FORMATION THEORY IN A CLASS OF LOCALLY FINITE GROUPS

by

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SUMMARY

Chapter 1 contains background material and definitions.
Chapter 2 gives some of the elementary properties of co-Černikov groups and has as its main result that the compact co-Černikov groups are precisely the pro-Černikov groups.
In chapter 3 the Sylow theory of pro-Černikov groups is developed. We define the concept of a generalised Sylow π-subgroup, for π a set of primes, and show that the theorems of Sylow and Hall extend to pro-Černikov groups whose Černikov factors are soluble.
In chapter 4, we discuss Sylow theory in the class, $\mathfrak{X}$, of countable, locally finite-soluble groups satisfying min-p for all primes p. We prove that the Sylow generating bases of such a group are locally conjugate by showing that these are generalised Sylow bases in the group's pro-Černikov completion. Since they are then conjugate in the completion, the result follows by observing that conjugacy in the completion implies local conjugacy in the group itself.
Chapter 5 is concerned with chief factors and $\mathfrak{F}$-normalisers and uses many standard proofs.
In chapter 6 we show that every $\mathfrak{X}$-group possesses $\mathfrak{F}$-projectors when $\mathfrak{F}$ is a co-Hopfian saturated formation (that is, $\mathfrak{F}$ contains no non co-Hopfian groups), by descending the derived series and keeping a check on certain subgroups that arise. We then show that the $\mathfrak{F}$-projectors are isomorphic and have a certain "local conjugacy" property. The $L\mathfrak{N}$-projectors are discussed in some detail.
In chapter 7 the $\mathfrak{F}$-abnormal subgroups are defined and give an alternative characterisation of the $\mathfrak{F}$-projectors.
In chapter 8, we ask what restrictions there are on an $\mathfrak{X}$-group if it possesses only countably many Sylow bases. In that case the Sylow bases are all conjugate and the group is poly-locally nilpotent.

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To my parents, I owe a great debt. It is them who made my stay at University possible. To my special friends, David Harman and Jane Silverthorne I express special thanks, for listening and helping during my deepest depressions. I should like to thank my other close friends, Denis, Jane, Jennie, John and Paul for all the fun we have had.

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DECLARATION

The work in this thesis is, to the best of my knowledge, original, except for the results which are attributed to others. Parts of it have been published as Warwick Notes; other parts have, or shall be, submitted for publication in the journals.

NOTATION

$H \leq G$	:	H is a subgroup of a group G.
$H \lhd G$	:	H is a normal subgroup of G.
H char G	:	H is a characteristic subgroup of G.
$\|G : H\|$	:	The index of a subgroup H in G.
$\langle X \rangle$	:	The group generated by a set X.
$C_G(H)$	:	The centraliser of a subset H in G.
$N_G(H)$	:	The normaliser of a subset H in G.
$\mathrm{Cr}_{i \in I} X_i$	:	The cartesian product of sets X_i.
$\mathrm{Dr}_{i \in I} X_i$	:	The direct product of groups X_i.
G'	:	The derived subgroup of G.
$G^{(n)}$	:	The n^{th} term of the derived series.
$O_\pi(G)$	:	The largest normal π-subgroup of G.
$O_{\pi,\pi'}(G)/O_\pi(G)$	:	$O_{\pi'}(G/O_\pi(G))$.
G^o	:	The radicable part of G.
$\rho(G)$	:	The Hirsch-Plotkin radical of G.
$\rho_\sigma(G)$	:	The σ^{th}-term of the upper locally nilpotent series.
$G^{\mathfrak{Z}}$	:	The $\mathfrak{Z}$-residual of a group G.
$A_G(H)$	:	The group of automorphisms of H induced by G on H.
G] H	:	The semidirect product of groups G and H.
$\mathrm{Syl}_\pi G$	:	The set of maximal π-subgroups of G.
$Q_\pi(G,\mathcal{N})$	:	The largest normal generalised π-subgroup of a co-Černikov group $(G,\mathcal{N})$.
$\mathrm{Syl}_\pi(G,\mathcal{N})$	:	The set of generalised Sylow π-subgroups of $(G,\mathcal{N})$.
$\mathrm{Max}_\pi(G,\mathcal{N})$	:	The set of maximal generalised π-subgroups of $(G,\mathcal{N})$.

C_{p^∞}	: The Prüfer p-group.
$[x,y]$	: The commutator, $x^{-1}y^{-1}xy$, of elements x and y.
x^g	: The element $g^{-1}xg$.
$[H,K]$	: The group $\langle [h,k] : h \in H, k \in K \rangle$, for subsets H, K of G.
H^g	: The set $\{h^g : h \in H\}$, for H a subset of G.
$X \leq_c Y$	: X is a closed subgroup of Y.
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$\overline{X}$	: The closure of a set X in some bigger space.
$\complement X$	: The complement of a set X.
$\pi(G)$	: The set of prime numbers dividing the orders of elements of G.
$\mathbb{P}$	: The set of prime numbers.
π'	: The complement of a set of primes in $\mathbb{P}$.
$\mathbb{N}_o$	: The set of natural numbers.
$\mathbb{F}_q$	: The field with q elements.
ω	: The first infinite ordinal.
$\bigoplus_{i \in I} V_i$	: The direct sum of vector spaces V_i.

CHAPTER 1. INTRODUCTION

In this chapter we give some of the notation, terminology and basic well known results that will be used in this thesis. Our notation is standard, but is listed on a previous page.

1.1. Basic Group Theory

We shall denote groups by capital Roman letters and elements of groups will be denoted by small Roman letters. A *periodic group* is a group in which each element has finite order. If G is a periodic group, $\pi(G)$ will denote the set of primes dividing the orders of the elements of G. If $\pi \subseteq \mathbb{P}$, the set of all primes, π' will denote the set $\mathbb{P} \backslash \pi$. A group G will be called a *π-group* if G is periodic and $\pi(G) \subseteq \pi$. If $\pi = \{p\}$, a single prime, we shall omit the braces and simply refer to G as a p-group and $\mathbb{P} \backslash \{p\}$ will be denoted by p'.

If G is a group and $\pi \subseteq \mathbb{P}$, G contains maximal π-subgroups, by Zorn's Lemma. A maximal π-subgroup will be called a *Sylow π-subgroup* and the set of all Sylow π-subgroups of a group G will be denoted by $Syl_\pi G$. If $\pi = \{p\}$ we shall omit the bracers and refer to *Sylow p-subgroups* and the set $Syl_p G$. Thus, if G is a finite group, Sylow's theorems hold for the Sylow p-subgroups ([22] I.7.5.2) and if G is also soluble then the theorems of P. Hall hold for the Sylow π-subgroups, for all $\pi \subseteq \mathbb{P}$ (see [12]).

If X is a subset of a group G, $\langle X \rangle$ will denote the group generated by X.

By a *Sylow basis* of a periodic group G we mean a complete set $\underset{\sim}{S} = \{S_p\}$ of Sylow p-subgroups of G, one for each prime p, satisfying $S_pS_q = S_qS_p$ for all primes p, q. If, furthermore, $\langle S_p : p \in \pi\rangle \in \mathrm{Syl}_\pi G$ for each set of primes π we shall say $\underset{\sim}{S}$ is a Sylow generating basis of G. In many classes of groups these two concepts are the same. P. Hall [13] has shown that a finite soluble group possesses Sylow bases and that these are all conjugate in the sense that if $\underset{\sim}{S} = \{S_p\}$, $\underset{\sim}{T} = \{T_p\}$ are Sylow bases of the finite soluble group G then there exists $g \in G$ such that

$$g^{-1}S_pg = S_p{}^g = T_p \quad \text{for all} \quad p \in \pi(G).$$

A subgroup H of a group G is called pronormal in G if, for each $g \in G$, H and H^g are conjugate in $\langle H,H^g\rangle$.

By a group theoretical class we shall mean a class of groups which contains all groups isomorphic to any one of its members and also containing all groups of order one. German script will be used to denote group theoretical classes. We shall also use the concept of a closure operation introduced by P. Hall [14]. We refer the reader to [37] for further discussion of closure operations and group classes. We shall use the well known algebra of closure operations. The operations we shall need most frequently are Q, S, L, R, and P (in standard notation). Thus, if $\mathfrak{X}$ is a class of groups, $L\mathfrak{X}$ will denote the class of locally $\mathfrak{X}$-groups and $G \in L\mathfrak{X}$ if and only if every finite subset of G is contained in an $\mathfrak{X}$-subgroup of G. The other operations are defined as in [37].

One of the most important classes of periodic groups is the class of periodic locally soluble groups which we denote by $\mathfrak{G}$. A group in the class $\mathfrak{G}$ will often be referred to as

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One of the most important classes of periodic groups is the class of periodic locally soluble groups which we denote by $\mathfrak{S}$. A group in the class $\mathfrak{S}$ will often be referred to as a locally finite-soluble group, since it is easily seen that a $\mathfrak{S}$-group is locally finite. The alphabet of group theoretic classes that we shall use is as follows:

- $\mathfrak{F}$: The class of finite groups.
- $\mathfrak{A}$: The class of abelian groups.
- $\mathfrak{N}$: The class of nilpotent groups.
- $\mathcal{J}_\pi$: The class of π-groups.
- $\mathfrak{S}_\pi$: The class of periodic locally soluble π-groups.
- $\mathfrak{U}$: The class introduced by Gardiner, Hartley and Tomkinson [8].
- Σ : The class introduced by Baer [1].

In this thesis the basic "building blocks" for constructing locally finite groups are Černikov groups. If G is a Černikov group, G^o will denote the unique subgroup of G minimal with respect to having finite index in G. The subgroup G^o is the unique maximal radicable subgroup of G; that is, if $x \in G^o$ and $n \in \mathbb{N}_o$ then there exists $y \in G^o$ such that $y^n = x$. Thus it is well known ([7] 19.1) that G^o is the direct sum of finitely many Prüfer p-groups for (possibly) different primes p. In general, if G is a group possessing a unique maximal radicable abelian subgroup G^o then we shall call G^o the radicable part of G and the Sylow π-subgroup of G^o will be called the π-radicable part of G, for $\pi \subseteq \mathbb{P}$.

If G is an arbitrary group, a set Ω of subgroups of G will be called a local system for G if

(i) $G = \cup\{H : H \in \Omega\}$.

(ii) Given $H, K \in \Omega$, there exists $L \in \Omega$ such that

$H, K \leq L$.

If G is locally finite, G always has a local system consisting of finite subgroups and if G is also countable we can choose this local system to be totally ordered.

Following Prüfer we shall say a group G has finite rank r if every finitely generated subgroup can be generated by r elements and if r is the least positive integer with this property. If no such integer r exists the group has rank ∞. We shall write rank G = r or ∞.

If Ω is a totally ordered set and G is an arbitrary group, a series of type Ω of G is a set $\{U_\sigma, V_\sigma : \sigma \in \Omega\}$ of pairs of subgroups, indexed by Ω, and satisfying

(i) $V_\sigma \lhd U_\sigma$ for all $\sigma \in \Omega$.

(ii) $U_\tau \leq V_\sigma$ if $\tau < \sigma$.

(iii) $G\backslash\{1\} = \bigcup_{\sigma\in\Omega} (U_\sigma \backslash V_\sigma)$.

The groups U_σ/V_σ are called the factors of the series. Such a series is called a normal series if the subgroups U_σ and V_σ are all normal subgroups of G and is a chief series if, in addition, U_σ/V_σ is a minimal normal subgroup of G/V_σ for each $\sigma \in \Omega$. The factors are then called chief factors of G. If the factors of a series are all $\mathfrak{X}$-groups, for some fixed class $\mathfrak{X}$, then the series is called an $\mathfrak{X}$-series. Every normal series can be refined to a chief series, but Jordan-Holder theorems do not generally hold. In [37] there is a full discussion of series.

By the Hirsch-Plotkin theorem and Zorn's Lemma, every group possesses a unique maximal normal locally nilpotent subgroup, $\rho(G)$, which we call the Hirsch-Plotkin radical of G.

If we define $R_1 = \rho(G)$, $R_{\sigma+1}/R_\sigma = \rho(G/R_\sigma)$ for ordinals σ and $R_\tau = \bigcup_{\sigma<\tau} R_\sigma$ for limit ordinals τ, we can form the <u>upper locally nilpotent</u> (or <u>radical</u>) series of G,

$$1 \leq R_1 \leq R_2 \leq \dots \leq R_\sigma \leq \dots \quad .$$

A group G will be called <u>radical</u> if $G = R_\gamma$ for some ordinal γ. In particular if $G = R_n$, for some integer n, $G \in PL\mathfrak{N}$ and the least such n will be called the <u>Fitting length</u> of G.

If $\mathfrak{X}$ is an R-closed class of groups then each group G possesses a unique normal subgroup, denoted by $G^{\mathfrak{X}}$, minimal with respect to the factor group being an $\mathfrak{X}$-group. We call $G^{\mathfrak{X}}$ the <u>$\mathfrak{X}$-residual</u> of the group G.

An automorphism α of a group G is called a <u>locally inner automorphism</u> of G if, given any finite set of elements g_1, ..., g_n of G, there exists an element $x \in G$, depending on $\{g_1, \dots, g_n\}$ such that

$$g_i^{\alpha} = g_i^{x} \quad \text{for} \quad i = 1, \dots, n.$$

Two subgroups H and K of G are said to be <u>locally conjugate</u> in G if there exists a locally inner automorphism mapping H onto K. Two Sylow bases $\underset{\sim}{S} = \{S_p\}$ and $\underset{\sim}{T} = \{T_p\}$ are said to be <u>locally conjugate</u> if, for some locally inner automorphism α of G,

$$S_p^{\alpha} = T_p \quad \text{for each prime} \quad p \in \pi(G).$$

<u>1.2. Locally Finite Groups with min-p</u>

A group G is said to satisfy min, <u>the minimum condition on subgroups</u>, if every descending chain of subgroups of G terminates in finitely many steps. Thus every Černikov group satisfies min. If p is a prime and every p-subgroup of G has min then G is said to satisfy min-p, the minimum condition on p-subgroups. In this section we shall outline some of the main results concerning locally finite groups satisfying min-p, for some (or all) prime(s) p. This class of groups has been the subject of much research activity in recent years, principally because of the following conjecture of V. P. Šunkov:

1.2.1. Conjecture: Every locally finite group G satisfying min-p for all primes p is locally soluble-by-finite (that is, G contains a locally soluble subgroup of finite index).

The difficulty in trying to prove 1.2.1 is tied up with the fact that infinite simple groups may be involved. Once this restriction is removed it is possible to obtain a very precise structural result (which we state as 1.2.4) and this may provide the key to the more complex problem of 1.2.1. We remark that I. I. Pavlyuk, A. A. Šafiro and V. P. Šunkov [36] have obtained a special case of 1.2.1.

In this thesis we are more concerned with Sylow theory in locally finite groups with min-p. Since a locally finite p-group is locally nilpotent and a locally nilpotent group satisfying min is Černikov ([24] 1.G.4), it follows that the Sylow p-subgroups of a locally finite group satisfying min-p for some prime p are Černikov groups. B. A. F. Wehrfritz [48] has shown that a locally finite group with min-p for some prime p has some well behaved Sylow p-subgroups. Using his methods he obtained the following very important result of Kargapolov [23].

1.2.2. Theorem: Let G be a locally finite group satisfying min-p. Then $|G : O_{p'p}(G)| < \infty$ if and only if G does not involve an infinite simple group containing elements of order p.

Here, if π is a set of primes, $O_\pi(G)$ denotes the unique maximal normal π-subgroup of a group G, $O_{\pi,\pi'}(G)/O_\pi(G) = O_{\pi'}(G/O_\pi(G))$ and so on.

Wehrfritz ([49] theorem 8) has shown that the class of locally finite groups satisfying min-p for some prime p is QS-closed. It is then easy to deduce from our previous remarks:

1.2.3. Lemma: Let G be a locally finite-soluble group satisfying min-p for some prime p. Then $G/O_{p'}(G)$ is a Černikov group.

It is easy to deduce, from 1.2.2, the following major structural result:

1.2.4. Theorem: Let G be a locally finite-soluble group satisfying min-p for all primes p. Then G has a radicable part G^o. The factor group G/G^o is residually finite and the Sylow p-subgroups of G/G^o are finite, for all primes p.

We prove the following well known consequence.

1.2.5. Corollary: Suppose G is a locally finite-soluble group satisfying min-p for all primes p. If π is a finite set of primes then

(i) The Sylow π-subgroups of G are conjugate.

(ii) The Sylow π'-subgroups of G are conjugate.

Proof: (i) Since π is a finite set of primes, the Sylow

π-subgroups are Černikov groups. Since $O_\pi(G)$ must contain the π-radicable part of G, the result follows from Hall's theorem in the theory of finite soluble groups.

(ii) Since $O_{\pi'}(G) = \bigcap_{p\in\pi} O_{p'}(G)$ and the extension of a Černikov group by a Černikov group is again a Černikov group ([24], 1.E.7), $G/O_{\pi'}(G)$ is a soluble Černikov group and has finite Sylow π'-subgroups. The result follows by Hall's theorem. □

Actually, Šunkov [44] has shown that in a locally finite group satisfying min-p for all primes p, the Sylow p-subgroups are conjugate.

Throughout the rest of this thesis we shall let $\mathfrak{X}$ denote the class of countable locally finite-soluble groups satisfying min-p for all primes p and we shall let $\mathfrak{Y}$ denote the class of locally finite-soluble groups satisfying min-p for all primes p. R. Baer [1] (Folgerung 5.4) has shown that $\mathfrak{Y}$ contains uncountable groups. We remark that, for sets of primes not satisfying the hypotheses of 1.2.5, the conclusions of 1.2.5 go badly wrong and it is this which creates many of the difficulties in dealing with $\mathfrak{Y}$-groups.

If G' denotes the derived subgroup of a group G, $G^{(i+1)}$ denotes the derived subgroup of $G^{(i)}$ and $G^{(\omega)} = \bigcap_{i\geq 1} G^{(i)}$ then 1.2.3 implies $G^{(\omega)} = 1$ for $G \in \mathfrak{Y}$. Thus every $\mathfrak{Y}$-group always has its derived series terminating in the trivial group. In contrast, there are $\mathfrak{X}$-groups G whose Hirsch-Plotkin radical is trivial, containing a proper subgroup H isomorphic to G. A group is said to be <u>co-Hopfian</u> if it contains no proper subgroup isomorphic to itself. In much of this thesis we have been forced to consider only $\mathfrak{X}$-groups which are co-Hopfian.

If G is a countable locally finite-soluble group then it is well known that G has a Sylow generating basis (for a proof, see [18] (lemma 2.1)). In [11], P. A. Gol'berg extended the theorem of P. Hall on the conjugacy of the Sylow bases of a finite soluble group to the class of soluble Černikov groups. We shall repeatedly use this fact.

The example of Baer [1] (Satz 5.3) shows that in general a Sylow basis of an $\mathfrak{X}$-group need not be a Sylow generating basis, although for co-Hopfian $\mathfrak{X}$-groups the equivalence of these two concepts is easily seen to be the same. Unfortunately there are trivial examples of metabelian $\mathfrak{X}$-groups whose Sylow generating bases are not conjugate. However, Baer [1] (lemma 3.5) showed that for the class Σ of $\mathfrak{X}$-groups with finite Sylow p-subgroups for all primes p, the Sylow generating bases are locally conjugate. Massey [30] unsuccessfully tried to extend Baer's result to the class $\mathfrak{X}$, but obtained some partial results in that direction.

At this point we introduce the concept of an inverse limit. Suppose I is a non-empty partially ordered index set and let $\{S_i : i \in I\}$ be a family of non-empty sets indexed by I. Suppose that if $i \leq j$ $(i, j \in I)$ there is a map

$$\alpha_{ij} : S_j \to S_i \quad \text{such that}$$

(i) $\alpha_{ii} = 1_{S_i}$, the identity map on S_i.

(ii) If $i, j, k \in I$ with $i \leq j \leq k$ then $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$.

The <u>inverse limit</u> of the system $\{S_i, \alpha_{ij} : i, j \in I, i \leq j\}$

of sets and mappings is a subset S of the cartesian product $C = \underset{i \in I}{\mathrm{Cr}}\, S_i$ given by

$$S = \{(s_i) \in C : \text{if } i \leq j,\ \alpha_{ij}(s_j) = s_i\}.$$

We denote this inverse limit by $\varprojlim S_i$. If I is also <u>directed</u>, that is, given i, j $\in$ I there exists k $\in$ I such that i, j $\leq$ k, then $\{S_i, \alpha_{ij} : i, j \in I, i \leq j\}$ will be called an <u>inverse system</u>.

In particular, if each S_i is a group and α_{ij} a group homomorphism then $\varprojlim S_i$ is a group. If S_i is a topological space and α_{ij} is a continuous map then $\varprojlim S_i$ has a natural topological space structure. Unfortunately the inverse limit of an inverse system may be empty. For our next result we give conditions when the inverse limit must be non-empty. A proof of this result can be found in [39] (theorem 2.1). A topological space is said to be T_1 if points are closed.

<u>1.2.6. Theorem</u>: Let $\{S_i, \alpha_{ij} : i, j \in I, i \leq j\}$ be an inverse system of non-empty compact topological T_1-spaces and closed, continuous maps. Then

(a) $S = \varprojlim S_i \neq \emptyset$

(b) The maps α_{ij} can be assumed to be surjective.

(c) The image of the canonical projection $\beta_i : S \to S_i$ is $\beta_i(S) = \bigcap_{i \leq j} \alpha_{ij}(S_j)$.

(d) If $T \subseteq S$ then $\overline{T} = \varprojlim \overline{\beta_i(T)}$ and if $T \subseteq_c S$ then $T = \varprojlim \beta_i(T) = \varprojlim \overline{\beta_i(T)}$.

(e) S is compact.

Here S has been given its natural topology as a subspace of a product. The notation $T \subseteq_c S$ means T is a closed subset of S and $\overline{T}$ means the closure of T in S. As a particular case, we obtain the well known theorem of Kuroš [29] that the inverse limit of an inverse system of non-empty finite sets is non-empty.

Using 1.2.6 we shall obtain one of our main results, that the Sylow generating bases of an $\mathfrak{X}$-group are locally conjugate. Actually, two proofs of this result occur in this thesis. The second method is analogous to that of Baer who proved the result for Σ-groups.

The first method is analogous to that used in [19]. It is well known that a residually finite group can be embedded as a dense subgroup of a profinite group, an inverse limit of finite groups. In particular, if G is a Σ-group then it can be embedded as a dense subgroup of a prosoluble group, an inverse limit of finite soluble groups. As in [19] one can define a "generalised" Sylow basis of a prosoluble group and show that these are conjugate. By dropping back to the original group, conjugacy in the "completion" (as the prosoluble group is called) induces local conjugacy in the original Σ-group.

We generalise this idea as follows. By a <u>separating filter base</u> $\mathcal{N}$ of a group G we shall mean a set of normal subgroups satisfying:

(i) If $N \in \mathcal{N}$, G/N is a Černikov group.

(ii) If $L, M \in \mathcal{N}$ there exists $N \in \mathcal{N}$ such that $N \leq L \cap M$.

(iii) $\cap\{N : N \in \mathcal{N}\} = 1$.

Thus G possesses a separating filter base if and only if G is a residually Černikov group. We shall call G a <u>co-Černikov group relative to $\mathcal{N}$</u> and regard G as a topological space with

$$\{Hx \ : \ x \in G \text{ and there exists } N \in \mathcal{N} \text{ such that } N \leq H \leq G\}$$

as a closed sub-base. Thus the closed subsets of G are intersections of finite unions of certain cosets of G. We shall let $(G,\mathcal{N})$ denote that G is a co-Černikov group relative to $\mathcal{N}$ and the topology determined by $\mathcal{N}$ will be called a <u>co-Černikov topology</u>. Of course G will possess many such topologies, depending on $\mathcal{N}$. By a <u>pro-Černikov</u> group we shall simply mean an inverse limit of Černikov groups.

We shall show that every co-Černikov group can be embedded as a dense subgroup of a pro-Černikov group. In particular 1.2.3 implies that every $\mathfrak{D}$-group can be thought of as a dense subgroup of a pro-Černikov group. J. Parker [35] has already shown how useful this can be for the Σ-groups of Baer.

In chapters 2 and 3 we give some of the very elementary properties of co-Černikov groups. Many of the results we obtain are already well known for cofinite groups (see [19]). In chapter 4 the main result on Sylow generating bases is deduced.

1.3. Formation Theory

With hindsight, the theory of formations in finite soluble groups really began with the famous result of Sylow that every finite group has Sylow p-subgroups all of which are conjugate. This work was then extended by P. Hall, but the real impetus that made Formation theory into such an important part of the theory of finite soluble groups was provided by the result of R. W. Carter [3]. He showed that every finite soluble group possesses a unique conjugacy class of nilpotent self normalising subgroups, which subsequently became known as Carter subgroups. W. Gaschütz [9] formalised all this work using the concept of a "saturated formation". He showed that a finite soluble group always contained "$\mathfrak{F}$-projectors", for $\mathfrak{F}$ a saturated formation, and that these subgroups were always conjugate.

In order to generalise P. Hall's idea of a basis normaliser, R. W. Carter and T. O. Hawkes [4] used the formation theoretic idea to define the "$\mathfrak{F}$-normalisers" of a finite soluble group. These turned out to be conjugate subgroups and were intimately related to the $\mathfrak{F}$-projectors of Gaschütz; every $\mathfrak{F}$-normaliser was contained in an $\mathfrak{F}$-projector and every $\mathfrak{F}$-projector contained an $\mathfrak{F}$-normaliser. Moreover, in certain classes the two concepts were the same.

As far as we are concerned, the next important turn of events came when S. E. Stonehewer [40] generalised Carter's original result to the class of locally finite-soluble groups whose Hirsch-Plotkin radical had finite index. This result was generalised in [42] when a theory of saturated formations was obtained. In the locally finite case however it was convenient to define the formations "locally"; Gaschütz and Lubeseder [10] had already shown that this was equivalent to the original definition of Gaschütz in the case of finite soluble groups.

Stonehewer [41] also generalised Carter's result to the class of locally soluble FC-groups except that in this case conjugacy of the various subgroups was replaced by local conjugacy. Subsequently, M. J. Tomkinson [46] obtained the full formation theoretic generalisation of Stonehewer's result, at least for locally finite-soluble FC-groups.

A further generalisation of Carter's result was obtained by B. A. F. Wehrfritz [47] for the class of all homomorphic images of periodic soluble linear groups.

All the previous theories, except the FC-case, were unified in the paper of Gardiner, Hartley and Tomkinson [8], who introduced a class $\mathfrak{U}$ of locally finite groups in which the Sylow theory was very well behaved. This process of formalisation was carried a stage further by A. A. Klimowicz in a series of papers. In [25] he defined a class $\mathfrak{W}$ of locally finite groups. In each $\mathfrak{W}$-group the Sylow structure was "well behaved" and was permuted transitively by a group of automorphisms of the $\mathfrak{W}$-group. This theory accounted for all previous theories, even the FC-case (see [26]). In [27] and [28] Klimowicz, in an axiomatic setting, gave conditions for a group to possess $\mathfrak{F}$-projectors.

J. Parker then established a theory of formations in the class of cofinite groups in which the finite factor groups were soluble. His results were applied to the class of Σ-groups, although some topological restrictions were necessary. In chapters 5, 6 and 7 we show how to generalise Parker's results on Σ-groups, omitting most of the topological restrictions. Chapter 5 contains many standard proofs of results

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We shall now define our notation and terminology used in the formation theory that follows.

If H/K is a chief factor of a group G, $A_G(H/K)$ will denote the group of automorphisms induced by G on H/K. Thus $A_G(H/K) \cong G/C_G(H/K)$. If $G \in \mathfrak{G}$, p is a prime and $\mathfrak{R}$ is some class of groups then we define

$$C_G(\mathfrak{R},p) = \begin{cases} \cap\{C_G(H/K) : H/K \text{ is a p-chief factor of } G \text{ such that } A_G(H/K) \in \mathfrak{R}\}. \\ G \quad \text{if no such chief factor exists.} \end{cases}$$

We shall call $C_G(\mathfrak{R},p)$ the <u>($\mathfrak{R}$,p)-centraliser</u> of G.

If $\mathfrak{D}$ is a Q-closed subclass of $\mathfrak{G}$, a subclass $\mathfrak{B}$ of $\mathfrak{D}$ will be called a <u>($\mathfrak{D}$,p)-preformation</u> if

(i) $\mathfrak{B}$ is Q-closed.

(ii) If $G \in \mathfrak{D}$ then $G/C_G(\mathfrak{B},p) \in \mathfrak{B}$.

Also $\mathfrak{B}$ is a <u>$\mathfrak{D}$-formation</u> if

(i) $\mathfrak{B}$ is a Q-closed subclass of $\mathfrak{D}$.

(ii) $\mathfrak{D} \cap R\mathfrak{B} \leq \mathfrak{B}$.

Thus every $\mathfrak{D}$-formation is a ($\mathfrak{D}$,p)-preformation. If π is a non-empty set of primes, a <u>$\mathfrak{D}$-preformation function</u> f, defined on π, associates with each $p \in \pi$ a ($\mathfrak{D}$,p)-preformation f(p). The <u>saturated $\mathfrak{D}$-formation</u> defined locally by f is

$$\mathfrak{F} = \mathfrak{F}(f) = \mathfrak{D} \cap \mathfrak{G}_\pi \cap \bigcap_{p\in\pi} \mathfrak{G}_{p'}\mathfrak{G}_p f(p).$$

If f is a $\mathfrak{D}$-preformation function then the (f(p),p)-centraliser of a group G will usually be called the f(p)-centraliser of the group and will be denoted by C_p or $C_p(G)$.

If $\mathfrak{D}$ is any class of groups a $\mathfrak{D}$-projector of a group G is a $\mathfrak{D}$-subgroup H of G such that, whenever $H \leq K \leq G$, $L \triangleleft K$ and $K/L \in \mathfrak{D}$, then $K = HL$.

The rest of the terminology used will be explained as the need arises.

CHAPTER 2. CO-ČERNIKOV GROUPS

In this chapter we give the basic definitions and elementary properties of co-Černikov groups that will be used in the rest of this thesis. The main aim of this chapter will be to prove that the compact co-Černikov groups are precisely the pro-Černikov groups.

If G is a Černikov group then we can regard G as a topological space with $\{Hx : x \in G, H \leq G\}$ as a closed sub-base. We shall call this topology the coset topology of G. Thus G is a co-Černikov group relative to {1} and every co-Černikov topology on G gives rise to the coset topology. The coset topology is analogous to the $\mathcal{W}$-topology, a subtopology of the Zariski topology of an affine algebraic group, defined in [39] (p. 188) and to the coset topology of a finite dimensional Lie Algebra, as studied in [5]. We shall see that the coset topology makes G into a compact T_1-space. We require information about various maps between Černikov groups and the following lemma will be useful.

2.1. Lemma: Suppose G is a group and $\bigcup_{i=1}^{n} H_i x_i \subseteq \bigcup_{i=1}^{m} K_i y_i$ are finite unions of cosets of subgroups of G. Suppose further that each subgroup H_i contains no proper subgroups of finite index. Then, given an integer $s \leq n$, there exists an integer t, dependent on s, such that $H_s \leq K_t$ and $H_s x_s \subseteq K_t y_t$. Moreover, for this t, $H_s = K_t$ if and only if $H_s x_s = K_t y_t$.

Proof: Let $s \leq n$. Then

$$H_s x_s = H_s x_s \cap \left(\bigcup_{i=1}^{m} K_i y_i\right) = \bigcup_{i=1}^{m} (H_s x_s \cap K_i y_i).$$

Hence $$H_s = \bigcup_{i=1}^{m} (H_s \cap K_i y_i x_s^{-1}).$$

Since $H_s \cap K_i y_i x_s^{-1}$ is either empty or a coset of $H_s \cap K_i$, H_s is a finite union of cosets of the subgroups $H_s \cap K_i$ ($i = 1, \ldots, m$). By B. H. Neumann [34] (4.4), there exists t such that $|H_s : H_s \cap K_t| < \infty$ and by the assumptions on H_s it follows that

$$H_s \leq K_t\ , \quad \text{for this } t.$$

Also $$H_s x_s \cap K_t y_t \neq \emptyset \Rightarrow K_t x_s \cap K_t y_t \neq \emptyset$$

$$\Rightarrow K_t x_s = K_t y_t,$$

since the cosets of a subgroup are equal or disjoint. Hence

$$H_s x_s \subseteq K_t x_s = K_t y_t \quad \text{for this } t.$$

Suppose $H_s = K_t$. Then $\emptyset \neq H_s x_s \cap K_t y_t = H_s x_s \cap H_s y_t$. Hence, as above, $H_s x_s = H_s y_t = K_t y_t$.

Conversely, if $H_s x_s = K_t y_t$ then,

$$K_t y_t = H_s x_s \subseteq K_t x_s \text{ so again } K_t y_t = K_t x_s$$

and hence $H_s = K_t$ as required. □

We note that if $H \leq G$, a Černikov group, then there is a finite subgroup F of H so that $H = H^o F$, where H^o is the radicable part of H. Thus, if $x \in G$, $Hx = \cup\{H^o fx : f \in F\}$ and hence every closed set in G is an intersection of a finite union of cosets of radicable subgroups of G.

A topological space T is called Noetherian if every descending chain of closed subsets of T terminates in finitely many steps. This is equivalent to requiring that every ascending chain of open subsets of T terminates in finitely many steps, hence the terminology.

2.2. Lemma: Let G be a Černikov group with coset topology. Then every closed subset of G is a finite union of cosets of radicable subgroups of G.

Proof: We note first that the remark following 2.1 implies that every closed subset of G is an intersection of a finite union of cosets of radicable subgroups of G. To prove the result we first show that if $\mathcal{S}$ is a descending chain of finite unions of cosets of radicable subgroups of G then $\mathcal{S}$ terminates in finitely many steps.

Let G^o be the radicable part of G and put rank $G^o = r < \infty$. Then, if K is a radicable subgroup of G, rank $K \leq r$.

If $A = \bigcup_{i=1}^{n} H_i x_i$ is a typical term of $\mathcal{S}$ (with H_i radicable, $x_i \in G$ for $i = 1, \ldots, n$) then we can associate with A an ordered (r+1)-tuple, $(a_r, a_{r-1}, \ldots, a_o)$, in which a_j is the number of subgroups H_i $(1 \leq i \leq n)$ of rank j. So we obtain a map

$$\phi : \mathcal{S} \longrightarrow \underbrace{\mathbb{N}_o \times \ldots \times \mathbb{N}_o}_{(r+1)\text{ times}} .$$

We note that under the lexicographic ordering:

$$(a_r, a_{r-1}, \ldots, a_o) < (b_r, b_{r-1}, \ldots, b_o)$$

$\Leftrightarrow$ there exists j such that $a_j < b_j$ and

$a_i = b_i$ for all $i > j$,

$\underbrace{\mathbb{N}_o \times \ldots \times \mathbb{N}_o}_{(r+1)\text{ times}}$ is a well ordered set.

Without loss of generality we may assume:

If $\bigcup_{i=1}^{n} H_i x_i \in \mathcal{S}$ and $i \neq j$ then $H_i x_i \not\subseteq H_j x_j$ (*)

Suppose $B = \bigcup_{i=1}^{m} K_i y_i$ is another term of $\mathcal{S}$ with $A \subseteq B$. We shall show by induction on n that

$$A \subsetneq B \Rightarrow \phi(A) \lneq \phi(B).$$

The case n = 1 is clear, so we assume n > 1. Let H_s be a subgroup such that rank $H_s \geq$ rank H_i for i = 1, ..., n. We may assume that rank $H_s \geq$ rank K_i for i = 1, ..., m, otherwise $\phi(A) \lneq \phi(B)$ is clear. Then by 2.1 there exists a subgroup K_t such that $H_s = K_t$ and $H_s x_s = K_t y_t$. Now

$$C = \bigcup_{\substack{i=1 \\ i \neq s}}^{n} H_i x_i \subsetneq \bigcup_{\substack{i=1 \\ i \neq t}}^{m} K_i y_i = D \text{ by } (*),$$

so by induction, $\phi(C) \lneq \phi(D)$. It is now clear from the definition of the lexicographic ordering that $\phi(A) \lneq \phi(B)$.

It follows that $\mathcal{S}$ cannot be infinite and strictly descending otherwise ϕ could be applied to the terms of $\mathcal{S}$ to obtain an infinite, strictly descending chain of elements in a well ordered set, which is impossible. Hence $\mathcal{S}$ must be finite.

It now follows from the remarks at the beginning of the proof and the distributive laws of set theory that every closed subset of G is a finite union of cosets of radicable subgroups of G. □

<u>2.3. Corollary</u>: Suppose G is a Černikov group with coset topology. Then

(i) G is Noetherian.

(ii) G is compact.

<u>Proof</u>: (i) The argument in 2.2 shows that G is Noetherian.

(ii) Every Noetherian space is clearly compact.

<u>2.4. Lemma</u>: Let G, H be Černikov groups with coset topologies:

(i) If $K \leq G$ then $xK \subseteq_c G$, for all $x \in G$.

(ii) If $\phi : G \to G$ is defined by $\phi(x) = x^{-1}$, for all $x \in G$, then ϕ is closed and continuous.

(iii) If $\theta : G \to H$ is a homomorphism then θ is closed and continuous.

(iv) If $y \in G$ and $\alpha_y, \beta_y : G \to G$ are defined by $\alpha_y(x) = xy$, $\beta_y(x) = yx$, for all $x \in G$, then α_y and β_y are both closed and continuous.

<u>Proof</u>: (i) Note that $xK = K^{x^{-1}}x$, which is closed by definition.

(ii) Clear.

(iii) The result follows by 2.2.

(iv) Clear. □

One might hope that a Černikov group with coset topology was Hausdorff also, but the following example shows that this is generally not the case.

2.5. Example: Let $G \cong C_{p^\infty}$ and suppose $x \neq y \in G$. Suppose there exist open sets $U, V \neq \emptyset$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $\mathscr{C}U \cup \mathscr{C}V = G$. Since $\mathscr{C}U$, $\mathscr{C}V$ are closed, they are finite unions of cosets by 2.2. It follows by [34] (4.4) that G has a subgroup of finite index. Hence $G = \mathscr{C}U$ or $G = \mathscr{C}V$, a contradiction.

This example also shows that Černikov groups with coset topology need not be topological groups, in this topology. For, every co-Černikov group with a co-Černikov topology is evidently T_1; but in the above example G cannot be a topological group since this would contradict the equivalence of (i) and (v) in proposition 3($\mathcal{TG}$) of [20]. However, for our purposes, it is the compactness of a Černikov group that is important.

2.6. Proposition: Let $(G,\mathcal{N})$ be a co-Černikov group and let $\mathcal{P}$ be the closed sub-base determined by $\mathcal{N}$. If $H \leq G$ then

$$\overline{H} = \cap\{HN : N \in \mathcal{N}\} = \cap\{HK : K \in \mathcal{P} \text{ and } K \leq G\}.$$

In particular $\overline{H} \leq_c G$ and if $H \triangleleft G$ then $\overline{H} \triangleleft_c G$.

Proof: If $N \in \mathcal{N}$, $N \leq HN \leq G$ so $HN \leq_c G$ by definition of $(G,\mathcal{N})$. Hence $H \leq \cap\{HN : N \in \mathcal{N}\} \leq_c G$ so

$$\overline{H} \subseteq \cap\{HN : N \in \mathcal{N}\}.$$

Conversely, $\overline{H}$ is closed and hence $\overline{H} = \bigcap_{i \in I} \bigcup_{j=1}^{n_i} K_{ij}x_{ij}$ for some index set I, elements $x_{ij} \in G$, subgroups K_{ij} of G (with $N_{ij} \leq K_{ij}$ for some $N_{ij} \in \mathcal{N}$) and $n_i \in \mathbb{N}_0$. Put

$A_i = \bigcup_{j=1}^{n_i} K_{ij}x_{ij}$. Since $\mathcal{N}$ is a separating filter base it follows that for each $i \in I$ there exists $N_i \in \mathcal{N}$ such that

$$N_i \leq K_{ij} \quad \text{for} \quad j = 1, \ldots, n_i.$$

Now $H = \bigcup_{j=1}^{n_i} (H \cap K_{ij}x_{ij}) = \bigcup_{j=1}^{m_i} (H \cap K_{ij})y_{ij}$ say. Here the K_{ij} have been renumbered if necessary, $m_i \leq n_i$ is the number of non-empty intersections $H \cap K_{ij}x_{ij}$ and $y_{ij} \in H \cap K_{ij}x_{ij}$.

Hence,

$$HN_i = \bigcup_{j=1}^{m_i} (H \cap K_{ij})N_i y_{ij} \subseteq \bigcup_{j=1}^{m_i} K_{ij}N_i y_{ij} \subseteq A_i \text{ for all } i.$$

Therefore $\cap\{HN : N \in \mathcal{N}\} \subseteq \bigcap_{i \in I} A_i = \bar{H}$ and the result follows.

It is now clear from the definitions that

$\bar{H} = \cap\{HK : K \in \mathcal{P}, K \underline{<} G\}$. □

As an easy consequence we have

<u>2.7. Corollary</u>: If $(G,\mathcal{N})$ is a co-Černikov group then $H \leq_d G$ if and only if $G = NH$ for all $N \in \mathcal{N}$.

The following extension of 2.4 is easily established.

<u>2.8. Lemma</u>: Let $(G,\mathcal{N})$ and $(H,\mathcal{M})$ be co-Černikov groups.

(i) If $K \leq G$ and there exists $N \in \mathcal{N}$ such that $N \leq K$ then xK is a closed set, for all $x \in G$.

(ii) If $\phi : G \to G$ denotes inversion then ϕ is closed and continuous.

(iii) If α_y, $\beta_y : G \to G$ are defined by $\alpha_y(x) = xy$ and $\beta_y(x) = yx$, for $x, y \in G$, then α_y and β_y are both closed and continuous.

We shall fix the following notation for the rest of this chapter. Let $\{G_i, \theta_{ij} : i, j \in I\}$ be an inverse system of Černikov groups, with coset topologies, and group homomorphisms, indexed by a set I. Thus if $i \geq j$ there is a homomorphism $\theta_{ji} : G_i \to G_j$. Let $G = \varprojlim G_i$, a pro-Černikov group, and let $H = \underset{i \in I}{\mathrm{Cr}}\ G_i$. Give G and H their usual topologies.

Let $\alpha : G \to H$ denote the inclusion map, $\beta_i : H \to G_i$ the i^{th} projection map and $\gamma_i = \beta_i \circ \alpha$. Put $M_i = \ker \beta_i$, $N_i = \ker \gamma_i$ and $\mathcal{N} = \{N_i : i \in I\}$. The following properties are then immediately established.

<u>2.9. Lemma</u>: (i) $M_i \cong \underset{j \neq i}{\mathrm{Cr}}\ G_j$ for all $i \in I$.

(ii) $M_i \cap G = N_i$ for all $i \in I$.

(iii) $\bigcap_{i \in I} M_i = 1$.

(iv) $\bigcap_{i \in I} N_i = 1$.

(v) $N_i \leq N_j$ if $i \geq j$.

<u>2.10. Lemma</u>:

(i) The maps α, β_i, γ_i, θ_{ij} are continuous for $i, j \in I$.

(ii) The left and right translation maps in G and H are continuous.

(iii) For each $i \geq j$, θ_{ji} is a closed map.

(iv) If $M_i \leq L \leq H$ then $L \leq_c H$ for each $i \in I$.

(v) If $N_i \leq L \leq G$ then $L \leq_c G$ for each $i \in I$.

<u>Proof</u>: Parts (i), (ii) and (iii) are trivial using elementary topology and 2.4 (see [38] for example).

(iv) With the usual identifications $H = M_i \times G_i$ so $L = M_i \times (G_i \cap L)$ by Dedekind's Law. Since G_i has the coset topology and H has its usual topology, L is a cartesian product of closed sets so is closed.

(v) If $N_i \leq L \leq G$ then $M_i \leq M_iL \leq H$ so $M_iL \leq_c H$ by (iv). Since G has subspace topology, 2.9(ii) implies

$$L = M_iL \cap G \leq_c G$$

and the result follows. □

To prove the main result of this section we shall require 1.2.6.

2.11. Theorem: A group K is isomorphic to a pro-Černikov group if and only if for some separating filter base $\mathcal{M}$, $(K,\mathcal{M})$ is a compact co-Černikov group.

Proof: (⇒) With the usual notation we may write $K = G = \varprojlim G_i$. By 2.3(ii) and 2.4(iii), the hypotheses of 1.2.6 are satisfied and by 1.2.6(b) and (c) we may assume that the γ_i are surjective.

Let τ be the co-Černikov topology induced on G by $\mathcal{N}$ and let σ denote the natural subspace topology on G. Then by 1.2.6(e), (G,σ) is a compact space. To prove the result it now suffices to show that $\sigma = \tau$. By 2.10(v), if $N_i \leq L \leq G$ then $L \leq_c (G,\sigma)$. Hence sub-basic closed sets in τ are closed in σ so $\tau \subseteq \sigma$. On the other hand if $L \leq G_i$ and $g \in G_i$ then $(Lg \times M_i) \cap G$ is a sub-basic closed set in σ and

$$N_i = M_i \cap G \leq (L \times M_i) \cap G.$$

Thus $(L \times M_i) \cap G \leq_c (G,\tau)$. Since the γ_i are surjective, 2.8(iii) implies that $(Lg \times M_i) \cap G \subseteq_c (G,\tau)$ and hence $\sigma = \tau$, as required.

($\Leftarrow$) Let $(K,\mathcal{M})$ be a compact co-Černikov group, for some separating filter base $\mathcal{M}$. Put $\mathcal{M} = \{H_i : i \in J\}$ for some index set J and order J via:

$$j \leq i \Leftrightarrow H_i \leq H_j.$$

Thus, for $j \leq i$, there is a map $\psi_{ji} : K/H_i \to K/H_j$ and $\{K/H_i, \psi_{ji} : i, j \in J\}$ is an inverse system of Černikov groups and group homomorphisms. Put $L = \varprojlim K/H_i$ and define $\Phi : K \to L$ by

$$\Phi(g) = (gH_i) \in L, \text{ for each } g \in K.$$

Then Φ is clearly a monomorphism, since $\mathcal{M}$ is a filter base. It now suffices to show that Φ is a surjection so let $(g_iH_i) \in L$.

If $\{H_{i_j} : 1 \leq j \leq r\}$ is any finite set of elements of $\mathcal{M}$ there exists $H_k \in \mathcal{M}$ such that

$$H_k \leq H_{i_1} \cap \ldots \cap H_{i_r}, \text{ whence } i_1, \ldots, i_r \leq k.$$

Also $\psi_{ji}(g_iH_i) = g_iH_j = g_jH_j$ if $j \leq i$ so $g_kH_k \subseteq g_{i_j}H_{i_j}$ for $j = 1, \ldots, r$. Hence the set $\{g_iH_i\}_{i \in J}$ has the finite intersection property and since $(K,\mathcal{M})$ is compact, it follows that

$$\cap\{g_iH_i : i \in J\} \neq \emptyset.$$

If g is an element of this intersection then it is clear that $\Phi(g) = (g_i H_i)$. Also $\Phi(H_j) = L \cap \underset{i \neq j}{Cr}\, G/H_i$, so Φ is a homeomorphism. □

We now show that a co-Černikov group can always be embedded as a dense subgroup of a compact co-Černikov group.

2.12. Corollary: Let $(K, \mathcal{M})$ be a co-Černikov group. Then K can be embedded as a dense subgroup of a pro-Černikov group.

Proof: Let $\mathcal{M} = \{H_i : i \in J\}$ and let J be an index set ordered by $j \leq i$ if and only if $H_i \leq H_j$, for $i, j \in J$. Put $L = \varprojlim K/H_i$. By 2.11, L together with a suitable separating filter base, (which can easily be written down), is a compact co-Černikov group when, for each i, K/H_i is given its coset topology and L is given its natural topology.

The map $\Phi : K \to L$ given by $\Phi(g) = (gH_i)$ (for $g \in K$) is certainly an embedding of K in L since $\mathcal{M}$ is a separating filter base, so it suffices to prove $\Phi(K)$ is dense in L.

Let U be a basic open subset of L so that $U = L \cap \underset{i \in J}{Cr}\, X_i$ with $X_i \subseteq_o K/H_i$ and, for all but finitely many i, $X_i = K/H_i$. To show that $\Phi(K)$ is dense in L, we need to show that $U \cap \Phi(K) = \underset{i \in J}{Cr}\, X_i \cap \Phi(K) \neq \emptyset$. If $(g_i H_i) \in \underset{i \in J}{Cr}\, X_i \cap L$ and $i_1, \ldots, i_r$ are the indices for which $X_i \neq K/H_i$, there exists $m \in J$ such that

$$H_m \leq H_{i_1} \cap \ldots \cap H_{i_r} .$$

Then, for this m, $(g_m H_i) \in \Phi(K) \cap \underset{i \in J}{Cr}\, X_i$. Also Φ is closed since $\Phi(H_j) = \Phi(K) \cap \underset{i \neq j}{Cr}\, G/H_i$ is in the filter base for $\Phi(K)$. Similarly Φ is continuous.

Using the notation of 2.12 let $\phi_i : L \to K/H_i$ be the natural projection map. If Φ is the embedding defined in 2.12, we can prove, in a similar manner to the proof of that corollary, with the notation of 2.12

<u>2.13. Lemma</u>: For each $i \in J$, $\overline{\Phi(H_i)} = \ker \phi_i$.

Any compact co-Černikov group $(L,\mathcal{P})$ containing $(K,\mathcal{M})$ as a dense subgroup will be called a <u>completion</u> of $(K,\mathcal{M})$. (Thus the topology induced by $\mathcal{P}$ on K is the same as that induced by $\mathcal{M}$.) We shall prove in 2.23 a result analogous to theorem 2.1 of [19]. Before doing this we give some further elementary properties of co-Černikov groups. We first give two obvious methods of constructing co-Černikov groups from a given co-Černikov group $(K,\mathcal{M})$. If $L \leq K$ then let $L \cap \mathcal{M} = \{L \cap M : M \in \mathcal{M}\}$ and if $L \lhd K$ let $\mathcal{M}L/L = \{ML/L : M \in \mathcal{M}\}$.

<u>2.14. Proposition</u>: Suppose $(K,\mathcal{M})$ is a co-Černikov group and $L \leq K$. Then $(L, \mathcal{M} \cap L)$ is a co-Černikov group and the co-Černikov topology induced by $\mathcal{M} \cap L$ is the subspace topology.

The proof is trivial and is omitted.

<u>2.15. Proposition</u>: If $(K,\mathcal{M})$ is a co-Černikov group and $L \lhd_c K$ then $(K/L, \mathcal{M}L/L)$ is a co-Černikov group.

<u>Proof</u>: This follows since $\mathcal{M}L/L$ is a separating filter base for K/L by 2.6. The co-Černikov topology defined on K/L then has as a closed sub-base the set

$$\{(F/L)\cdot Lx : \text{there exists } M \in \mathcal{M} \text{ such that } ML \leq F\}. \qquad \square$$

It is easily seen that the co-Černikov topology defined on K/L in 2.15 is the quotient topology. Because of 2.14 and 2.15 one might ask whether the product topology on a cartesian product of co-Černikov groups yields a co-Černikov topology. A positive answer would give, together with 2.14, a direct proof of the necessity of 2.11. However the following easy example shows this is not true.

Let $K \cong C_{p^\infty}$, the unique infinite locally cyclic p-group. Let τ be the product topology of $K \times K$, induced by the coset topology on K and let σ be the coset topology defined on $K \times K$. Then $\tau \neq \sigma$. For let $A = \{(a,a) : a \in K\}$. Then A is certainly σ-closed but is not τ-closed. Otherwise there would exist subgroups B_i, C_i of K and elements x_i, $y_i \in K$ such that

$$A = \cap\{(\bigcup_{\text{finite}} (B_i x_i \times K)) \cup (\bigcup_{\text{finite}} (K \times C_i y_i))\}.$$

Then by [34] (4.4) there exists i such that either $|A : A \cap (B_i \times K)| < \infty$ or $|A : A \cap (K \times C_i)| < \infty$. Since A is radicable it follows that either $A \leq B_i \times K$ or $A \leq K \times C_i$ for this i and hence $A = K \times K$, a contradiction.

The following three results, although straightforward, are very important for the applications in chapters 3 and 4.

<u>2.16. Lemma</u>: Let $(K,\mathcal{M})$ be a co-Černikov group and suppose $L \triangleleft_c K$ with K/L a Černikov group. Then there exists $M \in \mathcal{M}$ such that $M \leq L$.

<u>Proof</u>: If $\mathcal{M} = \{H_i : i \in J\}$ then by 2.6,

$$L = \cap\{LH_i : i \in J\}.$$

But K/L has the minimal condition on subgroups and so there are subgroups $H_1, \ldots, H_n \in \mathcal{M}$ such that $L = \bigcap_{i=1}^{n} LH_i$. Since $\mathcal{M}$ is a separating filter base there is an $M \in \mathcal{M}$ such that

$$M \leq \bigcap_{i=1}^{n} H_i \leq L \text{ , as required.}$$ □

<u>2.17. Corollary</u>: If $(K,\mathcal{M})$ is a co-Černikov group, $L \triangleleft_c K$ with K/L Černikov and $L \leq M \leq K$ then $M \leq_c K$.

<u>Proof</u>: The proof is clear from 2.16 and the definition of the co-Černikov topology induced on K by $\mathcal{M}$.

<u>2.18. Lemma</u>: Let $(K,\mathcal{M})$ be a co-Černikov group and $L \leq K$. If $(L, L \cap \mathcal{M})$ is compact then $L \leq_c K$.

<u>Proof</u>: By 2.6, $\overline{L} = \bigcap\{LM : M \in \mathcal{M}\}$. If $x \in \overline{L}$ then $x \in LM$ for each $M \in \mathcal{M}$. Hence $L \cap xM \neq \emptyset$ and $L \cap xM \subseteq_c L$ since L has subspace topology. Moreover, since $\mathcal{M}$ is a separating filter base, $\{L \cap xM : M \in \mathcal{M}\}$ has the finite intersection property. Hence by the compactness of L,

$$\emptyset \neq \bigcap\{L \cap xM : M \in \mathcal{M}\} = L \cap \{x\}.$$

So we must have $x \in L$ and L is closed. □

Thus in a compact co-Černikov group, closed subgroups and compact subgroups are precisely the same thing.

We shall now prove our generalisation of 2.1 of [19]. We give some preliminary results first, all of which are well known in the cofinite case.

<u>2.19. Lemma</u>: Let $(K,\mathcal{M})$ be a co-Černikov group and $L \leq_c K$. Then $N_K(L) \leq_c K$.

Proof: If $M \in \mathcal{M}$, define $N_M/M = N_{K/M}(LM/M)$. We shall show $N_K(L) = \cap\{N_M : M \in \mathcal{M}\} \leq_c K$. If $x \in \cap\{N_M : M \in \mathcal{M}\}$ then $L^x M = LM$ for all $M \in \mathcal{M}$. Since $L, L^x \leq_c K$, 2.6 implies $L^x = L$. Thus $x \in N_K(L)$. The reverse inclusion is obvious. □

2.20. Lemma: Suppose $(K,\mathcal{M})$ is a co-Černikov group.

(i) If $N \triangleleft_c K$ then the natural map $\alpha : (K,\mathcal{M}) \to (K/N, \mathcal{M}N/N)$ is continuous.

(ii) If $(L,\mathcal{L})$ is a co-Černikov group and $\alpha : (K,\mathcal{M}) \to (L,\mathcal{L})$ is a continuous epimorphism then given $M \triangleleft_c L$ with L/M Černikov there exists $N \triangleleft_c K$ with K/N Černikov and $\alpha(N) = M$.

Proof: (i) This is clear from the definitions.
(ii) Let $\alpha^{-1}(M) = N$. Since α is continuous, $N \triangleleft_c K$. Since α is an epimorphism, $\alpha(N) = M$ and clearly K/N is a Černikov group. □

2.21. Lemma: Suppose $(K,\mathcal{M})$ is a co-Černikov group and A, B are subsets of K. Then

$$\bar{A}\,\bar{B} \subseteq \overline{AB}\ .$$

Proof: For each $b \in B$, $Ab \subseteq AB$ and since K is T_1, $\overline{A}b \subseteq \overline{AB}$.

Thus $$\bar{A}B \subseteq \overline{AB}\ .$$

Hence $$\overline{\bar{A}B} \subseteq \overline{AB}\ .$$

Applying the first part of the argument with A and B interchanged gives

$$\bar{A}\,\bar{B} \subseteq \overline{AB}\ .$$ □

2.22. Lemma: Let $(K,\mathcal{M})$ be a co-Černikov group and $(\overline{K},\mathcal{L})$ a completion of K (thus $\mathcal{L} \cap K$ induces the same topology on K as $\mathcal{M}$ does).

(i) If $\mathcal{P} = \{M \triangleleft_c K : K/M$ is Černikov$\}$ then

$$\overline{\mathcal{P}} = \{\overline{M} : M \in \mathcal{P}\} = \{M \triangleleft_c \overline{K} : \overline{K}/M \text{ is Černikov}\}.$$

(ii) If $M \in \mathcal{P}$ then $M = \overline{M} \cap K$.

Proof: (i) Let $\mathcal{L} = \{L_i : i \in J\}$ be the separating filter base. If $M \triangleleft_c \overline{K}$ and $\overline{K}/M$ is Černikov there exists $L_k \leq M$ by 2.16. Let $\mathcal{Q} = \{L_i \in \mathcal{L} : L_i \leq M\}$. Then $\mathcal{Q}$ is a separating filter base for M. For if $L_j \in \mathcal{L}$, $L_j \cap L_k$ contains some $L_i \in \mathcal{L}$. Thus $L_i \in \mathcal{Q}$. If $x \in \cap\{L_m : L_m \in \mathcal{Q}\}$ then $x \in L_i$ and hence $x \in L_j$. Thus $x \in \cap\{L_m : L_m \in \mathcal{L}\} = 1$ and hence $\mathcal{Q}$ is a filter base.

Now $N = M \cap K \triangleleft_c K$ and K/N is Černikov. Also $\overline{N} \leq M$. We show $\overline{N} = M$. For each $L_i \in \mathcal{Q}$, $\overline{K} = KL_i$ by 2.7 so by the Dedekind law

$$M = L_i(M \cap K) = L_iN .$$

Since $\mathcal{Q}$ is a separating filter base for M, $N \leq_d M$ and hence $M = \overline{N}$.

Suppose now $N \triangleleft_c K$ and K/N is Černikov. Since $\overline{N} \leq_c \overline{K}$, $N_{\overline{K}}(\overline{N}) \leq_c \overline{K}$. Also if $g \in K$ then $N = N^g \leq \overline{N}^g$. Hence $\overline{N} = \overline{N}^g$ and $K \leq N_{\overline{K}}(\overline{N})$. It follows, since $K \leq_d \overline{K}$, that $\overline{N} \triangleleft_c \overline{K}$. Moreover, since $\overline{K}$ is compact 2.20(i) implies $\overline{K}/\overline{N}$ is compact and since $\mathcal{L}\overline{N}/\overline{N}$ is a filter base for $\overline{K}/\overline{N}$ it follows, in a similar fashion to the proof of 2.11, that

$$\overline{K}/\overline{N} \cong \varprojlim (\overline{K}/\overline{N})/(L_i\overline{N}/\overline{N}) \cong \varprojlim \overline{K}/L_i\overline{N}.$$

However $\bar{K} = KL_i$ and $L_i\bar{N} = L_iN$. Thus

$$\bar{K}/\bar{N} \cong \varprojlim KL_i/L_iN \cong \varprojlim K/(K \cap L_iN) = \varprojlim K/N(K \cap L_i). \quad (1)$$

(We here use standard facts about inverse limits.) However $(K \cap \mathcal{L})N/N$ is a filter base for the Černikov group K/N, which is its own completion. Thus the right hand side of (1) is isomorphic to K/N. Hence $\bar{K}/\bar{N}$ is a Černikov group and (i) follows.

(ii) Clearly $N \leq \bar{N} \cap K$. However K has subspace topology so there exists $C \leq_c \bar{K}$ such that $C \cap K = N$. Hence $\bar{N} \leq C$ and $\bar{N} \cap K \leq C \cap K = N$. The result follows. □

We can now prove the result we have been seeking. Our proof is similar to that of Hartley [19].

<u>2.23. Theorem</u>: Let $(K,\mathcal{M})$ be a co-Černikov group contained as a dense subgroup of the compact co-Černikov group $(\bar{K},\mathcal{P})$. Let $(L,\mathcal{L})$ be any compact co-Černikov group and $\alpha : (K,\mathcal{M}) \to (L,\mathcal{L})$ a continuous homomorphism. Then

(i) α can be uniquely extended to a continuous homomorphism $\bar{\alpha} : (\bar{K},\mathcal{P}) \to (L,\mathcal{L})$.

(ii) $\bar{\alpha}(\bar{K}) = \overline{\alpha(K)}$.

(iii) $\bar{\alpha}$ is injective if and only if α is an algebraic and topological embedding and in that case $\bar{\alpha}$ is an algebraic and topological isomorphism between $\bar{K}$ and $\overline{\alpha(K)}$.

<u>Proof</u>: To begin we prove that if $\bar{\alpha}$ is any continuous homomorphism extending α to $\bar{K}$ then $\bar{\alpha}(\bar{K}) = \overline{\alpha(K)}$. For, $\bar{K}$ is compact so $\bar{\alpha}(\bar{K})$ is compact and hence is closed by 2.18. Thus $\overline{\alpha(K)} \leq \bar{\alpha}(\bar{K})$.

Conversely, $K \leq \bar{\alpha}^{-1}(\overline{\alpha(K)})$ and since $\overline{\alpha(K)}$ is closed and $\bar{\alpha}$ is continuous, it follows that $\bar{K} \leq \bar{\alpha}^{-1}(\overline{\alpha(K)})$. Hence $\bar{\alpha}(\bar{K}) \leq \overline{\alpha(K)}$ and equality holds.

Thus (ii) is established and we may clearly also assume $\overline{\alpha(K)} = L$, so $\alpha(K)$ is dense in L.

We now show the existence of at most one continuous extension $\bar{\alpha}$ of α from K to $\bar{K}$. Let $\mathcal{Q}$ be the set of closed normal subgroups M of K such that K/M is Cernikov. If $g \in \bar{K}$, $\bar{\alpha}(g) \in \bar{\alpha}(g\bar{M})$ for all $M \in \mathcal{Q}$.

Thus $$\{\bar{\alpha}(g)\} \subseteq \bigcap_{M \in \mathcal{Q}} \bar{\alpha}(g\bar{M}) \tag{1}$$

We show the right hand side of (1) has just a single point. For, if $x \in \bigcap_{M \in \mathcal{Q}} \bar{\alpha}(g\bar{M})$ then $g \in \bigcap_{M \in \mathcal{Q}} \bar{\alpha}^{-1}(x)\bar{M}$. Since $\bar{\alpha}$ is continuous and L is a T_1-space, 2.6 and 2.22 imply $g \in \bar{\alpha}^{-1}(x)$. Hence $x = \bar{\alpha}(g)$ so the right hand side of (1) has a single point.

Since $\bar{K} = K\bar{M}$, $g\bar{M} \cap K \neq \emptyset$ and if $x \in g\bar{M} \cap K$

$$x\bar{M} \cap K = x(\bar{M} \cap K) = xM \quad \text{by } 2.22(ii).$$

Thus $x\bar{M} = \overline{(g\bar{M} \cap K)}$. Since $\bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\bar{M} \cap K)}$ has a single point, by the argument below and

$$\bar{\alpha}(g\bar{M}) = \bar{\alpha}(\overline{g\bar{M} \cap K}) \subseteq \overline{\bar{\alpha}(g\bar{M} \cap K)} = \overline{\alpha(g\bar{M} \cap K)}.$$

it follows that $$\{\bar{\alpha}(g)\} = \bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\bar{M} \cap K)}. \tag{2}$$

We have now determined $\bar{\alpha}$ uniquely in terms of α.

To show $\bar{\alpha}$ exists, we shall show that the right hand side of (2) is a single point for all $g \in \bar{K}$.

If $M \in \mathcal{Q}$ then 2.22(i) and $K \leq_d \overline{K}$ imply $g\overline{M} \cap K \neq \emptyset$ for each $g \in \overline{K}$. The set $\{g\overline{M} \cap K : M \in \mathcal{Q}\}$ therefore has the finite intersection property and hence so does the set $\{\overline{\alpha(g\overline{M} \cap K)} : M \in \mathcal{Q}\}$. It follows, by the compactness of L that

$$\bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\overline{M} \cap K)} \neq \emptyset .$$

Suppose $x, y \in \bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\overline{M} \cap K)}$. Since $g\overline{M} \cap K \neq \emptyset$, there exists $h \in K$ such that

$$g\overline{M} \cap K = h\overline{M} \cap K = h(\overline{M} \cap K) = hM.$$

Thus $x, y \in \overline{\alpha(hM)} = \alpha(h)\overline{\alpha(M)}$, since α is a homomorphism. Hence $xy^{-1} \in \overline{\alpha(M)}$ for all $M \in \mathcal{Q}$. However $\bigcap_{M \in \mathcal{Q}} \overline{\alpha(M)} = 1$ by 2.20(ii) and 2.22(i). Hence $x = y$, so $\bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\overline{M} \cap K)}$ has exactly one point, as required. Thus we define $\overline{\alpha} : (\overline{K}, \mathcal{P}) \to (L, \mathcal{L})$ by $\overline{\alpha}(g) = \bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\overline{M} \cap K)}$, for each $g \in \overline{K}$.

If $g \in K$ then $\overline{\alpha(g\overline{M} \cap K)} = \overline{\alpha(gM)} = \alpha(g)\overline{\alpha(M)}$. Thus $\alpha(g) \in \bigcap_{M \in \mathcal{Q}} \overline{\alpha(g\overline{M} \cap K)}$ so $\alpha(g) = \overline{\alpha}(g)$ and $\overline{\alpha}$ extends α.

We now show $\overline{\alpha}$ is a homomorphism. Suppose $g, h \in \overline{K}$ and $M \in \mathcal{Q}$. Since α is a homomorphism it is clear that

$$\alpha(g\overline{M} \cap K)\cdot\alpha(h\overline{M} \cap K) \subseteq \alpha(gh\overline{M} \cap K).$$

Hence by 2.21,

$$\overline{\alpha(g\overline{M} \cap K)}\cdot\overline{\alpha(h\overline{M} \cap K)} \subseteq \overline{\alpha(gh\overline{M} \cap K)}.$$

Intersecting over all $M \in \mathcal{Q}$ gives $\{\overline{\alpha}(g)\}\cdot\{\overline{\alpha}(h)\} \subseteq \{\overline{\alpha}(gh)\}$ and hence $\overline{\alpha}(gh) = \overline{\alpha}(g)\overline{\alpha}(h)$ as required.

Now we show $\bar{\alpha}$ is continuous. Since $\bar{\alpha}$ is a homomorphism and because of 2.20(ii), 2.22(i) and the definition of the co-Černikov topology on L, it is sufficient to show

$$\bar{\alpha}(\bar{M}) \leq \overline{\alpha(M)} \quad \text{for all} \quad M \in \mathcal{Q} . \qquad (3)$$

Suppose $g \in \bar{M}$, $N \in \mathcal{Q}$ and $N \leq M$. Then

$$\overline{\alpha(g\bar{N} \cap K)} \subseteq \overline{\alpha(\bar{M} \cap K)} = \overline{\alpha(M)} \quad \text{by 2.22(ii).}$$

Intersecting over all such N, we obtain

$$\bar{\alpha}(g) \in \overline{\alpha(M)}.$$

Hence (3) follows.

If $\bar{\alpha}$ is an injection, it is a closed continuous bijection between the compact groups $\bar{K}$ and $\overline{\alpha(K)}$, by (ii), and hence is a topological and algebraic isomorphism. Hence α is an algebraic and topological embedding.

Finally we suppose α is an algebraic and topological embedding. Let $g \in \ker \bar{\alpha}$. Then by definition of $\bar{\alpha}$,

$$1 \in \overline{\alpha(g\bar{M} \cap K)} \quad \text{for all} \quad M \in \mathcal{Q} .$$

Since α determines a topological isomorphism between K and $\alpha(K)$, $\alpha(g\bar{M} \cap K) \subseteq_c \alpha(K)$. Thus

$$\overline{\alpha(g\bar{M} \cap K)} \cap \alpha(K) = \alpha(g\bar{M} \cap K) \quad \text{for each} \quad M \in \mathcal{Q} .$$

Hence $1 \in \alpha(g\bar{M} \cap K)$ and since α is injective $1 \in g\bar{M} \cap K$. Thus $g \in \bar{M}$ for each $M \in \mathcal{Q}$, so intersecting over all such M and using 2.22(i), it follows that $g = 1$, so $\bar{\alpha}$ is injective. □

CHAPTER 3. SYLOW THEORY IN PRO-ČERNIKOV GROUPS

In this chapter we shall show that the classical theorems of Sylow and Hall in finite group theory can be extended to the class of pro-Černikov groups. The theory we develop is analogous to that established by E. Bolker [2] for profinite groups. Our approach, however, uses many of the methods of J. Parker [35] and B. Hartley [19]. The main difference is that instead of using the theorem of Kuroš [29] on inverse limits of finite sets, we have been forced to use 1.2.6. This involves some technicalities in ensuring that the correct topologies are induced, but these are easily overcome.

Some discussion is also included concerning formation theory in pro-Černikov groups. We shall briefly indicate that if projectors are defined in the correct way (that is, via the Černikov factor groups) then much of the well known theory established by Gaschütz [9] can be extended to subclasses of pro-Černikov groups.

To begin, we generalise the idea of a π-group. Let π be a set of primes. A co-Černikov group $(G,\mathcal{N})$ will be called a <u>generalised π-group</u> if G/N is a π-group (in the usual sense) for all $N \triangleleft_c G$ with G/N Černikov. This is analogous to the concept used in [19], although there the term "generalised" is omitted.

<u>3.1. Lemma</u>: Let $(G,\mathcal{N})$ be a co-Černikov group. Then $(G,\mathcal{N})$ is a generalised π-group if and only if G/N is a π-group for all $N \in \mathcal{N}$.

<u>Proof</u>: This follows from the definitions and 2.16.

The idea of a generalised π-group certainly depends on the filter base involved. For example, if $\langle g \rangle$ is the infinite cyclic group it has the filter bases $\mathcal{M} = \{\langle g^{2^i} \rangle : i \geq 1\}$ and $\mathcal{N} = \{\langle g^{3^i} \rangle : i \geq 1\}$. However $(\langle g \rangle, \mathcal{M})$ is a generalised 2-group and $(\langle g \rangle, \mathcal{N})$ is a generalised 3-group. We give several elementary properties of generalised π-groups. A generalised π-group $(H, \mathcal{M})$ that is a subgroup of a co-Černikov group $(G, \mathcal{N})$ will be called a generalised π-subgroup of $(G, \mathcal{N})$.

3.2. Lemma: Let $(G, \mathcal{N})$ be a co-Černikov group and π a set of primes.

(i) If $(G, \mathcal{N})$ is a generalised π-group and $H \leq G$ then $(H, H \cap \mathcal{N})$ is a generalised π-group.

(ii) If $(G, \mathcal{N})$ is a generalised π-group and $K \lhd_c H \leq G$ then $(H/K, (H \cap \mathcal{N})K/K)$ is a generalised π-group.

(iii) If $\{(H_i, H_i \cap \mathcal{N}) : i \in I\}$ is a set of generalised π-subgroups of $(G, \mathcal{N})$, totally ordered by inclusion, then $(\bigcup_{i \in I} H_i, (\bigcup_{i \in I} H_i) \cap \mathcal{N})$ is a generalised π-subgroup of $(G, \mathcal{N})$.

(iv) If $(H, H \cap \mathcal{N})$ is a generalised π-subgroup of $(G, \mathcal{N})$ then so is $(\overline{H}, \overline{H} \cap \mathcal{N})$.

(v) The product of every set of normal generalised π-subgroups of $(G, \mathcal{N})$ is a normal generalised π-subgroup of $(G, \mathcal{N})$.

Proof: (i) If $N \in \mathcal{N}$ then $H/H \cap N \cong HN/N \leq G/N$, a π-group by hypothesis. Hence, since $(H, H \cap \mathcal{N})$ is a co-Černikov group, it is a generalised π-group by 3.1.

(ii) The proof is similar to (i) using 2.15 and 3.1.

(iii) Suppose $L \lhd_c \bigcup_{i \in I} H_i = M$ say, with M/L a Černikov group. For each $i \in I$, $L \cap H_i \lhd_c H_i$ by 2.14. Moreover

$H_i/H_i \cap L$ is a Černikov π-group by hypothesis. Hence M/L is the union of the ascending chain of π-subgroups H_iL/L and so is a π-group. Thus $(\bigcup_{i\in I} H_i, (\bigcup_{i\in I} H_i) \cap \mathcal{N})$ is a generalised π-subgroup of $(G,\mathcal{N})$.

(iv) By 2.6, $\bar{H} \leq G$. Suppose $N \lhd_c \bar{H}$ and $\bar{H}/N$ is a Černikov group. Then $H \cap N \lhd_c H$ and $H/H \cap N$ is a Černikov π-group by hypothesis. Since the closure of H in $\bar{H}$ is precisely $\bar{H}$, 2.7 and 2.16 imply $\bar{H} = HN$ so $\bar{H}/N$ is a π-group. Hence $(\bar{H}, \bar{H} \cap \mathcal{N})$ is a generalised π-subgroup of $(G,\mathcal{N})$.

(v) It suffices to show that if $(L, L \cap \mathcal{N})$ and $(M, M \cap \mathcal{N})$ are normal generalised π-subgroups of $(G,\mathcal{N})$ then so is $(LM, LM \cap \mathcal{N})$. If $N \in \mathcal{N}$ then $L/L \cap N$ and $M/M \cap N$ are π-groups. Hence $(LM \cap LN)/LM \cap N$ and $(LM \cap MN)/LM \cap N$ are π-groups. Thus their product $LM/LM \cap N$ is also a π-group and the result follows by 3.1. □

If $(G,\mathcal{N})$ is a co-Černikov group let $Q_\pi(G,\mathcal{N})$ denote the unique largest normal generalised π-subgroup of $(G,\mathcal{N})$. By 3.2(v) this concept is well defined.

<u>3.3. Lemma</u>: Let $(G,\mathcal{N})$ be a co-Černikov group and π a set of primes. Then

(i) $Q_\pi(G,\mathcal{N}) \lhd_c (G,\mathcal{N})$.

(ii) $O_\pi(G) \leq Q_\pi(G,\mathcal{N})$.

(iii) For all $p \in \pi$, $O_p(G,\mathcal{N}) \leq O_\pi(G,\mathcal{N})$.

The proof of this result is straightforward and is omitted.

When it is clear which filter base is being used we shall merely write $Q_\pi(G)$ for $Q_\pi(G,\mathcal{N})$. It is of some interest to us to know when a generalised π-group is actually a π-group in the usual sense.

3.4. Lemma: Suppose $(G,\mathcal{N})$ is a periodic co-Černikov group and π is a set of primes. Then

(i) G is a π-group if and only if $(G,\mathcal{N})$ is a generalised π-group.

(ii) $O_\pi(G) = Q_\pi(G)$.

Proof: (i) Suppose $x \in G$ is an element of prime order p with $p \notin \pi$ and $(G,\mathcal{N})$ is a generalised π-group.

Then $x^p = 1 = \cap\{N : N \in \mathcal{N}\}$. Hence $x^p \in N$ for all $N \in \mathcal{N}$. However G/N is a π-group by hypothesis so $x \in N$ for all $N \in \mathcal{N}$. Therefore $x = 1$ and $(G,\mathcal{N})$ is a π-group.

(ii) $Q_\pi(G)$ is a normal periodic generalised π-subgroup of $(G,\mathcal{N})$ and hence is a π-group by (i). Hence $Q_\pi(G) \leq O_\pi(G)$ and the result follows by 3.3(ii). □

Thus, for periodic co-Černikov groups, $Q_\pi(G)$ is independent of the separating filter base chosen.

A subgroup P of a co-Černikov group $(G,\mathcal{N})$ will be called a generalised Sylow π-subgroup of $(G,\mathcal{N})$ if

(i) $P \leq_c G$.

(ii) $PN/N \in Syl_\pi G/N$ for all $N \triangleleft_c G$ with G/N Černikov.

We shall denote the set of generalised Sylow π-subgroups of $(G,\mathcal{N})$ by $Syl_\pi(G,\mathcal{N})$. It is not immediately clear that a co-Černikov group possesses even generalised Sylow p-subgroups. However we shall show that a pro-Černikov group does possess them.

By 3.2(iii) and Zorn's Lemma, the co-Černikov group $(G,\mathcal{N})$ contains maximal generalised π-subgroups and these subgroups are closed by 3.2(iv). Let $Max_\pi(G,\mathcal{N})$ denote the set of maximal generalised π-subgroups of $(G,\mathcal{N})$. In prosoluble groups, the concept of a generalised Sylow π-subgroup and a maximal generalised π-subgroup are the same ([19] (lemma 6.1)). We at least have:

<u>3.5. Lemma</u>. Let $(G,\mathcal{N})$ be a co-Černikov group. Then $Syl_\pi(G,\mathcal{N}) \subseteq Max_\pi(G,\mathcal{N})$.

<u>Proof</u>: Suppose $P \in Syl_\pi(G,\mathcal{N})$ and $P \leq Q \in Max_\pi(G,\mathcal{N})$. Then $P, Q \leq_C G$ and $PN \leq QN$ for all $N \in \mathcal{N}$. Since $PN/N \in Syl_\pi G/N$ and QN/N is a π-group, we must have $PN = QN$ for all $N \in \mathcal{N}$. Hence by 2.6,

$$P = \cap\{PN : N \in \mathcal{N}\} = \cap\{QN : N \in \mathcal{N}\} = Q. \qquad \square$$

For our purposes we do not need to know whether $Syl_\pi(G,\mathcal{N}) = Max_\pi(G,\mathcal{N})$. Whilst this result is not true in general for co-Černikov groups (see 4.13), it would be interesting to know whether it is true for pro-Černikov groups $(G,\mathcal{N})$ in which G/N is soluble for each $N \in \mathcal{N}$. Unfortunately the proof of lemma 6.1 in [19] does not seem to extend to our situation. In chapter 4 we shall give some conditions when equality does hold in 3.5.

A locally finite group G is <u>Sylow π-integrated</u> (for some set of primes π) if the Sylow π-subgroups of every subgroup of G are conjugate. For co-Černikov groups whose Černikov factor groups are Sylow π-integrated it is easier to check that a given subgroup is a generalised Sylow π-subgroup, as the following result shows.

3.6. Lemma: Suppose $(G,\mathcal{N})$ is a co-Černikov group and π is a set of primes. Suppose that for each $N \in \mathcal{N}$, G/N is Sylow π-integrated. Then $P \in \mathrm{Syl}_\pi(G,\mathcal{N})$ if and only if

(i) $P \leq_c G$

(ii) $PN/N \in \mathrm{Syl}_\pi G/N$ for each $N \in \mathcal{N}$.

Proof: We suppose (i) and (ii) hold and that $M \triangleleft_c G$ with G/M a Černikov group. By 2.16 there exists $N \in \mathcal{N}$ such that $N \leq M$ so, by (ii), $P_N = PN/N \in \mathrm{Syl}_\pi G/N$. Now $G_N = G/N$ is Sylow π-integrated and $G_M = G/M \cong G_N/H$, where $H = M/N$. Thus if $Q/H \in \mathrm{Syl}_\pi G_N/H$ there exists $g \in G_N$ such that $Q = HP_N{}^g$ (by [16] lemma 2.1, for example). Hence $P_NH/H \in \mathrm{Syl}_\pi G_N/H$ and $PM/M \in \mathrm{Syl}_\pi G_M$ (since $P_NH/H \cong PM/M$), as required. The reverse implication is clear. □

Of course, no restrictions are needed if $\pi = \{p\}$, a single prime, since a Černikov group is always Sylow p-integrated.

To generalise the results of Sylow and Hall we require some preliminary results.

3.7. Lemma: Suppose G is a Černikov group and $H \leq G$. Let G have the coset topology and let G/H denote the space of cosets of H in G with quotient topology. Then the natural map $\alpha : G \to G/H$ is closed and continuous.

Proof: The map α is certainly continuous from the definition of the quotient topology on G/H. By 2.2 every closed set in G is a finite union of cosets of G so we need only show that if $K \leq G$ then $\alpha(xK)$ is closed in G/H for each $x \in G$.

Now $\alpha(xK) = \{xkH : k \in K\}$ and hence $\alpha^{-1}(\alpha(xK)) = xKH$. Of course, KH need not be a subgroup but K and H possess radicable parts which we denote by K^o and H^o respectively. Also K^oH^o is a subgroup since $K^o, H^o \leq G^o$, the radicable part of G. If $\{x_i\}_{i=1}^m$ is a left transversal to K^o in K and $\{y_j\}_{j=1}^n$ is a right transversal to H^o in H then

$$KH = \bigcup_{i,j} x_i K^o H^o y_j, \quad \text{so} \quad xKH = \bigcup_{i,j} xx_i K^o H^o y_j.$$

This set is closed in G by 2.4(iv) and the definition of the coset topology. It follows by definition of the quotient topology that α is a closed map. □

<u>3.8. Corollary</u>: Suppose G is a Černikov group and $H \leq K \leq G$. If G/K and G/H denote, respectively, the quotient spaces of cosets of K and H in G, induced by the coset topology on G, then the natural map $\beta : G/H \to G/K$ is closed and continuous.

<u>Proof</u>: Let $\alpha_H : G \to G/H$ and $\alpha_K : G \to G/K$ be the natural maps. By 3.7 these maps are closed and continuous and since $\beta \circ \alpha_H = \alpha_K$ it follows that β is closed and continuous. □

The following lemma is an extension of [19] (lemma 6.2). We give its proof for the sake of completeness.

<u>3.9. Lemma</u>: Let $(G,\mathcal{N})$ be a compact co-Černikov group and suppose that for each $N \in \mathcal{N}$, $X(N)$ is a closed set with the property:

$$\text{If } M, N \in \mathcal{N} \text{ and } M \leq N \text{ then } X(M)N = X(N). \qquad (*)$$

Let $X = \cap\{X(N) : N \in \mathcal{N}\}$. Then, for all $N \in \mathcal{N}$, $XN = X(N)$.

Proof: It is clear that $XN \subseteq X(N)$, for each $N \in \mathcal{N}$. Let $N \in \mathcal{N}$ be fixed. If $x \in X(N)$ and $M \in \mathcal{N}$ is such that $M \leq N$ then $x \in X(M)N$ by (*). Hence $xN \cap X(M) \neq \emptyset$. If $M_1, \ldots, M_r \leq N$ with $M_i \in \mathcal{N}$ then there exists $M_{r+1} \in \mathcal{N}$ such that $M_{r+1} \leq M_1 \cap \ldots \cap M_r$. By (*), $X(M_{r+1})M_i = X(M_i)$ for $i = 1, \ldots, r$. Therefore,

$$\emptyset \neq xN \cap X(M_{r+1}) \subseteq \bigcap_{i=1}^{r} (xN \cap X(M_i)).$$

If $\mathcal{M} = \{M \in \mathcal{N} : M \leq N\}$ then $\{xN \cap X(M) : M \in \mathcal{M}\}$ is a set of closed subsets of $(G,\mathcal{N})$ with the finite intersection property. Since G is compact, there exists $y \in G$ such that

$$y \in xN \cap \cap\{X(M) : M \in \mathcal{M}\} = xN \cap \cap\{X(M) : M \in \mathcal{N}\} = xN \cap X.$$

Hence $x \in XN$ and this proves the result. □

We now give our extension of Sylow's Theorem.

3.10. Theorem: Let $(G,\mathcal{N})$ be a compact co-Černikov group. Then G possesses generalised Sylow p-subgroups for each prime p.

Proof: Let p be a fixed prime and for each $N \in \mathcal{N}$ let $A(N) = \{$Sylow p-subgroups of $G/N\}$. Then $A(N) \neq \emptyset$ and the elements of $A(N)$ are conjugate since the Sylow p-subgroups of a Černikov group are conjugate.

For $N \in \mathcal{N}$, put $G_N = G/N$, give G_N its coset topology and let $P_N \in A(N)$. By the previous remark we may put the elements of $A(N)$ in 1-1 correspondence with the cosets of $N_{G_N}(P_N)$ in G_N. Suppose $N \leq M$ and $N, M \in \mathcal{N}$. Since the Sylow p-subgroups of a Černikov group are homomorphism invariant, the natural map

$\alpha_{MN} : G_N \to G_M$ induces a map $\beta_{MN} : A(N) \to A(M)$. Suppose $\beta_{MN}(P_N) = P_M$ and put $G_N^* = G_N/N_{G_N}(P_N)$ as a topological space with quotient topology. We define a map $\beta_{MN}^* : G_N^* \to G_M^*$ by:

$$\text{if } g \in G \text{ then } \beta_{MN}^*((gN)N_{G_N}(P_N)) = (gM)N_{G_M}(P_M).$$

This map is well defined since $\beta_{MN}(P_N) = P_M$. If $\gamma_N : G_N \to G_N^*$ is the natural map then clearly $\gamma_M \circ \alpha_{MN} = \beta_{MN}^* \circ \gamma_N$. Thus, as in 3.7, β_{MN}^* is closed and continuous. Hence, if A(M) and A(N) are given the topologies induced from G_M^* and G_N^* respectively, the map β_{MN} is closed and continuous. Since G_N is compact and T_1, 3.7 implies that G_N^* is compact and T_1 and hence A(N), with its induced topology, is compact and T_1, for each $N \in \mathcal{N}$. Our aim is to eventually use 1.2.6 applied to the sets A(N). However it is first necessary to check that we can choose the representatives $P_N \in A(N)$ consistently and to do this it suffices to show that if P, Q $\in$ A(N) then the topologies induced on A(N) by $G_N/N_{G_N}(P)$ and $G_N/N_{G_N}(Q)$ are the same. So let τ and σ be the topologies induced by $G_N/N_{G_N}(P)$ and $G_N/N_{G_N}(Q)$ on A(N), respectively. We identify an element $P^h \in (A(N),\tau)$ with the right coset $N_{G_N}(P)h$ (where $h \in G_N$). Since P and Q are conjugate in G_N there exists $g \in G_N$ such that $P^g = Q$.

Let $\{P^{h_i} : i \in J\}$ be a closed subset of $(A(N),\tau)$, for some index set J. Then by definition, $\bigcup_{i\in J} N_{G_N}(P)h_i \subseteq_c G_N$. Hence by 2.4(iv),

$$\bigcup_{i\in J} N_{G_N}(Q)g^{-1}h_i = g^{-1}(\bigcup_{i\in J} N_{G_N}(P)h_i) \subseteq_c G_N.$$

Therefore $\{Q^{g^{-1}h_i} : i \in J\} \subseteq_c (A(N),\sigma)$ whence $\{P^{h_i} : i \in J\} \subseteq_c (A(N),\sigma)$ and $\tau \subseteq \sigma$. It follows by symmetry that $\tau = \sigma$ and consequently the topologies induced on A(N) are the same.

All the hypotheses of 1.2.6 are now satisfied for the inverse system $\{A(N), \beta_{MN} : M, N \in \mathcal{N}\}$ so by 1.2.6(a),

$$\varprojlim A(N) \neq \emptyset .$$

Let $(P_N) = (Q_N/N) \in \varprojlim A(N)$. If $N \leq M$, $\beta_{MN}(P_N) = P_M$ and $Q_N M = Q_M$. Put $P = \cap\{Q_N : N \in \mathcal{N}\} \leq_c G$, since $Q_N \leq_c G$ by definition. By 3.9, $PN = Q_N$ for all $N \in \mathcal{N}$ and hence $PN/N \in \mathrm{Syl}_p G/N$. By 3.6 and the remark following it, $P \in \mathrm{Syl}_p(G,\mathcal{N})$ and this completes the proof. □

Let $\mathcal{W}$ denote the class of co-Černikov groups $(G,\mathcal{N})$ with the property that if $N \in \mathcal{N}$ then G/N is soluble. The above proof then yields:

<u>3.11. Theorem</u>: Let $(G,\mathcal{N})$ be a compact $\mathcal{W}$-group. Then G possesses generalised Sylow π-subgroups for all sets of primes π.

We now obtain the conjugacy of the various generalised Sylow subgroups.

<u>3.12. Theorem</u>: Let $(G,\mathcal{N})$ be a compact co-Černikov group. Then:
(i) The generalised Sylow p-subgroups of $(G,\mathcal{N})$ are conjugate.
(ii) If $(G,\mathcal{N}) \in \mathcal{W}$ then the generalised Sylow π-subgroups of $(G,\mathcal{N})$ are conjugate, for all sets of primes π.

<u>Proof</u>: Since the proofs of (i) and (ii) are essentially the same, we merely give the proof of (i).

Let P, Q $\in$ $Syl_p(G,\mathcal{N})$ so that, for each N $\in \mathcal{N}$, PN/N, QN/N $\in Syl_pG/N$. Put $X(N) = \{g \in G : P^gN = QN\} \neq \emptyset$ since the Sylow p-subgroups of a Černikov group are conjugate. Now,

$$g,\ h \in X(N) \Rightarrow P^gN = P^hN \Rightarrow gh^{-1} \in N_G(PN).$$

Hence $X(N) = N_G(PN)g$ and so X(N) is closed in $(G,\mathcal{N})$ by 2.8(iii) and the definition of the co-Černikov topology induced by $\mathcal{N}$. Moreover, if M, N $\in \mathcal{N}$ then $X(M \cap N) \subseteq X(M) \cap X(N)$ and so the set $\{X(N) : N \in \mathcal{N}\}$ has the finite intersection property. Since $(G,\mathcal{N})$ is compact, $\cap\{X(N) : N \in \mathcal{N}\}$ contains an element g. Hence if N $\in \mathcal{N}$, $P^gN = QN$ and since $P^g, Q \leq_c G$, 2.6 implies

$$P^g = \cap\{P^gN : N \in \mathcal{N}\} = \cap\{QN : N \in \mathcal{N}\} = Q,$$

so P and Q are conjugate. □

The above method of proof is of course well known in the prosoluble group case.

By a generalised Sylow basis of a co-Černikov group $(G,\mathcal{N})$ we shall mean a complete set of generalised Sylow p-subgroups $\{S_p\}$, one for each prime p, with the property that if π is a set of primes then $\langle S_p : p \in \pi\rangle$ is a generalised π-group. This is a somewhat more general definition than that given by Parker [35], although in the prosoluble case our definition and that of Parker coincide.

It is possible to prove, in a similar manner to the proof of 3.10

3.13. Theorem: Let $(G,\mathcal{N})$ be a compact $\mathcal{W}$-group. Then $(G,\mathcal{N})$ possesses generalised Sylow bases.

To complete our survey of Hall's results we prove:

3.14. Theorem: Let $(G,\mathcal{N})$ be a compact $\mathcal{W}$-group. Then the generalised Sylow bases of $(G,\mathcal{N})$ are conjugate.

Proof: Let $\underset{\sim}{S} = \{S_p\}$ and $\underset{\sim}{T} = \{T_p\}$ be generalised Sylow bases and for $N \in \mathcal{N}$ set

$$X(N) = \{g \in G : S_p{}^g N = T_p N \text{ for all primes } p\}.$$

Because $(G,\mathcal{N}) \in \mathcal{W}$ and Gol'berg's result holds, the Sylow bases of G/N are conjugate so $X(N) \neq \emptyset$. Moreover if $g \in X(N)$ then

$$X(N) = \bigcap_{p\in\mathbb{P}} N_G(S_pN)g,$$

so $X(N)$ is closed in $(G,\mathcal{N})$ and the sets $X(N)$ are easily seen to have the finite intersection property, as in the proof of 3.12. The result then follows easily. □

This result is crucial in the next chapter.

We now briefly discuss formation theory in pro-Černikov groups. In an analogous manner to Parker [35], if $\mathfrak{Z}$ is a class of groups, we define a generalised $\mathfrak{Z}$-projector of a pro-Černikov group $(G,\mathcal{N})$ to be a subgroup H of G satisfying:

(i) $H \leq_c G$.

(ii) HN/N is a $\mathfrak{Z}$-projector of G/N for all $N \in \mathcal{N}$.

It is then possible to define the idea of a saturated formation of compact $\mathcal{W}$-groups in the manner that Parker suggests, together with the topological restrictions that he imposes. One can then obtain, using the methods of 3.10 and 3.12,

3.15. Theorem: Suppose $(G,\mathcal{N})$ is a compact $\mathcal{W}$-group and $\mathfrak{F}$ a (locally defined) saturated formation of $\mathfrak{W}$-groups. Then G possesses generalised $\mathfrak{F}$-projectors and all of these are conjugate.

It is then possible to define $\mathfrak{F}$-normalisers and prove various cover-avoidance properties with the necessary topological restrictions that Parker imposes, at least for compact $\mathfrak{W}$-groups.

CHAPTER 4. SYLOW THEORY IN $\mathfrak{X}$-GROUPS

In this chapter we use the previous results on compact co-Černikov groups to show that the Sylow generating bases of an $\mathfrak{X}$-group are locally conjugate. As we mentioned in the introduction, every countable locally finite-soluble group possesses Sylow generating bases. Also if $G \in \mathfrak{Y}$ the results of Massey [30] (Theorem 1.1) show that G contains a countable subgroup B with the property that $Syl_p B \subseteq Syl_p G$ for all primes p. We shall call such a subgroup a basic subgroup of G and denote the set of all basic subgroups of G by Basic G. Thus a Sylow generating basis of B is a Sylow basis of G. As Massey shows, the basic subgroups of a $\mathfrak{Y}$-group are isomorphic, but need not be locally conjugate. We shall give an alternative proof of this in chapter 6. For an alternative treatment of the main result of this chapter we refer the reader to [6].

It is also worth remarking at this point that our definition of a Sylow generating basis is equivalent to the following, at least for $\mathfrak{X}$-groups:

$$\underset{\sim}{S} = \{S_p\} \text{ is a Sylow generating basis}$$

of $G \in \mathfrak{X}$ if and only if,

(i) $S_p S_q = S_q S_p$ for each pair of primes p, q.

(ii) $G = \langle S_p : p \in \mathbb{P} \rangle$.

This follows because condition (i) is equivalent to:

(iii) If π is a set of primes $\langle S_p : p \in \pi \rangle$ is a π-group.

The equivalence of (i) and (iii) is obtained using the method of [8] (lemma 2.5 and corollary 2.6). We shall also often write $S_\pi = \langle S_p : p \in \pi \rangle$, for π a set of primes.

Our first result shows that, for $\mathfrak{X}$-groups, all co-Černikov topologies are the same. Of course every $\mathfrak{X}$-group is a residually Černikov group and so is co-Černikov by the remarks in the introduction.

<u>4.1. Lemma</u>: Suppose $G \in \mathfrak{Y}$. Then all the co-Černikov topologies on G are the same.

<u>Proof</u>: Let $\mathcal{N}$ be a fixed but arbitrary filter base for G. We show that if $N \triangleleft G$ and G/N is Černikov then $N \triangleleft_c (G,\mathcal{N})$ and the result then follows.

Let π be the set of primes dividing the orders of elements in G/N. Then π is a finite set. By 1.2.3, $G/O_{\pi'}(G)$ is a Černikov group. Also $O_{\pi'}(G) \triangleleft_c G$ by 3.3(i) and 3.4(ii) and there exists $M \in \mathcal{N}$ such that $M \leq O_{\pi'}(G)$, by 2.16. It follows that $M \leq O_{\pi'}(G) \leq N$ and hence $N \triangleleft_c (G,\mathcal{N})$. □

We now show that, for $G \in \mathfrak{Y}$, equality holds in 3.5 provided π is a finite set.

<u>4.2. Lemma</u>: If $G \in \mathfrak{Y}$ and π is a finite set of primes then

$$\mathrm{Syl}_\pi(G,\mathcal{N}) = \mathrm{Max}_\pi(G,\mathcal{N}) = \mathrm{Syl}_\pi G$$

for an arbitrary filter base $\mathcal{N}$.

<u>Proof</u>: By 3.4(i), $\mathrm{Syl}_\pi G = \mathrm{Max}_\pi(G,\mathcal{N})$, so let $P \in \mathrm{Max}_\pi(G,\mathcal{N})$. Then $P \leq_c G$ by 3.2(iv). If $N \triangleleft G$ and G/N is a Černikov group then $N \triangleleft_c G$ by 4.1. Suppose $PN/N \leq Q/N \in \mathrm{Syl}_\pi G/N$. Then by a well known result ([24](1.D.4) essentially) there exists $R \in \mathrm{Syl}_\pi G$ such that $Q = RN$. However, since π is a finite set,

the Sylow π-subgroups are conjugate in G so $R = P^g$ for some $g \in G$.

Hence $PN \leq P^gN$ and since G is periodic it must be the case that $PN = P^gN$. Thus $P \in \mathrm{Syl}_\pi(G,\mathcal{N})$ and $\mathrm{Max}_\pi(G,\mathcal{N}) \subseteq \mathrm{Syl}_\pi(G,\mathcal{N})$. Since the reverse inclusion always holds, the lemma is proved. □

4.3. Lemma: Let $(G,\mathcal{N})$ be a co-Černikov group. If $X \subseteq G$ then $C_G(X) \leq_c G$.

Proof: This follows in a similar fashion to 2.19.

We shall now let $G \in \mathfrak{Y}$ be arbitrary. Suppose $\mathcal{N} = \{N_i : i \in I\}$ is an arbitrary, but fixed, separating filter base for G indexed by a set I. If $N_i \leq N_j$ there is a natural map

$$\psi_{ji} : G/N_i \to G/N_j$$

and $\{G/N_i, \psi_{ji} : i, j \in I\}$ is then an inverse system of Černikov groups and epimorphisms. We give each G/N_i the coset topology and put $H = \varprojlim G/N_i$.

Let $\phi_i : H \to G/N_i$ be the i^{th} projection map;

$\Phi : G \to H$ be the embedding $g \mapsto (gN_i)$

and put $\ker \phi_i = M_i$. Put $\mathcal{M} = \{M_i : i \in I\}$. Then $(H,\mathcal{M})$ is a compact co-Černikov group and has precisely the natural topology as a subspace of the product space $\mathrm{Cr}_{i\in I}\, G/N_i$, by 2.11 and its proof. By 2.12 and 2.13, $\overline{\Phi(G)} = H$ and $\overline{\Phi(N_i)} = M_i$ and we shall often identify subgroups of G with their images under Φ, when convenient.

The following result and its corollary are then the crucial observations for proving our main result. Using the above notation, our method will be to show that if $\underset{\sim}{S}$, $\underset{\sim}{T}$ are Sylow bases of $G \in \mathfrak{Y}$ then there exists $g \in H$ such that $\underset{\sim}{S}^g = \underset{\sim}{T}$. Then if $G \in \mathfrak{X}$ and $\underset{\sim}{S}$, $\underset{\sim}{T}$ are generating we shall show that $g \in N_H(G)$ and then show that any such g induces a locally inner automorphism of G.

4.4. Lemma: Let π be a finite set of primes and G be as above. If $P \in Syl_\pi G$ then $P \in Syl_\pi(H,\mathcal{M})$.

Proof: Since π is finite, P is a Černikov group and has coset topology. Thus P is a compact subgroup of H, so is a closed subgroup of H by 2.18.

Now $\Phi(N_i) \leq \Phi(G) \cap M_i$ for each $i \in I$ and moreover

$$(g_jN_j) \in \Phi(G) \cap M_i \Rightarrow (g_jN_j) = \Phi(g) \text{ for some } g \in G$$
$$\Rightarrow g_jN_j = gN_j \quad \text{for all } j \in I.$$

Also $(gN_j) \in M_i \Rightarrow g \in N_i$ and so $\Phi(g) \in \Phi(N_i)$. Hence $\Phi(G) \cap M_i = \Phi(N_i)$. By 2.7,

$$H = GM_i \quad \text{for each} \quad i \in I \ , \tag{1}$$

so $H/M_i \cong G/N_i$. By 3.6 it suffices to show $PM_i/M_i \in Syl_\pi H/M_i$ for each $i \in I$. Since P is a π-group so is PM_i/M_i and if $PM_i/M_i \leq P_i/M_i \in Syl_\pi H/M_i$ then by (1),

$$P_i = (P_i \cap G)M_i \ . \tag{2}$$

However $(P_i \cap G)M_i/M_i \cong (P_i \cap G)/N_i$, a π-group. Also $PN_i/N_i \leq (P_i \cap G)/N_i$ so $PN_i = P_i \cap G$, since $P \in Syl_\pi(G,\mathcal{N})$ by 4.2. Hence $P_i = PM_i$ from (2) and the result follows. □

4.5. Corollary: If $\underset{\sim}{S}$ is a Sylow basis of $G \in \mathfrak{Y}$ then $\underset{\sim}{S}$ is a generalised Sylow basis of H.

We are now able to prove:

4.6. Theorem: Let $G \in \mathfrak{X}$. Then the Sylow generating bases of G are locally conjugate.

Proof: We shall use the notation introduced before the proof of 4.4. Let $\underset{\sim}{S} = \{S_p\}$ and $\underset{\sim}{T} = \{T_p\}$ be Sylow generating bases of G. Then $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are generalised Sylow bases of the compact co-Černikov group $(H,\mathcal{M})$ by 4.5 and hence by 3.14 there exists $g \in H$ such that

$$S_p{}^g = T_p \quad \text{for each prime } p .$$

But G is generated by $\underset{\sim}{S}$ and by $\underset{\sim}{T}$ and hence $G^g = G$. Thus $g \in N_H(G)$ and it now suffices to show that every element of $N_H(G)$ induces a locally inner automorphism of G.

(i) For each finite set of primes π and each $P \in Syl_\pi G$, $N_H(P)/C_H(P)$ is a Černikov group.

For, $G/O_{\pi'}(G)$ is a Černikov group so there exists $N_i \in \mathcal{N}$ such that $N_i \leq O_{\pi'}(G)$ and hence $N_i \cap P = 1$. Thus

$$M_i \cap P = M_i \cap G \cap P = N_i \cap P = 1$$

and so $[M_i \cap N_H(P), P] \leq M_i \cap P = 1$. Hence $M_i \cap N_H(P) \leq C_H(P)$. Since H/M_i is Černikov it follows that $N_H(P)/C_H(P)$ is Černikov thus proving (i).

Now let $x_1, \ldots, x_n \in G$. Then there is a finite set π of primes and $P \in Syl_\pi G$ such that $x_1, \ldots, x_n \in P$.

(ii) We may assume $g \in N_H(P)$.

For, $P^g \leq G^g = G$ and $P^g \in \mathrm{Syl}_\pi G$. Since π is a finite set, there exists $h \in G$ such that $P^{gh} = P$. Hence $gh \in N_H(P)$ and if gh induces a locally inner automorphism of G so does g.

Let $K = N_G(P)$ and $L = N_H(P)$. We shall show that K is dense in L. Since $G \leq_d H$, 2.7 implies $H = GM_i$. Since P is pronormal in G and normalisers of pronormal subgroups are homomorphism invariant, we have $N_{H/M_i}(PM_i/M_i) = KM_i/M_i$. Also $LM_i/M_i \leq N_{H/M_i}(PM_i/M_i)$ and hence $KM_i = LM_i$. Therefore,

$$L = L \cap KM_i = K(L \cap M_i) \quad \text{for each} \quad i \in I \ . \tag{1}$$

But $(L, L \cap \mathcal{M})$ is a co-Černikov group and (1) is precisely the condition that K should be dense in L. Hence by (i), the fact that $C_H(P) \triangleleft_c N_H(P)$ and by 2.16, it follows that

$$L = KC_H(P).$$

So there exists $h \in N_G(P)$ and $k \in C_H(P)$ such that $g = kh$. Finally,

$${x_i}^g = {x_i}^{kh} = {x_i}^h \quad \text{for} \quad i = 1, \ldots, n$$

and hence g induces a locally inner automorphism of G. This completes the proof. □

I should like to thank Dr. M. J. Tomkinson for greatly simplifying my original proof of (i) in 4.6. In our paper [6] we have given examples to show that 4.6 is the best possible result.

If π is a set of primes, G is a group and $S \in \mathrm{Syl}_\pi G$ then S is said to <u>reduce</u> into a subgroup H of G if $S \cap H \in \mathrm{Syl}_\pi G$.

If Ω is a local system of G, S <u>reduces</u> into Ω if S reduces into each subgroup of Ω. If $\underset{\sim}{S} = \{S_p\}$ is a Sylow (generating) basis of G then $\underset{\sim}{S}$ <u>reduces</u> into $H \leq G$ if $\underset{\sim}{S} \cap H = \{S_p \cap H\}$ is a Sylow (generating) basis of H.

More generally, if Ω is a local system of the group G, $\underset{\sim}{S}$ is said to <u>reduce</u> into Ω if it reduces into each subgroup $H \in \Omega$. A Sylow basis $\underset{\sim}{T} = \{T_p\}$ of a subgroup H of G is said to <u>extend</u> to a Sylow basis $\underset{\sim}{S} = \{S_p\}$ of G if $S_p \cap H = T_p$ for all primes p and we shall write $\underset{\sim}{S} \cap H = \underset{\sim}{T}$. We shall now give some elementary properties concerning Sylow generating bases of $\mathfrak{X}$-groups.

<u>4.7. Lemma</u>: Suppose $G \in \mathfrak{X}$ and $\underset{\sim}{S}$ is a Sylow generating basis of G. Then $\underset{\sim}{S}$ reduces into some totally ordered local system of finite subgroups of G.

<u>Proof</u>: Given a local system of G it is possible to find a Sylow generating basis which reduces into it ([18] lemma 2.1). The result follows from this remark and 4.6. □

It is possible to prove 4.7 without having to appeal to 4.6; 4.6 merely shortens the argument. The following result is well known and is easily deduced from the structure theorems for $\mathfrak{Y}$-groups. Of course, a proof of it has essentially occurred in 4.2.

<u>4.8. Lemma</u>: Suppose $G \in \mathfrak{Y}$ and π is a finite set of primes. If $N \triangleleft G$ and $S \in Syl_\pi G$ then

(i) $SN/N \in Syl_\pi G/N$ and all the Sylow π-subgroups of G/N have this form.

(ii) $S \cap N \in Syl_\pi N$ and all the Sylow π-subgroups of N have this form.

Our next result is useful in proving an analogue of 4.8.

4.9. Lemma: Let $G \in \mathfrak{D}$ and π a finite set of primes. If $S \in Syl_{\pi}G$ and $T \in Syl_{\pi'}G$ then $G = ST$.

Proof: Let $N = O_{\pi'}(G)$. By 4.8(i), $SN/N \in Syl_{\pi}G/N$ and clearly $T/N \in Syl_{\pi'}G/N$. Since 4.9 holds in the special case of a Černikov group, it follows that $G/N = (SN/N)(T/N)$. Hence $G = ST$. □

Using 4.8 and 4.9 we can now prove the following well known fact. The straightforward proof is omitted.

4.10. Corollary: Suppose $G \in \mathfrak{D}$ and π is a finite set of primes. If $S \in Syl_{\pi'}G$ and $N \triangleleft G$ then

(i) $SN/N \in Syl_{\pi'}G/N$ and all the Sylow π'-subgroups of G/N have this form.

(ii) $S \cap N \in Syl_{\pi'}N$ and all the Sylow π'-subgroups of N have this form.

The following is presumably well known but we give a proof. Let π be a set of primes.

4.11. Lemma: Suppose $G \in \mathfrak{D}$ and $N \triangleleft G$. Let $S \in Syl_{\pi}G$ and suppose S reduces into the local system Ω of N consisting of finite subgroups. Then $S \cap N \in Syl_{\pi}N$.

Proof: Suppose $S \cap N \leq T \in Syl_{\pi}N$. Then for each $H \in \Omega$, $S \cap N \cap H = S \cap H \leq T \cap H$. By hypothesis, $S \cap H = T \cap H$. Since $N = \cup\{H : H \in \Omega\}$ we have

$$S \cap N = \cup\{S \cap H : H \in \Omega\} = \cup\{T \cap H : H \in \Omega\} = T$$

and the result follows. □

We show in 4.13 that some hypotheses are required in 4.11.

<u>4.12. Lemma</u>: Suppose $G \in \mathfrak{X}$ and $N \triangleleft G$. Let $\underset{\sim}{S} = \{S_p\}$ be a Sylow generating basis of G. Then:

(i) $\underset{\sim}{S}N/N = \{S_pN/N\}$ is a Sylow generating basis of G/N.

(ii) $\underset{\sim}{S} \cap N = \{S_p \cap N\}$ is a Sylow generating basis of N.

<u>Proof</u>: (i) Clearly, $G/N = \langle S_p : p \in \pi(G)\rangle N/N$

$= \langle S_pN/N : p \in \pi(G)\rangle$

and the result follows by 4.8 and the remarks before 4.1.

(ii) Let π be a set of primes and $S_\pi = \langle S_p : p \in \pi\rangle$. By 4.7, $\underset{\sim}{S}$ reduces into a totally ordered local system, $\Omega = \{G_i : i \geq 1\}$, of finite subgroups. Thus $S_p \cap G_i \in \mathrm{Syl}_p G_i$ and $\langle S_p \cap G_i : p \in \pi\rangle \in \mathrm{Syl}_\pi G_i$, whence

$$\langle S_p \cap G_i : p \in \pi\rangle = S_\pi \cap G_i . \qquad (*)$$

Hence S_π reduces into Ω. Therefore S_π reduces into $\Omega \cap N$, a local system of N, since $S_\pi \cap G_i \in \mathrm{Syl}_\pi G_i$ and $N \cap G_i \triangleleft G_i$ imply $S_\pi \cap N \cap G_i \in \mathrm{Syl}_\pi(G_i \cap N)$ from the theory of finite groups. Thus, by 4.11, $S_\pi \cap N \in \mathrm{Syl}_\pi N$.

But $\langle S_p \cap N : p \in \pi\rangle = S_\pi \cap N$ by a similar argument to that used to show (*) and it follows from 4.8(ii) that $\underset{\sim}{S} \cap N$ is a Sylow generating basis of N. □

The following example was constructed by Professor B. Hartley and we are grateful for his permission to include it here. It is a counter example to many natural questions. First, if $G \in \mathfrak{Y}$ and π is a set of primes, a π-subgroup B of G is called a <u>basic π-subgroup</u> of G if B is countable and $\mathrm{Syl}_p B \subseteq \mathrm{Syl}_p G$ for all $p \in \pi$. The set of basic π-subgroups of G will be denoted by $\text{Basic}_\pi G$. Massey [32] (Theorem 1.1) has shown that every $\mathfrak{Y}$-group contains basic π-subgroups. For other information concerning them the reader should consult [30].

4.13. Example:

Let $\{p_i, q_i : i \geq 1\}$ be an infinite set of distinct odd primes satisfying $q_i | (p_i - 1)$. By Dirichlet's Theorem on the primes in an arithmetic progression such a set exists. Let $G_i = A_i B_i$ be a non abelian group of order $p_i q_i$ with $|A_i| = p_i$, $|B_i| = q_i$ and $A_i \triangleleft G_i$. Put $A = \underset{i \geq 1}{\text{Dr}}\ A_i$ and $B = \underset{i \geq 1}{\text{Dr}}\ B_i$. Let $H = \underset{i \geq 1}{\text{Dr}}\ G_i = A \rtimes B$. The group A also has an automorphism $\gamma : a \mapsto a^{-1}$ of order 2 and γ commutes with the elements of B. Put $G = A \rtimes \langle B, \gamma \rangle$ and $C = \langle B, \gamma \rangle$, an abelian group. Clearly G is a metabelian $\mathfrak{X}$-group. Let $N = A\langle \gamma \rangle$. Then $N \triangleleft G$ since B normalises A and centralises $\langle \gamma \rangle$.

(1) The complements to A in G are conjugate.

For if $G = A \rtimes C_1$ then $N = A(C_1 \cap N)$ and since $A \cap C_1 = 1$, $C_1 \cap N = \langle \gamma_1 \rangle$ for some element γ_1 of order 2. Thus $\langle \gamma \rangle, \langle \gamma_1 \rangle \in \text{Syl}_2 N$ so there exists $a \in A$ such that $\gamma_1{}^a = \gamma$. Since $q_i \neq 2$, for each i, $C_A(\gamma) = 1$ and hence $C_G(\gamma) = C$. Also $C_1{}^a \leq C_G(\gamma_1{}^a)$ since C_1 is abelian so by the Dedekind law $C_1{}^a = C$ as required.

Let $\pi = \{2, q_1, q_2, \ldots\}$. Since

$$C_A(B_1) \underset{\neq}{>} C_A(B_1 \times B_2) \underset{\neq}{>} \cdots,$$

the set of centralisers, $\{C_A(D) : D \leq C\}$, does not satisfy min so, by Hartley [16] (lemma 4.3), G has $2^{\aleph_0}$ Sylow π-subgroups. Thus G cannot be a $\mathfrak{U}$-group in the sense of [8]. However the situation is much more drastic.

(2) G possesses non-isomorphic Sylow π-subgroups.

Let $\overline{A} = \mathrm{Cr}_{i\geq 1} A_i \leq \overline{H} = \mathrm{Cr}_{i\geq 1} G_i$, so $H \triangleleft \overline{H}$ and $\overline{A}$ normalises H. Let $\alpha \in \overline{A}\backslash A$ and consider B^α. We shall show $B^\alpha \in \mathrm{Syl}_\pi G$ which proves (2) since $C \in \mathrm{Syl}_\pi G$ and $C \not\cong B^\alpha$. It is clear that $AB = AB^\alpha = O_{2'}(G)$ and $|G : AB^\alpha| = 2$ so either $B^\alpha \in \mathrm{Syl}_\pi G$ or $B^\alpha \leq C_1$, a complement to A (for if $B^\alpha < C_1 \in \mathrm{Syl}_\pi G$ then $G = AC_1$ or $AB^\alpha = AC_1$ and the latter cannot happen).

If $B^\alpha \leq C_1$ then by (1) there exists $a \in A$ such that $B^{\alpha a} \leq C$. Thus $B^{\alpha a} = O_{2'}(C) = B$ and $[B,\alpha a] \leq B \cap \overline{A} = 1$. So $\alpha a \in C_{\overline{A}}(B)$. However, if $\beta = (a_1, a_2, \ldots) \in C_{\overline{A}}(B)$ and $a_i \neq 1$ then $\beta^b = \beta$ for all $b \in B_i$ which means G_i is abelian. Hence $C_{\overline{A}}(B) = 1$ and $\alpha a = 1$, a contradiction. Therefore $B^\alpha \in \mathrm{Syl}_\pi G$ and (2) follows.

(3) The Sylow bases of G are conjugate.

For let $\underset{\sim}{S} = \{S_p\}$, $\underset{\sim}{T} = \{T_p\}$ be Sylow bases of G. For $p \notin \pi$, $S_p = T_p \triangleleft G$. If $S_\pi = \langle S_p : p \in \pi\rangle$, $T_\pi = \langle T_p : p \in \pi\rangle$ then S_π and T_π, being complements to A, are conjugate. Since the complements are abelian, the Sylow bases $\{S_p\}_{p\in\pi}$, $\{T_p\}_{p\in\pi}$ are therefore conjugate, as required.

(4) $\mathrm{Syl}_\pi G \not\subseteq \mathrm{Basic}_\pi G$.

Any of the subgroups B^α above give this. This answers in the negative a question of Massey [30] (p. 99).

(5) The Sylow bases of the closed subgroups B^α and $O_{2'}(G)$ do not extend to Sylow bases of G.

This is clear and is of some interest later, since various local conjugacy problems require a knowledge of whether Sylow bases can be extended or not.

(6) 4.8 does not extend to infinite sets of primes.

For, $B^{\alpha} \cap N = 1 \notin Syl_{\pi}N$. Also $B^{\alpha}O_{2'}(G)/O_{2'}(G)$ is trivial so cannot be a Sylow π-subgroup of $G/O_{2'}(G)$ which is a 2-group. This also shows that 4.2 does not extend to infinite sets of primes.

(7) The product of an arbitrary Sylow π-subgroup and an arbitrary Sylow π'-subgroup need not equal G.

For, $A \in Syl_{\pi'}G$, $B^{\alpha} \in Syl_{\pi}G$ and $AB^{\alpha} \neq G$.

Thus 4.13 gives a counter example to many interesting conjectures, even for metabelian $\mathfrak{X}$-groups. We shall discuss 4.13 again later.

We shall now give some further straightforward results concerning the Sylow theory of $\mathfrak{Y}$-groups. In contrast to 4.13(5) we have:

4.14. Lemma: Let $G \in \mathfrak{X}$ and suppose $H \leq G$ is a Černikov group. Then every Sylow basis of H extends to a Sylow basis of G.

Proof: Let $\Omega = \{G_i : i \geq 1\}$ be a totally ordered local system of Černikov subgroups of G and suppose $G_1 = H$. In a similar manner to the proof of [18] (lemma 2.1) we can extend a Sylow basis of G_i to a Sylow basis of G_{i+1} and obtain a Sylow generating basis of G which reduces into each G_i. In particular this Sylow generating basis reduces into H. The proof that we can extend a Sylow basis of a subgroup of a soluble Černikov group to a basis of the whole group is the same as that for the finite case. □

The following two results are elementary and their proofs are omitted.

4.15. Lemma: Suppose $G \in \mathfrak{X}$ and $\underset{\sim}{S} = \{S_p\}$ is a set of p-subgroups of G, one for each prime p satisfying:

(i) $\underset{\sim}{S} \cap N$ is a Sylow generating basis of N.

(ii) $\underset{\sim}{S}N/N$ is a Sylow generating basis of G/N for some $N \triangleleft G$.

Then $\underset{\sim}{S}$ is a Sylow generating basis of G.

4.16. Lemma: Suppose $G \in \mathfrak{X}$ and $N \triangleleft G$ with G/N Černikov. If $\underset{\sim}{S}$ is a Sylow generating basis of G then $\underset{\sim}{S}N/N$ is a Sylow generating basis of G/N and all the Sylow generating bases of G/N can be obtained in this way.

The next two results are well known in other classes of groups and they are the starting point for the formation theory that follows.

4.17. Lemma: Let $G \in \mathfrak{X}$ and suppose $\underset{\sim}{S}$ is a Sylow generating basis of G. If $K \leq N_G(H)$, with $H, K \leq G$ and if $\underset{\sim}{S}$ reduces into H and K then $\underset{\sim}{S}$ reduces into HK.

Proof: We need to prove that if $\underset{\sim}{S} = \{S_p\}$ then

(i) $S_p \cap KH \in Syl_p KH$.

(ii) $\langle S_p \cap KH : p \in \mathbb{P}\rangle = KH$.

(i) Clearly $S_p \cap K \leq N_G(S_p \cap H)$ so $T_p = (S_p \cap K)(S_p \cap H)$ is a p-group. Also $S_p \cap H \in Syl_p H$ and $S_p \cap H \leq T_p \cap H$ so $S_p \cap H = T_p \cap H$. Now, the restriction to K of the natural homomorphism $KH \to KH/H$ is surjective. Hence $(S_p \cap K) \to (S_p \cap K)H/H \in Syl_p KH/H$ by 4.8. Thus

$$(S_p \cap K)H/H = T_pH/H \in \mathrm{Syl}_p KH/H.$$

Since $T_p \cap H \in \mathrm{Syl}_p H$, it follows by the Dedekind Modular Law that $T_p \in \mathrm{Syl}_p KH$ and $T_p = S_p \cap KH$.

(ii) We know $\langle S_p \cap H : p \in \mathbb{P}\rangle = H$, $\langle S_p \cap K : p \in \mathbb{P}\rangle = K$ and from the proof of (i) that $S_p \cap KH = (S_p \cap H)(S_p \cap K)$. Thus

$$\begin{aligned}\langle S_p \cap KH : p \in \mathbb{P}\rangle &= \langle (S_p \cap H)(S_p \cap K) : p \in \mathbb{P}\rangle \\ &\supseteq \langle S_p \cap H : p \in \mathbb{P}\rangle\langle S_p \cap K : p \in \mathbb{P}\rangle \\ &= KH\end{aligned}$$

The result follows. □

4.18. Lemma: Suppose $H, K \trianglelefteq G \in \mathfrak{D}$ and let p be a prime. Suppose $S \in \mathrm{Syl}_p G$ and $T \in \mathrm{Syl}_{p'} G$. Put $N = N_G(T \cap K)$. Then

(i) $HK \cap HT = H(K \cap T)$.

(ii) $HN = N_G(HK \cap HT)$.

(iii) S reduces into N.

Proof: (i) By 4.8(ii), $K \cap T \in \mathrm{Syl}_{p'} K$ so, by 4.8(i), $(K \cap T)H/H \in \mathrm{Syl}_{p'} KH/H$. Since $(HK \cap HT)/H$ is a p'-group, the result follows.

(ii) The Sylow p'-subgroups of a $\mathfrak{D}$-group are conjugate so are pronormal. The result therefore follows by (i) and the well known fact that the normaliser of a pronormal subgroup is preserved by homomorphisms.

(iii) Clearly $T \leq N$ and $G = ST$ by 4.9. Hence, $N = N \cap ST = (N \cap S)T$ by the modular law. It follows that $N \cap S \in \mathrm{Syl}_p N$. □

Finally in this chapter, we give a result which shows that under certain conditions Sylow π-subgroups of $G \in \mathfrak{D}$ are necessarily Basic π-subgroups of G.

4.19. Lemma: Let $G \in \mathfrak{D}$ and suppose $G = ST$ with $S \in Syl_{\pi}G$, $T \in Syl_{\pi'}G$ for some set of primes π. If $\sigma \subseteq \pi$ is a finite set of primes and $P \in Syl_{\sigma}S$ then $P \in Syl_{\sigma}G$.

Proof: We give the proof for several different cases.

Case (a): $T \triangleleft G$.

By 4.8(i), $PT/T \in Syl_{\sigma}G/T$. Let $P \leq Q \in Syl_{\sigma}G$. Then $PT = QT$ and $P = Q$ by the modular law.

Case (b): G is a Černikov group with finite Sylow σ-subgroups.

Let G^{o} be the radicable part of G. Then G^{o} is a σ'-group and $G/G^{o} = (SG^{o}/G^{o})(TG^{o}/G^{o})$. Since $PG^{o}/G^{o} \in Syl_{\sigma}SG^{o}/G^{o}$ and since the result holds for finite groups it follows that $PG^{o}/G^{o} \in Syl_{\sigma}G/G^{o}$. The result now follows as in case (a).

Case (c): G is a Černikov group.

Let $P \leq Q \in Syl_{\sigma}G$. Then Q^{o} is the σ-radicable part of G and G/Q^{o} satisfies the hypotheses of (b). However $S \in Syl_{\pi}G$ implies $Q^{o} \leq S^{o} \leq S$ and hence $Q^{o} \leq P$, since $P \in Syl_{\sigma}S$. By (b), $P/Q^{o} \in Syl_{\sigma}G/Q^{o}$ and the result follows.

Case (d): The general case.

$G/O_{\sigma'}(G)$ is a Černikov group since σ is a finite set. Also $PO_{\sigma'}(G)/O_{\sigma'}(G) \in Syl_{\sigma}(SO_{\sigma'}(G)/O_{\sigma'}(G))$ by 4.8(i). If $P \leq Q \in Syl_{\sigma}G$ then $PO_{\sigma'}(G) = QO_{\sigma'}(G)$ and the result follows as in case (a). □

It seems that there should be an easy proof of 4.19.

In chapter 6 we shall examine more closely the following situation:

Suppose $G \in \mathfrak{X}$ and π is a set of primes. Let $G = S_{\pi}S_{\pi'} = T_{\pi}T_{\pi'}$ with S_{π}, $T_{\pi} \in Syl_{\pi}G$ and $S_{\pi'}$, $T_{\pi'} \in Syl_{\pi'}G$. Does it follow that S_π has a Sylow generating basis which can be extended to a Sylow generating basis of G? This would characterise such Sylow π-subgroups. We shall show in chapter 6 that $S_\pi \cong T_\pi$. (This is well known of course (see Massey [30]).) We shall say that a π-subgroup S of a group G is <u>complemented</u> if there is a π'-subgroup T such that G = ST. Thus 4.19 shows that for $\mathfrak{X}$-groups every complemented π-subgroup is a basic π-subgroup. For the class $\mathfrak{X}$ we therefore obtain an affirmative answer to a question of Massey [30] (p. 92, problem III). Massey does not seem to make this observation.

CHAPTER 5. CHIEF FACTORS AND $\mathfrak{F}$-NORMALISERS

In this chapter we consider the concept of a saturated formation. Our definition follows the usual practice in infinite group theory of using the idea of a preformation to define the formation "locally". Having done this we can then define the concept of an $\mathfrak{F}$-normaliser, for $\mathfrak{F}$ a formation, and deduce many of the usual properties attributed to such subgroups. In particular we show that the $\mathfrak{F}$-normalisers have the usual cover-avoidance properties with regard to the chief factors. The later results of the chapter allow us to prove a generalisation of a well known complementation result of Higman [21]. Our result (5.19) has already been proved by Hartley [18] (Theorem 1) in the special case $\mathfrak{F} = L\mathfrak{N}$.

Many of the proofs of these results are standard, but for the sake of completeness we give a fairly full account of the proofs. As far as possible we have given our results for $\mathfrak{Y}$-groups. Our first results concern chief factors.

5.1. Theorem: If $G \in \mathfrak{Y}$ then the chief factors of G are finite elementary abelian p-groups.

We omit the proof of this result since it is a direct consequence of the min-p conditions and the work of McLain [33] on minimal normal subgroups of a locally soluble group.

The following result is due to Gardiner, Hartley and Tomkinson [8] (Corollary 3.3).

5.2. Lemma: If $G \in \mathfrak{S}$ and H/K is a p-chief factor of G then $O_p(G/C_G(H/K)) = 1$.

A subgroup L of a group G is said to <u>cover</u> a chief factor H/K of G if $H \leq KL$ and to <u>avoid</u> H/K if $H \cap L \leq K$. Our next result has also been proved for $\mathfrak{U}$-groups ([8] lemma 3.1) and is vital in what follows.

<u>5.3. Lemma</u>: Let $M \triangleleft G \in \mathfrak{D}$ and let p be a prime. Suppose $S \in Syl_{p'}G$, $N = N_G(S \cap M)$ and H/K is a chief factor of G. Then N covers H/K unless H/K is a p-chief factor not centralised by M in which case N avoids H/K.

<u>Proof</u>: By 4.18(i) and (ii) we may assume K = 1.

(a) If H is a p'-group, clearly $H \leq S \leq N$, so N covers H.

(b) If H is a p-group centralised by M then H centralises $S \cap M$ and hence $H \leq N$, so N covers H.

(c) If H is a p-group not centralised by M let $C_1 = C_G(H)$ and $C_2 = C_M(H)$. Then $C_1, C_2 \triangleleft G$ and $C_2 \lneq M$, by assumption. Also $1 \neq M/C_2 \cong MC_1/C_1 \triangleleft G/C_1$ and by 5.2,

$$O_p(G/C_1) = 1. \qquad (1)$$

Since H is finite by 5.1, MC_1/C_1 is a finite soluble group and hence contains a non trivial characteristic q-group Q/C_1. By (1), $q \neq p$. Now,

$$S \in Syl_{p'}G \Rightarrow S \cap M \in Syl_{p'}M \quad \text{by 4.10(ii)}$$

$$\Rightarrow (S \cap M)C_1/C_1 \in Syl_{p'}MC_1/C_1 \quad \text{by 4.10(i)}.$$

Hence $$Q \leq (S \cap M)C_1. \qquad (2)$$

Moreover $[H \cap N, S \cap M] \leq (S \cap M) \cap H = 1$ (since S is a p'-group and H is a p-group), and $[H \cap N, C_1] = 1$. Thus

$[H \cap N, Q] = 1$ by (2) so $Q \leq C_G(H \cap N)$. But $C_H(Q) \triangleleft G$ and $Q \neq C_1$ so $C_H(Q) \underset{\neq}{\triangleleft} H$. Therefore, since H is a minimal normal subgroup of G, $C_H(Q) = 1$. Hence, since $H \cap N \leq C_G(Q)$, $H \cap N = 1$ and N therefore avoids H. □

The following lemma is due to Gardiner, Hartley and Tomkinson [8] (lemma 3.6).

<u>5.4. Lemma</u>: Let $\mathfrak{K}$ be a QS-closed class of groups and let $\mathfrak{D}$ be an $\mathfrak{K}$-formation. Let $G \in \mathfrak{K}$ and suppose X is a $\mathfrak{D}$-projector of G. Suppose $\{H_i : i \in I\}$ is a collection of normal subgroups of G. Then

(i) $\bigcap_{i \in I} XH_i = X(\bigcap_{i \in I} H_i)$.

(ii) If Y is another $\mathfrak{D}$-projector of G such that X and Y are conjugate in $\langle X,Y \rangle$ and $\{U_\sigma, V_\sigma : \sigma \in \Omega\}$ is a normal series of G then there exists $\mu \in \Omega$ such that

$$U_\mu X = U_\mu Y$$

$$V_\mu X \neq V_\mu Y.$$

We use this to show:

<u>5.5. Lemma</u>: Let $M, M^* \triangleleft G \in \mathfrak{Y}$ and let S be a Sylow p'-subgroup of G for some prime p. Let $N = N_G(S \cap M)$ and $N^* = N_G(S \cap M^*)$. Then the following are equivalent:

(i) $N = N^*$

(ii) The sets of p-chief factors of G centralised by M and M^* respectively are the same.

(iii) In some chief series of G the sets of p-chief factors centralised by M and M^* respectively are the same.

Proof: (i) ⇒ (ii) by 5.3.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i). We first remark that the Sylow p'-subgroups of G are precisely the $\mathcal{S}_{p'}$-projectors of G by 4.10(i). Also MM*, M and M* centralise the same p-chief factors in some chief series Γ so without loss of generality, $M \leq M^*$, whence $N^* \leq N$. Put $\Gamma = \{U_\sigma, V_\sigma : \sigma \in \Omega\}$ and consider $\Gamma \cap M^* = \{U_\sigma \cap M^*, V_\sigma \cap M^* : \sigma \in \Omega\}$. Now $(U_\sigma \cap M^*)/(V_\sigma \cap M^*)$ is G-isomorphic to $(U_\sigma \cap M^*)V_\sigma/V_\sigma$ so if the former is non-trivial,

$$U_\sigma/V_\sigma \underset{G}{\cong} (U_\sigma \cap M^*)/(V_\sigma \cap M^*), \text{ a chief factor of } G.$$

If $(U_\sigma \cap M^*)/(V_\sigma \cap M^*)$ is a p-chief factor centralised by M then $M \leq C_G(U_\sigma/V_\sigma)$ so $M^* \leq C_G(U_\sigma/V_\sigma)$ and hence $M^* \leq C_G(U_\sigma \cap M^*/V_\sigma \cap M^*)$, by hypothesis. Thus M and M* both centralise $U_\sigma \cap M^*/V_\sigma \cap M^*$ or neither does. Suppose, for a contradiction, that $N^* \lneq N$ and that $g \in N \backslash N^*$. Since $(S \cap M^*)^g$, $S \cap M^* \in \mathrm{Syl}_{p'}M^*$ (by 4.10(i)), they are conjugate in the group they generate. By our initial remark and 5.4 there is a factor H/K of $\Gamma \cap M^*$ such that

$$H(S \cap M^*) = H(S \cap M^*)^g \qquad (1)$$

$$K(S \cap M^*) \neq K(S \cap M^*)^g. \qquad (2)$$

By (1) there exists $h \in H$ such that $(S \cap M^*)^{gh} = S \cap M^*$. Thus $gh \in N^*$ and $h \in (N \cap H)\backslash K$ by (2). It follows that N does not avoid H/K so by 5.3, H/K is either a p'-factor or a p-chief factor centralised by M. By assumption M* also centralises H/K in the latter case so N* covers H/K in either case by 5.3. Hence $h \in H \leq KN^*$ so h normalises $K(S \cap M^*)$. This now provides a contradiction to (2) since

$$K(S \cap M^*)^g = K(S \cap M^*)^{h^{-1}} = K(S \cap M^*). \qquad \square$$

We shall now let $\mathfrak{Z}$ denote a fixed QS-closed subclass of $\mathfrak{X}$. Let π be a fixed set of primes, f a $\mathfrak{Z}$-preformation function defined on π and $\mathfrak{F}$ the saturated $\mathfrak{Z}$-formation defined locally by f. Hence

$$\mathfrak{F} = \mathfrak{F}(f) = \mathfrak{S}_\pi \cap \mathfrak{Z} \cap \bigcap_{p\in\pi} \mathfrak{S}_{p'}\mathfrak{S}_p f(p).$$

Put $\mathfrak{Z}_\pi = \mathfrak{Z} \cap \mathcal{S}_\pi$ and let C_p denote the f(p)-centraliser of $\mathfrak{Z}$. Thus

$$C_p = \cap\{C_G(H/K) : H/K \text{ is a p-chief factor and } A_G(H/K) \in f(p)\}.$$

If $\underset{\sim}{S} = \{S_p\}$ is a Sylow generating basis of G and σ is a set of primes let $S_\sigma = \langle S_p : p \in \sigma\rangle \in Syl_\sigma G$ and define

$$D = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p).$$

A subgroup defined in this manner will be called an <u>$\mathfrak{F}$-normaliser</u> of G. In the case $\pi = \mathbb{P}$ and $f(p) = 1$ for each $p \in \pi$, $\mathfrak{F} = L\mathfrak{N}$ and D will then be called a <u>Basis Normaliser</u>.

Before giving some of the properties of $\mathfrak{F}$-normalisers we shall first show that $\mathfrak{F}$ is indeed a $\mathfrak{Z}$-formation. The following theorem will be of use in this.

<u>5.6. Theorem</u>: Let G be a periodic locally soluble group. Then

$$O_{p'p}(G) = \cap\{C_G(H/K) : H/K \text{ is a p-chief factor}\}$$

where the intersection may be taken over all p-chief factors of

G or over all those p-chief factors of G occurring in some chief series of G.

The proof of this result is due to Gardiner, Hartley and Tomkinson [8] (Theorem 3.8). We can then easily obtain

<u>5.7. Lemma</u>: Let $G \in \mathfrak{Z}_\pi$. Then the following are equivalent.

(i) $G \in \mathfrak{F}$.

(ii) $G/O_{p',p}(G) \in f(p)$ for all $p \in \pi$.

(iii) $A_G(H/K) \in f(p)$ for all $p \in \pi$ and all p-chief factors H/K of G.

<u>Proof</u>: (i) $\Leftrightarrow$ (ii) is clear from the definition of $\mathfrak{F}$.
(ii) $\Rightarrow$ (iii) is clear from the Q-closure of f(p) and 5.6.
(iii) $\Rightarrow$ (ii). Since f(p) is a $(\mathfrak{Z},p)$-preformation and $G \in \mathfrak{Z}$, $G/C_G(\mathfrak{Z},p) \in f(p)$. But $C_G(\mathfrak{Z},p) = O_{p',p}(G)$ so the result follows. □

<u>5.8. Lemma</u>: Let $G \in \mathfrak{Z}$. Let C_p be the f(p)-centraliser of G, let Γ be a chief series of G and let $C_p^* = \bigcap C_G(H/K)$, where the intersection is taken over all p-chief factors H/K in Γ such that $A_G(H/K) \in f(p)$. Then $C_p = C_p^*$.

<u>Proof</u>: Let $S_{p'}$ be a Sylow p'-subgroup of G and let $N_p = N_G(S_{p'} \cap C_p)$, $N_p^* = N_G(S_{p'} \cap C_p^*)$. Clearly $C_p \leq C_p^*$. Since $G/C_p \in f(p)$, every p-chief factor H/K in Γ centralised by C_p satisfies $A_G(H/K) \in f(p)$, so is centralised by C_p^* also. Hence C_p and C_p^* centralise the same p-chief factors of Γ so 5.5 implies $N_p = N_p^*$. Again, by 5.5, C_p and C_p^* centralise the same set of p-chief factors of G. It follows from the definition of C_p that $C_p^* \leq C_p$ and we have equality. □

<u>5.9. Lemma</u>: $\mathfrak{F}$ is a $\mathfrak{Z}$-formation.

<u>Proof</u>: $\mathfrak{F}$ is certainly Q-closed. Let $G \in \mathfrak{Z}$ and suppose $N_i \triangleleft G$ with $i \in I$, an index set. Suppose $G/N_i \in \mathfrak{F}$ and $\cap\{N_i : i \in I\} = 1$. Certainly, $G \in \mathfrak{Z}_\pi$ since $\mathfrak{S}_\pi$ is a $\mathfrak{S}$-formation. For $p \in \pi$, let C_p be the f(p)-centraliser of G. Then $G/C_p \in f(p)$ since f(p) is a $(\mathfrak{Z},p)$-preformation.

Suppose H/K is a p-chief factor of G with $N_i \leq K \leq H$ for some $i \in I$. Then $A_G(H/K) \in f(p)$, by 5.7(iii) and the fact that $G/N_i \in \mathfrak{F}$. Hence C_p centralises every p-chief factor of G/N_i and

$$C_pN_i/N_i \leq O_{p'p}(G/N_i) \quad \text{by 5.6.}$$

Thus $C_p/C_p \cap N_i \in \mathfrak{S}_{p'}\mathfrak{S}_p$ and since this is a $\mathfrak{S}$-formation, $C_p \in \mathfrak{S}_{p'}\mathfrak{S}_p$. Hence $G \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$ for all $p \in \pi$ and therefore $G \in \mathfrak{F}$. □

We shall assume that the $\mathfrak{Z}$-preformation function f is <u>integrated</u>; that is $f(p) \leq \mathfrak{F}$, for all $p \in \pi$. Using 5.9 it is easily seen that

<u>5.10. Corollary</u>: Every saturated $\mathfrak{Z}$-formation can be defined by an integrated $\mathfrak{Z}$-preformation function.

<u>Proof</u>: Define $\bar{f}(p) = f(p) \cap \mathfrak{F}$. Then $\bar{f}$ is an integrated $\mathfrak{Z}$-preformation function defining $\mathfrak{F}$. □

We now show that the group of automorphisms induced by $G \in \mathfrak{Z}$ on a p-chief factor is independent of the integrated $\mathfrak{Z}$-preformation function defining the formation $\mathfrak{F}$. The proof is as in the $\mathfrak{U}$-group case.

5.11. Lemma: Let f, $\bar{f}$ be two integrated $\mathfrak{Z}$-preformation functions defining the same saturated $\mathfrak{Z}$-formation $\mathfrak{F}$. Let $G \in \mathfrak{Z}$, $p \in \pi$ and H/K a p-chief factor of G with centraliser C. Then $G/C \in f(p)$ if and only if $G/C \in \bar{f}(p)$.

Proof: Let $G/C \in f(p)$, $S \in Syl_{p'}C$ and put $N = N_G(S)$. Since the Sylow p'-subgroups of C are conjugate in C, the Frattini argument implies $G = NC$. Thus

$$G/C = NC/C \cong N/N \cap C. \tag{1}$$

Now $S \in Syl_{p'}(N \cap C)$ and $S \lhd N \cap C$ so $(N \cap C)/S$ is a p-group. By (1), $(N/S)/(N \cap C/S) \cong N/N \cap C \in f(p)$ so

$$N/S \in \mathfrak{S}_p f(p). \tag{2}$$

Moreover $O_{p'}(N) = S$. For, H/K is a p-chief factor centralised by C so N covers H/K by 5.3. Thus

$$H = K(H \cap N) \quad \text{and} \quad H/K \underset{N}{\cong} H \cap N/K \cap N.$$

Since $O_{p'}(N)$ centralises the p-group $H \cap N/K \cap N$, it centralises H/K. Thus $O_{p'}(N) \leq C \cap N$ so $O_{p'}(N) = S$. Hence by (2), $N \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$. Also $N/S \in \mathfrak{F}$. For, $N/S \in \mathfrak{Z}$ since $G \in \mathfrak{Z}$ and $\mathfrak{Z}$ is QS-closed. Also N/S is a π-group since f is integrated and if $q \in \pi$ with $q \neq p$ then $N/S \in \mathfrak{S}_p f(p) \leq \mathfrak{S}_p \mathfrak{S}_{q'}\mathfrak{S}_q f(q) = \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$. Thus $N/S \in \mathfrak{F}$. But $\bar{f}$ defines $\mathfrak{F}$ so $N/S \in \mathfrak{S}_{p'}\mathfrak{S}_p \bar{f}(p)$ and since $O_{p'}(N) = S$, $N/S \in \mathfrak{S}_p \bar{f}(p)$. It follows that $N/N \cap C \in \mathfrak{S}_p \bar{f}(p)$ so $G/C \in \mathfrak{S}_p \bar{f}(p)$ by (1). By 5.2, $O_p(G/C) = 1$ so $G/C \in \bar{f}(p)$ and the result now follows by symmetry. □

If $\mathfrak{F} = \mathfrak{F}(f)$ is a saturated $\mathfrak{Z}$-formation and $G \in \mathfrak{Z}$ we shall say that a p-chief factor of G is <u>$\mathfrak{F}$-central</u> if $p \in \pi$ and $A_G(H/K) \in f(p)$; otherwise H/K is called <u>$\mathfrak{F}$-eccentric</u>. By 5.11 these concepts are independent of the $\mathfrak{Z}$-preformation function defining $\mathfrak{F}$. It follows that the f(p)-centraliser, being the intersection of the centralisers of the $\mathfrak{F}$-central p-chief factors of G, is independent of the way $\mathfrak{F}$ is defined.

Our next result gives an alternative way of recovering the $\mathfrak{F}$-normalisers.

<u>5.12. Lemma</u>: For $p \in \pi$, let $C_p^* \lhd G \in \mathfrak{Z}$ and suppose $G/C_p^* \in f(p)$. Let $\underset{\sim}{S}$ be a Sylow generating basis of G and put

$$D^* = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p^*); \quad D = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p).$$

Then $D^* \leq D$ and if $C_p^* \leq C_p$ for all $p \in \pi$ then $D = D^*$.

<u>Proof</u>: Let $C_p^{**} = C_pC_p^*$ and let H/K be a p-chief factor of G centralised by C_p^*. Then $A_G(H/K) \in f(p)$ and hence H/K is an $\mathfrak{F}$-central p-chief factor of G so is centralised by C_p. Hence C_p^{**} and C_p^* centralise the same p-chief factors of G so by 5.5

$$N_G(S_{p'} \cap C_p^*) = N_G(S_{p'} \cap C_p^{**}) \leq N_G(S_{p'} \cap C_p),$$

since $C_p \leq C_p^{**}$. Hence $D^* \leq D$.

If $C_p^* \leq C_p$ then $N_G(S_{p'} \cap C_p^*) \leq N_G(S_{p'} \cap C_p)$ as above and it follows that $D = D^*$. □

This is useful if f(p) is a $\mathfrak{Z}$-formation when we may put $C_p^* = R_p$, the f(p)-residual of the $\mathfrak{Z}$-group G.

We shall now give some properties of the $\mathfrak{F}$-normalisers. As usual f is an integrated $\mathfrak{Z}$-preformation function defining

$\mathfrak{F}$ and C_p denotes the f(p)-centraliser of a $\mathfrak{Z}$-group.

5.13. Lemma: Let $G \in \mathfrak{Z}$ and let $\underset{\sim}{S} = \{S_p\}$ be a Sylow generating basis of G. Let $N_p = N_G(S_{p'} \cap C_p)$ and let D be the $\mathfrak{F}$-normaliser of G associated with $\underset{\sim}{S}$. Then

(i) $S_p \cap N_p = S_p \cap D \in Syl_p D$.

(ii) $D = \langle S_p \cap D : p \in \pi \rangle$.

(iii) If $H \triangleleft G$ then DH/H is the $\mathfrak{F}$-normaliser of G/H associated with $\underset{\sim}{S}H/H$.

Proof: (i) If $q \neq p$ and $q \in \pi$ then $S_{q'} \leq N_q = N_G(S_{q'} \cap C_q)$. Also $S_p \leq S_{q'}$. Hence, if $q \neq p$, $S_p \cap N_p \leq N_q$. Therefore $S_p \cap N_p \leq S_\pi \cap \bigcap_{q \in \pi} N_q = D$. It follows that $S_p \cap N_p = S_p \cap D \in$ $Syl_p D$ since $S_p \cap N_p \in Syl_p N_p$ by 4.18(iii).

(ii) Let $g \in D$. Then there is a finite set of primes $\sigma \subseteq \pi$ such that

$$g \in S_\sigma \cap \bigcap_{p \in \pi} N_p = E, \quad \text{say.}$$

We show that $E = \langle S_p \cap E : p \in \sigma \rangle$. Indeed, if $p \in \sigma$, $S_p \cap N_p \leq S_\sigma$ and if $q \neq p$,

$$S_p \cap N_p \leq S_p \leq N_q.$$

Thus $S_p \cap N_p \leq E \cap S_p$ and since $S_p \cap N_p \in Syl_p N_p$ we must have $S_p \cap N_p = E \cap S_p \in Syl_p E$. Thus $\{E \cap S_p : p \in \sigma\}$ is a Sylow basis of the soluble Černikov group E. Hence $E = \langle S_p \cap E : p \in \sigma \rangle$.

Therefore $g \in E = \langle S_p \cap E : p \in \sigma \rangle \leq \langle S_p \cap D : p \in \pi \rangle$ and the result follows.

(iii) First, since $G/C_p \in f(p)$, $G/C_pH \in f(p)$. Also every

$\mathfrak{F}$-central p-chief factor of G/H corresponds to an $\mathfrak{F}$-central p-chief factor of G so is centralised by C_p. Hence C_pH/H is a subgroup of the f(p)-centraliser of G/H. Thus by 5.12 the set $\{C_pH/H\}$ defines the $\mathfrak{F}$-normaliser of G/H associated with $\underset{\sim}{S}H/H$. However, by 4.18(iii), $S_p \cap N_p \in Syl_pN_p$, so $(S_p \cap N_p)H/H \in Syl_pN_pH/H$ by 4.10(i). Thus

$$(S_p \cap N_p)H/H = (S_pH \cap N_pH)/H = (S_pH/H) \cap N_{G/H}((S_{p'} \cap C_p)H/H)$$

by 4.18(i) and (ii). By the above remarks and (ii) above the $\mathfrak{F}$-normaliser associated with the Sylow generating basis $\underset{\sim}{S}H/H$ is

$$\langle (S_p \cap N_p)H/H : p \in \pi \rangle = \langle S_p \cap N_p : p \in \pi \rangle H/H = DH/H$$

as required. □

The cover-avoidance property of the $\mathfrak{F}$-normalisers is now easily established.

<u>5.14. Theorem</u>: Let $G \in \mathfrak{Z}$ and π a set of primes. Let $\underset{\sim}{S} = \{S_p\}$ be a Sylow generating basis of G and let D be the corresponding $\mathfrak{F}$-normaliser, for $\mathfrak{F}$ a saturated $\mathfrak{Z}$-formation. Then D covers the $\mathfrak{F}$-central chief factors of G and avoids the $\mathfrak{F}$-eccentric ones.

<u>Proof</u>: Put $N_p = N_G(S_{p'} \cap C_p)$, where $S_{p'} = \langle S_q : q \neq p \rangle$ and C_p is the f(p)-centraliser of G. Then $D = S_\pi \cap \bigcap_{p \in \pi} N_p$.

If H/K is a π'-chief factor, $H/K \cap DK/K = 1$ so

$$H \cap DK = (H \cap D)K = K.$$

Thus $H \cap D = H \cap K$ and D avoids H/K.

If H/K is a p-group, for some $p \in \pi$, that is not centralised by C_p then N_p avoids H/K by 5.3 and a fortiori D avoids H/K.

Finally if H/K is a p-chief factor centralised by C_p then N_p covers H/K by 5.3. Hence $H = K(H \cap N_p)$ and

$$H/K = (H \cap N_p)K/K \leq (S_p \cap N_p)K/K \quad \text{since} \quad S_p \cap N_p \in Syl_p N_p$$

$$= (S_p \cap D)K/K \quad \text{by} \quad 5.13(i).$$

Hence $H \leq DK$ so D covers H/K. □

<u>5.15. Theorem</u>: Let $G \in \mathfrak{Z}$ and $\mathfrak{F}$ a saturated $\mathfrak{Z}$-formation. Then

(i) The $\mathfrak{F}$-normalisers of G are locally conjugate.

(ii) If $G \in \mathfrak{F}$ then the $\mathfrak{F}$-normalisers of G coincide with G.

(iii) If $H \triangleleft G$, $G/H \in \mathfrak{F}$ and D is an $\mathfrak{F}$-normaliser of G then $G = DH$.

(iv) The $\mathfrak{F}$-normalisers of G belong to $\mathfrak{F}$.

<u>Proof</u>: (i) This follows from 4.6.

(ii) If $G \in \mathfrak{F}$ then $C_p = O_{p'p}(G)$ is the f(p)-centraliser of G by 5.1. Thus

$$(S_{p'} \cap C_p)O_{p'}(G)/O_{p'}(G) \leq O_{p'p}(G)/O_{p'}(G), \text{ a p-group.}$$

Hence $S_{p'} \cap C_p \leq O_{p'}(G) \leq S_{p'} \cap C_p$. Therefore $S_{p'} \cap C_p \triangleleft G$ and the result follows.

(iii) Follows from (ii) and 5.13(iii).

(iv) Let D be the $\mathfrak{F}$-normaliser of G corresponding to the Sylow generating basis $\underset{\sim}{S}$. Then, for each $p \in \pi$, D normalises $S_{p'} \cap C_p$ and hence normalises $D \cap S_{p'} \cap C_p$. By 5.13(i) and (ii),

$\{S_p \cap D : p \in \pi\}$ is a Sylow generating basis of D so $S_{p'} \cap D \in Syl_{p'}D$. Hence $S_{p'} \cap D \cap C_p$ is a (normal) Sylow p'-subgroup of $D \cap C_p$ by 4.10(ii), whence

$$D \cap C_p \in \mathfrak{S}_{p'}\mathfrak{S}_p. \tag{1}$$

Since $G/C_p \in f(p) \leq \mathfrak{F}$, it follows by (iii) that $G = C_pD$. Hence $D/D \cap C_p \cong G/C_p \in f(p)$. Thus by (1), $D \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, for each prime $p \in \pi$. Moreover, $G \in \mathfrak{Z}$ implies $D \in \mathfrak{Z}$ since $S\mathfrak{Z} = \mathfrak{Z}$. Finally $D \in \mathfrak{S}_\pi$ by definition so $D \in \mathfrak{F}$. □

Our next task is to investigate the relationship between the $\mathfrak{F}$-normalisers of various subgroups of a $\mathfrak{Z}$-group G and the $\mathfrak{F}$-normalisers of the group itself. The following lemma is of use and can be found in [8] (lemma 4.7).

<u>5.16. Lemma</u>: Let G = RH be a periodic locally soluble group with $H \leq G$ and R a normal locally nilpotent subgroup of G. Let $\{U_\sigma, V_\sigma : \sigma \in \Omega\}$ be a chief series of G. Then, after suppressing trivial factors, $\{U_\sigma \cap H, V_\sigma \cap H : \sigma \in \Omega\}$ is a chief series of H and if $U_\sigma \cap H/V_\sigma \cap H$ is non-trivial then

$$A_H(U_\sigma \cap H/V_\sigma \cap H) \cong A_G(U_\sigma/V_\sigma).$$

<u>5.17. Theorem</u>: Let $G = RH \in \mathfrak{Z}$ with $H \leq G$, $R \triangleleft G$ and $R \in L\mathfrak{N}$. Let $\underset{\sim}{T} = \{T_p\}$ be a Sylow generating basis of H and $\underset{\sim}{R} = \{R_p\}$ the unique Sylow generating basis of R. Then $\underset{\sim}{S} = \{R_pT_p\}$ is a Sylow generating basis of G and if D, E are the $\mathfrak{F}$-normalisers of G associated with $\underset{\sim}{S}$ and of H associated with $\underset{\sim}{T}$ respectively then $E = D \cap H$.

<u>Proof</u>: It is clear that $\underset{\sim}{S}$ is a Sylow generating basis of G.

For $p \in \pi$ and $K \leq G$ let $C_p(K)$ denote the f(p)-centraliser of K. We show that $C_p(G) \cap H \leq C_p(H)$. For, let Γ be a chief series of G. By 5.16, $\Gamma \cap H$ is a chief series Γ' of H, every $\mathfrak{F}$-central p-chief factor of which is H-isomorphic to some $\mathfrak{F}$-central p-chief factor of Γ. Thus $C_p(G) \cap H$ centralises every $\mathfrak{F}$-central p-chief factor in Γ' so by 5.8,

$$C_p(G) \cap H \leq C_p(H).$$

Now $R \leq C_p(G)$ by 5.6. Hence $G = C_p(G)H$ and $H/H \cap C_p(G) \in f(p)$. Thus by 5.12,

$$E = T_\pi \cap \bigcap_{p \in \pi} N_H(T_{p'} \cap H \cap C_p(G)).$$

$$= T_\pi \cap \bigcap_{p \in \pi} N_H(T_{p'} \cap C_p(G)).$$

Also
$$D \cap H = S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap C_p(G)) \cap H$$

$$= T_\pi \cap \bigcap_{p \in \pi} N_H(S_{p'} \cap C_p(G)).$$

Hence, if $g \in E$,

$$(S_{p'} \cap C_p(G))^g = R_{p'}(T_{p'} \cap C_p(G))^g = R_{p'}(T_{p'} \cap C_p(G))$$
$$= S_{p'} \cap C_p(G).$$

Therefore $g \in D \cap H$. Thus $E \leq D \cap H$.

Conversely, D normalises $S_{p'} \cap C_p(G)$ so $D \cap H$ normalises $S_{p'} \cap C_p(G) \cap H = T_{p'} \cap C_p(G)$. Thus $D \cap H \leq E$ and the result follows. □

5.18. Lemma: Let $G \in \mathfrak{Z}$ and let D be the $\mathfrak{F}$-normaliser of G associated with the Sylow generating basis $\underset{\sim}{S} = \{S_p\}$. If $D \leq H \leq G$ then

(i) $C_p(H) \leq H \cap C_p(G)$.

(ii) If $\underset{\sim}{S}$ reduces into H then D is contained in the $\mathfrak{F}$-normaliser associated with $\underset{\sim}{S} \cap H$.

Proof: (i) Since $G/C_p(G) \in f(p) \leq \mathfrak{F}$, $G = DC_p(G) = HC_p(G)$. Let L/M be an $\mathfrak{F}$-central p-chief factor of G. Then L/M is H-irreducible since $C_p(G)$ centralises it. Thus

$$L \cap H/M \cap H \underset{H}{\cong} (L \cap H)M/M$$

and if $L \cap H/M \cap H$ is non-trivial,

$$L \cap H/M \cap H \underset{H}{\cong} L/M. \qquad (1)$$

Hence $L \cap H/M \cap H$ is a chief factor of H. Now $H/H \cap C_p(G) \cong G/C_p(G) \in f(p)$ and since $C_p(G)$ centralises L/M, $H \cap C_p(G)$ centralises L/M. Hence $H \cap C_p(G)$ centralises $L \cap H/M \cap H$ by (1). Since f(p) is Q-closed, $A_H(L \cap H/M \cap H) \in f(p)$. Thus $L \cap H/M \cap H$ is $\mathfrak{F}$-central in H so is centralised by $C_p(H)$. By (1), $C_p(H)$ centralises L/M so $C_p(H) \leq C_p(G)$ as required.

(ii) Since $H/H \cap C_p(G) \in f(p)$ and $D \leq H$, 5.12 implies

$$D \leq S_\pi \cap H \cap \bigcap_{p \in \pi} N_H(S_{p'} \cap C_p(G) \cap H)$$

$$\leq S_\pi \cap H \cap \bigcap_{p \in \pi} N_H(S_{p'} \cap C_p(H))$$

and the latter is the $\mathfrak{F}$-normaliser of H associated with $\underset{\sim}{S} \cap H$. □

For the final result of this chapter we give the formation theoretic generalisation of Hartley's result [18] (Theorem 1).

5.19. Theorem: Let $G \in \mathfrak{Z}$ and $\mathfrak{F}$ a saturated $\mathfrak{Z}$-formation. Suppose $G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G, is abelian. Then $G^{\mathfrak{F}}$ is complemented in G. The complements are precisely the $\mathfrak{F}$-normalisers of G and are locally conjugate.

<u>Proof</u>: Let D be an $\mathfrak{F}$-normaliser of G corresponding to the Sylow generating basis $\underset{\sim}{S}$ and put $M = G^{\mathfrak{F}}$. Then $G/M \in \mathfrak{F}$ so $G = MD$ by 5.15(iii). If E is a complement of M then $E \in \mathfrak{F}$ and $G = ME$ imply E is contained in an $\mathfrak{F}$-normaliser by 5.17. So we only need show that the $\mathfrak{F}$-normalisers complement M.

Let $\{M_p\}$ denote the unique Sylow basis of M and for each $p \in \pi$ let $O_{p'}(D) = L_{p'}$. Since M is abelian, $M_{p'}[M_p,L_{p'}] \triangleleft G$. Let $x \mapsto x^*$ denote the natural homomorphism $G \to G^* = G/M_{p'}[M_p,L_{p'}]$. Suppose $q \in \pi$ and $q \neq p$. Since $D \in \mathfrak{F}$, $G/M_{p'} \in \mathfrak{S}_p\mathfrak{F}$. Thus $G^* \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$. On the other hand $L^*_{p'} \triangleleft G^*$ so $G^*/L^*_{p'} \in \mathfrak{S}_p f(p)$. Thus $G^* \in \mathfrak{F}$ and since $M = G^{\mathfrak{F}}$,

$$M \leq M_{p'}[M_p,L_{p'}] \leq M.$$

Thus $M_p = [M_p,L_{p'}]$. Let $x \in M_p$ so $x = \prod_{i=1}^{n} [x_i,y_i]$, with $x_i \in M_p$, $y_i \in L_{p'}$. Put $F = \langle y_i : i = 1, \ldots, n\rangle$, a finite p'-subgroup of $L_{p'}$ and let $E = \langle x_i^f : i = 1, \ldots, n;\ f \in F\rangle$ so that E is an F-invariant finite p-group. Since E is abelian, [22] (III.13.4(b)) implies

$$E = C_E(F) \times [E,F].$$

Hence if $x \in C_{M_p}(L_{p'})$ then $x \in C_E(F) \cap [E,F] = 1$. Thus $C_{M_p}(L_{p'}) = 1$. However $D \cap M_p \triangleleft D$ so $[D \cap M_p, L_{p'}] \leq D \cap M_p \cap L_{p'} = 1$ since M_p is a p-group. Hence $D \cap M_p \leq C_{M_p}(L_{p'})$ and thus $D \cap M_p = 1$. Since this is true for all $p \in \pi$, $D \cap M = 1$, as required. □

CHAPTER 6. $\mathfrak{F}$-PROJECTORS

In this chapter we shall show that provided the saturated formation $\mathfrak{F}$ satisfies certain conditions every $\mathfrak{Z}$-group possesses $\mathfrak{F}$-projectors. The conditions seem hard to remove because they restrict our attention to the cases when $\mathfrak{F}$ consists solely of co-Hopfian $\mathfrak{Z}$-groups. Once existence of $\mathfrak{F}$-projectors is established it is natural to ask whether one can obtain any sort of result concerning their conjugacy. Since the Sylow generating bases are locally conjugate it seems clear that we should ask for the projectors to be locally conjugate. However, we have been unable to show this even for $\mathfrak{F} = L\mathfrak{N}$. Instead, we obtain a different conjugacy property, which seems to occur repeatedly for $\mathfrak{X}$-groups.

To obtain the existence of $\mathfrak{F}$-projectors we first discuss the case $G \in (L\mathfrak{N})\mathfrak{F} \cap \mathfrak{Z}$. As usual $\mathfrak{F}$ is a saturated $\mathfrak{Z}$-formation.

<u>6.1. Lemma</u>: Suppose $G \in (L\mathfrak{N})\mathfrak{F} \cap \mathfrak{Z}$. Then the $\mathfrak{F}$-projectors of G are precisely the $\mathfrak{F}$-normalisers of G.

<u>Proof</u>: Let $R \triangleleft G$ with $R \in L\mathfrak{N}$ and $G/R \in \mathfrak{F}$. If D is an $\mathfrak{F}$-normaliser of G associated with the Sylow generating basis $\underset{\sim}{S} = \{S_p\}$ then by 5.15(iii),

$$G = RD. \tag{1}$$

Suppose $D \leq H$, $K \triangleleft H$ and $H/K \in \mathfrak{F}$. By (1), $H = H \cap RD = D(H \cap R)$. Also $H \cap R$ is a normal $L\mathfrak{N}$-subgroup of H and $\underset{\sim}{S}$ reduces into D by 5.13(i) and (ii). So by 5.17, if $\{R_p\}$ is the unique Sylow generating basis of R, we have $S_p = R_p(S_p \cap D)$. By 5.17 again, $\underset{\sim}{T} = \{(R_p \cap H)(S_p \cap D)\}$ is a Sylow generating basis of H. If E is the $\mathfrak{F}$-normaliser of H associated with $\underset{\sim}{T}$ then by 5.17,

$$E = D \cap H = D.$$

Thus D is an $\mathfrak{F}$-normaliser of H so by 5.15(iii), H = DK and hence D is an $\mathfrak{F}$-projector of G.

Conversely, if E is an $\mathfrak{F}$-projector of G then $G/R \in \mathfrak{F}$ implies G = RE. Let $\{E_p\}$ be a Sylow generating basis of E and $\{R_p\}$ the unique Sylow generating basis of R. Then $\{R_pE_p\}$ is a Sylow generating basis of G by 5.17 and if D is the associated $\mathfrak{F}$-normaliser of G, 5.17 implies

$$E = D \cap E \leq D.$$

Since $D \in \mathfrak{F}$ by 5.15(iv), it follows that E = D. □

We shall repeatedly use the following two results, the first of which is obvious. A proof of the second can be found in [8] (lemma 5.3).

<u>6.2. Lemma</u>: Suppose G is a group, $\mathfrak{D}$ is any Q-closed class and let L be a $\mathfrak{D}$-projector of G.

(i) If $L \leq H \leq G$ then L is a $\mathfrak{D}$-projector of H.

(ii) If $N \triangleleft G$ then LN/N is a $\mathfrak{D}$-projector of G/N.

<u>6.3. Lemma</u>: Let G be a group, $\mathfrak{D}$ any Q-closed class and $N \triangleleft G$. If M/N is a $\mathfrak{D}$-projector of G/N and L is a $\mathfrak{D}$-projector of M then L is a $\mathfrak{D}$-projector of G.

We can now prove:

6.4. Lemma: Let $G \in \mathfrak{Z} \cap (L\mathfrak{N})^n\mathfrak{F}$. Then G possesses $\mathfrak{F}$-projectors.

Proof: The proof is by induction on n. Let $1 = G_o \leq G_1 \leq \ldots \leq G_n \leq G_{n+1} = G$ be a normal series for G with $G/G_n \in \mathfrak{F}$ and $G_{i+1}/G_i \in L\mathfrak{N}$ (for $i = 0, \ldots, n-1$).

If $n = 1$ then $G \in (L\mathfrak{N})\mathfrak{F}$ and the result follows by 6.1 so we may assume $n > 1$ and that the result is true for groups in the class $(L\mathfrak{N})^{n-1}\mathfrak{F}$. Consider $G/G_1 \in (L\mathfrak{N})^{n-1}\mathfrak{F}$. By induction G/G_1 possesses an $\mathfrak{F}$-projector $H/G_1 \in \mathfrak{F}$. Hence $H \in (L\mathfrak{N})\mathfrak{F}$ and by the case $n = 1$, H possesses an $\mathfrak{F}$-projector K, say. By 6.3 K is an $\mathfrak{F}$-projector of G, as required. □

So far we have not had to put any restrictions on $\mathfrak{F}$, other than that it be a saturated formation. To cover the general case however we have found it necessary to restrict our attention although, as we shall show, every $\mathfrak{Z}$-group does possess $\mathfrak{F}$-projectors for the important classes $\mathfrak{F} = L\mathfrak{N}$, $(L\mathfrak{N})^k$, $\mathfrak{S}_\sigma$ (for σ a set of primes) and for $\mathfrak{F}$ the class of locally supersoluble groups. Consequently, we define a class $\mathfrak{D}$ to be co-Hopfian if every $\mathfrak{D}$-group is co-Hopfian. Our method for showing that $\mathfrak{F}$-projectors exist consists of descending the derived series in steps of two, keeping a check on various groups that occur; to this end we may assume G is not soluble by 6.4 so the derived series terminates in its ω^{th}-term. We shall require the following lemma. First recall $\mathfrak{F} = \mathfrak{S}_\pi \cap \mathfrak{Z} \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$.

<u>6.5. Lemma</u>: Let $G \in (L\mathfrak{N})\mathfrak{F} \cap \mathfrak{Z}$ and suppose $N \triangleleft G$ with $N \in L\mathfrak{N}$ and $G/N \in \mathfrak{F}$. Suppose σ is a finite set of primes such that $\sigma \subseteq \pi$ and N is a σ'-group. Then each Sylow σ-subgroup of G is contained in an $\mathfrak{F}$-normaliser of G.

<u>Proof</u>: Let $S \in \mathrm{Syl}_\sigma G$ and $\{S_p : p \in \sigma\}$ a Sylow basis of S. Since σ is finite, S is a Černikov group so by 4.14 we can extend $\{S_p : p \in \sigma\}$ to a Sylow generating basis $\underset{\sim}{S} = \{S_p\}$ of G. As usual let C_p be the $f(p)$-centraliser, $S_\pi = \langle S_p : p \in \pi\rangle$ and put $D = S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap C_p)$, the $\mathfrak{F}$-normaliser of G associated with $\underset{\sim}{S}$.

Since $G/N \in \mathfrak{F}$, $G = DN$ by 5.15(iii). If $p \in \sigma$, 4.19 implies that a Sylow p-subgroup of D is a Sylow p-subgroup of G, since N is a σ'-group. Also $S_p \cap D \in \mathrm{Syl}_p D$ by 5.13(i) so by the previous remark $S_p \cap D \in \mathrm{Syl}_p G$. Hence $S_p \cap D = S_p$ so $S_p \leq D$ and this holds for each $p \in \sigma$. The result now follows. □

We can now prove our main existence theorem.

<u>6.6. Theorem</u>: Let $G \in \mathfrak{Z}$ and suppose $\mathfrak{F}$ is a saturated formation. If $\mathfrak{F}$ is co-Hopfian then G possesses $\mathfrak{F}$-projectors.

<u>Proof</u>: Let $\mathfrak{F} = \mathfrak{Z} \cap \mathfrak{S}_\pi \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, where f is a preformation function defined on π. Let $\pi = \{p_1, p_2, \ldots\}$ and choose integers $0 < i_1 < i_2 < \ldots$ so that $G^{(i_m)}$ is a $\{p_1, \ldots, p_m\}'$-group. This is always possible since $G/O_{\sigma'}(G)$ is a soluble Černikov group for each finite set of primes σ. Put $j_m = i_m + 1$, $G_m = G^{(j_m)}$ and suppose that

$$L_1 \geqslant L_2 \geqslant \ldots \geqslant L_n$$

is a chain of subgroups with the properties:

(i) L_m/G_m is an $\mathfrak{F}$-projector of G/G_m ($m = 1, \ldots, n$).

(ii) $\{S_1, \ldots, S_m\}$ is a fixed "partial" Sylow basis of L_m ($1 \leq m \leq n$) with $S_m \in Syl_{p_m} L_m$ (for $m = 1, \ldots, n$). (By the word "partial," we mean $\{S_1, \ldots, S_m\}$ can be extended to a Sylow generating basis of L_m. This is possible by 4.14).

We now construct L_{n+1} as follows. By (i) $L_n/G^{(j_n+1)} \in (L\mathfrak{N})\mathfrak{F}$. Let $\sigma = \{p_1, \ldots, p_n\}$ so that $G_n/G^{(j_n+1)}$ is a σ'-group by choice of i_n. By 6.5, $L_n/G^{(j_n+1)}$ has an $\mathfrak{F}$-normaliser $D/G^{(j_n+1)}$ so that

$$S_1, \ldots, S_n \leq D.$$

Moreover $D/G^{(j_n+1)}$ is an $\mathfrak{F}$-projector of $G/G^{(j_n+1)}$ by 6.1 and 6.3.

Continuing in this manner we obtain an $\mathfrak{F}$-projector $E/G^{(i_{n+1})}$ of $G/G^{(i_{n+1})}$ such that

$$S_1, \ldots, S_n \leq E.$$

Now $G^{(i_{n+1})}/G_{n+1}$ is a p'_{n+1}-group by choice of i_{n+1}. Since $\langle S_1, \ldots, S_n\rangle \in Syl_\sigma E$ by (ii) and σ is finite, 4.14 can be applied and we can extend $\{S_1, \ldots, S_n\}$ to a Sylow generating basis of E. In this basis let X be the Sylow p_{n+1}-subgroup. Then by 6.5 there is an $\mathfrak{F}$-normaliser L_{n+1}/G_{n+1} of E/G_{n+1} such that

$$S_1, \ldots, S_n, X \leq L_{n+1}.$$

Again by 6.1 and 6.3, L_{n+1}/G_{n+1} is an $\mathfrak{F}$-projector of G/G_{n+1}

and we may put $X = S_{n+1}$. Thus we have now constructed L_{n+1} and $\{S_1, \ldots, S_{n+1}\}$ with properties (i) and (ii).

Put $$L = \bigcap_{m \geq 1} L_m .$$

We shall show that L is an $\mathfrak{F}$-projector of G. Since $\mathfrak{F}$ is a formation, it is clear from (3) below that $L \in \mathfrak{F}$. Let $p \in \pi$ be any prime. Then for some n, $p = p_n$ and

$$S_n \leq L_m \text{ for all } m \geq n, \text{ by construction of } L_m.$$

<u>(1) $\{S_n\}$ is a Sylow basis of L.</u>

For, $S_n \in \mathrm{Syl}_{p_n} L_n$ so by the above remark and the definition of L, $S_n \in \mathrm{Syl}_{p_n} L$. Also if $p, q \in \pi$ then $p = p_m$, $q = p_n$ for some m, n and by the choice of S_m, S_n we have $S_mS_n = S_nS_m$. Thus (1) holds.

<u>(2) $L_n = L_mG_n$ for all $m \geq n$.</u>

For, L_mG_n/G_n is an $\mathfrak{F}$-projector of G/G_n by the homomorphism invariance of $\mathfrak{F}$-projectors and is contained in the $\mathfrak{F}$-group L_n/G_n so we must have equality.

We now show:

<u>(3) $L_n = LG_n$ for all n.</u>

If (3) does not hold then for some integer n,

$$LG_n < L_n. \tag{4}$$

We claim $$LG_m < L_m \quad \text{for all } m \geq n. \tag{5}$$

Otherwise, for some $m > n$, $LG_m = L_m$. Multiplying by G_n gives

$$LG_n = L_mG_n = L_n \quad \text{by (2)}$$

and this contradicts (4). Hence (5) holds.

Now LG_n/G_n is a soluble group and $\{S_r\}G_n/G_n$ is a Sylow basis of it by 4.12(i) and hence is generating. Since $LG_n < L_n$, (1) implies that for some prime $p = p_m$,

$$S_mG_n/G_n \not\in Syl_pL_n/G_n. \qquad (6)$$

(Otherwise $\{S_r\}G_n/G_n$ is a Sylow generating basis of L_n/G_n so $L_n = LG_n$ contradicting (4)). Then $m > n$ otherwise there is a contradiction since $S_m \in Syl_{p_m}L_n$ by construction. By (2), $L_n = L_mG_n$ and $S_m \in Syl_pL_m$ by construction so $S_mG_n/G_n \in Syl_p(L_mG_n/G_n)$ by 4.8(i). This contradicts (6) and hence (3) follows.

Suppose now $H/K \in \mathfrak{F}$ with $K \triangleleft H$ and $L \leq H$. Then, for each n, $HG_n/KG_n \in \mathfrak{F}$ since $\mathfrak{F}$ is Q-closed and

$$LG_n/G_n = L_n/G_n \leq HG_n/G_n.$$

Thus $LKG_n = HG_n$ for all n. Let $\{H_p\}$ be a Sylow generating basis of H. Then for each prime p, there is an integer n such that G_n is a p'-group.

By 4.8(i), $H_pG_n/G_n \in Syl_pLKG_n/G_n$ and there is a Sylow p-subgroup $(LK)_p$ of LK such that

$$H_pG_n = (LK)_pG_n.$$

Since G_n is a p'-group, $(LK)_p$ is a Sylow p-subgroup of H_pG_n, by 4.19. Hence $(LK)_p$ and H_p are conjugate in H_pG_n. Since the Sylow p-subgroups of H are conjugate and G is periodic, it follows that

$$(LK)_p \in Syl_pH.$$

This holds for each prime p so LK ∈ Basic H. Thus LK/K ∈ Basic H/K and since $\mathfrak{F}$ is co-Hopfian, LK = H. Thus L is an $\mathfrak{F}$-projector of G. □

6.7. Corollary: Suppose G ∈ $\mathfrak{Z}$ is a radical group. Then G possesses $\mathfrak{F}$-projectors for each saturated formation $\mathfrak{F}$.

This follows from the well known result of Baer [1] that every radical Σ-group is co-Hopfian.

We can also obtain projectors for various non co-Hopfian classes.

6.8. Lemma: Let G ∈ $\mathfrak{Z}$ and σ a set of primes. Suppose S is a complemented σ-subgroup of G. Then S is a $\mathfrak{S}_\sigma$-projector of G.

Proof: Suppose $K \triangleleft H \leq G$, $H/K \in \mathfrak{S}_\sigma$ and $S \leq H$. Put G = ST, with T a σ'-group. Then

$$H = H \cap ST = S(H \cap T).$$

Thus $H/K = (SK/K)\cdot(H \cap T)K/K \in \mathfrak{S}_\sigma$. Since T is a σ'-group it follows that H = SK and S is a $\mathfrak{S}_\sigma$-projector of G. □

The existence of Sylow generating bases now gives

6.9. Theorem: Suppose G ∈ $\mathfrak{Z}$. Then G possesses $\mathfrak{S}_\sigma$-projectors for each set of primes σ.

6.10. Remarks.

(i) In [35], Parker defined the concept of a generalised $\mathfrak{F}$-projector for a cofinite group G, whose topology was defined by a separating filter base $\mathcal{N}$ and he asked whether the generalised $\mathfrak{F}$-projectors were $\mathfrak{F}$-projectors in the usual sense, for the class Σ of locally finite-soluble groups with finite Sylow p-subgroups for all primes p. If $\mathfrak{F}$ is co-Hopfian then this is the case. For suppose L is a generalised $\mathfrak{F}$-projector of $G \in \Sigma$ and that $L \leq H \leq G$. Suppose $K \triangleleft H$ and $H/K \in \mathfrak{F}$. Then for each $N \in \mathcal{N}$, $HN/KN \in \mathfrak{F}$ and LN/N is an $\mathfrak{F}$-projector of G/N. Thus for each $N \in \mathcal{N}$, HN = KLN. It follows by the last part of the argument in 6.6 that H = LK, since $\mathfrak{F}$ is co-Hopfian. Hence L is an $\mathfrak{F}$-projector of G.

Since Parker has shown ([35] p. 81) that countable Σ-groups possess generalised $\mathfrak{F}$-projectors, this gives an alternative proof of 6.6, because every $\mathfrak{X}$-group is abelian-by-Σ.

(ii) It is easily shown using 4.13 that if $G \in \mathfrak{X}$, $\mathcal{N} = \{N_i\}$ is a separating filter base for G and L_i/N_i is a $L\mathfrak{N}$-projector of G/N_i with $L_i \geq L_{i+1}$ then it need not follow that $\bigcap_{i \geq 1} L_i$ is a $L\mathfrak{N}$-projector of G. Indeed, with the notation of 4.13, put $N_i = \operatorname{Dr}_{j \geq i+1} G_j$ and $L_i = \langle \gamma \rangle (B_1 \times B_2 \ldots \times B_i)^{a_1 a_2 \ldots a_i} N_i$ (i = 1, 2, ...), where a_i is a non trivial element of A_i. It is easily seen that $\bigcap_{i \geq 1} L_i$ has no Sylow 2-subgroup, whereas every $L\mathfrak{N}$-projector has a non-trivial Sylow 2-subgroup.

Thus an existence proof similar to the approach of Tomkinson [46] (Theorem 7.6) does not appear to be available.

Having obtained the existence of $\mathfrak{F}$-projectors we now examine what sort of conjugacy properties they have. We have been unable to show that the $\mathfrak{F}$-projectors are locally conjugate as one would perhaps hope. Instead we shall say that two subgroups H, K of a group G are <u>σ-conjugate</u> if

(i) $H \cong K$.

(ii) Every Sylow σ-subgroup of H is conjugate (in G) to a Sylow σ-subgroup of K.

If H and K are σ-conjugate for each finite set of primes σ we shall say H and K are finitely conjugate. We shall show that if $G \in \mathfrak{Z}$ and H, K are $\mathfrak{F}$-projectors of G then H and K are finitely conjugate.

6.11. Lemma: Suppose $G \in \mathfrak{Z}$ and $\mathfrak{F}$ is a saturated $\mathfrak{Z}$-formation. Suppose H, K are $\mathfrak{F}$-projectors of G. If σ is a finite set of primes then the Sylow σ-subgroups of H and K are conjugate in G.

Proof: Let $H_\sigma \in Syl_\sigma H$ and $K_\sigma \in Syl_\sigma K$. Put $N = O_{\sigma'}(G)$. Then G/N is a Černikov group and its $\mathfrak{F}$-projectors are therefore conjugate (for example, see [8] (theorem 5.4)). By 6.2(ii) therefore, there exists $g \in G$ such that

$$H^g N = KN.$$

Now by 4.9 if $H_{\sigma'} \in Syl_{\sigma'} H$, $K_{\sigma'} \in Syl_{\sigma'} K$ then $H = H_\sigma H_{\sigma'}$ and $K = K_\sigma K_{\sigma'}$. Thus

$$KN = H_\sigma{}^g(H_{\sigma'}{}^g N) = K_\sigma(K_{\sigma'} N).$$

Since the expression in brackets is a σ'-group it follows that $H_\sigma{}^g, K_\sigma \in Syl_\sigma KN$. Since σ is a finite set, there exists $h \in KN$ such that

$$H_\sigma{}^{gh} = K_\sigma$$

and this proves the result. □

Our next result can be deduced directly from Massey [31] (Theorem A). However we give a different proof because it gives another use of the coset topology and also a slicker proof of 4.6, which appears as a corollary.

6.12. Theorem: Let G be a periodic group possessing subgroups $E \in \mathfrak{X}$ and $F \in \mathfrak{Y}$. If the Sylow σ-subgroups of E and the Sylow σ-subgroups of F are conjugate in G, for each finite set of primes σ, then F contains a subgroup isomorphic to E.

Proof: Let $\{E_p\}$ be a Sylow generating basis of E and let $\{F_p\}$ be a Sylow basis for F. For each finite set of primes σ let

$$E_\sigma = \langle E_p : p \in \sigma \rangle, \quad F_\sigma = \langle F_p : p \in \sigma \rangle.$$

Then $E_\sigma \in \mathrm{Syl}_\sigma E$ and $F_\sigma \in \mathrm{Syl}_\sigma F$ since σ is a finite set.

Let Λ_σ = {Isomorphisms $\alpha : E_\sigma \to F_\sigma$ induced by elements of G, satisfying $\alpha(E_p) = F_p$ for all $p \in \sigma$}.

Then $\Lambda_\sigma \neq \emptyset$. For, by hypothesis there exists $g \in G$ such that $E_\sigma{}^g = F_\sigma$. Thus $\{E_p{}^g\}_{p\in\sigma}$ is a Sylow basis of F_σ. But by Gol'berg [11] the Sylow bases of the Černikov group F_σ are conjugate in F_σ. Hence there exists $h \in F_\sigma$ so that

$$E_p{}^{gh} = F_p \quad \text{for all} \quad p \in \sigma.$$

Let λ_{gh} denote the inner automorphism of G induced by gh. Then

$$\lambda_{gh}(E_p) = E_p{}^{gh} = F_p$$

and hence $\lambda_{gh}|_{E_\sigma} \in \Lambda_\sigma$.

Suppose τ is a finite set of primes and $\sigma \subseteq \tau$. Define a map $\theta_{\sigma\tau} : \Lambda_\tau \to \Lambda_\sigma$ by:

If $\alpha \in \Lambda_\tau$ then $\theta_{\sigma\tau}(\alpha) = \alpha|_{E_\sigma}$, the restriction of α to E_σ.

Then $\alpha|_{E_\sigma} \in \Lambda_\sigma$, clearly, and it is easily seen that $\{\Lambda_\sigma, \theta_{\sigma\tau} : \sigma \subseteq \tau$ are finite sets of primes$\}$ is an inverse system of sets and mappings. Unfortunately the sets Λ_σ need not be finite so we endow Λ_σ with a suitable topology, to facilitate the use of 1.2.6.

Suppose $\alpha, \beta \in \Lambda_\sigma$. Then there exist $x, y \in G$ so that

$$\alpha = \lambda_x|_{E_\sigma}, \quad \beta = \lambda_y|_{E_\sigma}.$$

Thus $\beta^{-1} \circ \alpha = \lambda_{xy^{-1}}|_{E_\sigma}$ and $\beta^{-1} \circ \alpha$ is a periodic automorphism of the Černikov group E_σ. Moreover, $\beta^{-1}\alpha(E_p) = E_p$ so $xy^{-1} \in \bigcap_{p\in\sigma} N_G(E_p)$. Conversely, if $x \in \bigcap_{p\in\sigma} N_G(E_p)$ and $\alpha \in \Lambda_\sigma$ then $\alpha \circ \lambda_x|_{E_\sigma} \in \Lambda_\sigma$. Let $K_\sigma = (\bigcap_{p\in\sigma} N_G(E_p))/C_G(E_\sigma)$, a Černikov group by [24] (1.F.3). Fix $\alpha \in \Lambda_\sigma$. Then the above remarks imply that there is a bijection

$$\gamma_\sigma : \Lambda_\sigma \to K_\sigma.$$

In fact, if $\beta \in \Lambda_\sigma$ then $\beta^{-1} \circ \alpha = \lambda_x|_{E_\sigma}$ for some $x \in \bigcap_{p\in\sigma} N_G(E_p)$ so define $\gamma_\sigma(\beta) = xC_G(E_\sigma)$. Give K_σ the coset topology and give Λ_σ the topology induced by the coset topology via the map γ_σ^{-1}. Then Λ_σ is compact, since γ_σ is a homeomorphism and K_σ is compact.

We check that if β is another element of Λ_σ then the topology induced on Λ_σ by K_σ, using β as the fixed element, is the same as the one previously obtained. Let τ_1 be the topology induced when α is the fixed element and let τ_2 be the topology induced when β is the fixed element of Λ_σ. Then there exists

$x \in \bigcap_{p\in\sigma} N_G(E_p)$ so that $\alpha = \beta \circ \lambda_x|_{E_\sigma}$. Let $\{\delta_i : i \in I\}$ be a τ_1-closed subset of Λ_σ. Then, for certain elements $x_i \in \bigcap_{p\in\sigma} N_G(E_p)$, $\delta_i = \alpha \circ \lambda_{x_i}|_{E_\sigma}$. Thus $\delta_i = \beta \circ \lambda_{x_i x}|_{E_\sigma}$. Since $\{\delta_i : i \in I\}$ is τ_1-closed, the set $\{x_i C_G(E_\sigma) : i \in I\}$ is closed in K_σ. Hence the set $\{x_i x C_G(E_\sigma) : i \in I\}$ is closed by 2.4(iv). Hence by definition $\{\beta \circ \lambda_{x_i x}|_{E_\sigma} : i \in I\} = \{\delta_i : i \in I\}$ is a τ_2-closed set and $\tau_1 \subseteq \tau_2$. It follows by symmetry that $\tau_1 = \tau_2$ so the topologies induced on Λ_σ are the same.

Now suppose τ is a finite set of primes and that $\sigma \subseteq \tau$. If $\gamma_\sigma : \Lambda_\sigma \to K_\sigma$, $\theta_{\sigma\tau} : \Lambda_\tau \to \Lambda_\sigma$ are as defined above and if $\delta_{\sigma\tau} : K_\tau \to K_\sigma$ is the natural homomorphism then

$$\delta_{\sigma\tau} \circ \gamma_\tau = \gamma_\sigma \circ \theta_{\sigma\tau}.$$

By 2.4(iii), $\delta_{\sigma\tau}$ is closed and continuous as a mapping of Černikov groups with coset topology. Since γ_τ, γ_σ are homeomorphisms, it therefore follows that $\theta_{\sigma\tau}$ is a closed continuous mapping. Hence by 1.2.6(a), $\Lambda = \varprojlim \Lambda_\sigma \neq \emptyset$. Suppose $(\alpha_\sigma) \in \Lambda$ and define $\beta : E \to F$ by:

If $x \in E$ and σ is a finite set of primes such that $x \in E_\sigma$ then $\beta(x) = \alpha_\sigma(x) \in F_\sigma \leq F$.

Then β is well defined, because if $\sigma \subseteq \tau$, a finite set of primes, then $\alpha_\tau|_{E_\sigma} = \alpha_\sigma$ and hence $\alpha_\tau(x) = \alpha_\sigma(x)$. Also, if $x, y \in E$ there is a finite set of primes σ so that $x, y \in E_\sigma$. Hence

$$\beta(xy) = \alpha_\sigma(xy) = \alpha_\sigma(x)\alpha_\sigma(y) = \beta(x)\beta(y).$$

Therefore β is a homomorphism. In a similar manner β is easily seen to be injective and this proves the result. □

6.13. Corollary: Suppose the hypotheses of 6.12 hold and suppose $F \in \mathfrak{X}$. Then E and F are isomorphic.

Proof: In the proof of 6.12 use a Sylow generating basis $\{F_p\}$ of F. Since α_σ is then an isomorphism for each finite set of primes σ it follows that β is a surjection. □

Our result also gives a proof that the basic subgroups of a $\mathfrak{Y}$-group are isomorphic. This result has also appeared in [32] (Theorem 1.1). Also we have an alternative proof of 4.6 since the hypotheses of that theorem imply that, in that case, β is a locally inner automorphism.

One presumes that the proof of 4.6 in [6] can be extended to prove 6.12.

6.14. Corollary: Suppose $G \in \mathfrak{Z}$ and $\mathfrak{F}$ is a saturated $\mathfrak{Z}$-formation. If H, K are $\mathfrak{F}$-projectors of G then H and K are finitely conjugate.

It does not seem clear whether finite conjugacy characterises $\mathfrak{F}$-projectors; that is, if H is an $\mathfrak{F}$-projector and K is finitely conjugate to H, is K also an $\mathfrak{F}$-projector? This seems unlikely because the fact that two subgroups are finitely conjugate tells us very little about the rest of the group.

J. Parker [35] (proposition 5.3.11) has shown that if $\mathfrak{F}$ contains the class of finite nilpotent groups and $G \in \Sigma$ then the set of generalised $\mathfrak{F}$-projectors into which some Sylow generating basis of G reduces is permuted transitively by the group of locally inner automorphisms of G. Thus the problem of showing that the $\mathfrak{F}$-projectors are locally conjugate reduces to showing that we can always extend some Sylow generating basis of the $\mathfrak{F}$-projector to a Sylow generating basis of the group, at least for Σ-groups. We have been unable to resolve this even in the case $\mathfrak{F} = L\mathfrak{N}$.

In the case $\mathfrak{F} = L\mathfrak{N}$, we shall call the $\mathfrak{F}$-projectors the <u>Carter subgroups</u> of $G \in \mathfrak{Z}$. As usual, these are self normalising locally nilpotent subgroups but this does not characterise them, even though locally nilpotent $\mathfrak{Z}$-groups satisfy the normaliser condition (see [8] (lemma 5.8)). For, in the notation of 4.13, whereas $\langle B,\gamma\rangle$ is a basis normaliser and hence a Carter subgroup of G, $N_G(B^\alpha) = B^\alpha$ so since $\langle B,\gamma\rangle$ and B^α are not isomorphic, B^α cannot possibly be a Carter subgroup. (Of course, since $G \in \mathfrak{Z}$, the Sylow p-subgroups of G are hypercentral ([24] 1.E.5) and hence the Carter subgroups are hypercentral.)

We shall say that a subgroup M of $G \in \mathfrak{X}$ is <u>abnormal closed</u> if for all $g \in G$, $g \in \overline{\langle M,M^g\rangle}$, the closure of $\langle M,M^g\rangle$ in the unique co-Černikov topology of G. We shall say M is <u>quasi-abnormal closed</u> if whenever $M \leq K \leq_c G$ then $N_G(K) = K$. This terminology is slightly different to that of Parker [35] (p. 71). We shall say that $M \leq G \in \mathfrak{X}$ is <u>abnormal</u> if $g \in \langle M,M^g\rangle$ for all $g \in G$ and <u>quasi-abnormal</u> if $N_G(K) = K$ whenever $M \leq K \leq G$.

Unfortunately, we have not been able to obtain the usual characterisation of Carter subgroups as being the abnormal locally nilpotent subgroups. Instead we have:

<u>6.15. Lemma</u>: Let $G \in \mathfrak{X}$ and suppose E is a closed locally nilpotent subgroup of G (with its unique co-Černikov topology).

Then the following are equivalent.

(i) E is a Carter subgroup of G.

(ii) E is abnormal closed in G.

(iii) E is quasi-abnormal closed in G.

Proof: (i) ⇒ (ii). If $K \triangleleft_c G$ and G/K is Černikov then EK/K is a Carter subgroup of G/K. Since Carter subgroups of Černikov groups are abnormal ([8] lemma 5.6),

$$gK \in \langle EK/K, E^gK/K \rangle = \langle E,E^g \rangle K/K \text{ for all } g \in G.$$

Hence $g \in \cap\{\langle E,E^g\rangle K : K \triangleleft G, G/K \text{ is Černikov}\}$

$= \overline{\langle E,E^g \rangle}$ by 2.6.

Hence E is abnormal closed.

(ii) ⇒ (iii). Suppose $E \leq F \leq_c G$. If $g \in N_G(F)$ then $E^g \leq F^g = F$. Thus $g \in \overline{\langle E,E^g\rangle} \leq \overline{F} = F$ and $N_G(F) = F$.

(iii) ⇒ (i). If E is quasi-abnormal closed in G then EK/K is quasi-abnormal in G/K for all $K \triangleleft G$ with G/K Černikov. By [8] (lemma 5.6), EK/K is a Carter subgroup of G/K. Since $E \leq_c G$, E is a generalised $L\mathfrak{N}$-projector of G and hence is a Carter subgroup of G by 6.10(i) and its obvious extension to $\mathfrak{X}$-groups. □

The hypothesis that E be closed in 6.15 is not unreasonable since every $\mathfrak{F}$-projector is closed if $\mathfrak{F}$ is a saturated formation. For if E is an $\mathfrak{F}$-projector and $N \triangleleft G \in \mathfrak{X}$ with G/N Černikov then $EN = \overline{E}N$. Since $\mathfrak{F}$ is R-closed it follows that $\overline{E} \in \mathfrak{F}$ and hence $E = \overline{E}$.

CHAPTER 7. $\mathfrak{F}$-ABNORMAL SUBGROUPS

In this chapter we obtain a different characterisation of the $\mathfrak{F}$-projectors obtained in chapter 6. First we prove some results concerning maximal subgroups, which must be well known, although do not seem to be written down. Our other results are analogous to those in the case of finite group theory. We have made no attempt to deal with the concepts of $\mathfrak{F}$-critical subgroups, $\mathfrak{F}$-ascendabnormal subgroups or $\mathfrak{F}$-subabnormal subgroups, as defined in Hartley [15]. The results that follow have essentially been obtained by Parker [35] for the class Σ.

As usual, we shall let $\mathfrak{F}$ denote a saturated formation, defined locally by an integrated preformation function f. A subgroup M of $G \in \mathfrak{X}$ will be called <u>p-maximal</u> if M is maximal in G (which we denote by $M <\cdot G$) and $|G : M| = p^r$ for some prime p and integer r.

<u>7.1. Lemma</u>: If $G \in \mathfrak{X}$ and G is not a radicable abelian group then G possesses maximal subgroups. Moreover if $M <\cdot G$ then either

(i) M is p-maximal for some prime p.

or (ii) $|G : M|$ is infinite and M is a basic subgroup of G.

<u>Proof</u>: Since G is not radicable, it has a proper radicable part G^o by 1.2.4. Also G/G^o has finite Sylow p-subgroups and for some prime p, $|G/G^o : O_{p'}(G/G^o)| \neq 1$, but is finite. Let $(M/G^o)/O_{p'}(G/G^o)$ be a maximal subgroup of $(G/G^o)/O_{p'}(G/G^o)$. Then clearly $M <\cdot G$.

If $M <\cdot G$ put $\pi(G) = \{p_1, p_2, \ldots\}$, $\pi_i = \{p_1, \ldots, p_i\}$ and

$N_i = O_{\pi_i'}(G)$. Then either

(a) For some i, $N_i \leq M$.

(b) $N_i M = G$ for all i.

(a) If $N_i \leq M$ then since G/N_i is Černikov it has a proper radicable part H/N_i say. If $H \leq M$ then M/H is a maximal subgroup of the finite soluble group G/H so (i) follows from well known facts. Otherwise $H/N_i \not\leq M/N_i$ so there is a prime p and a Sylow p-subgroup K/N_i of H/N_i such that $K/N_i \not\leq M/N_i$. Hence, since K/N_i is a union of finite characteristic subgroups there must be one of these, L/N_i, not contained in M/N_i. Thus, since $M \lessdot G$, $M/N_i \cdot L/N_i = G/N_i$. Hence $|G : M| = |L : L \cap M| = p^r$ since L/N_i is a p-group. Thus (i) follows.

(b) If $MN_i = G$ for all i, let $G_p \in \mathrm{Syl}_p G$. For some i, N_i is a p'-group so there exists $M_p \in \mathrm{Syl}_p M$ such that $M_p N_i = G_p N_i$. Thus $M_p \in \mathrm{Syl}_p G$ by 4.19 and it follows that $M \in \mathrm{Basic}\, G$. Then it is clear that $|G : M|$ is infinite. Hence (ii) follows. □

Thus in the co-Hopfian case every maximal subgroup is p-maximal for some prime p. However in the non co-Hopfian case it can happen that a basic subgroup is maximal (Hartley [17]). One wonders whether this characterises the non co-Hopfian case and it would be interesting to know the answer to this.

One of the key results that we shall require is our next proposition. The result is a well known theorem, due to Galois, in the finite case.

<u>7.2. Proposition</u>: Suppose $G \in \mathfrak{X}$ and M is a p-maximal subgroup of G. If $K = \mathrm{core}_G M$, G/K has a unique minimal normal subgroup

H/K which is elementary abelian. Also H/K is complemented by M/K in G/K.

Proof: Since G/K is a finite group which has a maximal subgroup with trivial core, the result follows from the theory of finite groups. For a proof of the result in the finite case the reader should consult [22] (II.3.2). □

We note that H/K in 7.2 is also self centralising. The above result is in strict contrast to the case when M <· G and M is a basic subgroup since if K = $core_G M$, M/K cannot even be supplemented by a minimal normal subgroup of G/K since these have finite order whereas $|G : M|$ is infinite.

A maximal subgroup M of G $\in \mathfrak{X}$ will be called $\mathfrak{F}$-normal in G if

(i) M is p-maximal in G for some prime p.

(ii) $M/Core_G M \in f(p)$.

Otherwise we shall say M is $\mathfrak{F}$-abnormal in G. An arbitrary subgroup H of G will be called $\mathfrak{F}$-abnormal in G if whenever $H \leq M <\cdot L \leq G$ then M is $\mathfrak{F}$-abnormal in L.

This generalises the concept of $\mathfrak{F}$-abnormality in finite groups to $\mathfrak{X}$-groups. We shall show that the concept of $\mathfrak{F}$-normality is independent of the (integrated) preformation function defining $\mathfrak{F}$. This follows in a similar fashion to the case of finite groups since

7.3. Lemma: A subgroup M is an $\mathfrak{F}$-normal p-maximal subgroup of G $\in \mathfrak{X}$ if and only if M complements an $\mathfrak{F}$-central p-chief factor of G.

Proof: ($\Rightarrow$) Let $K = Core_G M$. Then by 7.2 and the remark following it, M/K complements a self centralising minimal normal subgroup H/K. Thus $G/K = H/K \cdot M/K$ and $C_{G/K}(H/K) = H/K$ so $(G/K)/(H/K) \cong G/H \cong M/K \in f(p)$. Thus H/K is $\mathfrak{F}$-central.

The converse follows similarly. □

With the aid of 5.11 we have,

7.4. Lemma: If f_1, f_2 are integrated $(\mathfrak{Z},p)$-preformation functions defining $\mathfrak{F}$ and $M <\cdot G$ then:

$M/Core_G M \in f_1(p)$ if and only if $M/Core_G M \in f_2(p)$.

This shows that the concept of $\mathfrak{F}$-abnormality is independent of f. We shall write $H \rtimes_{\mathfrak{F}} G$ to denote H is $\mathfrak{F}$-abnormal in G, $H \triangleleft_{\mathfrak{F}} G$ to denote H is $\mathfrak{F}$-normal in G and $H \rtimes G$ to denote H is abnormal in G. We remark that the concepts of $L\mathfrak{N}$-abnormality and quasi-abnormality are easily seen to be the same. The following result is trivial.

7.5. Lemma: (i) If $H \rtimes_{\mathfrak{F}} G$ and $H \leq L \leq G$ then $H \rtimes_{\mathfrak{F}} L$ and $L \rtimes_{\mathfrak{F}} G$.

(ii) If $H \rtimes_{\mathfrak{F}} G$ and $N \triangleleft G$ then $HN/N \rtimes_{\mathfrak{F}} G/N$.

We begin our characterisation with the following straightforward result.

7.6. Lemma: If $G \in \mathfrak{X}$ is co-Hopfian, $\{N_i : i \in I\}$ is a separating filter base for G and either

(i) $HN_i = G$ for all $i \in I$

or (ii) $HG^{(j)} = G$ for all $j \geq 1$

then $H = G$.

Proof: The proof of (ii) is similar to the proof of (i) so we simply prove (i). Let $p \in \pi(G)$. By 2.16 and 3.4 there exists $i \in I$ such that $N_i \leq O_{p'}(G)$. Thus $H_p \in Syl_p H$ implies $H_p \in Syl_p G$ by 4.19. Hence $H \in$ Basic G. Since G is co-Hopfian the result follows. □

7.7. Lemma: Suppose $G \in \mathfrak{Z}$ and $E \leq G$. If E is an $\mathfrak{F}$-projector of G then $E \rtimes_{\mathfrak{F}} G$.

Proof: Suppose that the lemma is false. Then there exist subgroups M, L of G such that

$$E \leq M <\cdot L \leq G$$

and M is $\mathfrak{F}$-normal in L. Thus M is p-maximal in L for some prime p and if $K = Core_L M$, M/K complements the unique minimal subgroup H/K of L/K, which is $\mathfrak{F}$-central in L by 7.3. By the remark after 7.2, $L/H \cong A_L(H/K) \in f(p) \leq \mathfrak{F}$. Therefore L/K has a chief series passing through H/K in which every factor is $\mathfrak{F}$-central. By 5.6, $(L/K)/O_{p'p}(L/K) \in f(p)$ and by 5.7, $L/K \in \mathfrak{F}$. Since E is an $\mathfrak{F}$-projector of L, L = KE and hence L = M which is a contradiction. □

We now prove the result that enables us to prove the characterisation sought.

7.8. Theorem: Suppose $G \in \mathfrak{Z}$ and $\mathfrak{F}$ is a saturated formation.

(i) If $\mathfrak{F}$ is co-Hopfian then G has no proper $\mathfrak{F}$-abnormal subgroups if and only if $G \in \mathfrak{F}$.

(ii) If $\mathfrak{F}$ is not co-Hopfian and $G \in \mathfrak{F}$ is not co-Hopfian, all the basic subgroups of G are $\mathfrak{F}$-abnormal.

Proof: (i) ($\Rightarrow$). Let $\pi(G) = \{p_1, p_2, \ldots\}$, $\pi_i = \{p_1, \ldots, p_i\}$ and $N_i = O_{\pi_i'}(G)$. Since G has no proper $\mathfrak{F}$-abnormal subgroups, G/N_i cannot have such subgroups either. Thus by [15] (lemma 3.4), $G/N_i \in \mathfrak{F}$, for each $i \geq 1$. Since $\mathfrak{F}$ is a $\mathfrak{Z}$-formation it follows that $G \in \mathfrak{F}$. (We remark that the co-Hopficity restrictions are not required here.)

($\Leftarrow$) Suppose now that $G \in \mathfrak{F}$ and $H \lneq G$ is a proper $\mathfrak{F}$-abnormal subgroup of G. With the notation above, $HN_i \leq G$ and $HN_i/N_i \rightarrowtail_{\mathfrak{F}} G/N_i \in \mathfrak{F}$ since $\mathfrak{F}$ is Q-closed. By [15] (lemma 3.4), it follows that $G = HN_i$ for each $i \geq 1$. By 7.6(i) it follows that $G = H$. Hence G cannot have proper $\mathfrak{F}$-abnormal subgroups.

(ii) If $G \in \mathfrak{F}$ has a proper basic subgroup H then $H \in \mathfrak{F}$ since $G \cong H$ and H must be $\mathfrak{F}$-abnormal since any subgroup containing H must be a basic subgroup of G and basic subgroups always have infinite index in any group that contains them (Baer [1] Folgerung 3.8). □

We now have

7.9. Theorem: Suppose $G \in \mathfrak{Z}$ and $\mathfrak{F}$ is co-Hopfian. Then the $\mathfrak{F}$-projectors of G are precisely the $\mathfrak{F}$-abnormal $\mathfrak{F}$-subgroups of G.

Proof: Suppose E is an $\mathfrak{F}$-abnormal $\mathfrak{F}$-subgroup of G and suppose $E \leq L \leq G$, $K \triangleleft L$ and $L/K \in \mathfrak{F}$. Then by 7.5, EK/K is $\mathfrak{F}$-abnormal in L/K. Hence by 7.8(i), $EK = L$. Hence E is an $\mathfrak{F}$-projector of G. The result now follows by 7.7. □

CHAPTER 8. SOME CONJUGACY THEOREMS

Having obtained the local conjugacy of the Sylow generating bases of $\mathfrak{X}$-groups, a natural task is to see whether anything can be said if the Sylow generating bases are actually conjugate. For $\mathfrak{X}$-groups this means there are countably many Sylow generating bases so one asks whether the converse holds.

In this chapter we discuss this type of situation quite thoroughly. Our results are not just restricted to Sylow bases however. We shall also see that a countable number of Carter subgroups implies that the Carter subgroups are all conjugate, at least in the PL$\mathfrak{N}$ case and this leads us inevitably to some rather bold conjectures.

Perhaps some of our motivation for the results we obtain comes from the following result of Hartley [16] (lemma 6.2). It is easily extended to the class $\mathfrak{X}$.

8.1. Lemma: Let $G \in \Sigma$ and suppose $\mathrm{Syl}_\pi G$ is countable for all sets of primes π. Then $G \in \mathfrak{U}$.

We first give some preliminary results. We order the primes naturally, put $\pi(G) = \{p_1, p_2, \ldots\}$ and $S_i \in \mathrm{Syl}_{p_i} G$ for a group G.

8.2. Lemma: Suppose $G \in \mathfrak{X}$ and that $\underset{\sim}{S} = \{S_i : i \geq 1\}$ is a Sylow basis of G satisfying:

(*) There exists an integer n such that if $\underset{\sim}{T}$ is a Sylow basis of G containing $\{S_1, \ldots, S_n\}$ then $\underset{\sim}{S} = \underset{\sim}{T}$.

Then every Sylow basis of G satisfies (*).

Proof: Let $\underset{\sim}{U} = \{U_i : i \geq 1\}$ be a fixed Sylow basis of G and $\underset{\sim}{V} = \{V_i : i \geq 1\}$ another Sylow basis of G with $U_i = V_i$ for $i = 1, \ldots, n$. We need to show $\underset{\sim}{U} = \underset{\sim}{V}$. Now $\langle U_1, \ldots, U_n \rangle$ and $\langle S_1, \ldots, S_n \rangle$ are Sylow σ-subgroups of G for some finite set of primes σ so these groups are conjugate Černikov subgroups of G. By the well known result of Gol'berg [11] it follows that

$$U_i^{\,g} = S_i \quad \text{for} \quad i = 1, \ldots, n.$$

By hypothesis, $\underset{\sim}{S} = \underset{\sim}{U}^g = \underset{\sim}{V}^g$ and the result follows. □

Our next lemma shows that if $G \in \mathfrak{X}$ is a group with no Sylow basis satisfying condition (*) in 8.2 then G possesses an uncountable number of Sylow bases.

8.3. Lemma: Suppose $G \in \mathfrak{X}$ and $\underset{\sim}{S} = \{S_i : i \geq 1\}$ is a particular Sylow basis of G. Suppose for each positive integer n there exists a Sylow basis $\underset{\sim}{S}^n$ such that $\{S_1, \ldots, S_n\} \subseteq \underset{\sim}{S}^n$ but $\underset{\sim}{S}^n \neq \underset{\sim}{S}$. Then G possesses at least $2^{\aleph_0}$ Sylow bases.

Proof: We define a "partial" Sylow basis to be a subset of a Sylow basis. For each integer $n \geq 1$ we shall inductively construct an integer i_n and 2^n distinct partial Sylow bases $X_k = \{T_1^{\,k}, \ldots, T_{i_n}^{\,k}\}$ such that

(i) $i_n < i_{n+1}$

(ii) There exists m such that $T_m^{\,k} \neq T_m^{\,\ell}$ for $k \neq \ell$.

(iii) $\{T_1^{\,k}, \ldots, T_{i_{n-1}}^{\,k}\}$ is contained in two distinct X_ℓ.

To begin, the hypotheses imply the existence of a Sylow basis $\underset{\sim}{S}^1$ such that $S_1 \in \underset{\sim}{S}^1$ but $\underset{\sim}{S} \neq \underset{\sim}{S}^1$. Choose i_1 to be the least integer such that $\{S_1, \ldots, S_{i_1-1}\} \subseteq \underset{\sim}{S}^1$, but $S_{i_1} \notin \underset{\sim}{S}^1$. Then $\{S_1, \ldots, S_{i_1}\}$ and $\{S_1, \ldots, S_{i_1-1}, S_{i_1}{}^1\}$ satisfy conditions (i)-(iii) where, if S_{i_1} is the Sylow p_{i_1}-subgroup of $\underset{\sim}{S}$, $S_{i_1}{}^1$ is the Sylow p_{i_1}-subgroup of $\underset{\sim}{S}^1$.

Suppose $i_1, \ldots, i_n$ have been chosen and $X_1, \ldots, X_{2^n}$ are distinct partial Sylow bases so that $X_k = \{T_1{}^k, \ldots, T_{i_n}{}^k\}$ $(k = 1, \ldots, 2^n)$. Then 8.2 implies all Sylow bases of G satisfy the hypotheses of 8.3 so X_k is contained in two distinct Sylow bases of G. Let these be $\underset{\sim}{U}^k = \{U_i{}^k\}$ and $\underset{\sim}{V}^k = \{V_i{}^k\}$. Let $m_k > i_n$ be the least integer such that $V_{m_k}{}^k \neq U_{m_k}{}^k$. Put $i_{n+1} = \max\{m_k : 1 \leq k \leq 2^n\}$, $Y_k = \{U_1{}^k, \ldots, U_{i_{n+1}}{}^k\}$ and $Z_k = \{V_1{}^k, \ldots, V_{i_{n+1}}{}^k\}$, for $k = 1, \ldots, 2^n$. Then $X_k \subseteq Y_k \cap Z_k$ and $Y_k \neq Z_k$ so (iii) is satisfied. Clearly (i) is satisfied and if $Y_k = Y_\ell$, $Z_k = Y_\ell$ or $Z_k = Z_\ell$, for some k, ℓ, we contradict the fact that X_k satisfies (ii). Thus we have now constructed 2^{n+1} distinct partial Sylow bases and the construction proceeds. This allows us to construct $2^{\aleph_0}$ chains of partial Sylow bases, by (iii), the union of any one of which is a Sylow basis. By (ii) these are all distinct and thus we construct $2^{\aleph_0}$ Sylow bases as required. □

This enables us to prove the following result for $\mathfrak{X}$-groups. It is a partial generalisation of a result of Baer [1] (Satz 7.9).

<u>8.4. Theorem</u>: Let $G \in \mathfrak{X}$. Then the following conditions on G are equivalent.

(i) The Sylow bases of G are conjugate.

(ii) G possesses countably many Sylow bases.

(iii) There exists a Sylow basis $\underset{\sim}{S} = \{S_i : i \geq 1\}$ possessing a finite subset $\{S_1, \ldots, S_n\}$ satisfying the following condition:

(*) If $\underset{\sim}{T}$ is a Sylow basis of G containing $\{S_1, \ldots, S_n\}$ then $\underset{\sim}{T} = \underset{\sim}{S}$.

(iv) There exists a Sylow basis $\underset{\sim}{S} = \{S_i : i \geq 1\}$ possessing a finite subset $\{S_1, \ldots, S_n\}$ such that

$$N_G(\underset{\sim}{S}) = \bigcap_{i \geq 1} N_G(S_i) = \bigcap_{i=1}^{n} N_G(S_i).$$

<u>Proof</u>: (i) ⇒ (ii). Since G is countable this is clear.

(ii) ⇒ (iii). If no such Sylow basis exists with a finite subset satisfying property (*) then 8.3 implies G has $2^{\aleph_0}$ Sylow bases contrary to (ii). Hence (iii) follows from (ii).

(iii) ⇒ (iv). By (iii) there exists a Sylow basis $\underset{\sim}{S} = \{S_i : i \geq 1\}$ satisfying (*). Suppose $x \in \bigcap_{i=1}^{n} N_G(S_i)$ and consider the Sylow basis $\underset{\sim}{S}^x$. Then $\{S_1, \ldots, S_n\} \subseteq \underset{\sim}{S}^x$ so by (*) we must have $\underset{\sim}{S}^x = \underset{\sim}{S}$. Hence $x \in \bigcap_{i \geq 1} N_G(S_i)$. Since the reverse inclusion is obvious we must have equality.

(iv) ⇒ (i). Let $\underset{\sim}{T} = \{T_i : i \geq 1\}$ be a Sylow basis of G. We show that $\underset{\sim}{T}$ is conjugate to $\underset{\sim}{S}$. Consider the Černikov group $H = \langle S_1, \ldots, S_n, T_1, \ldots, T_n \rangle$. Since we can extend the Sylow bases $\{S_i\}_{i=1}^n$, $\{T_i\}_{i=1}^n$ to Sylow bases of H and since the Sylow bases of a Černikov group are conjugate (Gol'berg [11]), there exists $h \in H$ such that

$$S_i = T_i^h \quad \text{for} \quad i = 1, \ldots, n.$$

Suppose $j \geq n + 1$. Then by the above argument there exists $g \in \langle S_1, \ldots, S_n, S_j, T_1, \ldots, T_n, T_j \rangle$ such that

$$S_i = T_i^g \quad \text{for} \quad i = 1, \ldots, n, j.$$

Hence $h^{-1}g \in \bigcap_{i=1}^{n} N_G(S_i) = \bigcap_{i \geq 1} N_G(S_i)$. Therefore $S_j^{h^{-1}g} = S_j = T_j^g$ so $S_j = T_j^h$. Since $j \geq n + 1$ was arbitrary it follows that $\underset{\sim}{S} = \underset{\sim}{T}^h$ and $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are conjugate. □

We now investigate further the situation when the Sylow bases of an $\mathfrak{X}$-group are conjugate. The example given in 4.13 shows that in this case $\mathfrak{X}$-groups are very far from being $\mathfrak{U}$-groups in the sense of [8]. However this condition does place quite severe restrictions on the structure of an $\mathfrak{X}$-group as is evident from the following result.

8.5. Theorem: Let $G \in \mathfrak{X}$ and suppose the Sylow bases of G are conjugate. Then $G \in PL\mathfrak{N}$.

Proof: Let $\underset{\sim}{S} = \{S_i\}$ be a Sylow basis of G satisfying 8.4(iv). Let n be the integer obtained in 8.4 and let $A = \langle S_1, \ldots, S_n \rangle$, so that $A \in Syl_\pi G$ for some finite set of primes π. Let $H = O_{\pi'}(G)$, so that G/H is a Černikov group. Then A is a π-group of operators acting on H.

Consider the group $K = HA$. This is a countable locally finite group and the complements to H in K are the Sylow π-subgroups of K all of which are conjugate, since π is finite. Thus by [16] (lemma 4.3), there is a finite subgroup B of A such that $C_H(B) = C_H(A)$.

Suppose L is a finite subgroup of H. Then $M = L^B = \langle \ell^b : \ell \in L, b \in B \rangle$ is a finite soluble group and is, of course, B-invariant. By a theorem of Thompson [45] (corollary),

$$h(M) \leq 5^{|B|} h(C_M(B)),$$

where h(M) denotes the Fitting length of M. However, $\bigcap_{i \geq 1} N_G(S_i)$ is clearly locally nilpotent and it contains $C_H(A)$. Hence $C_M(B)$ is nilpotent and we therefore have

$$h(M) \leq 5^{|B|}.$$

Therefore the Fitting lengths of finite subgroups of H are bounded so $H \in PL\mathfrak{N}$ by [8] (lemma 3.10). Since G/H is soluble it follows that $G \in PL\mathfrak{N}$, as required. □

There are many easy examples to show that the converse of 8.5 is false, as given in [6]. One wonders what further restrictions on an $\mathfrak{X}$-group are necessary for the converse to hold.

The following example gives a method for constructing $\mathfrak{X}$-groups with conjugate Sylow bases, whose Fitting lengths are arbitrary positive integers. I should like to thank Professor Hartley for his permission to include these examples here.

<u>8.6. Example</u>:

Let π be a set of primes and suppose H is a locally finite π-group which has a countable number $\{N_i : i \geq 1\}$ of normal subgroups of finite index such that $\bigcap_{i \geq 1} N_i = 1$. Let p be a prime and suppose $\{q_i : i \geq 1\}$ is an infinite set of primes, none of which is in π, satisfying $q_i \equiv 1 (\text{mod } p)$ for all $i \geq 1$. This is possible by Dirichlet's theorem on the number of primes in an arithmetic progression. Let $\mathbb{F}_{q_i}$ be the field with q_i elements and construct a module V_i, faithful for H/N_i, over the field $\mathbb{F}_{q_i}$ (for example $V_i = \mathbb{F}_{q_i}(H/N_i)$, the group ring of H/N_i over $\mathbb{F}_{q_i}$, will do).

We can view V_i as an H-module and ker(H on V_i) = $\{h \in H : vh = v, \text{ for all } v \in V_i\} = N_i$, by definition of the H-action. Put $V = \oplus V_i$. This is a faithful H-module since

$$\begin{aligned} vh = v \text{ for all } v \in V &\Rightarrow v\cdot h = v \text{ for all } v \in V_i, \text{ for all } i \geq 1 \\ &\Rightarrow v\cdot(hN_i) = v \\ &\Rightarrow h \in N_i \text{ for all } i \geq 1 \\ &\Rightarrow h = 1. \end{aligned}$$

Since $q_i \equiv 1(\text{mod } p)$, $\mathbb{F}_{q_i}$ contains a primitive p^{th}-root of unity, which we shall denote by α_i. If $\langle x \rangle$ is a group of order p, x acts on V_i naturally if we define

$$v\cdot x = \alpha_i v \quad \text{for} \quad v \in V_i.$$

Since x acts like a scalar, it commutes with the action of the elements of H and it follows that $K = H \times \langle x \rangle$ acts naturally on V. Put $G = V] K$.

<u>(1) The complements to V in G are conjugate</u>.

For $C_V(x) = 0$ so $C_G(x) = K$. If L is a complement to V then L must contain a conjugate x^g of x. Also $L \leq C_G(x^g) = K^g$, since L is abelian, so that $L = K^g$ by the Dedekind law.

Let $\sigma = \pi \cup \{p\}$. Then K is a σ-group whereas V is a σ'-group. Each Sylow generating basis $\underset{\sim}{S}$ of G determines a complement to V in G and this complement is conjugate to K.

(2) If the Sylow bases of H are conjugate, the Sylow bases of G are conjugate.

For, let $\underset{\sim}{S}$, $\underset{\sim}{T}$ be Sylow bases of G. Then, for $p \in \sigma'$, the Sylow p-subgroups are unique. If $S_\sigma = \langle S_q \in \underset{\sim}{S} : q \in \sigma \rangle$ and $T_\sigma = \langle T_q \in \underset{\sim}{T} : q \in \sigma \rangle$ then S_σ and T_σ are conjugate to K so

$$S_\sigma{}^g = T_\sigma{}^h = K \quad \text{for some} \quad g, h \in G.$$

Thus $S_q{}^g, T_q{}^h \leq K$ for $q \in \sigma$ so as $\{S_q \in \underset{\sim}{S} : q \in \sigma\}$ and $\{T_q \in \underset{\sim}{T} : q \in \sigma\}$ are Sylow bases for S_σ and T_σ respectively, it follows that the Sylow bases of G are conjugate.

Since $G \in \mathfrak{X}$ the construction can be repeated to obtain $\mathfrak{X}$-groups whose Fitting length is any positive integer. □

We now discuss the imposition, on an $\mathfrak{X}$-group, that there be only countably many Carter subgroups. We begin with what must be a well known result.

8.7. Lemma: Suppose G is an arbitrary group whose finite factors are soluble. Suppose G has only finitely many $L\mathfrak{N}$-projectors. Then these are all conjugate.

Proof: Let L be an $L\mathfrak{N}$-projector of G. Then L is self normalising and $|G : N_G(L)| < \infty$. Hence $|G : L| < \infty$. Therefore there exists $N \triangleleft G$ such that $|G : N| < \infty$ and $N \leq L$. If M is any other $L\mathfrak{N}$-projector of G then MN/N and L/N are Carter subgroups of G/N by 6.2 and hence are conjugate. Hence there exists $g \in G$

such that $M^g \leq M^g N = L$. Thus $M^g = L$ and the result follows. □

The following lemma is a triviality by 6.3.

8.8. Lemma: Let $G \in \mathfrak{X}$ and suppose $N \triangleleft G$. Then every Carter subgroup of G/N has the form LN/N for some Carter subgroup L of G.

In the theory of finite groups, it is well known that if $G \in \mathfrak{N}^2$ then there is a unique Sylow basis reducing into each basis normaliser. Our immediate aim is to extend this result to $\mathfrak{X}$-groups.

8.9. Lemma: Let $G \in \mathfrak{X} \cap (L\mathfrak{N})^2$. Then there exists a unique Sylow basis reducing into each basis normaliser.

Proof: Let $M = \rho(G)$, the Hirsch-Plotkin radical of G, and suppose $\underset{\sim}{S} = \{S_p\}$ and $\underset{\sim}{T} = \{T_p\}$ are Sylow bases reducing into the basis normaliser D of G. Then $S_p \cap D = T_p \cap D \in \mathrm{Syl}_p D$ and $S_p \cap M = T_p \cap M \in \mathrm{Syl}_p M$ by 4.8(ii). Furthermore $G = MD$ by 5.15(iii). Hence $\{(S_p \cap M)(S_p \cap D)\}$ and $\{(T_p \cap M)(T_p \cap D)\}$ are Sylow bases of G by 5.17 and it follows that

$$S_p = (S_p \cap M)(S_p \cap D) = (T_p \cap M)(T_p \cap D) = T_p$$

for all primes p. Hence $\underset{\sim}{S} = \underset{\sim}{T}$ and the result now follows from 5.13(ii). □

We can now obtain, at least for $PL\mathfrak{N}$-groups, the result we wish to establish.

8.10. Theorem: Suppose $G \in \mathfrak{X} \cap PL\mathfrak{N}$. If G possesses countably many Carter subgroups then the Carter subgroups of G are conjugate.

Proof: Suppose the result is false and let G be a counter example of minimal Fitting length, and suppose

$$1 < G_1 < G_2 < \dots < G_n = G \text{ is a series,}$$

so that $\rho(G) = G_1$ and $\rho(G/G_i) = G_{i+1}/G_i$ for $i \geq 1$. Then G/G_1 satisfies the hypotheses of the theorem by 8.8 and has Fitting length n - 1. Thus by choice of G, if E, F are Carter subgroups of G, there exists $g \in G$ such that

$$E^g G_1 = FG_1 = H \quad \text{say.}$$

Now $H \in (L\mathfrak{N})^2$ and H has countably many Carter subgroups since a Carter subgroup of H is a Carter subgroup of G by 6.3. But a Carter subgroup of H is a basis normaliser of H, so H possesses countably many basis normalisers. By 8.9, H possesses countably many Sylow bases so the Sylow bases of H are conjugate by 8.4. Hence the basis normalisers of H are conjugate. Therefore there exists $h \in H$ such that $E^{gh} = F$. Hence E and F are conjugate. Since E and F were arbitrary the choice of G is contradicted. The result follows. □

It seems likely that the following conjecture holds.

8.11. Conjecture: Suppose $G \in \mathfrak{X}$ has countably many Carter subgroups. Then the Carter subgroups of G are conjugate and $G \in PL\mathfrak{N}$.

Since we know of no counter examples and since Stonehewer [43] (theorem D) has proved a somewhat analogous result for locally finite-soluble FC-groups we also wildly conjecture:

<u>8.12. Conjecture</u>: If $G \in \mathfrak{X}$ then the following are equivalent.

(i) The Sylow bases of G are conjugate.

(ii) G has countably many Sylow bases.

(iii) The basis normalisers of G are conjugate.

(iv) G has countably many basis normalisers.

(v) The Carter subgroups of G are conjugate.

(vi) G has countably many Carter subgroups.

REFERENCES

1. R. Baer, "Lokal endlich-auflösbare Gruppen mit endlichen Sylowuntergruppen", J. Reine Angew. Math., 239/240 (1970), 109-144.

2. E. Bolker, "Inverse limits of solvable groups", Proc. Am. Math. Soc., 14(1963), 147-152.

3. R. W. Carter, "Nilpotent self-normalising subgroups of Soluble groups", Math. Z., 75(1961), 136-139.

4. R. W. Carter and T. O. Hawkes, "The $\mathfrak{F}$-Normalisers of a finite soluble group", J. Algebra, 5(1967), 175-202.

5. C. Christodolou, Ph.D. Thesis, (University of Warwick 1979).

6. M. R. Dixon and M. J. Tomkinson, "The local conjugacy of some Sylow bases in a class of locally finite groups", J. London Math. Soc. to appear.

7. L. Fuchs, Abelian Groups (Pergamon Press Ltd., 1960).

8. A. D. Gardiner, B. Hartley and M. J. Tomkinson, "Saturated formations and Sylow structure in locally finite groups", J. Algebra, 17(1971), 177-211.

9. W. Gaschütz, "Zur theorie der endlichen auflösbaren Gruppen", Math. Z., 80(1963), 300-305.

10. W. Gaschütz and U. Lubeseder, "Kennzeichnung gesättigter Formationen", Math. Z., 82(1963), 198-199.

11. P. A. Gol'berg, "Sylow bases of infinite groups", Mat. Sbornik, N.S. 32(1953), 465-476.

12. P. Hall, "A note on soluble groups", J. London Math. Soc., 3(1928), 98-105.

13. P. Hall, "On the Sylow systems of a soluble group", Proc. London Math. Soc. (2), 43(1937), 316-323.

14. P. Hall, "On non-strictly simple groups", Proc. Camb. Phil. Soc., 59(1963), 531-553.

15. B. Hartley, "$\mathfrak{F}$-abnormal subgroups of certain locally finite groups", Proc. London Math. Soc. (3), 23(1971), 128-158.

16. B. Hartley, "Sylow subgroups of locally finite groups", Proc. London Math. Soc. (3), 23(1971), 159-192.

17. B. Hartley, "Some examples of locally finite groups", Arch. Math., 23(1972), 225-231.

18. B. Hartley, "Splitting over the locally nilpotent residual for a class of locally finite groups", Quat. J. Math. Oxford (2), 27(1976), 395-400.

19. B. Hartley, "Profinite and residually finite groups", Rocky Mountain Journal of Mathematics, 7(1977), 193-217.

20. P. Higgins, An Introduction to Topological Groups (L.M.S. Lecture Note Series, 15. C.U.P. 1974).

21. G. Higman, "Complementation of abelian normal subgroups", Publ. Math. Debrecen, 4(1956), 455-458.

22. B. Huppert, Endliche Gruppen I, (Die Grundlehren der Mathematischen Wissenschaften Band 134. Springer-Verlag, Berlin, Heidelberg, New York 1967).

23. M. I. Kargapolov, "Locally finite groups having normal systems with finite factors", Sibirsk. Mat. Z., 2(1961), 853-873.

24. O. H. Kegel and B. A. F. Wehrfritz, Locally Finite Groups (North Holland Publishing Co. 1973).

25. A. A. Klimowicz, "Formation theory in locally finite groups", Proc. London Math. Soc. (3), 30(1975), 257-286.

26. A. A. Klimowicz, "Sylow structure and basis normalisers in a class of locally finite groups", J. London Math. Soc. (2), 13(1976), 69-79.

27. A. A. Klimowicz, "Sufficient conditions for the existence of $\mathfrak{F}$-projectors", J. London Math. Soc. (2), 13(1976), 424-426.

28. A. A. Klimowicz, "Criteria for a satisfactory formation theory", Arch. Math., 29(1977), 116-117.

29. A. G. Kuroš, Theory of Groups, Vol. 2 (translation K. A. Hirsch, Chelsea Publishing Co., New York, 1956).

30. N. Massey, Locally Finite Groups with Minimality Conditions, M. Phil. Thesis (University of London, 1974).

31. N. Massey, "Locally finite groups with min-p for all p. I", J. London Math. Soc. (2), 12(1975), 7-14.

32. N. Massey, "Locally finite groups with min-p for all p. II", J. London Math. Soc. (2), 12(1975), 15-23.

33. D. H. McLain, "On locally nilpotent groups", Proc. Camb. Phil. Soc., 52(1956), 5-11.

34. B. H. Neumann, "Groups covered by permutable subsets", J. London Math. Soc., 29(1954), 236-248.

35. J. Parker, A Topological Approach to a Class of Residually Finite Groups, Ph.D. Thesis (University of Warwick 1973).

36. I. I. Pavlyuk, A. A. Šafiro and V. P. Šunkov, "On locally finite groups with the condition of primary minimality for subgroups", Alg. i logika, 15(1974), 324-336.

37. D. J. S. Robinson, Finiteness Conditions and Generalised Soluble Groups. Part I. (Ergebnisse der Mathematik und ihrer Grenzgebiete Band 62. Springer-Verlag, Berlin, Heidelberg, New York 1972).

38. G. F. Simmons, Introduction to Topology and Modern Analysis (McGraw-Hill Book Co. Inc. 1963).

39. I. Stewart, "Conjugacy theorems for a class of locally finite Lie algebras", Comp. Math., 30(1975), 181-210.

40. S. E. Stonehewer, "Abnormal subgroups of a class of periodic locally soluble groups", Proc. London Math. Soc. (3), 14(1964), 520-536.

41. S. E. Stonehewer, "Locally soluble FC-groups", Arch. Math., 16(1965), 158-177.

42. S. E. Stonehewer, "Formations and a class of locally soluble groups", Proc. Camb. Phil. Soc., 62(1966), 613-635.

43. S. E. Stonehewer, "Some finiteness conditions in locally soluble groups", J. London Math. Soc., 43(1968), 689-694.

44. V. P. Šunkov, "On the conjugacy of the Sylow p-subgroups in SF-groups", Alg. i logika, 10(1971), 587-598.

45. J. G. Thompson, "Automorphisms of soluble groups", J. Algebra, 1(1964), 259-267.

46. M. J. Tomkinson, "Formations of locally soluble FC-Groups", Proc. London Math. Soc. (3), 19(1969), 675-708.

47. B. A. F. Wehrfritz, "Sylow theorems for periodic linear groups", Proc. London Math. Soc. (3), 18(1968), 125-140.

48. B. A. F. Wehrfritz, "Sylow subgroups of locally finite groups with min-p", J. London Math. Soc. (2), 1(1969), 421-427.

49. B. A. F. Wehrfritz, "On locally finite groups with min-p", J. London Math. Soc. (2), 3(1971), 121-128.

APPENDIX

Since this thesis was completed I have proved the following generalisation of 8.10:

Theorem: Suppose $G \in \mathfrak{X}$. Then G possesses countably many Carter subgroups if and only if the Carter subgroups of G are conjugate.

The method of proof is somewhat similar to the existence proof of the Carter subgroups. Full details of the proof will appear in a future paper.

Note that, in the example on page 59, B^{α} is a $\mathfrak{G}_{\sigma}$-projector, for $\sigma = \{q_1, q_2, \ldots\}$. Moreover, B^{α} is not locally conjugate to B, otherwise there would be a Sylow basis of G reducing into B^{α}.

In the example on page 109, let $\{q_i : i \geq 1\}$ be an infinite set of primes such that $q_i \equiv 1 \pmod{p}$, and choose H to be a locally finite π-group, where π is infinite, $p \notin \pi$ and $q_i \notin \pi$ for each i.