summarize necessary conditions to eliminate the would-be BD ghost. In Sec. IV, we investigate a kinetic Lagrangian including a nonminimal coupling and derive a degeneracy condition by using $3 + 1$ decomposition. In Sec. V, we derive background equations and quadratic actions for perturbations in the two theories obtained in Sec. IV and show that all perturbations around Friedmann-Lemaître-Robertson-Walker (FLRW) background are free of ghost and gradient instabilities. Section VI is devoted to summary.

II. BREAKING TRANSLATION INVARIANCE AND NONMINIMAL COUPLING

In this section, we briefly introduce the ghost-free massive gravity and argue the existence of a nonminimal coupling in a global Lorentz invariant massive gravity. Let us start with the ghost-free dRGT massive gravity, which is given by [7,8]

$$S_{\mathrm{dRGT}} = \int d^4x\sqrt{-g}\frac{M_p^2}{2}\left[R - 2m^2\sum_{n=0}^{3}\beta_n e_n\left(\sqrt{g^{-1}f}\right)\right]$$
$$+ S_{\mathrm{m}}[g,\psi], \tag{1}$$

where β_n are constant parameters, S_{m} is the matter action, and the dRGT potential terms are built out of e_n,

$$e_0(\mathcal{Q}) = 1, \tag{2}$$

$$e_1(\mathcal{Q}) = [\mathcal{Q}], \tag{3}$$

$$e_2(\mathcal{Q}) = \frac{1}{2!}([\mathcal{Q}]^2 - [\mathcal{Q}^2]), \tag{4}$$

$$e_3(\mathcal{Q}) = \frac{1}{3!}([\mathcal{Q}]^3 - 3[\mathcal{Q}][\mathcal{Q}^2] + 2[\mathcal{Q}^3])), \tag{5}$$

$$e_4(\mathcal{Q}) = \frac{1}{4!}([\mathcal{Q}]^4 - 6[\mathcal{Q}]^2[\mathcal{Q}^2] + 3[\mathcal{Q}^2]^2 + 8[\mathcal{Q}][\mathcal{Q}^3] - 6[\mathcal{Q}^4]). \tag{6}$$

Here, $[\mathcal{Q}]$ denotes the trace of the matrix $\mathcal{Q}$, a square root of the matrix represents the matrix satisfies $\sqrt{\mathcal{Q}}^\mu{}_\rho\sqrt{\mathcal{Q}}^\rho{}_\nu = \mathcal{Q}^\mu{}_\nu$, and $f_{\mu\nu}$ is the fiducial metric, which is defined through

$$f_{\mu\nu} \equiv \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b, \tag{7}$$

and η_{ab} is the Minkowski metric with $a, b = 0, 1, 2, 3$. The action (1) is manifestly invariant under the Poincaré symmetry in the internal field space. Once we abandon the global translation invariance $\phi^a \to \phi^a + c$, the scalar function $X = \eta_{ab}\phi^a\phi^b = \phi^a\phi_a$ can promote the constant parameters β_n to be functions of X, and the resultant theory, i.e., the generalized massive gravity, still enjoys a global Lorentz invariance [21]. In the present paper, we seek further extensions to such a global Lorentz invariant massive gravity.

Let us first consider the conformal transformation utilizing the scalar X,

$$\tilde{g}_{\mu\nu} = G(X)g_{\mu\nu}. \tag{8}$$

Performing this transformation to the action (1), the gravitational part of the Lagrangian becomes

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_p^2}{2}R[\tilde{g}] - M_p^2m^2\sum_{n=0}^{3}\beta_n e_n\left(\sqrt{\tilde{g}^{-1}f}\right)\right]$$
$$= \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}G\left(R + \frac{3}{2}\nabla^\mu\log G\nabla_\mu\log G\right) - M_p^2m^2\sum_{n=0}^{3}\tilde{\beta}_n(X)e_n\left(\sqrt{g^{-1}f}\right)\right], \tag{9}$$

where we defined the rescaled parameters as

$$\tilde{\beta}_n(X) \equiv \beta_n G(X)^{\frac{4-n}{2}}. \tag{10}$$

The kinetic part of the Lagrangian contains the nonminimal coupling with the conformal factor and its counterterm, which can be rewritten as

$$\frac{3}{2}\nabla^\mu \log G(X)\nabla_\mu \log G(X) = \frac{6G_X^2}{G^2}\phi_a\phi_b g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{11}$$

Thus, once the translation invariance is broken, the Einstein-Hilbert term can be nonminimally coupled with ϕ^a with a suitable counterterm. Furthermore, the parameters $\tilde{\beta}_n$ after the transformation are the arbitrary functions of X, and this is reminiscent of the generalized massive gravity. The conformal scaling in the mass terms can be interpreted as a redefinition of the fiducial metric, $\tilde{g}^{-1}f \to g^{-1}(G^{-1}f)$. This implies that the fiducial metric no longer needs to be of the form (7), and one can deform the fiducial metric by appropriately contracting the Lorentz indices by η_{ab} and ϕ_a and introducing arbitrary functions of X.

III. DISFORMAL DEFORMATIONS OF THE FIDUCIAL METRIC

As we have seen in the previous section, a conformal transformation of the physical metric would lead us to a

new extension of massive gravity including a nonminimal coupling. Another effect of the conformal transformation is to rescale the fiducial metric by a conformal factor that depends on the Stückelberg fields. In this section we explore this option further by starting with the most general deformation of the fiducial metric,

$$\tilde{f}_{\mu\nu,I} = L_{ab,I}\partial_\mu\phi^a\partial_\nu\phi^b, \tag{12}$$

with

$$L_{ab,I} = C_I(X)\eta_{ab} + D_I(X)\phi_a\phi_b, \tag{13}$$

where I is a label that will be assigned to each mass term. We then construct the following square-root matrix:

$$Q^\mu{}_{\nu,I} \equiv \left(\sqrt{g^{-1}\tilde{f}_I}\right)^\mu{}_\nu. \tag{14}$$

Here, if all $C_I = 1$ and $D_I = 0$, this reduces to the square-root matrix in dRGT massive gravity. Then, we consider the action with the Einstein-Hilbert term and mass terms,

$$S = \int d^4x \sqrt{-g}\frac{M_p^2}{2}[R[g] - 2m^2\mathcal{L}_{\text{mass}}], \tag{15}$$

where

$$\mathcal{L}_{\text{mass}} = \beta(X)[Q_\beta] + \gamma_1(X)[Q_{\gamma_1}]^2 + \gamma_2(X)[Q_{\gamma_2}^2] + \delta_1(X)[Q_{\delta_1}]^3 + \delta_2(X)[Q_{\delta_2}][Q_{\delta_2}^2] + \delta_3(X)[Q_{\delta_3}^3]$$
$$+ \sigma_1(X)[Q_{\sigma_1}]^4 + \sigma_2(X)[Q_{\sigma_2}]^2[Q_{\sigma_2}^2] + \sigma_3(X)[Q_{\sigma_3}^2]^2 + \sigma_4(X)[Q_{\sigma_4}][Q_{\sigma_4}^3] + \sigma_5(X)[Q_{\sigma_5}^4]. \tag{16}$$

In general, the four scalar components in the Stückelberg fields ϕ^a bring an extra degree of freedom, i.e., the BD ghost. To avoid this, the action should be arranged such that one of the component has no dynamics. This can be actually realized by the degeneracy of the kinetic matrix of the Stückelberg fields [24]. Then, it generates a primary and subsequent constraints, and the BD ghost can be successfully eliminated. We here adopt this approach to obtain BD ghost-free conditions rather than the decoupling limit approach [7].

The majority of the nonperturbative proofs of the absence of ghost in dRGT theory relies on the unitary gauge.[1] There are several nonperturbative proofs of the absence of the BD ghost in the presence of the Stückelberg fields [24–29], but these rely on specific techniques not applicable to the case at hand. Due to the square-root form of the building block tensor (14) it is highly challenging to

derive the degeneracy conditions exactly. However, around fixed backgrounds we can obtain necessary (but not sufficient) degeneracy conditions as follows.[2] In the mass term, we introduced 20 arbitrary functions to represent the 10 disformal transformations. The following conditions reduce the number of independent functions down to 4:

$$\frac{D_\beta}{C_\beta} = \frac{D_{\gamma_1}}{C_{\gamma_1}} = \frac{D_{\gamma_2}}{C_{\gamma_1}} = \frac{D_{\delta_1}}{C_{\delta_1}} = \frac{D_{\delta_2}}{C_{\delta_1}} = \frac{D_{\delta_3}}{C_{\delta_1}} = \frac{D_{\sigma_1}}{C_{\sigma_1}} = \frac{D_{\sigma_2}}{C_{\sigma_1}}$$
$$= \frac{D_{\sigma_3}}{C_{\sigma_1}} = \frac{D_{\sigma_4}}{C_{\sigma_1}} = \frac{D_{\sigma_5}}{C_{\sigma_1}},$$
$$C_{\gamma_1} = C_{\gamma_2},$$
$$C_{\delta_1} = C_{\delta_2} = C_{\delta_3},$$
$$C_{\sigma_1} = C_{\sigma_2} = C_{\sigma_3} = C_{\sigma_4} = C_{\sigma_5}, \tag{17}$$

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[1]When translation invariance is broken, the unitary gauge, $\phi^a = \delta_\mu^a x^\mu$, is no longer a convenient choice. Instead, the gauge fixing with $\phi^0 = \sqrt{t^2 + x^i x_i}$ and $\phi^i = x^i$ so that $\phi^a\phi_a = -t^2$ would be convenient as introduced in [21].

[2]These conditions were derived by combining the degeneracy conditions around Bianchi type V and a static nondiagonal metric with a preferred direction. The details of the calculation can be found in Appendix A.

which implies that all mass terms are built out of a universal field space metric $\eta_{ab} + \frac{D}{C}\phi_a\phi_b$ with three independent conformal factors for each orders. Using these relations, the conditions that the mass functions need to satisfy can be written as

$$(C_{\gamma_1} + XD_{\gamma_1})(\gamma_1 + \gamma_2) = 0,$$
$$(C_{\delta_1} + XD_{\delta_1})^{3/2}(\delta_1 + \delta_2 + \delta_3) = 0,$$
$$\sqrt{C_{\delta_1}}(C_{\delta_1} + XD_{\delta_1})(3\delta_1 + \delta_2) = 0,$$
$$(C_{\sigma_1} + XD_{\sigma_1})^2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) = 0,$$
$$\sqrt{C_{\sigma_1}}(C_{\sigma_1} + XD_{\sigma_1})^{3/2}(4\sigma_1 + 2\sigma_2 + \sigma_4) = 0,$$
$$C_{\sigma_1}(C_{\sigma_1} + XD_{\sigma_1})(6\sigma_1 + \sigma_2) = 0,$$
$$C_{\sigma_1}(C_{\sigma_1} + XD_{\sigma_1})(3\sigma_1 + \sigma_2 + \sigma_3) = 0. \tag{18}$$

There are two ways to satisfy these conditions. For $(1 + D_I X/C_I) \neq 0$, the first solution is

$$\gamma_1 = -\gamma_2, \qquad \delta_1 = -\frac{\delta_2}{3} = \frac{\delta_3}{2},$$
$$\sigma_1 = -\frac{\sigma_2}{6} = \frac{\sigma_3}{3} = \frac{\sigma_4}{8} = -\frac{\sigma_5}{6}, \tag{19}$$

which simply is the dRGT tuning with mass term,[3]

$$\mathcal{L}_{\text{mass}} = \beta_1(X)e_1(Q) + \beta_2(X)e_2(Q)$$
$$+ \beta_3(X)e_3(Q) + \beta_4(X)e_4(Q), \tag{20}$$

where $Q \equiv \eta_{ab} + D(X)\phi_a\phi_b$ and by absorbing the conformal factors at each order into the mass function, we reduce the number of arbitrary functions down to 4 with the following definitions:

$$\beta_1 \equiv C_\beta\beta, \qquad \beta_2 \equiv 2C_{\gamma_1}\gamma_1, \qquad \beta_3 \equiv 3!C_{\delta_1}^{3/2}\delta_1,$$
$$\beta_4 \equiv 4!C_{\sigma_1}^{3/2}\sigma_1, \qquad D \equiv \frac{D_{\gamma_1}}{C_{\gamma_1}}. \tag{21}$$

In other words, there can only be one field space metric, disformally related to the original Minkowski metric. The case with a field space-metric that can explicitly depend on the ϕ^a was already argued to be ghost-free in [26]. Thus, starting from dRGT theory (1) with constant mass parameters, one can consider deformations of the field space metric with different conformal coefficients at different orders and generate the generalized massive gravity action.

The second option for satisfying Eq. (18) is to fix the disformal term in the field space metric as

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[3]With this tuning, the quartic mass term $\sqrt{-g}\beta_4(X)e_4(Q) = \sqrt{-\tilde{f}}\beta_4(X)$ becomes a boundary term, similarly to constant mass dRGT.

$$\frac{D_{\gamma_1}}{C_{\gamma_1}} = -\frac{1}{X}, \tag{22}$$

which leaves all of the mass functions unconstrained. This condition is equivalent to having a field space metric proportional to a projection tensor P_{ab} defined as

$$P_{ab} \equiv \left(\eta_{ab} - \frac{\phi_a\phi_b}{X}\right), \tag{23}$$

which projects onto surfaces in the field space defined by normal vector ϕ^a, while the conformal factors can be absorbed in the individual mass functions. This precise combination guarantees that the derivative of one of the directions is absent in $\tilde{f}_{\mu\nu}$. The mass term constructed with this projection tensor cannot be combined with (unprojected) dRGT-type terms where the degeneracy disagrees with the one imposed by the projection. As we will see in the next section, the projected mass terms naturally lack the BD mode, and are reminiscent of some of the Lorentz violating massive gravity theories [30,31]. The difference however is that the time direction in our theory remains unspecified, thanks to the explicit dependence on ϕ^a allowed by the broken translation invariance.

IV. EVADING BD GHOST IN 3 + 1 DECOMPOSITION

In this section, we extend the previous analysis to include a nonminimal coupling. To see the kinetic structure of the nonminimal coupling, we consider the following action:

$$S = \int d^4x\sqrt{-g}\frac{M_p^2}{2}[G(X)R[g] + F(X)[Y] + A(X)[W]], \tag{24}$$

where we defined W and Y as

$$W^\mu_\nu = (g^{-1}f)^\mu_\nu, \qquad Y^\mu_\nu \equiv g^{\mu\nu}\phi_a\phi_b\partial_\mu\phi^a\partial_\nu\phi^b. \tag{25}$$

These two terms are responsible for the degeneracy of the kinetic matrix with the nonminimal coupling $G(X)R$. Other higher order candidates such as $[W^2]$ and $[Y^2]$ as well as the terms involving the square-root tensor cannot be counterterms for the nonminimal coupling. This is because the nonminimal coupling term is quadratic in the extrinsic curvature and contains a linear mixing between $\dot{\phi}_a$ and the extrinsic curvature [see (27) below]. Thus the counterterms need to be quadratic in $\dot{\phi}_a$. The F term is the one that arises from the conformal transformation of the physical metric (11). F and A terms can be included in γ_2 term of (15), and these terms can be generated by the conformal and disformal transformation of the fiducial metric.

We investigate the degeneracy using a 3 + 1 decomposition. This is similar to the analysis performed to find

degenerate higher order scalar tensor theories [32–34], but the difference is that we are interested in finding degeneracies between metric variables and four scalar fields ϕ^a in our case.

Let us define normal vector n^μ of each spacelike hypersurface Σ_t, which satisfies $n^\mu n_\mu = -1$. The induced metric $\gamma_{\mu\nu}$ on a spacelike hypersurface Σ_t is then defined by $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. The derivative of the normal vector can be expressed as $\nabla_\mu n_\nu = -n_\mu a_\nu + K_{\mu\nu}$, where a_μ is the acceleration vector, $a^\mu := n^\nu \nabla_\nu n^\mu$, and $K_{\mu\nu}$ is the extrinsic curvature, $K_{\mu\nu} := \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \nabla_\rho n_\sigma$. By using the normal vector and the induced metric, the partial derivative of four scalar fields can be decomposed as

$$\partial_\mu \phi^a = -n_\mu \dot{\phi}^a + D_\mu \phi^a, \qquad (26)$$

where we defined $\dot{\phi}^a := n^\mu \partial_\mu \phi^a$ and $D_\mu \phi^a := \gamma_\mu{}^\nu \partial_\nu \phi^a$.

Then the kinetic part of the Lagrangian (24) can be expressed as

$$\mathcal{L}_{\text{kin}} = \mathcal{A}^{ab} \dot{\phi}_a \dot{\phi}_b + \mathcal{C}^{a\mu\nu} \dot{\phi}_a K_{\mu\nu} + \mathcal{F}^{\mu\nu\rho\sigma} K_{\mu\nu} K_{\rho\sigma}, \qquad (27)$$

with

$$\mathcal{A}^{ab} = -A\eta^{ab} - F\phi^a \phi^b, \qquad \mathcal{C}^{a\mu\nu} = -4\eta^{ab} G_X \phi_b \gamma^{\mu\nu},$$

$$\mathcal{F}^{\mu\nu\rho\sigma} = G(\gamma^{\mu(\rho}\gamma^{\sigma)\nu} - \gamma^{\mu\nu}\gamma^{\rho\sigma}), \qquad (28)$$

where we have used $R = {}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - Kn^\mu)$, and ${}^{(3)}R$ stands for the three-dimensional Ricci scalar composed by the spatial metric, $\gamma_{\mu\nu}$. Now the canonical momenta are given by

$$\pi^a = \frac{\delta \mathcal{L}}{\delta \dot{\phi}_a}$$
$$= 2\mathcal{A}^{ab}\dot{\phi}_b + \mathcal{C}^{a\mu\nu}K_{\mu\nu} + (\text{no time derivative terms}), \quad (29)$$

$$\pi^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta K_{\mu\nu}}$$
$$= \mathcal{C}^{a\mu\nu}\dot{\phi}_a + 2\mathcal{F}^{\mu\nu\rho\sigma}K_{\rho\sigma} + (\text{no time derivative terms}). \qquad (30)$$

The existence of a primary constraint is ensured if and only if the linear combination of the canonical momenta,

$$\Psi \equiv \alpha_1 \phi^a \pi_a + \alpha_2 \gamma^{\mu\nu} \pi_{\mu\nu}, \qquad (31)$$

where α_1 and α_2 are constants, is independent of the velocities $K_{\mu\nu}$ and $\dot{\phi}^a$, i.e.,

$$\frac{\partial \Psi}{\partial(\phi_a \dot{\phi}^a)} = 0, \qquad \frac{\partial \Psi}{\partial K} = 0. \qquad (32)$$

These equations provide the degeneracy condition,

$$GA + GFX - 6G_X^2 X = 0. \qquad (33)$$

As long as this condition is satisfied, the primary (and subsequent) constraint should exist, and the absence of the BD ghost is ensured. One should note that this agrees with the condition (A10) in Appendix A, in the absence of γ_1 and γ_2.

We now discuss how to include the mass terms obtained in the previous section to this construction. We start with the first option (19). After solving the degeneracy condition (33) in terms of F, the kinetic Lagrangian can be written as

$$\mathcal{L}_{\text{kin}} = \mathcal{F}^{\mu\nu\rho\sigma}\left(K_{\mu\nu} + \frac{G_X}{G}\phi_a \dot{\phi}^a \gamma_{\mu\nu}\right)\left(K_{\rho\sigma} + \frac{G_X}{G}\phi_b \dot{\phi}^b \gamma_{\rho\sigma}\right)$$
$$- A\dot{\phi}_a \dot{\phi}^a. \qquad (34)$$

Therefore, in the absence of A, the F term can be absorbed into the extrinsic curvature by a field redefinition, and we can safely add the mass terms of the generalized massive gravity with disformal field space metric, as discussed in the previous section. On the other hand, in the presence of A term, the degeneracy condition is not compatible with the dRGT potential terms since A term can be included in γ_2 in (15), and it changes the ratio D/C. Therefore, $A = 0$ should be required under the condition $\gamma_1 + \gamma_2 = 0$ to ensure the absence of the BD ghost. Therefore, the ghost-free extension of generalized massive gravity with the nonminimal coupling is given by

$$S = \int d^4x \sqrt{-g} \frac{M_p^2}{2}\left[GR[g] + \frac{6G_X^2}{G}[Y] \right.$$
$$\left. - 2m^2 \sum_{n=0}^{3}\beta_n(X)e_n(\sqrt{g^{-1}\tilde{f}})\right] + S_m[g, \psi]. \qquad (35)$$

Let us now discuss another option (22) obtained in the previous section. As mentioned in the previous section, this is nothing but the projection onto the ϕ^a direction. With this projection operator, $P_{ab} = \eta_{ab} - \phi_a \phi_b / X$, one can construct the following fiducial metric $\bar{f}_{\mu\nu}$ and the building block tensor $Z^\mu{}_\nu$:

$$\bar{f}_{\mu\nu} = P_{ab}\partial_\mu \phi^a \partial_\nu \phi^b, \qquad Z^\mu{}_\nu = (g^{-1}\bar{f})^\mu{}_\nu. \qquad (36)$$

With the trace of this tensor $Z^\mu{}_\nu$ and the degeneracy condition (33), we can rewrite the Lagrangian as

$$GR + F[Y] + A[W] = GR + \frac{6G_X^2}{G}[Y] + A[Z], \qquad (37)$$

where we have used the relation $Z = W - Y/X$. The potential term $[Z]$ can be further generalized, and we arrive at ghost-free massive gravity, which is distinct from (35),

$$S = \int d^4x \sqrt{-g} \frac{M_p^2}{2} \left[GR + \frac{6G_X^2}{G} [Y] \right.$$
$$\left. + m^2 \mathcal{U}(X, [Z], [Z^2], [Z^3]) \right] + S_m[g, \psi], \quad (38)$$

where $\mathcal{U}$ is now an arbitrary function of X, $[Z]$, $[Z^2]$ and $[Z^3]$. Note that higher order terms $[Z^n]$ with $n \geq 4$ can be also included in $\mathcal{U}$, but they can be always reduced to lower order terms by Cayley-Hamilton theorem. The absence of the BD ghost can be simply understood as follows. The contribution to the momenta π^a from the potential $\mathcal{U}$ is proportional to the projection tensor P_{ab}, and thus $\mathcal{U}$ does not contribute to (31) automatically. Thus, it is manifest that the potential $\mathcal{U}$ including higher order is still compatible with the condition (31), and the action (38) is therefore ghost-free. In the Appendix B, we show an explicit derivation of (38) starting with the most general mass terms up to quadratic order composed of W and Y. Finally, let us mention the relation between the action with (22) in the previous section and the action (38). The trace of any square-root tensor can be written in terms of one without square-roots (see Ref. [35] for details). This allows us to rewrite (15) in terms of $[Z^n]$, and the theory (15) with (22) is manifestly included in (38).

V. COSMOLOGICAL BACKGROUND AND PERTURBATIONS

In the previous section, we have proposed two distinct types of the extended theories (35) and (38). It is now interesting to ask whether all the polarization modes are healthy or not on physically relevant backgrounds. To this end, we study FLRW cosmologies in the obtained theories (35) and (38) and derive the conditions for avoiding the ghost and gradient instabilities. Following [9], we choose the Stückelberg fields as[4]

$$\phi^0 = f(t)\sqrt{1 + \kappa(x^2 + y^2 + z^2)}, \quad \phi^i = f(t)\sqrt{\kappa}x^i, \quad (39)$$

and the physical metric is chosen to be compatible with this choice, i.e., an open FLRW metric. We remark that due to the explicit dependence on $\phi^a \phi_a$ in the action, the requirement of homogeneity allows only for an open universe solution. Including perturbations, the physical metric is given by

$$ds^2 = -(1 + 2\Phi)dt^2 + a(t)(\partial_i B + B_i)dtdx^i$$
$$+ a(t)^2(\Omega_{ij} + h_{ij}), \quad (40)$$

where Ω_{ij} is the spatial metric with a constant curvature $-\kappa < 0$,

$$\Omega_{ij} = \delta_{ij} - \frac{\kappa x^i x^j}{1 + \kappa x^k x^k}, \quad (41)$$

and we decompose spatial components of metric perturbations as follows:

$$h_{ij} = 2\psi \Omega_{ij} + \left(D_i D_j - \frac{1}{3} \Omega_{ij} D_k D^k \right) E$$
$$+ \frac{1}{2}(D_i E_j + D_j E_i) + \gamma_{ij}. \quad (42)$$

Here, D_i is the covariant derivative compatible with the Ω_{ij} metric. The vector perturbations are divergence-free $D^i E_i = D^i B_i = 0$, and the tensor perturbation is divergence and trace-free $D^i \gamma_{ij} = \Omega^{ij} \gamma_{ij} = 0$. For the four scalar fields ϕ^a, we fix the gauge such that $\phi^a = \langle \phi^a \rangle$, i.e., no perturbation. As for the matter sector, we introduce a k-essence field to mimic an irrotational perfect fluid,

$$S_m = \int d^4x \sqrt{-g} P(\Theta), \quad \Theta \equiv -\frac{1}{2} \partial_\mu \chi \partial^\nu \chi, \quad (43)$$

where χ is a scalar field, which can be decomposed into the background χ_0 and perturbation $\delta\chi$ as

$$\chi = \chi_0 + M_p \delta\chi. \quad (44)$$

Throughout this section, we use the equivalent energy density, pressure, and the sound speed of the k-essence field, defined as

$$\rho_\chi = 2P_\Theta \Theta - P, \quad p_\chi = P, \quad c_\chi^2 = \frac{P_\Theta}{2P_{\Theta\Theta}\Theta + P_\Theta}, \quad (45)$$

where $P_\Theta = \partial P/\partial \Theta$ and $P_{\Theta\Theta} = \partial^2 P/\partial \Theta^2$. Then the equation of motion for χ_0 is given by

$$\dot{\rho}_\chi + 3H(\rho_\chi + p_\chi) = 0. \quad (46)$$

Hereafter, we also use

$$\xi \equiv \frac{\sqrt{\kappa}f}{a}, \quad (47)$$

in replacement of f. In deriving second order actions in each sector, we expand tensor, vector, and scalar perturbations in terms of harmonics, for example,

$$\gamma_{ij} = \int k^2 dk \gamma_{|\vec{k}|} Y_{ij}(\vec{k}, \vec{x}), \quad (48)$$

$$B_i = \int k^2 dk B_{V,|\vec{k}|} Y_i(\vec{k}, \vec{x}), \quad (49)$$

$$\Phi = \int k^2 dk \Phi_{S,|\vec{k}|} Y(\vec{k}, \vec{x}), \quad (50)$$

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[4]The requirement of homogeneity and isotropy constrains the Stückelberg configuration uniquely. However, if one relaxes the requirement, it is possible to realize backgrounds that are approximately FLRW within the observable patch [36], which allows more general configuration of the scalar fields.

with $D_l D^l Y_{ij} = -k^2 Y_{ij}$ and $D^i Y_{ij} = \Omega^{ij} Y_{ij} = 0$ for tensor harmonics, $D_l D^l Y_i = -k^2 Y_i$ and $D^i Y_i = 0$ for vector harmonics, and $D_l D^l Y = -k^2 Y$ for scalar harmonics. Other perturbations, E_i, B, ψ, and E are defined in a similar manner.

A. Nonminimal generalized massive gravity

In this subsection, we focus on the nonminimally coupled generalized massive gravity (35), with the general reference metric,

$$S = \int d^4x \sqrt{-g} \frac{M_p^2}{2} \left[GR[g] + \frac{6G_X^2}{G} [Y] \right.$$
$$\left. - 2m^2 \sum_{n=0}^{3} \beta_n(X) e_n \left(\sqrt{g^{-1}\tilde{f}} \right) \right] + S_m[g,\chi], \quad (51)$$

where Y is defined in Eq. (25) and the disformal fiducial metric is

$$\tilde{f}_{\mu\nu} = (\eta_{ab} + D\phi_a\phi_b)\partial_\mu\phi^a\partial_\nu\phi^b. \quad (52)$$

For the field configuration (39), the background line element for the fiducial metric is given by

$$\tilde{f}_{\mu\nu}dx^\mu dx^\nu = -\dot{f}^2(1-f^2D)dt^2 + \kappa f^2 dx^i \Omega_{ij} dx^j, \quad (53)$$

so the disformal part of the field space metric D shifts the lapse function of the fiducial metric to $\dot{f} \to \dot{f}\sqrt{1-f^2D}$, while the scale factor in the fiducial metric continues to be $\sqrt{\kappa}f$.

1. Background equations

We define the following functions which will be useful later on:

$$\rho_{m,g} \equiv \beta_0 + 3\xi\beta_1 + 3\xi^2\beta_2 + \xi^3\beta_3,$$
$$J \equiv \beta_1 + 2\xi\beta_2 + \xi^2\beta_3,$$
$$\rho_{m,f} \equiv \frac{1}{\xi^3}(\beta_1 + 3\xi\beta_2 + 3\xi^2\beta_3),$$
$$\Gamma \equiv \xi\beta_1 + \xi^2\beta_2 + r\xi^2(\beta_2 + \xi\beta_3), \quad (54)$$

where r quantifies the alignment of the light cone defined by the $\tilde{f}$–metric with respect to g,

$$r \equiv \frac{\dot{f}\sqrt{1-Df^2}}{\xi}. \quad (55)$$

With this definition, we can also express the time derivative of the ratio of the two scale factors as

$$\dot{\xi} = \xi \left[\frac{\sqrt{\kappa}r}{a\sqrt{1 - \frac{a^2\xi^2 D}{\kappa}}} - H \right]. \quad (56)$$

We now calculate the background equations of motion by varying with respect to N (which can be reintroduced by rescaling all time derivatives as well as the volume element), a, f and matter perturbations, obtaining

$$3G\left[\left(H + \frac{\dot{G}}{2G} \right)^2 - \frac{\kappa}{a^2} \right] = \frac{\rho_\chi}{M_p^2} + m^2\rho_{m,g}, \quad (57)$$

$$- 2G\left[\partial_t\left(H + \frac{\dot{G}}{2G} \right) + \frac{\kappa}{a^2} \right] + \dot{G}\left(H + \frac{\dot{G}}{2G} \right)$$
$$= \frac{\rho_\chi + p_\chi}{M_p^2} + m^2 J\xi(1-r), \quad (58)$$

$$\dot{\rho}_\chi = -3H(\rho_\chi + p_\chi), \quad (59)$$

where ρ_χ and P_χ are the energy density and pressure of the matter fluid. We also defined c_s as the propagation speed of the fluid. The equation of motion for the Stückelberg fields on the other hand is given by

$$3m^2 J\xi(1-r)\left(H + \frac{\dot{G}}{2G} \right) + m^2\dot{\rho}_{m,g}$$
$$- \frac{\dot{G}}{2G}\left(\frac{\rho_\chi - 3p_\chi}{M_p^2} + 4m^2\rho_{m,g} \right) = 0. \quad (60)$$

In standard dRGT, the analogue of this equation forces the function J to vanish around self-accelerating backgrounds. In our case, this is no longer true. In addition to the effect of the conformal factor $\dot{G}$, there is also the effect of the generalized mass terms encoded in $\dot{\rho}_{m,g}$. To make the latter effect explicit, we rewrite Eq. (60) in terms of derivatives with respect to $\phi^a\phi_a$,

$$3m^2 Jr\xi\left(\sqrt{1 - \frac{a^2\xi^2 D}{\kappa}}H - \frac{\sqrt{\kappa}}{a} \right) + \frac{2m^2 ar\xi^2}{\sqrt{\kappa}}\rho_{m,g}^{(1)}$$
$$- \frac{ar\xi^2 G'}{\sqrt{\kappa}M_p^2 G}[\rho_\chi - 3p_\chi + m^2 M_p^2(3J\xi(r-1) + 4\rho_{m,g})] = 0, \quad (61)$$

where $G' = G'(\phi^a\phi_a)$, and we defined

$$\rho_{m,g}^{(1)} \equiv \beta_0' + 3X\beta_1' + 3X^2\beta_2' + X^3\beta_3'. \quad (62)$$

From Eq. (61), we see that in standard dRGT, $D = 0$, both G and β_n are constant; thus we either have $H = \sqrt{\kappa}/a$ (i.e., physical metric is Minkowski in open chart) or $J = 0$ [9].

In the perturbation calculation, we solve the set of background equations (57)–(60) for $\rho_{m,g}$, $\dot{H}$, $\dot{\rho}$ and $\dot{\rho}_{m,g}$ and evaluate the quadratic action on shell.

2. Tensor sector

For the tensor modes, one can obtain the action quadratic in perturbations as (after expanding in harmonics)

$$S^{(2)} = \frac{M_p^2}{8} \int d^3k \, dt \, a^3 G \left[|\dot{\gamma}|^2 - \left(\frac{k^2 - 2\kappa}{a^2} + \frac{m^2\Gamma}{G} \right) |\gamma|^2 \right]. \tag{63}$$

The dispersion relation for the canonical mode is

$$\omega_T^2 = \frac{k^2 - 2\kappa}{a^2} + \frac{m^2\Gamma}{G} - \frac{3H\dot{G}}{2G} + \frac{\dot{G}^2}{4G^2} - \frac{\ddot{G}}{2G}. \tag{64}$$

In contrast to the Horndeski theory in which the first derivative of the scalar field is nonminimally coupled to gravity [37,38], the propagation speed of the tensor mode is the same as the speed of light, and the nonminimal coupling simply shift the mass of graviton.

3. Vector sector

For the vector modes, we first solve for the shift perturbations,

$$B_i = \left[1 + \frac{2a^2\xi}{G(k^2 + 2\kappa)} \frac{m^2 J}{1 + r} \right]^{-1} \frac{a\dot{E}_i}{2}. \tag{65}$$

Substituting it back to the action, we find the following form for the reduced action:

$$S^{(2)} = \frac{M_p^2}{8} \int d^3k \, dt \, a^3 \mathcal{T}_V$$
$$\times \left[|\dot{E}_i|^2 - \left(c_V^2 \frac{k^2 + 2\kappa}{a^2} + \frac{m^2\Gamma}{G} \right) |E_i|^2 \right], \tag{66}$$

where the kinetic term is

$$\mathcal{T}_V \equiv \left(\frac{2}{G(k^2 + 2\kappa)} + \frac{r + 1}{m^2 a^2 \xi J} \right)^{-1}, \tag{67}$$

and the propagation speed is given by

$$c_V^2 = \frac{\Gamma(1 + r)}{2\xi J}. \tag{68}$$

In the UV, the kinetic term is

$$\mathcal{T}_V \Big|_{k \to \infty} = \frac{m^2 a^2 \xi J}{r + 1}. \tag{69}$$

In the dRGT limit, i.e., $G \to 1, J \to 0, D \to 0$, the sound speed reduces to

$$c_v^2 \Big|_{dRGT} = \frac{3m^2 M_p^2 H^2 \Gamma}{2\xi^2 [G'(\rho_\chi - 3P_\chi + 4m^2 M_p^2 \rho_{m,g}) - 2m^2 M_p^2 \rho_{m,g}^{(1)}]}$$
$$\to \infty, \tag{70}$$

where we assumed

$$\mathcal{O}\left(\frac{G^{(n)}}{G} \right) \sim \mathcal{O}\left(\frac{\rho_{m,g}^{(n)}}{\rho_{m,g}} \right) \sim \mathcal{O}\left(\frac{\Gamma^{(n)}}{\Gamma} \right) \ll 1, \quad \text{for } n \geq 1, \tag{71}$$

and $H \gg \sqrt{\kappa}/a$.

4. Scalar sector

Finally, we calculate the quadratic action for scalar perturbations $(\Phi, \psi, B, E, \delta\chi)$. Two of these scalar perturbations are nondynamical, and we can integrate out Φ and B. One of the remaining variables corresponds to the BD ghost. Then, one can check that the determinant of the remaining kinetic matrix composed of ψ, E, and $\delta\chi$ vanishes, and this implies the absence of the BD ghost. We then introduce the new variable $\widetilde{\delta\chi}$ to remove the would-be BD ghost,

$$\widetilde{\delta\chi} \equiv \psi + \frac{k^2}{6} E - \frac{M_p}{\dot{\chi}_0} \left(H + \frac{\dot{G}}{2G} \right) \delta\chi. \tag{72}$$

Replacing $\delta\chi$ in favor of $\widetilde{\delta\chi}$, the resulting action becomes independent of $\dot{\psi}$. We then integrate out ψ and end up with an action containing only 2 degrees of freedom $\widetilde{\delta\chi}$ and E. One of these corresponds to the matter perturbations while the other is the scalar polarization of graviton. Then, the quadratic action in the scalar sector can be formally written as

$$S_S^{(2)} = \frac{M_p^2}{2} \int d^3k \, dt \, a^3$$
$$\times \left(\dot{\Psi}^\dagger \mathcal{K} \dot{\Psi} + \frac{1}{2} \dot{\Psi}^\dagger \mathcal{G} \Psi + \frac{1}{2} \Psi^\dagger \mathcal{G}^T \dot{\Psi} - \Psi^\dagger \mathcal{M} \Psi \right), \tag{73}$$

where $\Psi \equiv (E, \widetilde{\delta\chi})$ and $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{M}$ are the real 2×2 kinetic, mixing, and mass matrices, and $\mathcal{K}$ and $\mathcal{M}$ are symmetric. The eigenvalues of the kinetic matrix in the subhorizon limit $k \to \infty$ yield the conditions to avoid ghostlike instability. These can be determined as

$$\kappa_1 = \mathcal{K}_{11}, \qquad \kappa_2 = \frac{\det \mathcal{K}}{\mathcal{K}_{11}}. \tag{74}$$

The eigenvalues of the kinetic matrix in the subhorizon limit yield the conditions to avoid ghostlike instability. The first eigenvalue in this limit is simply

$$\kappa_1 = \frac{\rho_\chi + P_\chi}{c_\chi^2 (H + \frac{\dot{G}}{2G})^2} + \mathcal{O}\left(\frac{k}{aH}\right)^{-2}, \qquad (75)$$

and it can be identified as the matter perturbation. It is not a ghost as long as the null energy condition is satisfied. The second eigenvalue is more complicated, but in subhorizon limit it can be written as

$$\kappa_2 = \frac{m^2 r J^2 \xi^2}{2G} + \frac{\kappa J r \xi}{a^2} + \left(H + \frac{\dot{G}}{2G}\right)^2 (2\Gamma + J\xi)$$
$$+ \frac{\rho_\chi + P_\chi}{2m^2 M_p^2} \left[\frac{m^2 J r \xi}{G} + (1 - 3c_s^2)\frac{\dot{G}}{G}\left(H + \frac{\dot{G}}{2G}\right)\right]$$
$$- \xi\left(H + \frac{\dot{G}}{2G}\right)\left(4HJ + \frac{J\dot{\xi}}{\xi} + \dot{J}\right) + \mathcal{O}\left(\frac{k}{aH}\right)^{-2}. \qquad (76)$$

At this point, we see that the subhorizon limit and dRGT limit do not commute. Sending $J, \dot{J}, \dot{G} \to 0$,

$$\kappa_2\big|_{k\to\infty,dRGT} = 2H^2\Gamma. \qquad (77)$$

If we reverse the order of the limits, we then get

$$\kappa_2\big|_{dRGT,k\to\infty} \to 0. \qquad (78)$$

This is compatible with what was found in the generalized dRGT without conformal transformation [22].

Finally, we calculate the sound speeds of the scalar degrees of freedom. Unlike the minimally coupled case where one sound speed coincides with c_s, this is no longer true when nonminimal coupling is introduced. The full expression for the scalar sound speeds are reported in Appendix C. As in the vector sector, in the dRGT limit, these reduce to

$$C_1^2\big|_{dRGT} = c_s^2,$$
$$C_2^2\big|_{dRGT} = \frac{2m^2 M_p^2 H^2 \Gamma}{\xi^2[G'(\rho - 3P + 4m^2 M_p^2 \rho_{m,g}) - 2m^2 M_p^2 \rho_{m,g}^{(1)}]}$$
$$= \frac{4}{3}c_v^2\big|_{dRGT} \to \infty. \qquad (79)$$

As expected the strong coupling problem of the constant mass theory manifests itself as sound speed that diverge in the exact dRGT limit.

All of the expressions for this theory recovers the results of Ref. [22] in the limit $G \to 1$ and $\dot{\xi} \to \xi(\sqrt{\kappa}r/a - H)$ (or equivalently, $D \to 0$).

B. Projected massive gravity theory

In this subsection, we study the action (38) up to the quadratic mass terms,

$$S = \int d^4x \sqrt{-g}\frac{M_p^2}{2}\left[GR + \frac{6G_X^2}{G}[Y]\right.$$
$$\left. + m^2(A[Z] + B_1[Z]^2 + B_2[Z^2])\right] + S_m[g,\chi], \qquad (80)$$

where G, A, B_1, and B_2 are functions of X.

1. Background equations

The background equations for gravity are given by

$$3G\left[\left(H + \frac{\dot{G}}{2G}\right)^2 - \frac{\kappa}{a^2}\right] = \frac{\rho_\chi}{M_p^2} + \frac{\rho_g}{M_p^2}, \qquad (81)$$

$$-2G\left[\partial_t\left(H + \frac{\dot{G}}{2G}\right) + \frac{\kappa}{a^2}\right] + \dot{G}\left(H + \frac{\dot{G}}{2G}\right)$$
$$= \frac{\rho_\chi + p_\chi}{M_p^2} + \frac{\rho_g + p_g}{M_p^2}, \qquad (82)$$

where we introduced the effective density and pressure for the mass terms as

$$\rho_g = -\frac{3}{2}M_p^2 m^2 \xi^2 (A + (3B_1 + B_2)\xi^2), \qquad (83)$$

$$p_g = \frac{1}{2}M_p^2 m^2 \xi^2 (A - (3B_1 + B_2)\xi^2). \qquad (84)$$

The equation of motion of the Stückelberg fields is given by

$$\dot{\rho}_g + 3H(\rho_g + p_g) - \frac{\dot{G}}{2G}(\rho_g - 3p_g + \rho_\chi - 3p_\chi) = 0. \qquad (85)$$

Again, three of these equations are independent, and we solve these equations (46), (82), and (85) for $\dot{H}$, $\dot{\rho}_g$, and ρ_χ for deriving the quadratic action.

2. Tensor sector

For the tensor modes, one can obtain the action quadratic in perturbations as

$$S_T^{(2)} = \frac{M_p^2}{8}\int d^3k \, dt \, a^3 G\left[|\dot{\gamma}|^2 - \left(\frac{k^2 - 2\kappa}{a^2} + \frac{M_{GW}^2}{G}\right)|\gamma|^2\right]. \qquad (86)$$

The dispersion relation for the canonical mode is

$$\omega_T^2 = \frac{k^2 - 2\kappa}{a^2} + \frac{M_{GW}^2}{G} - \frac{3H\dot{G}}{2G} + \frac{\dot{G}^2}{4G^2} - \frac{\ddot{G}}{2G}, \qquad (87)$$

where we defined the mass of the tensor modes in the minimal coupling case,

$$M_{\rm GW}^2 = \frac{2(\rho_g + p_g - 2m^2 M_p^2 B_2 \xi^4)}{M_p^2}. \qquad (88)$$

As in the theory (51), the propagation speed of the tensor mode is exactly the same as the speed of light.

3. Vector sector

Next, we expand the action up to quadratic order in the vector sector. Since the shift perturbation is nondynamical, we first solve the constraint equation as

$$B_i = \frac{M_p^2 G(k^2 + 2\kappa)}{2M_p^2 G(k^2 + 2\kappa) + 4a^2(\rho_g + p_g)} a\dot{E}_i. \qquad (89)$$

Plugging this back into the action, we find the following form for the reduced action:

$$S_V^{(2)} = \frac{M_p^2}{8} \int d^3k \, dt \, a^3 \mathcal{T}_V$$
$$\times \left[|\dot{E}_i|^2 - \left(c_V^2 \frac{k^2 + 2\kappa}{a^2} + \frac{M_{\rm GW}^2}{G} \right) |E_i|^2 \right], \qquad (90)$$

where the kinetic coefficients is given by

$$\mathcal{T}_V \equiv \frac{a^2 G(k^2 + 2\kappa)(\rho_g + p_g)}{M_p^2 G(k^2 + 2\kappa) + 2a^2(\rho_g + p_g)}, \qquad (91)$$

and the propagation speed for the vector mode is given by

$$c_V^2 = \frac{M_p^2 M_{\rm GW}^2}{2(\rho_g + p_g)}. \qquad (92)$$

The propagation speed of the vector polarization modes can in general differ from the speed of light. From this expression, we find that the ghost instability can be avoided when $\rho_g + p_g > 0$. In addition, the gradient instability in the vector sector is absent if $M_{\rm GW}^2 \geq 0$, and it coincides with the condition for avoiding the tachyonic instability in the tensor modes, when $\dot{G} = 0$.

4. Scalar sector

Now, let us move on to the scalar perturbations. As in the case of the previous theory class (51), we can integrate out the nondynamical variables Φ, B, and ψ by using the same $\widetilde{\delta\chi}$ defined by (72), and the reduced action can be written in the form of (73). The eigenvalues are given by

$$\kappa_1 = \frac{1}{(H + \frac{\dot{G}}{2G})^2} \left[\frac{2c_\chi^2}{\rho_\chi + P_\chi} - \frac{(1 - 3c_\chi^2)^2 \dot{G}^2}{G^2(H + \frac{\dot{G}}{2G})} \left(\frac{\dot{G}}{G} [9c_\chi^2 \rho_\chi - 3(4 - 3c_\chi^2)P_\chi + \rho_g - 3P_g] + 2[H(\rho_g + 9P_g) + 3\dot{P}_g] \right)^{-1} \right]^{-1}$$
$$+ \mathcal{O}\left(\frac{k}{aH} \right)^{-2},$$
$$\kappa_2 = \frac{a^2(\rho_g + P_g)}{8} k^2 - \frac{3a^4(\rho_g + P_g)^2}{16M_p^2 G} + \mathcal{O}\left(\frac{k}{aH} \right)^{-2}. \qquad (93)$$

The first eigenvalue κ_1 can be identified as the matter perturbation which can be easily seen when $G = 1$,

$$\kappa_1 = \frac{\rho_\chi + p_\chi}{2c_\chi^2 H^2} + \mathcal{O}\left(\frac{k}{aH} \right)^{-2}. \qquad (94)$$

Therefore, the scalar graviton is free from ghost when $\rho_g + p_g > 0$. The full expression of the propagation speeds are summarized in the Appendix C.

5. Concrete model

In this subsection, we give a concrete model in the projected theory (38). Let us first choose the simplest functions,

$$G = 1, \quad A = a_1, \quad B_1 = b_1, \quad B_2 = b_2, \qquad (95)$$

where a_1, b_1, and b_2 are constants. Then, the equation of motion for f gives the constraint equation,

$$\xi(H\xi + \dot{\xi})[a_1 + 2(3b_1 + b_2)\xi^2] = 0. \qquad (96)$$

Assuming that ξ is nonzero, the first solution $H\xi + \dot{\xi} = 0$ gives $\xi \propto 1/a$. In this case, ρ_g behaves as the sum of the spacial curvature and the radiation, so we do not discuss this solution here. The second one gives

$$\xi = \pm\sqrt{\frac{-a_1}{2(3b_1 + b_2)}}. \qquad (97)$$

In order for ξ to be real, we impose $a_1/(3b_1 + b_2) < 0$. Since ξ is a constant, ρ_g and p_g now becomes constants,

$$\rho_g = -p_g = \frac{3a_1^2 m^2 M_p^2}{8(3b_1 + b_2)}. \qquad (98)$$

Since the mass term exactly behaves as a cosmological constant, the kinetic terms of the vector and scalar graviton

modes vanish in this case ($\rho_g + p_g = 0$). This strong coupling behavior can be avoided once the X-dependence in the arbitrary functions G, A_1, B_1, or B_2 is taken into account. Here, let us consider the nonminimal coupling case,

$$G = 1 + gm^2 X, \tag{99}$$

where we assume $g \ll 1$. Then, we expand ξ in terms of this small parameter g,

$$\xi = \xi_0 + g\xi_1 + \mathcal{O}(g^2). \tag{100}$$

Here, ξ_0 is given by the positive sign of (97),

$$\xi_0 = \sqrt{\frac{-a_1}{2(3b_1 + b_2)}}. \tag{101}$$

From the equation of motion for f, we obtain

$$\xi_1 = -\frac{a^2 \xi_0}{6a_1 \kappa M_p^2}(\rho_\chi - 3p_\chi - 3a_1 m^2 M_p^2 \xi_0^2), \tag{102}$$

and the energy density and pressure can be then expressed as

$$\rho_g = -\frac{3}{4}a_1 m^2 M_p^2 \xi_0^2 + \mathcal{O}(g^2), \tag{103}$$

$$p_g = \frac{3}{4}a_1 m^2 M_p^2 \xi_0^2 + 2a_1 m^2 M_p^2 \xi_0 \xi_1 g + \mathcal{O}(g^2). \tag{104}$$

Then, the background equations become

$$3\left[\left(1 - 3gm^2\xi_0^2\frac{a^2}{\kappa}\right)H^2 - \frac{\kappa}{a^2}\right] \simeq \frac{\rho_\chi}{M_p^2} + \frac{\tilde{\rho}_g}{M_p^2}, \tag{105}$$

$$-2\left[\left(1 - 2gm^2\xi_0^2\frac{a^2}{\kappa}\right)\dot{H} + \frac{\kappa}{a^2} - gm^2\xi_0^2\frac{a^2}{\kappa}H^2\right]$$
$$\simeq \frac{\rho_\chi + p_\chi}{M_p^2} + \frac{\tilde{\rho}_g + \tilde{p}_g}{M_p^2}, \tag{106}$$

where

$$\tilde{\rho}_g = \rho_g - 3gm^2 M_p^2 \xi_0^2, \tag{107}$$

$$\tilde{p}_g = p_g + gm^2 M_p^2 \xi_0^2. \tag{108}$$

Now we would like to derive the conditions for avoiding ghost and gradient instabilities. As mentioned the above, the positivity of the cosmological constant requires that $\rho_g > 0$, and ξ_0 has to be real,

$$a_1 < 0, \qquad 3b_1 + b_2 > 0. \tag{109}$$

All modes are ghost-free when $\rho_g + p_g > 0$, which gives

$$\xi_1 g < 0. \tag{110}$$

Substituting the background solution, the propagation speed of the vector mode is given by

$$c_V^2 = -\frac{b_2 \xi_0^3}{a_1 \xi_1 g} + \mathcal{O}(g^0), \tag{111}$$

and the propagation speeds of the scalar modes can be now simplified as

$$c_1^2 = c_\chi^2, \qquad c_2^2 = -\frac{4b_2 \xi_0^3}{3a_1 \xi_1 g} + \mathcal{O}(g^0). \tag{112}$$

Combining these conditions, we obtain

$$a_1 < 0, \quad 3b_1 + b_2 > 0, \quad \xi_1 g < 0, \quad b_2 < 0. \tag{113}$$

Therefore, all modes are free of ghost and gradient instabilities when the conditions (113) are satisfied.

VI. SUMMARY

In the present paper, we studied a generalization of massive gravity with the broken translation invariance. Introducing a deformation to the fiducial metric $\tilde{f}_{\mu\nu} = (\eta_{ab} + D\phi_a\phi_b)\partial_\mu\phi^a\partial_\nu\phi^b$ is essential to find extended theories beyond the dRGT massive gravity. Starting with arbitrary mass functions, we found two potential ways to avoid the BD ghost. The first case is the extension of the generalized massive gravity and any detuning of the quadratic dRGT potential requires a nonminimal coupling with curvature. The action for this theory is given by

$$S = \int d^4x\sqrt{-g}\frac{M_p^2}{2}\left[GR + \frac{6G_X^2}{G}[Y]\right.$$
$$\left. - 2m^2\sum_{n=0}^{3}\beta_n(X)e_n\left(\sqrt{g^{-1}\tilde{f}}\right)\right] + S_m[g,\psi]. \tag{114}$$

The second theory can be constructed using the fiducial metric $\bar{f}_{\mu\nu} = P_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$, where we use the projection tensor $P_{ab} = \eta_{ab} - \phi_a\phi_b/X$, which manifestly eliminates one of the Stückelberg fields along ϕ^a. The action for the projected theory is given by

$$S = \int d^4x\sqrt{-g}\frac{M_p^2}{2}\left[GR + \frac{6G_X^2}{G}[Y]\right.$$
$$\left. + m^2\mathcal{U}(X, [Z], [Z^2], [Z^3])\right] + S_m[g,\psi], \tag{115}$$

where Y and Z are defined in (25). In the form that we proposed, this theory can also have the same nonminimal

coupling, but the mass term is no longer of the form of the dRGT potential terms. In addition, the potential term is an arbitrary function of X and $[Z^n]$. We have systematically proved the absence of the BD ghost in this theory. The projected theory action (115) is actually not the most general, since there remains some freedom to include further nonminimal coupling without generating the BD ghost. For instance, the term $G^{\mu\nu}\bar{f}_{\mu\nu}$ was considered in Ref. [30]. The possibility of other nonminimal coupling terms can be easily seen by considering general disformal transformations of the metric tensor $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = Cg_{\mu\nu} + DZ_{\mu\nu} + E(Z^2)_{\mu\nu} + F(Z^3)_{\mu\nu}$, where all coefficients are functions of X, $[Z]$, $[Z^2]$, $[Z^3]$. Such a transformation would generate the $G^{\mu\nu}\bar{f}_{\mu\nu}$ coupling, as well as many others, and we will report this in a later study.

We have then studied open-FLRW cosmologies of these obtained theories. In both cases, all perturbations are free of ghost and gradient instabilities. In addition, we have found that the structure of the nonminimal coupling does not change the propagation speed of the tensor modes while the vector and scalar graviton propagates either subluminal or superluminal speed.

These new theories are the first ones where the kinetic term of a massive graviton is no longer Einstein-Hilbert term due to the nonminimal coupling, and it might bring a new phenomenology of large scale structure. For instance, the translation breaking will be manifested as time variation in coupling constants. For solar system tests, we expect that the theory (35) exhibits Vainshtein mechanism due to its connection to generalized Galileon theories in the decoupling limit. The phenomenology of this theory will be investigated in a future publication [39]. Conversely, the theory (38) is disconnected from the dRGT construction, thus the existence and/or necessity of a screening mechanism needs to be confirmed.

APPENDIX A: DEGENERACY CONDITIONS AROUND FIXED BACKGROUNDS

Due to the complexity of the action (15), we here simplify the derivation of degeneracy conditions, instead of using $3 + 1$ decomposition. An obvious first choice is the homogeneous and isotropic background. However, even for constant mass parameters, one cannot even deduce the full degeneracy conditions that yield the dRGT form. In this appendix, we instead consider two backgrounds that have fewer symmetries than the cosmological background. We first consider a homogeneous background with broken isotropy, which yields explicit necessary conditions. We next study an inhomogeneous background, which gives tighter degeneracy conditions that we could not write down explicitly. However, we are able to check some options for the relations between functions. We summarize our findings at the end of the appendix, which form the basis of the conditions (17)–(18) quoted in the main text.

The action we consider in this appendix is

$$S = \int d^4x \sqrt{-g} \frac{M_p^2}{2} [G(x)R[g] + F(X)[Y] + A(X)[W] - 2m^2 \mathcal{L}_{\text{mass}}], \tag{A1}$$

where we defined W and Y in Eq. (25), while $\mathcal{L}_{\text{mass}}$ is given in Eq. (16).

1. Degeneracy around anisotropic background

We first start with the Bianchi type-V spacetime, which is the simplest anisotropic background that is compatible with uniform $\phi^a\phi_a$. The physical metric is given by

$$ds^2 = -dt^2 + a(t)^2 dx^2 + e^{2\alpha x}(b(t)^2 dy^2 + c(t)^2 dz^2). \tag{A2}$$

The scalar field configuration is chosen to be

$$\phi^0 = f(t)\left[\cosh(\alpha x) + \frac{\alpha^2(y^2 + z^2)e^{\alpha x}}{2}\right],$$

$$\phi^1 = f(t)\left[\sinh(\alpha x) - \frac{\alpha^2(y^2 + z^2)e^{\alpha x}}{2}\right],$$

$$\phi^A = f(t)\alpha x^A e^{\alpha x}, \tag{A3}$$

where $A = 2, 3$. With this choice, we have $X = \eta_{ab}\phi^a\phi^b = -f^2$ and

$$f_{\mu\nu}dx^\mu dx^\nu = -\dot{f}^2(t)dt^2 + \alpha^2 f(t)^2 dx^2 + \alpha^2 f(t)^2 e^{2\alpha x}(dy^2 + dz^2). \tag{A4}$$

This is simply the Minkowski metric written in a chart that is compatible with the Bianchi type-V form. We can also define the fiducial metric obtained from a transformed field space metric (12),

$$\tilde{f}_{\mu\nu,I}dx^\mu dx^\nu = -(C_I - f^2 D_I)\dot{f}^2 dt^2 + \alpha^2 C_I f(t)^2 dx^2 + \alpha^2 f(t)^2 C_I e^{2\alpha x}(dy^2 + dz^2). \tag{A5}$$

Then, we have the diagonal matrix,

$$(g^{-1}\tilde{f}_I)^{\mu}{}_{\nu} = \begin{pmatrix} (C_I + D_I X)\dot{f}^2 & 0 & 0 & 0 \\ 0 & \alpha^2 C_I \frac{f^2}{a^2} & 0 & 0 \\ 0 & 0 & \alpha^2 C_I \frac{f^2}{b^2} & 0 \\ 0 & 0 & 0 & \alpha^2 C_I \frac{f^2}{c^2} \end{pmatrix}. \tag{A6}$$

We can then evaluate the action (15) for this background. We vary the action in the minisuperspace approximation $\mathcal{S}$ with respect to the variables a, b, c and f, and obtain four dynamical equations,

$$E_n \equiv (E_a, E_b, E_c, E_f), \qquad E_a \equiv \frac{\delta\mathcal{S}}{\delta a}, \qquad E_b \equiv \frac{\delta\mathcal{S}}{\delta b},$$

$$E_c \equiv \frac{\delta\mathcal{S}}{\delta c}, \qquad E_f \equiv \frac{\delta\mathcal{S}}{\delta f}. \tag{A7}$$

Since these equations of motion contain the second time derivative of $f(t)$, to ensure the absence of the BD ghost,

$f(t)$ should be nondynamical. Therefore, we require that the kinetic matrix,

$$K_{mn} = \frac{\partial E_m}{\partial \ddot{q}_n}, \tag{A8}$$

is degenerate. Here we have defined the variables as $q_n \equiv (a, b, c, f)$. Assuming $G \neq 0$, the determinant of the kinetic matrix is given by

$$\det K = \mathcal{D}_1 + 6\dot{f}^2 \mathcal{D}_2 - 2m^2 \alpha f \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(3\dot{f}\mathcal{D}_3 + \mathcal{D}_5)$$

$$- 6m^2 \dot{f}\mathcal{D}_4$$

$$- 4m^2\alpha^2 f^2 \left[\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right)\mathcal{D}_6\right.$$

$$\left. + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\mathcal{D}_7\right], \tag{A9}$$

where

$$\mathcal{D}_1 = A + FX - 6X\frac{G_X^2}{G} - 2m^2[(C_{\gamma_1} + XD_{\gamma_1})\gamma_1 + (C_{\gamma_2} + XD_{\gamma_2})\gamma_2],$$

$$\mathcal{D}_2 = X^2 B_1 - 2m^2[(C_{\sigma_1} + XD_{\sigma_1})^2\sigma_1 + (C_{\sigma_2} + XD_{\sigma_2})^2\sigma_2 + (C_{\sigma_3} + XD_{\sigma_3})^2\sigma_3 + (C_{\sigma_4} + XD_{\sigma_4})^2\sigma_4 + (C_{\sigma_5} + XD_{\sigma_5})^2\sigma_5],$$

$$\mathcal{D}_3 = 4\sqrt{C_{\sigma_1}}(C_{\sigma_1} + XD_{\sigma_1})^{3/2}\sigma_1 + 2\sqrt{C_{\sigma_2}}(C_{\sigma_2} + XD_{\sigma_2})^{3/2}\sigma_2 + \sqrt{C_{\sigma_4}}(C_{\sigma_4} + XD_{\sigma_4})^{3/2}\sigma_4,$$

$$\mathcal{D}_4 = (C_{\delta_1} + XD_{\delta_1})^{3/2}\delta_1 + (C_{\delta_2} + XD_{\delta_2})^{3/2}\delta_2 + (C_{\delta_3} + XD_{\delta_3})^{3/2}\delta_3,$$

$$\mathcal{D}_5 = 3\sqrt{C_{\delta_1}}(C_{\delta_1} + XD_{\delta_1})\delta_1 + \sqrt{C_{\delta_2}}(C_{\delta_2} + XD_{\delta_2})\delta_2,$$

$$\mathcal{D}_6 = 6C_{\sigma_1}(C_{\sigma_1} + XD_{\sigma_1})\sigma_1 + C_{\sigma_2}(C_{\sigma_2} + XD_{\sigma_2})\sigma_2,$$

$$\mathcal{D}_7 = 3C_{\sigma_1}(C_{\sigma_1} + XD_{\sigma_1})\sigma_1 + C_{\sigma_2}(C_{\sigma_2} + XD_{\sigma_2})\sigma_2 + C_{\sigma_3}(C_{\sigma_3} + XD_{\sigma_3})\sigma_3. \tag{A10}$$

For the kinetic matrix to be noninvertible, all seven of these functions should be zero,

$$\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \mathcal{D}_4 = \mathcal{D}_5 = \mathcal{D}_6 = \mathcal{D}_7 = 0. \tag{A11}$$

Here, the tadpole term, $\beta(X)[Q_\beta]$, in (16) does not contribute to the kinetic matrix since it is linear in $\dot{f}$ in this background. One can confirm that when the above conditions are imposed, the study of linear perturbations does not reveal any new information on degeneracy. Note that we can obtain the dRGT tuning in the translation invariant case, $F = A = D_I = 0$ and $G = C_I = 1$. In a FLRW background, where $\alpha = 0$ and $a = b = c$, the conditions $\mathcal{D}_6 = 0$ and $\mathcal{D}_7 = 0$ are combined into a single condition, that is, $\mathcal{D}_6 + \mathcal{D}_7 = 0$. This demonstrates that the FLRW background is not adequate to reveal all of the dRGT tuning. Although these conditions are sufficient to eliminate the BD ghost in the Bianchi type-V background, the

BD ghost reappears in more general backgrounds as we show in the next subsection.

2. Degeneracy around inhomogeneous background

So far, we have considered degeneracy conditions around a homogeneous but anisotropic background. It is therefore a legitimate question whether these conditions are sufficient to ensure nonlinear degeneracy (or equivalently, degeneracy around arbitrary backgrounds). We here consider a fixed physical metric given by [24,29]

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{A12}$$

where we use spherical coordinates $dx^i = (dr, d\theta, d\phi)$ and consider flat hypersurfaces $h_{ij} = \text{diag}(1, r^2, r^2 \sin^2\theta)$. We also use a shift vector that is aligned with the radial direction $N^i = (l, 0, 0)$. As for the scalar field configuration, we consider

$$\phi^0 = f(t)\sqrt{1 + \kappa r^2} + \delta\phi^0,$$
$$\phi^1 = f(t)\sqrt{\kappa}r \sin\theta \cos\phi + \delta\phi^1,$$
$$\phi^2 = f(t)\sqrt{\kappa}r \sin\theta \sin\phi + \delta\phi^2,$$
$$\phi^3 = f(t)\sqrt{\kappa}r \cos\theta + \delta\phi^3, \qquad (A13)$$

where $\delta\phi^a$ are perturbations. With this choice, the background $\tilde{f}_{\mu\nu}$ is diagonal

$$\tilde{f}_{\mu\nu,I} = \text{diag}\left[-(C_I - f^2 D_I)\dot{f}^2, \frac{\kappa C_I f^2}{1 + \kappa r^2},\right.$$
$$\left. \kappa C_I r^2 f^2, \kappa C_I r^2 f^2 \sin^2\theta\right] + \mathcal{O}(\delta\phi^a), \qquad (A14)$$

while the physical metric has a 2×2 nondiagonal block in the (t, r) space. This example, although not necessarily corresponding to any solution of the equations of motion,

nevertheless provides a background which is minimally nondiagonal, potentially revealing new degeneracy conditions not covered by (A10). Since the background is not a consistent parameterisation of the degrees of freedom, we need to look at the action quadratic in perturbations to deduce the conditions on degeneracy.

At the background level, we have

$$(g^{-1}\tilde{f}_I)^\mu_{\ \nu} = \begin{pmatrix} (C_I - f^2 D_I)\dot{f}^2 & \frac{\kappa l C_I f^2}{1 + \kappa r^2} & 0 & 0 \\ -l(C_I - f^2 D_I)\dot{f}^2 & \frac{\kappa(1 - l^2)C_I f^2}{1 + \kappa r^2} & 0 & 0 \\ 0 & 0 & \kappa C_I f^2 & 0 \\ 0 & 0 & 0 & \kappa C_I f^2 \end{pmatrix}$$
$$+ \mathcal{O}(\delta\phi^a). \qquad (A15)$$

We start by diagonalizing this tensor. The background eigenvalues for the (t, r) nondiagonal block are

$$\lambda_{1,2}^I \equiv \frac{1}{2}\left[\frac{\kappa(1 - l^2)C_I f^2}{1 + \kappa r^2} + (C_I - f^2 D_I)\dot{f}^2\right]\left(1 \pm \sqrt{1 - \frac{4\kappa(1 + \kappa r^2)C_I f^2(C_I - f^2 D_I)\dot{f}^2}{[\kappa(1 - l^2)C_I f^2 + (1 + \kappa r^2)(C_I - f^2 D_I)\dot{f}^2]^2}}\right). \qquad (A16)$$

We can find perturbation corrections to the eigenvalues of $g^{-1}\tilde{f}$ by solving

$$\det(g^{-1}\tilde{f}_I - \mathbb{1}\ell) = 0, \qquad (A17)$$

perturbatively up to second order in perturbations. Unfortunately, this process is rather bulky for presentation. To simplify the process, we fix the angles $\theta = \pi/2$, $\phi = 0$ and assume all perturbations are time dependent only, since we are eventually interested in terms quadratic in time derivatives. In the end, we formally have

$$(R^T g^{-1}\tilde{f}R)^\mu_{\ \nu} = \ell(\mu)\delta_{\nu\mu}, \qquad (A18)$$

where R is an orthogonal matrix and $\ell(\mu)$ denotes the μth eigenvalue. Then, observing that

$$R^T\sqrt{g^{-1}\tilde{f}}R = \sqrt{R^T g^{-1}\tilde{f}R}, \qquad (A19)$$

we deduce that $\sqrt{\ell(\mu)}$ are the eigenvalues of $\sqrt{g^{-1}\tilde{f}}$, so they can directly be used when calculating the various traces.

Using *Mathematica*, we calculate the kinetic matrix and degeneracy conditions generated by only the mass terms, and hereafter we thus set $\dot{G} = F = A = 0$. Note that the inclusion of the nonminimal coupling is justified by $3 + 1$ decomposition in Sec. IV. The expression of the degeneracy conditions is cumbersome, and it is difficult to solve these equations exactly in general. Thus, we here systematically

assign random values to the function, and then we can confirm the degeneracy or nondegeneracy of the system to obtain the conditions.[5] Using this approach, we find that the following conditions complement (A11):

$$C_{\gamma_1} = C_{\gamma_2}, \qquad C_{\delta_1} = C_{\delta_2} = C_{\delta_3},$$
$$C_{\sigma_1} = C_{\sigma_2} = C_{\sigma_3} = C_{\sigma_4} = C_{\sigma_5}, \qquad (A20)$$

and

$$\frac{D_I}{C_I} = D(X) \quad \text{for any label } I, \qquad (A21)$$

where D is a single function. These conditions imply that only a single field space metric is allowed, whilst the conformal factors C_I can be absorbed in the definitions of the mass function. This can be seen as generalizing the original field space metric η_{ab} to $\tilde{\eta}_{ab}(\phi^a)$.

APPENDIX B: DERIVATION OF THE GENERAL ACTION FOR THE PROJECTED THEORY

In this appendix, we derive the action (38) in a systematic way. In contrast to the square root structure of the dRGT mass terms, we here construct mass terms by using W and Y, defined in Eq. (25), as building blocks. Let

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[5]If there is degeneracy, it is not possible to conclusively demonstrate this using perturbative methods around fixed backgrounds.

us consider the most general mass terms up to quadratic order in W and Y,

$$\mathcal{L}_{\mathrm{mass}} = B_1[W]^2 + B_2[W^2] + B_3[Y]^2 + B_4[Y^2]$$
$$+ B_5[W][Y] + B_6[WY], \tag{B1}$$

where B_i are function of X. Then we consider the following action including the nonminimal coupling:

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_p^2}{2}[G(X)R[g] + F(X)[Y]$$
$$+ A(X)[W] + m^2\mathcal{L}_{\mathrm{mass}}]. \tag{B2}$$

After $3+1$ decomposition, the canonical momenta are given by

$$\pi^a = \frac{\delta\mathcal{L}}{\delta\dot{\phi}_a}$$
$$= -4G_X K\phi^a - 2A\dot{\phi}^a - 2F\phi^a\phi^b\dot{\phi}_b + 4(B_1 + B_2)\dot{\phi}^a\dot{\phi}_b\dot{\phi}^b + 2(B_5 + B_6)(\phi^b\phi^c\dot{\phi}^a\dot{\phi}_b\dot{\phi}_c + \phi^a\phi^b\dot{\phi}_b\dot{\phi}_c\dot{\phi}^c)$$
$$+ 4(B_3 + B_4)(\phi^a\phi^b\phi^c\phi^d\dot{\phi}_b\dot{\phi}_c\dot{\phi}_d - \phi^a\phi^b\phi^c\phi^d\dot{\phi}_b(D_\mu\phi_d)(D^\mu\phi_c))$$
$$- 4B_1\dot{\phi}^a(D_\mu\phi_b)(D^\mu\phi^b) - 4B_2\dot{\phi}^b(D_\mu\phi_b)(D^\mu\phi^a) - 2B_5(\phi^a\phi^b\dot{\phi}_b(D_\mu\phi_c)(D^\mu\phi^c) + \phi^b\phi^c\dot{\phi}^a(D_\mu\phi_c)(D^\mu\phi_b))$$
$$- 2B_6(\phi^b\phi^c\dot{\phi}_b(D_\mu\phi_c)(D^\mu\phi^a) + \phi^a\phi^b\dot{\phi}^c(D_\mu\phi_c)(D^\mu\phi_b)), \tag{B3}$$

$$\pi^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta K_{\mu\nu}} = 2G(K^{\mu\nu} - \gamma^{\mu\nu}K) - 4G_X\gamma^{\mu\nu}\phi^a\dot{\phi}_a. \tag{B4}$$

As in Sec. IV, we consider the linear combination of the canonical momenta,

$$\Psi \equiv \alpha_1\phi^a\pi_a + \alpha_2\gamma^{\mu\nu}\pi_{\mu\nu}, \tag{B5}$$

where α_1 and α_2 are constants. To ensure the existence of a primary constraint on arbitrary backgrounds, Ψ should be independent of K, $\phi_a\dot{\phi}^a$, $w_1 \equiv \phi^a\dot{\phi}^b D_\mu\phi_a D^\mu\phi_b$, $w_2 \equiv \phi_a\dot{\phi}^a D_\mu\phi_b D^\mu\phi^b$, and $w_3 \equiv \phi_a\phi_b\phi_c\dot{\phi}^a D_\mu\phi^b D^\mu\phi^c$, i.e.,

$$\frac{\partial\Psi}{\partial(\phi_a\dot{\phi}^a)} = 0, \qquad \frac{\partial\Psi}{\partial K} = 0, \qquad \frac{\partial\Psi}{\partial w_1} = 0,$$
$$\frac{\partial\Psi}{\partial w_2} = 0, \qquad \frac{\partial\Psi}{\partial w_3} = 0, \tag{B6}$$

which gives the five conditions,

$$G_X X\alpha_1 + G\alpha_2 = 0,$$
$$(A + FX)\alpha_1 + 6G_X\alpha_2 = 0,$$
$$B_5 + B_6 + 2(B_3 + B_4)X = 0,$$
$$6B_1 + 4B_2 + 5B_5 X + 4X(B_6 + (B_3 + B_4)X) = 0,$$
$$10B_1 + 12B_2 + X(7B_5 + 8B_6 + 4(B_3 + B_4)X) = 0. \tag{B7}$$

Solving these equations, we find

$$F = \frac{6G_X^2}{G} - \frac{A}{X}, \qquad B_6 = -\frac{2B_2}{X},$$
$$B_5 = -\frac{2B_1}{X}, \qquad B_4 = -\frac{B_1 + B_2 - B_3 X^2}{X^2}. \tag{B8}$$

As one can see, the degeneracy conditions for G, A, F and B_i do not mix, and this implies that they can be imposed at each order. In the translation invariant case ($B_{3,4,5,6} = 0$ and $B_{1,2} = \mathrm{const}$), we have $B_1 = B_2 = 0$. Thus these translation-breaking terms are crucial to ensure the degeneracy. Once we impose these degeneracy conditions, the mass term $\mathcal{L}_{\mathrm{mass}}$ is characterized by only the arbitrary functions B_1 and B_2, while the B_3 term is canceled due to the above conditions. We can then rewrite the mass term in terms of only the projection tensor, and it can be written as $\mathcal{L}_g = B_1[Z]^2 + B_2[Z^2]$. Thus, besides the nonminimal coupling and its counterterms, the mass terms can be described by the traces of the matrix $Z^\mu{}_\nu$. With the same procedure, one can easily show that the cubic mass terms are described by all the possible combinations of the traces of $Z^\mu{}_\nu$ with three arbitrary functions. The higher order extension can be also possible, and we finally arrive at the action (38).

APPENDIX C: SCALAR SOUND SPEEDS IN THE EXTENDED THEORIES WITH NONMINIMAL COUPLING

In this section, we present the full expressions of the sounds speeds of scalar perturbations in the nonminimally coupled theories defined by the actions (51) and (80). After the nondynamical degrees of freedom are integrated out, the reduced scalar action contains two propagating degrees, one corresponding to the matter perturbations and the other

to the scalar graviton polarization. Considering a monochromatic wave and taking the subhorizon limit, we can solve the equations of motion, which can formally be reduced to the following algebraic equation for the sound speed C:

$$\left(\frac{C^2}{c_s^2} - 1\right)(\mathcal{A}C^2 - \mathcal{B}) - \mathcal{D} = 0. \tag{C1}$$

For the first class of theory (51), the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ are given by

$$\mathcal{A} = \frac{3(\rho_\chi + P_\chi)}{m^2 M_p^2 r^2}\left[\frac{J\xi r}{(H + \frac{\dot{G}}{2G})^2}\left(\frac{\kappa}{a^2} + \frac{m^2 J\xi}{2G}\right) - \frac{J\xi}{(H + \frac{\dot{G}}{2G})}\left(H - \frac{3\dot{G}}{2G} + \frac{\dot{J}}{J} + \frac{\dot{\xi}}{\xi}\right) + 2(\Gamma - J\xi)\right]$$
$$+ \frac{3}{(2GH + \dot{G})r^2}\left(\frac{\rho_\chi + P_\chi}{m^2 M_p^2}\right)^2\left(\frac{m^2 J\xi r}{H + \frac{\dot{G}}{2G}} + (1 - 3c_s^2)\dot{G}\right),$$

$$\mathcal{B} = \left(\frac{\rho_\chi + P_\chi}{m^2 M_p^2(H + \frac{\dot{G}}{2G})r}\right)^2\left((1 - 3c_s^2)^2\frac{\dot{G}^2}{4G^2} - \frac{m^2 r}{2G}(2\Gamma - 3J\xi)\right)$$
$$- \frac{J\xi(\rho_\chi + P_\chi)}{m^2 M_p^2 r(H + \frac{\dot{G}}{2G})^2}\left\{\frac{\ddot{\xi}}{\xi} - \frac{1 + r}{r}\left(H + \frac{\dot{\xi}}{\xi}\right)^2\right.$$
$$+ \left[\frac{4(H + \frac{\dot{G}}{G})[J\xi(r - 1) + (r + 1)\Gamma]}{Jr\xi} - \frac{\dot{G}[J\xi(7r - 10) + 4(r + 1)\Gamma]}{2GJr\xi} - \frac{2\dot{J}}{Jr} - \frac{\dot{r}}{r}\right]\left(H + \frac{\dot{\xi}}{\xi}\right)\right\}$$
$$+ \frac{(\rho_\chi + P_\chi)\xi}{4m^2 M_p^2 J(H + \frac{\dot{G}}{2G})^2}\left\{\frac{4(H + \frac{\dot{G}}{2G})^2}{r^2\xi^2}[4(1 + r)(\Gamma - J\xi)^2 + Jr\xi(10\Gamma - 7J\xi)]\right.$$
$$+ \frac{4J}{r^2\xi}\left(H + \frac{\dot{G}}{2G}\right)\left[-2(\Gamma - J\xi)\left(\dot{r} + \frac{2(1 + r)\dot{J}}{J} - \frac{(3 + r)\dot{G}}{G}\right) + 2r(\dot{\Gamma} - \dot{J}\xi) + \frac{Jr\xi}{2}\left(\frac{7\dot{G}}{G} - \frac{8\dot{J}}{J}\right)\right]$$
$$+ \frac{4J^2}{r}\left(\frac{2\ddot{G}}{G} - \frac{\ddot{J}}{J}\right) + \frac{2J(r - 6)\dot{J}\dot{G}}{Gr^2} - \frac{2J^2\dot{r}}{r^2}\left(\frac{3\dot{G}}{G} - \frac{2\dot{J}}{J}\right) + \frac{J^2(9 - 10r)\dot{G}^2}{G^2 r^2}$$
$$\left. + \frac{4(1 + r)\dot{J}^2}{r^2} + \frac{4\kappa J}{a^2 r\xi}[2(r - 1)\Gamma + 3J\xi] + \frac{4m^2 J^2(3r - 1)}{Gr}(\Gamma - J\xi) + \frac{2m^2 J^3\xi(2r + 1)}{Gr}\right\},$$

$$\mathcal{D} = \left(\frac{(1 - 3c_s^2)\dot{G}(\rho_\chi + P_\chi)}{m^2 M_p^2 r(2GH + \dot{G})}\right)^2. \tag{C2}$$

For the second class, i.e., the projected theory (80), these coefficients are given by

$$\mathcal{A} \equiv 9c_s^2(\rho_g + p_g)(\rho_\chi + p_\chi)(\dot{G} + 2GH)[(9c_s^2\rho_\chi + 3(3c_s^2 - 4)p_\chi + \rho_g - 3p_g)\dot{G} + 2G(3\dot{p}_g + H(\rho_g + 9p_g))],$$

$$\mathcal{B} \equiv -9c_s^2(1 - 3c_s^2)^2(\rho_\chi + p_\chi)^2(\rho_\chi + p_\chi + \rho_g + p_g)\dot{G}^2 - \frac{\mathcal{A}^2}{81c_s^2(\rho_g + p_g)^2(\rho_\chi + p_\chi)(\dot{G} + 2GH)^2}$$
$$+ \left[\frac{(13\rho_g + 21p_g)(\dot{G} + 2GH) + 6(3c_s^2 - 1)(\rho_\chi + p_\chi)\dot{G}}{9(\rho_g + p_g)(\dot{G} + 2GH)} - \frac{2}{3}\left(2 - \frac{M_p^2 M_{GW}^2}{\rho_g + p_g}\right)\right]\mathcal{A},$$

$$\mathcal{D} \equiv -(1 - 3c_s^2)^2(\rho_\chi + p_\chi)^2\dot{G}^2[(9c_s^2 - 1)\rho_g + 9(c_s^2 - 1)p_g - 6M_p^2 M_{GW}^2]. \tag{C3}$$

The solutions of Eq. (C1) can then be written as

$$C^2 = \frac{1}{2}\left[c_s^2 + \frac{\mathcal{B}}{\mathcal{A}} \pm \left(c_s^2 - \frac{\mathcal{B}}{\mathcal{A}}\right)\sqrt{1 + 4c_s^2\frac{\mathcal{D}}{\mathcal{A}}\left(c_s^2 - \frac{\mathcal{B}}{\mathcal{A}}\right)^{-2}}\right]. \tag{C4}$$

The conditions for avoiding gradient instability simply require a real sound speed, i.e., $C^2 > 0$ for both roots. Due to the presence of the nonminimal coupling, it is not straightforward to distinguish between the matter perturbation and the scalar graviton. However, we observe that in the case $c_s^2 = 1/3$, the sound speeds in both theories become relatively simple with

$$C_1^2\big|_{c_s^2 = 1/3} = \frac{1}{3}, \qquad C_2^2\big|_{c_s^2 = 1/3} = \frac{\mathcal{B}}{\mathcal{A}}. \tag{C5}$$

[1] M. Fierz and W. Pauli, Proc. R. Soc. A **173,** 211 (1939).

[2] V. I. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **12,** 447 (1970) [JETP Lett. **12,** 312 (1970)], http://www.jetpletters.ac.ru/ps/1734/article_26353.shtml.

[3] H. van Dam and M. J. G. Veltman, Nucl. Phys. **B22,** 397 (1970).

[4] A. I. Vainshtein, Phys. Lett. **39B,** 393 (1972).

[5] D. G. Boulware and S. Deser, Phys. Rev. D **6,** 3368 (1972).

[6] P. Creminelli, A. Nicolis, M. Papucci, and E. Trincherini, J. High Energy Phys. 09 (**2005**) 003.

[7] C. de Rham and G. Gabadadze, Phys. Rev. D **82,** 044020 (2010).

[8] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106,** 231101 (2011).

[9] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmol. Astropart. Phys. 11 (**2011**) 030.

[10] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmol. Astropart. Phys. 03 (**2012**) 006.

[11] A. De Felice, A. E. Gumrukcuoglu, and S. Mukohyama, Phys. Rev. Lett. **109,** 171101 (2012).

[12] G. D'Amico, G. Gabadadze, L. Hui, and D. Pirtskhalava, Phys. Rev. D **87,** 064037 (2013).

[13] Q.-G. Huang, Y.-S. Piao, and S.-Y. Zhou, Phys. Rev. D **86,** 124014 (2012).

[14] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 02 (2012) 126.

[15] S. Folkerts, A. Pritzel, and N. Wintergerst, arXiv:1107.3157.

[16] K. Hinterbichler, J. High Energy Phys. 10 (2013) 102.

[17] R. Kimura and D. Yamauchi, Phys. Rev. D **88,** 084025 (2013).

[18] C. de Rham, A. Matas, and A. J. Tolley, Classical Quantum Gravity **31,** 165004 (2014).

[19] X. Gao, Phys. Rev. D **90,** 064024 (2014).

[20] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (Amsterdam) **305,** 96 (2003).

[21] C. De Rham, L. Keltner, and A. J. Tolley, Phys. Rev. D **90,** 024050 (2014).

[22] M. Kenna-Allison, A. E. Gumrukcuoglu, and K. Koyama, Phys. Rev. D **101,** 084014 (2020).

[23] S. F. Hassan, R. A. Rosen, and A. Schmidt-May, J. High Energy Phys. 02 (2012) 026.

[24] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Lett. B **711,** 190 (2012).

[25] J. Kluson, J. High Energy Phys. 06 (2012) 170.

[26] S. F. Hassan, A. Schmidt-May, and M. von Strauss, Phys. Lett. B **715,** 335 (2012).

[27] J. Kluson, Phys. Rev. D **86,** 124005 (2012).

[28] J. Kluson, Int. J. Mod. Phys. A **28,** 1350160 (2013).

[29] T. Kugo and N. Ohta, Prog. Theor. Exp. Phys. (**2014**), 43B04.

[30] C. Lin, Phys. Lett. B **727,** 31 (2013).

[31] C. Lin, Phys. Lett. B **738,** 386 (2014).

[32] D. Langlois and K. Noui, J. Cosmol. Astropart. Phys. 02 (2016) 034.

[33] M. Crisostomi, K. Koyama, and G. Tasinato, J. Cosmol. Astropart. Phys. 04 (2016) 044.

[34] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, J. High Energy Phys. 12 (2016) 100.

[35] A. E. Gumrukcuoglu and K. Koyama, Phys. Rev. D **99,** 084004 (2019).

[36] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, and A. Tolley, Phys. Rev. D **84,** 124046 (2011).

[37] G. W. Horndeski, Int. J. Theor. Phys. **10,** 363 (1974).

[38] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. **126,** 511 (2011).

[39] M. Kenna-Allison, A. E. Gumrukcuoglu, and K. Koyama (to be published).

[40] C. Pitrou, X. Roy, and O. Umeh, Classical Quantum Gravity **30,** 165002 (2013).