PROFINITE LIE ALGEBRAS

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Introduction

It is well-known that the theory of finite dimensional Lie algebras is similar in many ways to the theory of finite groups. Noting the way that finite groups are generalised to certain periodic FC-groups (see Baer, [2], for example), we can likewise generalise finite dimensional Lie algebras; alternatively we can study ideally finite Lie algebras by using ideas from the theory of periodic FC-groups, drawing analogies as far as possible (see Stewart, [15], for example). We can also extend the theory of finite groups to profinite groups (see for example Hartley, [6]) using the ideas and methods used for finite groups and periodic FC-groups. In this thesis we study profinite Lie algebras from two viewpoints: by using the ideas and methods of ideally finite Lie algebras, and by analogy with profinite groups.

Some of the ideas can be traced to the corresponding situation for algebraic groups, that is pro-affine algebraic groups (see Hochschild, [7], and Hochschild and Mostow, [8]), for example the definition of coset topology in Hochschild and Mostow, p1130, suggests the definition of the affine topology.

In the first four chapters we use a topological approach similar to that used in Hartley, [6], and Parker, [13] to study cofinite and profinite classes of algebras, and from chapter 5 onwards we concentrate our attention on profinite Lie algebras. Also, when no extra effort is involved we prove results more generally for cofinite Lie algebras, or for cofinite algebras.

Chapters 1 and 2 deal with the definitions of cofinite and profinite algebras, along with some basic results. The definition of a cofinite algebra is chosen so that certain properties of cofinite groups also hold for cofinite algebras (see for example Hartley, [6], and Parker, [13]), in particular that a compact cofinite algebra may be regarded as an inverse limit of finite dimensional algebras when given a suitable topology.

In chapter 3 we look at the profinite completion of a cofinite algebra, showing that it is unique up to a topological isomorphism (i.e. a map that is both a homeomorphism and an algebra isomorphism), and that continuous algebra homomorphisms of a cofinite algebra can be extended to continuous algebra homomorphisms of the profinite completion.

In chapter 4 we give some examples of profinite algebras, and indicate how further profinite algebras may be obtained; also we give an alternative construction of the profinite completion of a vector space, using the dual space and the double dual. This form of the profinite completion is different to that given in chapter 3.

The first four chapters apply to any class of algebras.

In chapter 5 we look at the cofinitely soluble radical and the cofinitely nilpotent radical of a cofinite Lie algebra; these correspond to the soluble and nilpotent radicals in finite dimensional Lie algebras, and are the maximal cofinitely soluble and cofinitely nilpotent ideals.

Chapter 6 deals with pro-semisimple Lie algebras. These may be expressed either as inverse limits of finite dimensional semisimple Lie algebras, or more usefully as a Cartesian sum of finite dimensional simple Lie algebras. As a consequence of the latter result we consider sums of finite dimensional simple Lie algebras.

We generalise the definitions of Levi, Borel and Cartan subalgebras in chapters 7,8 and 9 respectively from finite dimensions to profinite and cofinite Lie algebras, proving existence in the profinite case and giving equivalent definitions: a Levi subalgebra Λ of a profinite Lie algebra L is a closed subalgebra whose factors $(\Lambda + K)/K$ by closed ideals K of finite codimension are Levi subalgebras of L/K, or alternatively a Levi subalgebra of a profinite Lie algebra is a closed maximal semisimple subalgebra; B is a Borel subalgebra of the profinite Lie algebra L if it is closed and $(B + K)/K$ is a Borel subalgebra of L/K for each closed ideal K of L of finite codimension; C is a Cartan subalgebra if it is closed and each $(C + K)/K$ is a Cartan subalgebra of L/K. Also, in

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In chapter 10 we show that each of these three classes of subalgebras is conjugate under a certain group $\mathcal{E}(L)$ of automorphisms of the profinite Lie algebra L. This group may be regarded as an inverse limit of such groups of automorphisms of finite dimensional Lie algebras.

In chapter 11 we look at the Fitting decomposition of a profinite L-module: a profinite L-module is isomorphic, as an L-module, to the Cartesian sum of its weight spaces, when L satisfies certain conditions relating to nilpotency. We also particularise to a cofinite Lie algebra L considered as a C-module for a Cartan subalgebra C of L, and show that in this case if L is also pro-semisimple then its weight spaces are one-dimensional.

In chapter 12 we look at cofinitely cleft Lie algebras. We define a cofinite version of the Chevalley-Jordan decomposition of a linear map on a cofinite algebra fixing the closed ideals of finite codimension, and show that this is unique. We define tori in a cofinite Lie algebra and use them to find an equivalent form of Cartan subalgebras of a profinitely cleft Lie algebra i.e. they are precisely the an infinite dimensional profinite Lie algebra L the Borel subalgebras have the same dimension as L; the same result does not hold for the Cartan subalgebras of L, although it is true when L is pro-semisimple.

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In chapter 13 we show that any profinite Lie algebra may be embedded in a profinitely cleft Lie algebra.

Many of the results of this thesis are easy generalisations of analogous results in finite dimensions, and are often proved by passing to finite dimensional factors.

I would like to express my deep gratitude to both of my supervisors, Ian Stewart and, during his absences, Brian Hartley, for their help and guidance, as well as their unlimited patience. Without them this thesis would not have reached its final form.

Chapter 1 Cofinite Algebras

Section 1 Some Preliminary Definitions

In a residually finite algebra there may be many different finite residual systems. By defining a suitable topology on the algebra we may focus our attention on a particular finite residual system, and use the topological structure invoked to study the algebra.

In this chapter we define this topology as the cofinite topology and derive some basic results. The definition corresponds to the definition of a cofinite group given in Hartley [6], p194, and is defined so that compact cofinite algebras will correspond to inverse limits of finite dimensional algebras, as in the case of profinite groups (see for example Hartley, p196).

Throughout this thesis, by an algebra we will mean any algebra (not necessarily a Lie algebra or an associative algebra) unless otherwise indicated, and the ground field, denoted by F, will be algebraically closed of characteristic zero.

Recall (see for example Kelley, pp65,66):

Definition

A partially ordered set I is directed if for $i, j \in I$ there exists an element k of I such that $i \leq k$ and $j \leq k$.

A subset J of I is cofinal in I if for each $i \in I$ there exists $j \in J$ such that $i \leq j$.

Definition

(See for example Stewart [15], p37). An algebra A is residually finite if it has a system of ideals $\{K_i : i \in I\}$ such that:

(i) A/K_i is finite dimensional for each $i \in I$,

(ii) $\bigcap_{i \in I} K_i = 0$.

We say that $\{K_i : i \in I\}$ is a finite residual system for A if it satisfies (i) and (ii) above, together with

(iii) For each $i,j \in I$ there exists $k \in I$ such that $K_k \leq K_i \cap K_j$.

We can partially order I by taking $i \leq j$ whenever $K_j \leq K_i$, thus making I into a directed set. Henceforth we shall assume that I is thus directed whenever we consider a finite residual system, and we shall say that $\{K_i : i \in I\}$ is directed.

We shall also call a finite residual system of ideals $\{K_i : i \in J\}$ cofinal in a finite residual system $\{K_i : i \in I\}$ if J is cofinal in I.

The following definition corresponds to the coset topology defined for groups in Hochschild and Mostow [8], p1130.

Definition

Let V be a vector space, with a topology defined on it. Then the topology on V is affine if $\{x + U : x \in V, U \text{ is a vector subspace of } V\}$ is a subbase of closed sets.

We can also talk of the affine topology on an algebra A; this is simply the affine topology on A when A is regarded as a vector space.

Section 2 Cofinite Algebras

Definition 1.1

An algebra A (not necessarily associative) is a cofinite algebra if it has a topology such that

C1 The intersection of all closed ideals of A of finite codimension is trivial.

C2 If H is a closed vector subspace of A of finite codimension then H contains a closed ideal of finite codimension.

C3 If H, K are vector subspaces of A of finite codimension such that H is closed and such that H is contained in K, then K is closed.

C4 The set $\{x + K : x \in A, K \text{ is a closed vector subspace of } A \text{ of finite codimension}\}$ is a subbase of closed sets.

Condition C1 ensures that A is residually finite, while C2 and C4 imply that the topology is determined by a particular finite residual system of A (i.e. the set of closed ideals of finite codimension).

Condition C3 allows us to induce the affine topology on the finite dimensional quotients of A when factoring by closed ideals of finite codimension (see 1.13 for the definition of this induced topology).

Notation

Let A be a cofinite algebra. Then $\mathcal{K}(A)$ (or $\mathcal{K}$ when there is no ambiguity) will denote the set of closed ideals of A of finite codimension, and $\mathcal{J}(A)$ (or $\mathcal{J}$) will denote the set of closed vector subspaces of A of finite codimension.

Remark

If A is a finite dimensional cofinite algebra, then the topology on A is the affine topology, and conversely if A is a finite dimensional algebra with the affine topology, then A is a cofinite algebra.

Proposition 1.2

Let A be a cofinite algebra, x a fixed element of A, λ a fixed element of the ground field F. Then the following maps on A are continuous:

$$\theta_1 : y \mapsto x + y$$

$$\theta_2 : y \mapsto \lambda y$$

$$\theta_3 : y \mapsto xy$$

Proof: Let z + K be a closed subbasic subset of A, so that $K \in \mathfrak{J}(A)$, the set of closed vector subspaces of A of finite codimension. Then it suffices to show that $\theta_i^{-1}(z + K)$ is closed for i = 1,2,3, by Simmons, theorem E, p102, [14].

But $\theta_1^{-1}(z + K) = (z - x) + K$, which is a closed subset of A, since $K \in \mathfrak{J}(A)$, and $\theta_2^{-1}(z + K) = \lambda^{-1}z + K$, also a closed subset of A. So θ_1 and θ_2 are continuous.

Now $K \in \mathfrak{J}(A)$, so K contains some $H \in \mathcal{K}(A)$, by C2. But H is an ideal of A, so $xH \subseteq H \subseteq K$. Thus $H \subseteq \theta_3^{-1}(K)$, whence $\theta_3^{-1}(K)$ is closed, by C3. Hence $\theta_3^{-1}(z + K) = u + \theta_3^{-1}(K)$, where u is any element of $\theta_3^{-1}(z)$. □

Proposition 1.3

Let A be a residually finite algebra with a finite residual system $\mathcal{C}$. Then there is a unique way to make A into a cofinite algebra such that $\mathcal{C}$ is cofinal in $\mathcal{J}$, the set of closed vector subspaces of A of finite codimension.

Proof: To make A into a cofinite algebra, take as a closed subbase the set $\{x + U: x \in A, K \subseteq U \text{ for some } K \in \mathcal{C}\}$. That this is the unique way of making A into a cofinite algebra such that $\mathcal{C}$ is cofinal in $\mathcal{J}$ is clear. □

We can see from this that two finite residual systems of a residually finite algebra will yield the same cofinite topology if they are both cofinal in their union.

The next lemma shows that if a vector subspace of a vector space V is contained in a finite union of affine subspaces of V, then it is contained in one of these affine subspaces.

Lemma 1.4

Let V be a vector space, U a vector subspace and suppose that $U \subseteq \bigcup_{i=1}^{n} (x_i + V_i)$, where $x_i \in V$ and each V_i is a vector subspace of V.

Then there exists $t \in \{1,2,\ldots,n\}$ such that $U \leq V_t$ and $x_t \in V_t$.

Further if $U = \bigcup_{i=1}^{n} (x_i + V_i)$, then $U = V_t$ for some t.

Proof: We can see that $U = \bigcup_{i=1}^{n} (U \cap (x_i + V_i))$

$$= \bigcup_{i=1}^{m} (y_i + W_i)$$

for some $m \leq n$, where $W_i = U \cap V_i$ and each y_i is such that $y_i + U = U$ and $y_i + V_i = x_i + V_i$. Note that each W_i is contained in U.

We shall prove by induction on the number k of distinct subspaces W_i involved in this expression that $U = W_t$ for some t.

In the case where $k = 1$, $U = \bigcup_{i=1}^{m} (y_i + W)$. If $U \neq W$, then $U/W = \{y_1 + W, \ldots\ldots, y_m + W\}$. Thus U/W is a finite vector space, which is impossible, as the ground field F has characteristic zero. So we must have $U = W$.

Now suppose that the result holds for the case $k = r - 1$. We shall show that the result holds for the case $k = r$. We can re-order the W_i's so that

$$U = (y_1 + W_1) \cup \ldots\ldots (y_j + W_1) \cup \bigcup_{i=j+1}^{m} (y_i + W_i)$$

where each W_i is distinct from W_1 when $i > j$.

If $W_1 = U$ the result follows, so assume that $W_1 \neq U$; then there exists $z \in U$ such that $z + W_1 \neq y_i + W_1$ for every i in I_j, where I_j denotes the set $\{1,2,\ldots,j\}$.

$$\begin{aligned} \text{Now } z + W_1 &= (z + W_1) \cap U \\ &= \bigcup_{i=1}^{m} ((z + W_1) \cap (y_i + W_i)) \\ &= \bigcup_{i=j+1}^{m} ((z + W_1) \cap (y_i + W_i)) \\ &\subseteq \bigcup_{i=j+1}^{m} (y_i + W_i) \end{aligned}$$

So for each $s \in I_j$, $y_s + W_1 = z + (y_s - z) + W_1$

$$\subseteq (y_s - z) + \bigcup_{i=j+1}^{m} (y_i + W_i)$$

So $y_s + W_1 = \bigcup_{i=j+1}^{m} (y_{is} + W_i)$, where $y_{is} = y_s + y_i - z$.

Thus $U = \bigcup_{s=1}^{j} \bigcup_{i=j+1}^{m} (y_{is} + W_i) \cup \bigcup_{i=j+1}^{m} (y_i + W_i)$

By the induction hypothesis $U = W_t$ for some $t \in I_m$. But $W_t = U \cap V_t$, whence $U = V_t$, as required.

The second part of the lemma follows immediately from the first part. $\square$

Recall that a topological space X is a T_1-space if each singleton set is closed (see for example Kelley, [11], p56).

Proposition 1.5

Suppose that A is a cofinite algebra. Then

(i) A is a T_1-space,

(ii) A is connected.

Proof: (i) The singleton set $\{0\}$ is closed, by C1 of definition 1.1. Also, for any $x \in A$, x is the inverse image of 0 under the map $y \to y - x$. So by proposition 1.2, the singleton set $\{x\}$ is closed.

(ii) Suppose, for a contradiction, that $A = U \cup V$ is a disconnection of A, where U and V are disjoint. Then U and V are both closed.

Therefore $U = \bigcap_{j \in J} \bigcup_{r=1}^{n_r} (x_{jr} + K_{jr})$ for some index set J and where $x_{jr} \in A$, $K_{jr} \in \mathfrak{J}(A)$ for all j,r.

Then $U \subseteq \bigcup_{r=1}^{n_r} (x_{jr} + K_{jr}) \neq A$ for a suitable choice of $j \in J$.

So $y_s + W_1 = \bigcup_{i=j+1}^{m} (y_{is} + W_i)$, where $y_{is} = y_s + y_i - z$.

Thus $U = \bigcup_{s=1}^{j} \bigcup_{i=j+1}^{m} (y_{is} + W_i) \cup \bigcup_{i=j+1}^{m} (y_i + W_i)$

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Then $U \subseteq \bigcup_{r=1}^{n_r} (x_{jr} + K_{jr}) \neq A$ for a suitable choice of $j \in J$.

Similarly we can take V to be a subset of $\bigcup_{s=1}^{m} (y_s + K_s)$, which does not equal A.

Thus $U \cup V \subseteq \bigcup_{r=1}^{n_j} (x_{jr} + K_{jr}) \cup \bigcup_{s=1}^{m} (y_s + K_s)$

So by lemma 1.4 $U \cup V \neq A$. This gives the required contradiction, so A is connected. □

The proof of proposition 1.5(ii) also shows that a cofinite algebra A cannot be Hausdorff, for if U and V are open subsets of A, then U' and V', their complements, are closed, and as in the above proof $U' \cup V' \neq A$, so that $U \cap V$ is non-empty; hence we cannot separate points by disjoint open sets.

The following result, which is the analogue of Parker, [13], theorem 2.1.1, p8, will be used extensively in later chapters. A bar over a set will denote closure.

<u>Proposition 1.6</u>

Let A be a cofinite algebra and let U be a vector subspace of A. Suppose that $\mathcal{J}'$ is a cofinal subset of $\mathcal{J} = \mathcal{J}(A)$. Then

(i) $\overline{U} = \bigcap_{K \in \mathcal{J}} (U + K) = \bigcap_{K \in \mathcal{J}'} (U + K)$

(In particular $\overline{U} = \bigcap_{K \in \mathcal{K}} (U + K)$)

(ii) $\overline{x + U} = x + \overline{U} = \bigcap_{K \in \mathcal{J}} (x + (U + K)) = \bigcap_{K \in \mathcal{K}} (x + (U + K))$

<u>Proof:</u> (i) (a) U + K is closed for each $K \in \mathcal{J}$, by C3, so that $\bigcap_{K \in \mathcal{J}} (U + K)$ is closed and contains U, so must contain $\overline{U}$.

Therefore

$$\overline{U} \subseteq \bigcap_{K \in \mathcal{J}} (U + K)$$

(b) Conversely, since $\overline{U}$ is closed, $\overline{U} = \bigcap_{j \in J} \bigcup_{r=1}^{n_j} (x_{rj} + K_{rj})$

for some J, n_j, with $K_{rj} \in \mathfrak{J}$ for all r,j. We want to show that $\overline{U}$ is the intersection of some of these K_{rj}'s.

Fix $j \in J$. Then $\overline{U} \subseteq \bigcup_{r=1}^{n_j} (x_{rj} + K_{rj})$

So $U = U \cap \overline{U} = \bigcup_{r=1}^{n_j} (U \cap (x_{rj} + K_{rj}))$

If $U \cap (x_{rj} + K_{rj})$ is empty, discard it from this union. There is some r for which $U \cap (x_{rj} + K_{rj})$ is non-empty, otherwise U would be empty.

Renumber the r's so that $U = \bigcup_{r=1}^{m} (U \cap (x_{rj} + K_{rj}))$ with each term in the union being non-empty. Then for each r there exists $x_r \in U \cap (x_{rj} + K_{rj})$ such that $x_r + U = U$ and $x_r + K_{rj} = x_{rj} + K_{rj}$

Hence $U = \bigcup_{r=1}^{m} (x_r + (U \cap K_{rj}))$, whence by lemma 1.4, $U = U \cap K_{rj}$ for some $r = r(j)$, so $U \subseteq K_{rj}$. Also x_r will be an element of $U \cap K_{rj}$, which is contained in K_{rj}. Therefore $x_{rj} \in K_{rj}$ for $r = r(j)$.

Now K_{rj} is in $\mathfrak{J}$, so is closed; hence $\overline{U} \subseteq K_{rj}$.

Thus $\overline{U} \subseteq \bigcap_{j \in J} K_{r(j)j}$

$= \bigcap_{j \in J} (x_{rj} + K_{rj})$

$\subseteq \bigcap_{j \in J} (\bigcup_{r=1}^{n_j} (x_{rj} + K_{rj}))$

$= \overline{U}$

Therefore $\overline{U} = \bigcap_{j \in J} K_{r(j)j}$ where each K_{rj} is an element of $\mathfrak{J}$

and each K_{rj} contains U.

Therefore $\overline{U} \supseteq \bigcap_{\substack{L \in \mathcal{J} \\ L \supseteq U}} L = \bigcap_{K \in \mathcal{J}} (U + K)$, so $\overline{U} = \bigcap_{K \in \mathcal{J}} (U + K)$, since U + K is closed.

(c) We want to show that $\bigcap_{K \in \mathcal{J}} (U + K) = \bigcap_{K \in \mathcal{J}'} (U + K)$ for any subset $\mathcal{J}'$ of $\mathcal{J}$ which is cofinal in $\mathcal{J}$.

That the left hand side is contained in the right hand side is clear.

Conversely, for $K \in \mathcal{J}$ there exists $H_K \in \mathcal{J}'$ such that $H_K \leq K$, since $\mathcal{J}'$ is cofinal in $\mathcal{J}$; so $U + H_K \leq U + K$.

Thus

$$\bigcap_{K \in \mathcal{J}} (U + K) \supseteq \bigcap_{K \in \mathcal{J}} (U + H_K)$$

$$\supseteq \bigcap_{K \in \mathcal{J}'} (U + K)$$

(ii) Let θ: A $\rightarrow$ A be the translation map $y \mapsto x + y$ for a fixed $x \in A$. Then θ is a homeomorphism, by proposition 1.2, and so $\overline{\theta(U)} = \theta(\overline{U})$ i.e. $\overline{x + U} = x + \overline{U}$

$$= x + \bigcap_{K \in \mathcal{J}} (U + K)$$

$$= \bigcap_{K \in \mathcal{J}} (x + (U + K)) \quad \square$$

<u>Corollary 1.7</u>

Let A be a cofinite algebra and let U be a vector subspace of A. Then U is closed if and only if $U = \bigcap_{K \in \mathcal{J}'} K$ for some subset $\mathcal{J}'$ of $\mathcal{J}$.

<u>Proof:</u> If U is an intersection of elements in $\mathcal{J}$ then it is clearly closed.

Conversely, if U is closed then $U = \overline{U} = \bigcap_{K \in \mathfrak{I}} (U + K)$, and since each U + K is in $\mathfrak{I}$, by C3, the result follows from proposition 1.6. □

When A is compact we can get a stronger result than proposition 1.6, analogous to Parker, [13], proposition 2.1.2, p9:

Proposition 1.8

Let A be a compact cofinite algebra, and let H be a closed vector subspace of A. Let $\{K_i : i \in I\}$ be a directed set of closed vector subspaces (i.e. for r,s ∈ I there exists t ∈ I such that $K_t \leq K_r \cap K_s$). Then

$$\bigcap_{i \in I} (H + K_i) = H + \bigcap_{i \in I} K_i$$

Proof: We follow the proof given in Parker. That the right hand side is contained in the left hand side is clear, so let x be an element of $\bigcap_{i \in I} (H + K_i)$ (which is clearly non-empty). Then for each i ∈ I, $(x + H) \cap K_i$ is non-empty. Also $\{(x + H) \cap K_i : i \in I\}$ satisfies the finite intersection property, since $\{K_i : i \in I\}$ is directed; so as each $(x + H) \cap K_i$ is closed and A is compact we see that $\bigcap_{i \in I} ((x + H) \cap K_i)$ is non-empty. This implies that $(x + H) \cap \bigcap_{i \in I} K_i$ is non-empty, so there exists $y \in \bigcap_{i \in I} K_i$ such that $y \in x + H$, whence $x \in H + \bigcap_{i \in I} K_i$. □

Theorem 1.9

Let A be a compact cofinite algebra, and let U be a vector subspace of A. Then U is compact in the relative topology if and only if U is closed.

Proof: If U is closed, then it is compact since it is a closed subset of a compact space.

Conversely, suppose that U is compact; we show that $U = \overline{U}$, whence U is closed.

$\overline{U} = \bigcap_{K \in \mathfrak{J}} (U + K)$, by proposition 1.6(i), so $\overline{U} \subseteq U + K$ for each $K \in \mathfrak{J}$. Pick any $x \in \overline{U}$. We shall show that $x \in U$. For each $K \in \mathfrak{J}$ there exists $h \in K$ and $u \in U$ such that $x = u + h$. Then $u = x - h$, so $u \in U \cap (x + K)$, showing that for each $K \in \mathfrak{J}$, $(x + K) \cap U$ is non-empty. Now for any $K_1, K_2, \ldots\ldots, K_n$ in $\mathfrak{J}$ there exists K in $\mathfrak{J}$ such that $K \leqslant K_1 \cap K_2 \cap \ldots\ldots \cap K_n$ so $x + K \subseteq \bigcap_{i=1}^{n} (x + K_i)$.

Then

$$\emptyset \neq (x + K) \cap U \subseteq ((x + K_1) \cap U) \cap \ldots\ldots \cap ((x + K_n) \cap U)$$

Thus $\{(x + K) \cap U : K \in \mathfrak{J}\}$ is a set of closed subsets of U with the finite intersection property, so by the compactness of U,

$\bigcap_{K \in \mathfrak{J}} (x + K) \cap U \neq \emptyset$

$$\text{I.e. } U \cap \bigcap_{K \in \mathfrak{J}} (x + K) \neq \emptyset$$

But $\bigcap_{K \in \mathfrak{J}} (x + K) = \{x\}$ by proposition 1.6(ii), since $\bigcap_{K \in \mathfrak{J}} K = 0$, so $U \cap \{x\}$ is non-empty, as required. $\square$

Corollary 1.10

Let A be a cofinite algebra, U a vector subspace of A. If U is compact in the relative topology then U is closed in A..

<u>Proof:</u> This is the same as the second part of the proof of theorem 1.9. □

<u>Corollary 1.11</u>

Let A and U be cofinite algebras and let f : A → U be a continuous and linear map and suppose that A is compact. Then f maps closed vector subspaces to closed vector subspaces.

<u>Proof:</u> Since f is linear it maps vector subspaces to vector subspaces. Also, since f is continuous it maps compact subsets to compact subsets. The result then follows from corollary 1.10. □

Section 3 Subspaces and Quotients of Cofinite Algebras

Proposition 1.12

Let A be a cofinite algebra and let U be a subalgebra of A. Then U is cofinite in the relative topology.

Proof: The set $\{x + K: K \in \mathfrak{J}\}$ is a closed subbase for A, so $\{U \cap (x + K): K \in \mathfrak{J}\}$ is a closed subbase for U in the relative topology. But $U \cap (x + K)$ is either empty, hence closed, or $U \cap (x + K) = y + (U \cap K)$ for some $y \in U$, and any affine subspace of the form $y + (U \cap K)$ can be obtained in this way, i.e. $y + (U \cap K) = (y + U) \cap (y + K) = U \cap (y + K)$.

So $\{y + (U \cap K): K \in \mathfrak{J}\}$ is a closed subbase for U and if $V \in \mathfrak{J}(U)$ an argument similar to that in proposition 1.6(b) will show that V is an intersection of finitely many $U \cap K$'s for $K \in \mathfrak{J}(A)$, so V has the form $U \cap K$ for some $K \in \mathfrak{J}(A)$. Hence $\mathfrak{J}(U) = \{U \cap K : K \in \mathfrak{J}(A)\}$. Further, if V is an ideal of U, then $V = U \cap K$ and K contains some H from $\mathcal{K}(A)$, so that V contains $U \cap H$. Hence $\{U \cap K: K \in \mathcal{K}(A)\}$ is cofinal in $\mathcal{K}(U)$.

This verifies C4.

C1 is clear.

We now check C2. If H is in $\mathfrak{J}(U)$, then $H = U \cap K$ for some $K \in \mathfrak{J}(A)$, so K contains some K' from $\mathcal{K}(A)$, whence H contains $U \cap K'$, which is in $\mathcal{K}(U)$.

Now we verify C3. Suppose H and V are vector subspaces of U with H contained in V and suppose further that H lies in $\mathfrak{J}(U)$. Then $H = U \cap K$ for some $K \in \mathfrak{J}(A)$, so $V \supseteq U \cap K$, whence by the modular law $V = V + (U \cap K) = U \cap (V + K)$. But $K \in \mathfrak{J}(A)$, so $V + K \in \mathfrak{J}(A)$, whence V is in $\mathfrak{J}(U)$. □

Definition 1.13

Let A be a cofinite algebra and let U be a closed ideal of A. Then we can give A/U the cofinite topology generated by

$$\mathcal{K}(A/U) = \{(K + U)/U: K \in \mathfrak{J}(A)\}$$

I.e. take $\{(x + U) + (K + U)/U: x \in A, K \in \mathcal{J}(A)\}$ as a closed subbase of A/U.

We shall call this the <u>(cofinite) topology induced from A.</u>

<u>Note 1.14</u>

We have to check that this topology is in fact a cofinite topology. C2, C3, and C4 are clear. We now check C1.

Now $\mathcal{K}(A/U) = \{(K + U)/U: K \in \mathcal{K}(A)\}$

So $$\bigcap_{H \in \mathcal{K}(A/U)} H = \bigcap_{K \in \mathcal{K}(A)} (K + U)/U$$

$$= \frac{\bigcap_{K \in \mathcal{K}(A)} (K + U)}{U}$$

= U/U by proposition 1.6(i), as required. □

<u>Proposition 1.15</u>

Let A be a compact cofinite algebra and suppose that U is a closed ideal of A. Give A/U the cofinite topology induced from A. Then

(i) A/U is compact

(ii) If H is a closed vector subspace of A then (H + U)/U is a closed vector subspace of A/U.

<u>Proof:</u> (i) Let $\theta : A \to A/U$ be the natural projection map taking any element x of A to x + U. Then for a subbasic closed set x + U + (K + U)/U, the inverse image under θ is x + (K + U), which is closed in A. Hence θ is continuous, by Simmons, theorem E, p102, [14]. Thus A/U is the continuous image of the compact space A, so is compact.

(ii) H is closed in A, so is compact in the relative topology, by theorem 1.9. Therefore $\theta(H)$ is compact, since θ is continuous. This implies that $\theta(H)$ is closed, by theorem 1.9, whence $(H + U)/U$ is closed in A/U. □

Proposition 1.16

Let A be a cofinite algebra, U a vector subspace of A. Then U is dense in A if and only if $U + K = A$ for each $K \in \mathfrak{J}(A)$.

Proof: U is dense in A if and only if $\overline{U} = A$. But $\overline{U} = \bigcap_{K \in \mathfrak{J}} (U + K)$.

So U is dense if and only if $\bigcap_{K \in \mathfrak{J}} (U + K) = A$, i.e. $U + K = A$ for each $K \in \mathfrak{J}$. □

Note 1.17

Proposition 1.16 holds for any cofinal subset $\mathfrak{J}'$ replacing $\mathfrak{J}$, and in particular proposition 1.16 holds if we replace $\mathfrak{J}$ by $\mathfrak{K}$.

To see that this is so we may use proposition 1.6(i), replacing $\overline{U} = \bigcap_{K \in \mathfrak{J}} (U + K)$ with

$\overline{U} = \bigcap_{K \in \mathfrak{J}'} (U + K)$ in the above proof.

Proposition 1.18

Let A be a cofinite algebra and let H be a dense subalgebra of A. If K is an ideal of H, then $\overline{K}$ is an ideal of A.

Proof: Since K is an ideal of H, $K + K_i$ is an ideal of $H + K_i$, for each $K_i \in \mathfrak{K}(A)$. Now H is dense in A, so $A = H + K_i$ for all i, and so $K + K_i$ is an ideal of A. But $\overline{K} = \bigcap_{i \in I} (K + K_i)$, so $\overline{K}$ is an ideal of A. □

Chapter 2 Profinite Algebras

Section 1 Finite Dimensional Algebras

Definition 2.1

A profinite algebra is a compact cofinite algebra.

Remark

We shall show that this definition is equivalent to the alternative definition: that a profinite algebra is an algebra which is isomorphic to an inverse limit of finite dimensional algebras each having the affine topology, the inverse limit having the relative topology induced from the Tychonoff topology on the product of these finite dimensional algebras.

We therefore consider finite dimensional algebras with the affine topology.

Proposition 2.2

Let A be a finite dimensional algebra with the affine topology. Then

(i) A is a T_1-space;

(ii) A is connected;

(iii) A is compact.

Proof: (i) and (ii) follow immediately from proposition 1.5.
(iii) Let $\mathcal{C} = \{x_i + B_i : i \in I\}$ be any class of subbasic closed sets in A having the finite intersection property. To show that A is compact it suffices to show that any such class has non-empty intersection, by Simmons, [14], theorem F, p112.

Let C be the intersection of the elements of $\mathcal{C}$. We have to show that C is non-empty. Suppose instead that C is empty. For any subset I' of I, $\bigcap_{i \in I'} B_i$ is finite dimensional, so we can choose a finite subset J of I such that $\bigcap_{i \in J} B_i$ has minimal dimension. Renumber I so that $J = \{1,2,\ldots\ldots,n\} = I_n$, and let $C' = \bigcap_{i \in I_n} (x_i + B_i)$, which is non-empty, by the finite intersection property. But C is empty, so there exists j in I such that the set $C'' = C' \cap (x_j + B_j)$ is a proper subset of C'. Without loss of generality we may assume that $j = n + 1$. Now C'' is non-empty, so there exists $x \in A$ such that

$$C'' = x + \bigcap_{i \in I_{n+1}} B_i \qquad (= \bigcap_{i \in I_{n+1}} (x_i + B_i))$$

where $x + B_i = x_i + B_i$ for each $i \in I_{n+1}$. This implies that

$$C' = \bigcap_{i \in I_n} (x + B_i)$$

$$= x + \bigcap_{i \in I_n} B_i$$

Therefore

$$x + \bigcap_{i \in I_{n+1}} B_i = C'' \subset C' = x + \bigcap_{i \in I_n} B_i$$

So the dimension of $B_1 \cap B_2 \cap \ldots\ldots \cap B_{n+1}$ is smaller than that of $B_1 \cap B_2 \cap \ldots\ldots\ldots \cap B_n$, contradicting the minimality of the dimension of the latter. Hence C cannot be empty. □

Lemma 2.3

Let A be a finite dimensional algebra with the affine topology. Then any closed set in A has the form $\bigcup_{r=1}^{n} (x_r + K_r)$, where each K_r is a vector subspace of A.

Proof: From the definition of affine topology, the set $\{x + K : x \in A, K \text{ is a vector subspace of } A\}$ is a closed subbase for A, so any closed subset C of A has the form

$$C = \bigcap_{i \in I} \bigcup_{j=1}^{n_i} (x_{ij} + K_{ij})$$

where each K_{ij} is a vector subspace of A, for some index set I. We show that we can in fact take I to be a finite set. Well-order I to get a (not necessarily countable) chain of closed sets $C_1 \supseteq C_2 \supseteq \ldots\ldots \supseteq C_r \supseteq \ldots\ldots$ where $C_r = \bigcap_{s=1}^{r} \bigcup_{j=1}^{n_s} (x_{sj} + K_{sj})$. Now A is finite dimensional, so is noetherian by Stewart, [15], theorem 2.1, p23, so this chain must terminate. Hence C is a finite intersection of finite unions of affine subspaces of A, which is a finite union of finite intersections of affine subspaces. The result follows from the fact that a finite intersection of affine subspaces is also an affine subspace. □

Lemma 2.4

Let U,V be finite dimensional vector spaces with the affine topology and let $\theta : U \to V$ be a linear map. Then θ is closed and continuous.

Proof: Straightforward. □

Section 2 Inverse Limits of Topological Spaces

The topological results of this section are essential to the rest of the thesis.

Notation

The product space of the topological spaces $\{X_i : i \in I\}$ will be written $\prod_{i\in I} X_i$, and the Cartesian sum of the vector spaces $\{V_i : i \in I\}$ will be written $Cr_{i\in I} V_i$. When each V_i has a topological structure the two products will coincide, in which case we will choose the notation according to the context, so that for example we shall use $\prod_{i\in I} V_i$ when considering the topological structure of the Cartesian sum of the V_i's.

An element of either of these products will be denoted by $(x_i)_{i\in I}$, $(x_i)_i$ or (x_i) .

The following definitions may be found in Eilenberg and Steenrod, [4], p215.

Definitions

Let $\{X_i : i \in I\}$ be a class of sets indexed by a directed set I, and suppose that there are maps $\pi_{ij} : X_i \to X_j$ whenever $i \geqslant j$, satisfying

(a) For $i \geqslant j \geqslant k$, $\pi_{jk}\pi_{ij} = \pi_{ik}$,

(b) π_{ii} is the identity map on X_i.

Then $\{X_i ; \pi_{ij} : i,j \in I\}$ is called an <u>inverse system</u> (or a <u>projective system</u>).

The set $\{(x_i) \in \prod_{i\in I} X_i : x_j = \pi_{ij}(x_i) \text{ for } i \geqslant j\}$ is called the <u>inverse limit</u> of the inverse system $\{X_i; \pi_{ij}: i,j \in I\}$, and is written $\varprojlim\{X_i; \pi_{ij}: i,j \in I\}$, $\varprojlim\{X_i; \pi_{ij}\}$ or $\varprojlim\{X_i\}$.

When each X_i is a topological space we shall give $\varprojlim\{X_i\}$ the relative topology induced by the Tychonoff topology on $\prod_{i\in I} X_i$.

The following result from Stewart, [16], theorem 2.1, p186, will be used repeatedly, and is stated without proof:

<u>Theorem 2.5</u>

Let $\{X_i; \pi_{ij}\}$ be an inverse system of topological spaces X_i such that

(a) Each X_i is non-empty, compact and T_1;

(b) The maps π_{ij} are closed and continuous.

Then

(i) $X = \varprojlim\{X_i; \pi_{ij}\}$ is non-empty;

(ii) The image of the canonical projection $p_j : X \to X_j$ is $p_j(X) = \bigcap_{i \geq j} \pi_{ij}(X_i)$

(iii) If bars denote closures then for $A \subseteq X$,

$$\overline{A} = \varprojlim\{\overline{p_i(A)}\}$$

and if A is closed then

$$A = \varprojlim\{p_i(A)\} = \varprojlim\{\overline{p_i(A)}\}.$$

(iv) X is compact in the aforementioned topology. □

The following, from Engelking, [5], pp89-91, is useful for showing that certain inverse limits of topological spaces are homeomorphic:

Let $\underline{X} = \{X_i; \pi_{ij}; i, j \in I\}$ and $\underline{Y} = \{Y_\alpha; \rho_{\alpha\beta}: \alpha, \beta \in \Lambda\}$ be two inverse systems. A mapping of $\underline{X}$ into $\underline{Y}$ is a family $\{\varphi, f_\alpha\}$ consisting of a monotone function φ from Λ into I and of continuous maps $f_\alpha : X_{\varphi(\alpha)} \to Y_\alpha$, for each $\alpha \in \Lambda$, such that

$\rho_{\alpha\beta} f_\alpha = f_\beta \rho_{\varphi(\alpha)\varphi(\beta)}$

i.e. such that the diagram

$$\begin{array}{ccc} X_{\varphi(\alpha)} & \xrightarrow{f_\alpha} & Y_\alpha \\ \downarrow{\scriptstyle \pi_{\varphi(\alpha)\varphi(\beta)}} & & \downarrow{\scriptstyle \pi_{\alpha\beta}} \\ X_{\varphi(\beta)} & \xrightarrow{f_\beta} & Y_\beta \end{array}$$

is commutative for each pair α, β of elements of Λ with $\beta \leq \alpha$.

Every mapping of an inverse system $\underline{X}$ into an inverse system $\underline{Y}$ induces a continuous map from the inverse limit of $\underline{X}$ into the inverse limit of $\underline{Y}$.

Consider the map $f : \varprojlim\{X_i; \pi_{ij}\} \to \varprojlim\{Y_\alpha; \rho_{\alpha\beta}\}$ defined as follows: let $x = (x_i) \in \varprojlim\{X_i\}$. For any $\alpha \in \Lambda$, define $y_\alpha = f_\alpha(x_{\varphi(\alpha)})$. Then f is defined by $f(x) = (y_\alpha)_{\alpha\in\Lambda}$, and is called the limit mapping induced by $\{\varphi, f_\alpha\}$.

Theorem 2.6

Let $\underline{X} = \{X_i; \pi_{ij}: i,j \in I\}$, $\underline{Y} = \{Y_\alpha; \rho_{\alpha\beta}: \alpha,\beta \in \Lambda\}$ be two inverse systems and suppose that $\{\varphi, f_\alpha\}$ is a mapping of the inverse system $\underline{X}$ into the inverse system $\underline{Y}$. If $\varphi(\Lambda)$ is cofinal in I and $f_\alpha : X_{\varphi(\alpha)} \to Y_\alpha$ is a homeomorphism for every $\alpha \in \Lambda$, then the limit mapping $f : \varprojlim\{X_i\} \to \varprojlim\{Y_\alpha\}$ is a homeomorphism. □

Corollary 2.7

Let $\underline{X} = \{X_i; \pi_{ij}: i,j \in I\}$ be an inverse system and suppose that J is cofinal in I. Then the map $f : \varprojlim\{X_i; \pi_{ij}: i,j \in I\} \to \varprojlim\{X_i; \pi_{ij}: i,j \in J\}$ given by $(x_i)_{i\in I} \mapsto (x_i)_{i\in J}$, is a homeomorphism. □

Corollary 2.8

Let $\underline{X} = \{X_i; \pi_{ij}: i,j \in I\}$, $\underline{Y} = \{Y_i; \rho_{ij}: i,j \in I\}$ be two inverse systems with the same index set I. If each $f_i : X_i \to Y_i$ is a homeomorphism, then the limit mapping $f : \varprojlim\{X_i\} \to \varprojlim\{Y_i\}$ given by $f((x_i)_i) = (f_i(x_i))_i$, is a homeomorphism. □

We have the analogue of corollary 2.8 for inverse limits of finite dimensional algebras. It is easy enough to extend this to a result analogous to theorem 2.6, though we do not do so, as the simpler case will suffice for our needs.

Proposition 2.9

Let $A = \varprojlim\{A_i; \pi_{ij}: i,j \in I\}$ and let $B = \varprojlim\{B_i; \rho_{ij}: i,j \in I$ be two inverse limits having the same index set. Suppose that for each $i \in I$ there is an algebra isomorphism $\theta_i : A_i \to B_i$ such that the following diagram commutes whenever $j \leqslant i$:

$$\begin{array}{ccc} A_i & \xrightarrow{\theta_i} & B_i \\ {\scriptstyle \pi_{ij}}\downarrow & & \downarrow{\scriptstyle \rho_{ij}} \\ A_j & \xrightarrow{\theta_j} & B_j \end{array}$$

Then the map $\theta : A \to B$ defined by $\theta((x_i)_i) = (\theta_i(x_i))_i$ is an algebra isomorphism.

Proof: The verification that θ is an algebra homomorphism and that θ is injective is straightforward.

To show that θ is surjective, let $y = (y_i)$ be an element of B. Then $\rho_{ij}(y_i) = y_j$. For each $i \in I$, let $x_i = \theta_i^{-1}(y\ )$. Then

$\theta_j(\pi_{ij}(x_i)) = \rho_{ij}(\theta_i(x_i))$, from the commuting diagram above.

$= \rho_{ij}(y_i)$

$= y_j$

Thus $\pi_{ij}(x_i) = \theta_j^{-1}(y_j) = x_j$ and so $(x_i) \in A$, and (y_i) is the image under θ of (x_i). $\square$

Section 3 Inverse Limits of Finite Dimensional Algebras

Notation

Whenever we consider $\varprojlim\{A_i; \pi_{ij}\}$ for finite dimensional algebras A_i we shall assume that each π_{ij} is an algebra homomorphism.

The following result shows that we may replace an inverse system by one whose maps are all surjective.

Proposition 2.10

Suppose $A = \varprojlim\{A_i; \pi_{ij}: i,j \in I\}$, where each A_i is a finite dimensional algebra. Then for all $i,j \in I$ such that $j \leq i$ there exist subalgebras B_i of A_i and surjective maps $\rho_{ij} : B_i \to B_j$ such that $A = \varprojlim\{B_i; \rho_{ij}\}$ and such that ρ_{ij} is the restriction of π_{ij} to B_i.

Proof: For each $j \in I$ let $p_j : A \to A_j$ be the projection map $(x_i) \mapsto x_j$, and let $B_j = p_j(A)$; also let ρ_{ij} be the restriction of π_{ij} to B_i. Let $B = \varprojlim\{B_i; \rho_{ij}\}$. We show that $B = A$.

It is clear that B is a subset of A, so it suffices to show that A is contained in B. Give each A_i the affine topology. Then each A_i is T_1, non-empty and compact, by 2.2, and each π_{ij} is closed and continuous, by 2.4. So we may apply theorem 2.5 to see that for each $j \in I$, $B_j = p_j(A) = \bigcap_{i \geq j} \pi_{ij}(A_i)$.

Let $x = (x_i) \in A$ and fix $j \in I$. Then for any $i \geq j$,

$x_j = \pi_{ij}(x_i) \in \pi_{ij}(A_i)$. Thus $x_j \in \pi_{ij}(A_i)$ for all $i \geq j$, and so $x_j \in B_j$, whence $x \in \varprojlim\{B_i; \rho_{ij}\} = B$. Therefore $A \subseteq B$, as required.

We now show that ρ_{ij} is surjective. Let $x = (x_i) \in A$. Then

$$\pi_{ij}(p_i(x)) = \pi_{ij}(x_i) = x_j = p_j(x)$$

So $\pi_{ij}(p_i(A)) = p_j(A)$, whence the restriction to $B_i = p_i(A)$, that is ρ_{ij}, is surjective. □

We next show that an inverse limit A of finite dimensional algebras may be embedded by a topological isomorphism into an inverse limit of finite dimensional quotients of A.

Proposition 2.11

Let $A = \varprojlim\{A_i; \rho_{ij}: i,j \in I\}$, where each A_i is a finite dimensional algebra with the affine topology.

Then there exists a finite residual system $\{K_i: i \in I\}$ of A such that the embedding $\varphi : A \to P = \varprojlim\{A/K_i; \pi_{ij}: i,j \in I\}$ given by $x \mapsto (x + K_i)_i$ is a topological isomorphism when each A/K_i is given the affine topology and A,P have the relative topologies induced from the Tychonoff topologies on $\prod_{i\in I} A_i$, $\prod_{i\in I} A/K_i$ respectively.

Proof: By theorem 2.6 we may assume that each ρ_{ij} is surjective. For each $j \in I$ define $p_j : A \to A_j$ by $(x_i) \mapsto x_j$. Then each p_j is surjective, by theorem 2.5(ii). Now let K_j be the kernel of p_j, i.e. the set $\{(x_i) \in A: x_j = 0\}$. Then $\{K_i: i \in I\}$ will be the required finite residual system (it is clear that this set is in fact a finite residual system).

For each $j \in I$, define $\varphi_j : A/K_j \to A_j$ by $(x_i) + K_j \to x_j$. Then each φ_j is an algebra isomorphism. Let $P = \varprojlim\{A/K_i : \pi_{ij}\}$, where $\pi_{ij} : A/K_i \to A/K_j$ for each $i \geq j$ and $\pi_{ij}(x + K_i) = x + K_j$. Define $\varphi : A \to P$ by $x \mapsto (x + K_i)_i$.

We show firstly that φ is an algebra isomorphism. That φ is an algebra monomorphism is clear, so it suffices to show that φ is surjective. Let $(x_i + K_i) \in P$; we want to find an element $y \in A$ such that $(x_i + K_i) = (y + K_i)$. Now $\pi_{ij}(x_i + K_i) = x_i + K_j$ and $\pi_{ij}(x_i + K_i) = x_j + K_j$. Therefore $x_i - x_j \in K_j$ whenever $j \leq i$. Now for each $i \in I$ suppose that $x_i = (y_{ir})_{r \in I}$, where $x_i \in A$ and $y_{ir} \in A_r$ for each r and for any $i \in I$. Let $y = (y_{rr})_{r \in I}$. Then $y \in A$, and we show that $y + K_i = x_i + K_i$ for every i, i.e. that the i-coordinate of y equals the i-coordinate of x_i. But the i-coordinate of y is y_{ii}, which is also the i-coordinate of x_i. So $(x_i + K_i) = (y + K_i)$, which is the image of y under φ. Thus φ is a surjection, and hence an isomorphism.

Now give each A/K_i the affine topology, and give A and P the relative topologies induced from the Tychonoff topologies on $\prod_{i \in I} A_i$ and $\prod_{i \in I} A/K_i$ respectively. We show that φ is a homeomorphism. For any $j,k \in I$ with $k \leq j$ the following diagram commutes:

$$\begin{array}{ccc} A/K_j & \xrightarrow{\ \varphi_j\ } & A_j \\ \pi_{jk}\downarrow & & \downarrow \rho_{jk} \\ A/K_k & \xrightarrow{\ \varphi_k\ } & A_k \end{array}$$

where

$$\begin{array}{ccc} (x_i) + K_j & \longmapsto & x_j \\ \big\downarrow & & \big\downarrow \\ (x_i) + K_k & \longmapsto & x_k \end{array}$$

Now each φ_j is a homeomorphism, so by corollary 2.8 φ is a homeomorphism, and hence a topological isomorphism. $\square$

The following result relates a cofinite topology on $\varprojlim\{A_i\}$ with the affine topology on each A_i.

Proposition 2.12

Let $A = \varprojlim\{A_i; \pi_{ij}\}$, where each A_i is a finite dimensional algebra, and where each π_{ij} is surjective.

(i) If A is cofinite relative to the finite residual system $\{K_i: i \in I\}$ defined in proposition 2.11, then the topology induced on each A_i from A (in the sense of definition 1.13) by the canonical projection $p_j : A \to A_j$ is the affine topology.

(ii) If each A_i has the affine topology, then the relative topology on A induced by the Tychonoff topology on $\prod_{i\in I} A_i$ is a cofinite topology. Further, A is compact in this topology.

Proof: (i) The topology induced on each A_i is a cofinite topology, by the note 1.14, so is the affine topology, since each A_i is finite dimensional.

(ii) (a) Each π_{ij} is closed, by lemma 2.4, and since each A_i is compact and T_1, by proposition 2.2, we deduce that A is compact, by theorem 2.5(iv).

The rest of the proof will entail finding a suitable closed subbase for A, from which we extract a finite residual system, and then check the axioms of definition 1.1 to show that A is cofinite relative to this finite residual system.

(b) Let $P = \prod_{i \in I} A_i$. We shall find a closed subbase for P, from which we shall obtain a closed subbase for A.

Now $\{C_j \times \prod_{i \neq j} A_i : C_j$ is a closed subset of A_j, $j \in I\}$ is a closed subbase for P. But each C_j is a finite union of affine subspaces of A_j, by lemma 2.3, so $\{(x + B_j) \times \prod_{i \neq j} A_i : B_j$ is a vector subspace of A_j, $x \in A_j$, $j \in I\}$ is a closed subbase for P. Therefore $\{A \cap ((x + B_j) \times \prod_{i \neq j} A_i) : B_j$ is a vector subspace of A_j, $x \in A_j$, $j \in I\}$ is a closed subbase for A. We want to simplify the elements of this set.

Now $(x + B_j) \times \prod_{i \neq j} A_i = (y_i) + \prod_{i \in I} C_i$, for some $(y_i) \in A$ such that $y_j = x$, where $C_j = B_j$ and $C_i = A_i$ when $i \neq j$. Also, it is clear that any such $(y_i) \times \prod_{i \in I} C_i$ can be obtained in this way from some $(x + B_j) \times \prod_{i \neq j} A_i$.

So

$$A \cap ((x + B_j) \times \prod_{i \neq j} A_i) = (y_i) + (A \cap \prod_{i \in I} C_i)$$

Thus $\{(x_i) + (A \cap \prod_{i \in I} C_i)\}$ is a closed subbase for A. To simplify this further, we claim that $A \cap \prod_{i \in I} C_i = \underleftarrow{\lim}\{B_i ; \rho_{ij}\}$ for some vector subspaces B_i of A_i for each i, where each ρ_{ij} is the restriction of π_{ij} to B_i. To see this, note that $A \cap \prod_{i \in I} C_i$

equals $\varprojlim\{C_i; \bar{\pi}_{ij}\}$, where $\bar{\pi}_{ij}$ is the restriction of π_{ij} to C_i, and is not necessarily surjective. By proposition 2.10, $A \cap \prod_{i\in I} C_i = \varprojlim\{B_i; \rho_{ij}\}$ for some $\{B_i: i \in I\}$ with ρ_{ij} surjective for $j \leq i$. Thus $\{(x_i) + \varprojlim\{B_i ; \rho_{ij}\}: (x_i) \in A, \varprojlim\{B_i\}$ has finite codimension in $A, \pi_{ij}(B_i) = B_j$ for all $i \geq j\}$ is a closed subbase for A ---------- (1)

(c) Let $\mathcal{C} = \{B = \varprojlim\{B_i\}: B$ is an ideal of finite codimension in $A, \pi_{ij}(B_i) = B_j\}$. Then $\mathcal{C}$ is a finite residual system for A.

We now show that A is cofinite relative to $\mathcal{C}$. C1 is clear. To verify C2, suppose H is a closed vector subspace of finite codimension in A. We want to show that H contains some $C \in \mathcal{C}$. Let $\mathcal{K} = \{B = \varprojlim\{B_i\}: B$ is a vector subspace of A of finite codimension$\}$, and let $B = \varprojlim\{B_i\}$ be a typical element of $\mathcal{K}$. We show firstly that B contains some $C \in \mathcal{C}$, and secondly that $H \in \mathcal{K}$, from which C2 will follow.

For each $j \in I$ let $R_j = \prod_{i\in I} C_i$, where $C_i = A_i$ when $i \neq j$ and $C_j = 0$, and let $H_j = \prod_{i\in I} C_i$ where $C_i = A_i$ for $i \neq j$ and $C_j = B_j$. Further, let $K_j = A \cap R_j$. Then $B = A \cap \prod_{i\in I} B_i = A \cap \bigcap_{j\in I} H_j = \bigcap_{j\in I} (A \cap H_j)$. Now B has finite codimension in A, so there exist $i_1, i_2, \ldots\ldots, i_n$ in I such that

$$B = (A \cap H_{i_1}) \cap \ldots\ldots\ldots \cap (A \cap H_{i_n})$$

$$= A \cap (H_{i_1} \cap \ldots\ldots \cap H_{i_n})$$

$$\supseteq A \cap (R_{i_1} \cap \ldots\ldots \cap R_{i_n})$$

which is an element of $\mathcal{C}$, since $\varprojlim\{B_i'\} \cap \ldots \cap \varprojlim\{B_i^n\} = \varprojlim\{B_i' \cap \ldots B_i^n\}$

Thus any element of $\mathcal{K}$ contains an element of $\mathcal{C}$. We now show that $H \in \mathcal{K}$. By (1),

$$H = \bigcap_{\lambda \in \Lambda} \bigcup_{r=1}^{n_\lambda} (x_{r\lambda} + K_{r\lambda}) \text{ with } K_{r\lambda} \in \mathcal{K} \text{ for each } r,$$

So for each $\lambda \in \Lambda$, $H \subseteq \bigcup_{r=1}^{n_\lambda} (x_{r\lambda} + K_{r\lambda})$. Then $H \subseteq K_{r_\lambda \lambda}$ for some $r = r_\lambda$, by lemma 1.4.

Hence $H \subseteq \bigcap_{\lambda \in \Lambda} K_{r_\lambda \lambda}$

$$\subseteq \bigcap_{\lambda \in \Lambda} \bigcup_{r=1}^{n_\lambda} (x_{r\lambda} + K_{r\lambda})$$

$$= H$$

Therefore $H = \bigcap_{\lambda \in \Lambda} K_{r_\lambda \lambda}$. But H has finite codimension in A, so there exist $J_1, \ldots\ldots, J_m \in \{K_{r_\lambda \lambda} : \lambda \in \Lambda\}$ such that $H = \bigcap_{s=1}^{m} J_s$. But each J_s has the form $\varprojlim\{K_{is}\}$, for some vector subspaces K_{is} of A_i, for each i, since each J_s is in $\mathcal{K}$.

Thus $H = \varprojlim\{K_{i1}\} \cap \ldots\ldots \cap \varprojlim\{K_{im}\}$

$$= \varprojlim\{K_{i1} \cap \ldots\ldots \cap K_{im}\}$$

$\in \mathcal{K}$, as required.

Note that we have also shown that if H is a closed vector subspace of finite codimension in A then H has the form $\varprojlim\{K_i\}$

———— (2)

To verify C3, suppose that H is a closed vector subspace of finite codimension in A and that K is a vector subspace of A containing H. Now $H = \varprojlim\{B_i\}$ for some vector subspaces B_i of A_i, by (2). We shall prove that one of these B_i's determines H, i.e. that $H = (B_j \times \prod_{i \neq j} A_i) \cap A$, for each B_i determines a subalgebra of A containing H; from these we may choose a descending chain, which must terminate at H since H has finite codimension in A. To see this, firstly choose any j_1 in I. For $i \geqslant j_1$ define $B_{i1} = \pi_{ij_1}^{-1}(B_{j_1})$. For any $r \in I$ there exists $s \in I$ such that $s \geq r$ and $s \geqslant j_1$, so let $B_{r1} = \pi_{sr}(B_{s1})$. We show that these B_{ri}'s are consistently defined, i.e. that $\pi_{ri}(B_{r1}) = B_{i1}$ whenever $i \leqslant r$. Because of the definition of the B_{r1}'s it suffices to prove this for the case $j_1 \leqslant i \leqslant r$. Let $x \in B_{r1}$. Then $x \in \pi_{rj_1}^{-1}(B_{j_1})$, and so $\pi_{rj_1}(x) \in B_{j_1}$.

$$
\begin{aligned}
\text{Now } \pi_{ij_1}(\pi_{ri}(x)) &\in \pi_{ij_1}(\pi_{ri}(B_{r1})) \\
&= \pi_{rj_1}(B_{r1}) \\
&= B_{j_1}
\end{aligned}
$$

So $\pi_{ri}(x) \in \pi_{ij_1}^{-1}(B_{j_1}) = B_{i1}$. Therefore $\pi_{ri}(B_{r1}) \subsetneq B_{i1}$.

Conversely, let $y \in \pi_{ri}^{-1}(B_{i1})$. Then $\pi_{ri}(y) \in B_{i1}$. This implies that $\pi_{ij_1}(\pi_{ri}(y)) \in \pi_{ij_1}(B_{i1}) = B_{j_1}$. But $\pi_{ij_1}(\pi_{ri}(y)) = \pi_{rj_1'}(y)$, and so $y \in \pi_{rj_1}^{-1}(B_{j_1}) = B_{r1}$. Thus $\pi_{ri}^{-1}(B_{i1}) \leqslant B_{r1}$, i.e. $B_{i1} \subsetneq \pi_{ri}(B_{r1})$; therefore $B_{i1} = \pi_{ri}(B_{r1})$, as required.

To form the descending chain, let $H_1 = \varprojlim\{B_{i1}\}$. Then $H \leqslant H_1$, since $B_i \leqslant B_{i1}$ for every $i \in I$. We can see that

$$
H_1 = A \cap (B_{j_1} \times \prod_{i \neq j} A_i) \qquad \text{———} \qquad (3)
$$

If $H_1 \neq H$, then there exists $j_2 \in I$ such that $Bj_2 < B_{j_21}$. So define $B_{i2} = \pi_{ij_2}^{-1}(B_{j_2})$ whenever $j_2 \leqslant i$, and $B_{r2} = \pi_{ir}(B_{i2})$

for some $i \geq r, j_2$ if $j_2 \not\leq r$. Then, as before, $\pi_{ri}(B_{r2}) = B_{i2}$ whenever $i \leq r$. Let $H_2 = \varprojlim\{B_{i2}\}$. Then $H_1 > H_2 \geq H$. Continuing in this way we get a descending chain $A \geq H_1 > H_2 > \ldots\ldots \geq H$, and since H has finite codimension in A this chain is finite, so $H = H_n$ for some n. Then by (3), $H = A \cap (B_j \times \prod_{i \neq j} A_i)$ for some $j \in I$. For this fixed j we now show that $K = A \cap (P_j \times \prod_{i \neq j} A_i)$ where P_j is the image of the restriction to K of the canonical projection map $p_j : A \to A_j$, i.e. $P_j = p_j(K)$, and since $P_j \times \prod_{i \neq j} A_i$ is closed in $\prod_{i \in I} A_i$, K will be closed in A.

$$
\begin{aligned}
\text{Now}\quad K &= K + H \\
&= K + (A \cap (B_j \times \prod_{i \neq j} A_i)) \\
(4) \text{ ———— } \quad &= A \cap (K + (B_j \times \prod_{i \neq j} A_i)) \text{ by the modular law.}
\end{aligned}
$$

We claim that

$$K + (B_j \times \prod_{i \neq j} A_i) = P_j \times \prod_{i \neq j} A_i$$

for P_j defined as above. For let $(x_i) + (y_i)$ be an element of the expression on the left hand side of the equation. Then $x_j \in P_j$ and $y_j \in B_j \subseteq p_j(H) \subseteq p_j(K)$, so $(x_i) + (y_i) = (x_i + y_i)$, which is in the expression on the right hand side, since $x_j + y_j \in P_j$.

Conversely, if (x_i) is in the right hand side, then $x_j \in P_j$, so there exists $(y_i) \in K$ such that $y_j = x_j$. So $(x_i) = (y_i) + (x_i - y_i)$, which lies in $K + (B_j \times \prod_{i \neq j} A_i)$ since $x_j - y_j = 0 \in B_j$, proving the claim.

Thus $K + (B_j \times \prod_{i\neq j} A_i) = P_j \times \prod_{i\neq j} A_i$, which is closed in $\prod_{i\in I} A_i$, so K is closed in A in the relative topology, by (4), as required.

C4 follows from (1) and (2). □

Notation 2.13

Let $A = \varprojlim\{A_i; \pi_{ij}\}$, where each A_i is a finite dimensional algebra, and give each A_i the affine topology. Then the topology induced on A as in 2.12(ii) will be called the usual (cofinite) topology.

Corollary 2.14

Let $A = \varprojlim\{A_i : i \in I\}$, where each A_i is a finite dimensional algebra, and give A the usual cofinite topology as defined above. Then H is a closed vector subspace of A if and only if $H = \varprojlim\{H_i\}$ for some choice H_i of vector subspaces of A_i, for each $i \in I$.

Proof: Suppose that H is closed. Then by corollary 1.7, $H = \bigcap_{j\in J} K_j$ for some subset J of I, where each K_j is in $\mathfrak{J}(A)$. By (2) from the proof of proposition 2.12(ii), $K_j = \varprojlim\{K_{ij}\}$ for some vector subspace K_{ij} of A_i for every $i \in I$.

So $H = \bigcap_{j\in J} \varprojlim\{K_{ij}\} = \varprojlim\{\bigcap_{j\in J} K_{ij}\}$.

Conversely suppose that $H = \varprojlim\{H_i\}$. For each i, H_i is closed in A_i, since A_i has the affine topology. So $\prod_{i\in I} H_i$ is closed in $\prod_{i\in I} A_i$ in the Tychonoff topology. Then $A \cap \prod_{i\in I} H_i$ is closed in A in the relative topology. But $A \cap \prod_{i\in I} H_i = H$, so H is closed in A. □

Remark 2.15

In proposition 2.12, $P = \prod_{i\in I} A_i$ is not necessarily cofinite in the Tychonoff topology. For example, consider $P = A \times A$ where A is a one-dimensional vector space. Then $\{(x + K_j) \times A;$ $A \times (y + H_j)$: K_j, H_j are vector subspaces of A, $x \in A$, $y \in A\}$ is a closed subbase for P, as in the proof of proposition 2.12(ii). So the closed vector subspaces are $\{0\} \times \{0\}$, $\{0\} \times A$, $A \times \{0\}$ and P itself.

Now fix $x \in A$ and let $H = \{0\} \times \{0\}$ and $K = \{a(x,x): a \in F\}$. K is the one-dimensional vector subspace spanned by $(x,x) \in P$. Then K is a vector subspace of P containing H and H is closed with finite codimension in P, but K is not closed. Hence C3 of definition 1.1 is not satisfied.

Section 4 Connections Between Inverse Limits and Profinite Algebras

Lemma 2.16

Let A be a residually finite algebra with a finite residual system $\{K_i : i \in I\}$. Let $P = \varprojlim\{A/K_i ; \pi_{ij} : i,j \in I\}$, where $\pi_{ij}(x + K_i) = x + K_j$. Let $Q = \{(x + K_i) : x \in A\}$. Give each A/K_i the affine topology and give P the usual cofinite topology (see notation 2.13).

Then Q is dense in P.

Proof: For each $j \in I$ let $p_j : P \to A/K_j$ be the projection $(x_i + K_i)_i \mapsto x_j + K_j$ and let H_j be the kernel of p_j. Then $\{H_i : i \in I\}$ is a finite residual system for P and determines the cofinite topology of P, from proposition 2.12. It suffices to show that $\bigcap_{i \in I} (Q + H_i) = P$, since by proposition 1.6(i) $\overline{Q} = \bigcap_{i \in I} (Q + H_i)$.

Fix $j \in I$ and let x be an element of P. Then there exist elements x_i in A such that $x = (x_i + K_i)_i$

$$= (x_j + K_i)_i + (x_i - x_j + K_i)_i$$

Now $(x_j + K_i)_i \in Q$ and $(x_i - x_j + K_i)_i \in P$ because for $i \geq r$ $\pi_{ir}(x_i - x_j + K_i) = x_r - x_j + K_r$. Also $p_j((x_i - x_j + K_i)_i)$ equals $x_j - x_j + K_j$, which equals K_j. So $(x_i - x_j + K_i)_i$ lies in H_j, whence $x \in Q + H_j$ and so $P = Q + H_j$ for all j in I. Hence $P = \bigcap_{i \in I} (Q + H_i)$, as required. $\square$

Theorem 2.17

An algebra A is algebraically isomorphic to $\varprojlim\{A_i\}$, where each A_i is a finite dimensional algebra, if and only if A can be given a cofinite topology relative to which A is profinite.

Proof: One implication was proved in proposition 2.12(ii).

Conversely, suppose A is profinite. Let $\{K_i : i \in I\} = \mathcal{K}(A)$. This is a finite residual system for A. Define p_j, H_j, Q and P as in lemma 2.16, and let $\varphi : A \to P$ be the embedding $x \mapsto (x + K_i)$. We shall show that φ is a topological isomorphism.

(a) φ is clearly an algebra homomorphism.

(b) φ is injective, for if $\varphi(x) = \varphi(y)$, then $(x + K_i) = (y + K_i)$, so that $x - y$ lies in every K_i, hence in the intersection of the K_i's, so that $x - y = 0$, whence $x = y$.

(c) Let φ' be the restriction of φ to A. To show that φ' is continuous, consider a subbasic closed set $y + R$ of P, where R contains some H_j. Fix j. Since $y \in P$, $y \in Q + H_j$, by lemma 2.16, so there exists $z \in Q$ such that $y - z \in H_j \subseteq R$. This implies that $y + R = z + R$, so we may consider y to be an element of Q.

Now $\varphi^{-1}(y + R) = x + \varphi^{-1}(R)$, where $x \in A$ and is such that $\varphi(x) = y$. Also, $\varphi^{-1}(R) = K$, say, a vector subspace of A, and K contains $\varphi^{-1}(H_j)$, which contains K_j. So $x + K$ is closed, since A has cofinite topology relative to $\mathcal{K}(A)$. Therefore φ is continuous, so the restriction φ' of φ to A is continuous.

(d) To show that φ' is closed, let $x + K$ be a subbasic closed set in A. Then K contains some K_j. Now $\varphi(x + K) = \varphi(x) + \varphi(K)$, so it suffices to show that $\varphi(K)$, which equls $\varphi'(K)$, is closed in Q, since φ' is bijective.

We claim that $\varphi(K_j) = Q \cap H_j$. For consider $y \in Q \cap H_j$. Then $y = (x + K_i)_i = \varphi(x)$ for some $x \in A$. Since $y \in H_j$ we see that $x + K_j = 0$, so that $x \in K_j$, and hence $y \in \varphi(K_j)$.

The converse, that $\varphi(K_j)$ is contained in $Q \cap H_j$, is clear. So $K \supseteq K_j$, which implies that $\varphi(K) \supseteq \varphi(K_j) = Q \cap H_j$. But Q has the relative topology in P, so $\{Q \cap H_i : i \in I\}$ generates the topology in Q, and each $Q \cap H_i$ has finite codimension in Q. Therefore $\varphi(K)$, and hence $\varphi'(K)$, is a closed vector subspace of Q.

(e) We have thus shown that φ' is a topological isomorphism, so it now suffices to show that Q = P. Consider $y \in P$. For each $i \in I$ there exist $q_i \in Q$, $h_i \in H_i$ such that $y = q_i + h_i$, so $q_i = y - h_i \in y + H_i$. Thus $Q \cap (y + H_i)$ is non-empty for every $i \in I$. But $y + H_i$ is closed in P, so $Q \cap (y + H_i)$ is closed in Q for every $i \in I$.

Now for any $i_1, i_2, \ldots\ldots, i_n \in I$

$$\bigcap_{r=1}^{n} (Q \cap (y + H_{i_r})) = Q \cap (y + \bigcap_{r=1}^{n} H_{i_r})$$

$$\supseteq Q \cap (y + H_j)$$

for some $j \leqslant i_1, \ldots\ldots, i_n$.

Therefore $\bigcap_{r=1}^{n} (Q \cap (y + H_{i_r}))$ is non-empty, so $\{Q \cap (y + H_i) : i \in I\}$ has the finite intersection property. But Q is homeomorphic to A and hence is compact, so $\bigcap_{i\in I} (Q \cap (y + H_i))$ is non-empty i.e. $Q \cap \bigcap_{i\in I} (y + H_i)$ is non-empty. Thus $Q \cap \{y\}$ is non-empty, since $\bigcap_{i\in I} H_i = 0$. It follows that $y \in Q$, so that $P \subseteq Q$ and hence $P = Q$, as required. □

Corollary 2.18

Let A be a residually finite algebra with a finite residual system $\{K_i : i \in I\}$. Let $P = \varprojlim\{A/K_i; \pi_{ij} : i,j \in I\}$, where $\pi_{ij} : x + K_i \mapsto x + K_j$ for $j \leqslant i$. Give P the usual cofinite topology (as defined in notation 2.13), and give A the cofinite topology relative to $\{K_i : i \in I\}$.

(i) Then the map $\varphi : A \to P$ given by $x \mapsto (x + K_i)_i$ is a topological and algebraic embedding.

(ii) Further, φ is surjective (and hence is a topological isomorphism) if and only if A is compact.

Proof: The proof of (i) follows from the proof of theorem 2.17.

To prove (ii), suppose that φ is surjective. Then A is homeomorphic to P, and since P is compact, by theorem 2.17, it follows that A is compact.

Conversely suppose that A is compact. Then $\varphi(A)$ is compact, since φ is a topological embedding. So $\varphi(A)$ is closed in P, by theorem 1.9, and $\varphi(A) = \overline{\varphi(A)} = P$, by lemma 2.16. Therefore φ is surjective. □

Remark 2.19

We can now see that, up to a topological isomorphism, profinite algebras and inverse limits of finite dimensional algebras (when given the usual cofinite topology) are the same thing.

Proposition 2.20

Let A be a cofinite algebra with $\chi(A) = \{K_i : i \in I\}$, and let $\varphi : A \to P = \varprojlim\{A/K_i;\ \pi_{ij} : i,j \in I\}$ be the embedding defined in corollary 2.18. Further, let K be a vector subspace of A and let ρ_{ij} be the restriction of π_{ij} to $(K + K_i)/K_i$ for $j \leq i$.

Then

$$\overline{\varphi(K)} = \varprojlim\{(K + K_i)/K_i;\ \rho_{ij}\}$$

If, further, A is compact, then

$$\varphi(\overline{K}) = \overline{\varphi(K)} = \varprojlim\{(K + K_i)/K_i\}$$

Proof: Let $H = \varprojlim\{(K + K_i)/K_i\}$. Then $\varphi(K)$ is a vector subspace of H. By corollary 2.14 H is closed in P, so $\overline{\varphi(K)}$ is contained in H.

Conversely, $\overline{\varphi(K)}$ is closed in P, so $\overline{\varphi(K)} = \varprojlim\{H_i/K_i;\ \tau_{ij}\}$ for some vector subspaces H_i of A, where τ_{ij} is the restriction of π_{ij} to H_i/K_i. Now let $x \in K$. Then $(x + K_i) = \varphi(x) \in \overline{\varphi(K)} = \varprojlim\{H_i/K_i\}$. Therefore $x + K_i \in H_i/K_i$ for each i in I and so $x \in H_i$, whence $K \subseteq H_i$ for each $i \in I$. Thus $K + K_i \subseteq H_i$ for all $i \in I$, so that

$$\overline{\varphi(K)} \subseteq H = \varprojlim\{(K + K_i)/K_i\} \subseteq \varprojlim\{H_i/K_i\} = \overline{\varphi(K)}$$

So we see that $\overline{\varphi(K)} = H$.

Finally, if A is compact, then φ is a homeomorphism, by corollary 2.18, so that $\varphi(\overline{K}) = \overline{\varphi(K)}$. □

Proposition 2.21

Assuming the same notation as in proposition 2.20, suppose further that A is compact and that K is a closed vector subspace of A. Then the map $\theta : K \to \varprojlim\{K/(K \cap K_i); \rho_{ij}\}$ is a topological isomorphism, where for $i,j \in I$ such that $j \leqslant i$, $\rho_{ij} : x + (K \cap K_i) \mapsto x + (K \cap K_j)$ and $\theta : x \mapsto (x + (K \cap K_i))_i$.

Proof: Since K is closed in A, K is compact, by theorem 1.9. So K is profinite with $\mathcal{X}(K) = \{K \cap K_i : i \in I\}$, and we may argue as in corollary 2.18. □

Section 5 Inverse Limits of Coset Varieties

In the sequal we shall often need to use a result on algebraic groups (theorem 2.22). The following may be found in Stewart [16], pp188,189.

Definition

Let G be an affine algebraic group over F, an algebraically closed field of characteristic zero, and let $\mathcal{Z}$ denote the Zariski topology on G. Then the $\mathcal{W}$-topology on G is defined as follows: a closed subbase consists of all cosets xH of $\mathcal{Z}$-closed subgroups H of G.

A closed set in the $\mathcal{W}$-topology has the form $x_1H_1 \cup \ldots \cup x_nH_n$, where $x_1, \ldots, x_n \in G$ and $H_1, \ldots, H_n$ are algebraic subgroups of G.

Note

If we take G to be a vector space, with addition as the group operation, then the $\mathcal{W}$-topology is merely the affine topology.

Let G be an affine algebraic group and let H be a $\mathcal{Z}$-closed subgroup of G. Let $\alpha : G \to K$ be an algebraic group morphism such that $\alpha(H)$ is contained in a $\mathcal{Z}$-closed subgroup L of K. Then α induces a map $\bar{\alpha} : G/H \to K/L$, whereby $gH \mapsto \alpha(g)L$. Give G/H and K/L the quotient topologies relative to $\mathcal{W}$, and call these topologies $\mathcal{W}$-topologies on G/H and K/L.

Definition

A coset variety over F is any closed subset of a homogeneous space G/H, where G is an affine algebraic group and H is a $\mathcal{Z}$-closed subgroup of G.

A map $\tilde{\alpha}$: $G/H \to K/L$, or its restriction to a coset variety contained in G/H, is an affine map if it is induced as above from an algebraic group morphism $\alpha : G \to K$ such that $\alpha(H) \subseteq L$.

Theorem 2.22

Let $\{X_i; \pi_{ij}\}$ be an inverse limit system, where the X_i's are coset varieties over F equipped with the $\mathcal{W}$-topology, and the π_{ij}'s are affine maps. Suppose that each X_i is non-empty. Then the inverse limit $X = \varprojlim\{X_i\}$ is non-empty. □

We apply the above theory to the next result:

Proposition 2.23

Let $A = \varprojlim\{A_i; \pi_{ij}\}$, where each A_i is a finite dimensional algebra and suppose that for each $i \in I$ there exist vector subspaces H_i, K_i of A_i such that $\pi_{ij}(H_i) \subseteq H_j$ and $\pi_{ij}(K_i) \subseteq K_j$. Then

(i) $\varprojlim\{H_i + K_i; \rho_{ij}\} = \varprojlim\{H_i; \rho_{ij}'\} + \varprojlim\{K_i; \rho_{ij}''\}$ where ρ_{ij}, ρ_{ij}', ρ_{ij}'' are the restrictions of π_{ij} to $H_i + K_i$, H_i and K_i respectively.

(ii) If further $H_i \cap K_i = 0$ for each i, then $\varprojlim\{H_i\} \cap \varprojlim\{K_i\} = 0$.

Proof: (i) That the right hand side is a subset of the left hand side is clear.

Conversely, let $(h_i + K_i)$ be an element of the left hand side, where for each $i, h_i \in H_i$ and $k_i \in K_i$. Then $\pi_{ij}(h_i + K_i) = h_j + K_j$ for $j \leqslant i$. We want to use theorem 2.22 to find some $(h_i') \in \varprojlim\{H_i\}$ and $(k_i') \in \varprojlim\{K_i\}$ such that $(h_i + k_i) = (h_i') + (k_i')$.

Let $\mathcal{H}_i = h_i + (H_i \cap K_i)$ for each $i \in I$. Now each $H_i + K_i$ is an algebraic group, the group operation being addition, $H_i \cap K_i$ is a closed subgroup and each π_{ij} is an algebraic group morphism with $\pi_{ij}(H_i \cap K_i) \subseteq H_j \cap K_j$. So each $h_i + (H_i \cap K_i)$ is a coset variety. Let σ_{ij} be the restriction of π_{ij} to $h_i + (H_i \cap K_i)$. We want to show that $\sigma_{ij}(h_i + (H_i \cap K_i)) \subseteq h_j + (H_j \cap K_j)$. Now $\sigma_{ij}(h_i + k_i) = h_j + k_j$, so $\varrho_{ij}(h_i) - h_j = k_j - \pi_{ij}(k_i) \in K_j$. But $\sigma_{ij}(h_i) - h_j$ is in H_j, so is in $H_j \cap K_j$. Therefore $\sigma_{ij}(h_i)$ lies in $h_j + (H_j \cap K_j)$.

Thus

$$\begin{aligned}\sigma_{ij}(h_i + (H_i \cap K_i)) &= \sigma_{ij}(h_i) + \pi_{ij}(H_i \cap K_i) \\ &\leqslant \sigma_{ij}(h_i) + (H_j \cap K_j) \\ &= h_j + (H_j \cap K_j)\ \text{, as required.}\end{aligned}$$

So $\sigma_{ij} : \mathcal{H}_i \to \mathcal{H}_j$, and is an affine map, and so by proposition 2.21 we can deduce that $\varprojlim\{\mathcal{H}_i; \sigma_{ij}\}$ is non-empty. Let (h_i') be an element of $\varprojlim\{\mathcal{H}_i\}$. Then $\pi_{ij}(h_i') = \sigma_{ij}(h_i') = h_j'$ whenever $j \leqslant i$. Now for each i, $h_i' = h_i + x_i$ for some $x_i \in H_i \cap K_i$, so let $k_i' = h_i + k_i - h_i'$. Then $k_i' = k_i - x_i$ and so lies in K_i. Also, $\pi_{ij}(k_i') = \pi_{ij}(h_i + k_i) - \pi_{ij}(h_i') = h_j + k_j - h_j' = k_j'$. So $(k_i') \in \varprojlim\{K_i\}$. Also, $(h_i') \in \varprojlim\{H_i\}$.

where for each $i, h_i \in H_i$ and $k_i \in K_i$. Then $\pi_{ij}(h_i + K_i) = h_j + K_j$ for $j \leq i$. We want to use theorem 2.22 to find some $(h_i') \in \varprojlim\{H_i\}$ and $(k_i') \in \varprojlim\{K_i\}$ such that $(h_i + k_i) = (h_i') + (k_i')$.

Let $\mathcal{H}_i = h_i + (H_i \cap K_i)$ for each $i \in I$. Now each $H_i + K_i$ is an algebraic group, the group operation being addition, $H_i \cap K_i$ is a closed subgroup and each π_{ij} is an algebraic group morphism with $\pi_{ij}(H_i \cap K_i) \subseteq H_j \cap K_j$. So each $h_i + (H_i \cap K_i)$ is a coset variety. Let σ_{ij} be the restriction of π_{ij} to $h_i + (H_i \cap K_i)$. We want to show that $\sigma_{ij}(h_i + (H_i \cap K_i)) \subseteq h_j + (H_j \cap K_j)$. Now $\sigma_{ij}(h_i + k_i) = h_j + k_j$, so $\rho_{ij}(h_i) - h_j = k_j - \pi_{ij}(k_i) \in K_j$. But $\sigma_{ij}(h_i) - h_j$ is in H_j, so is in $H_j \cap K_j$. Therefore $\sigma_{ij}(h_i)$ lies in $h_j + (H_j \cap K_j)$.

Thus

$$\begin{aligned}\sigma_{ij}(h_i + (H_i \cap K_i)) &= \sigma_{ij}(h_i) + \pi_{ij}(H_i \cap K_i) \\ &\leq \sigma_{ij}(h_i) + (H_j \cap K_j) \\ &= h_j + (H_j \cap K_j) \text{ , as required.}\end{aligned}$$

So $\sigma_{ij} : \mathcal{H}_i \to \mathcal{H}_j$, and is an affine map, and so by proposition 2.21 we can deduce that $\varprojlim\{\mathcal{H}_i; \sigma_{ij}\}$ is non-empty. Let (h_i') be an element of $\varprojlim\{\mathcal{H}_i\}$. Then $\pi_{ij}(h_i') = \sigma_{ij}(h_i') = h_j'$ whenever $j \leq i$. Now for each i, $h_i' = h_i + x_i$ for some $x_i \in H_i \cap K_i$, so let $k_i' = h_i + k_i - h_i'$. Then $k_i' = k_i - x_i$ and so lies in K_i. Also, $\pi_{ij}(k_i') = \pi_{ij}(h_i + k_i) - \pi_{ij}(h_i') = h_j + k_j - h_j' = k_j'$. So $(k_i') \in \varprojlim\{K_i\}$. Also, $(h_i') \in \varprojlim\{H_i\}$.

Thus $(h_i + k_i) = (h_i' + k_i') = (h_i) + (k_i) \in \underleftarrow{\lim}\{H_i\} + \underleftarrow{\lim}\{K_i\}$, which completes the proof of (i).

The proof of (ii) is straightforward. □

Another application of theorem 2.22 shows that an inverse limit of finite dimensional algebras factored by an inverse limit of finite dimensional algebras is topologically isomorphic to an inverse limit of finite dimensional algebras, when each inverse limit is given the usual cofinite topology.

Proposition 2.24

Let $X = \underleftarrow{\lim}\{X_i; \pi_{ij}\}$, where each X_i is a finite dimensional algebra. Suppose that for each $i \in I$, Y_i is an ideal of X_i and that ρ_{ij} is the restriction of π_{ij} to Y_i, with $\rho_{ij}(Y_i) \subseteq Y_j$. Let $Y = \underleftarrow{\lim}\{Y_i; \rho_{ij}\}$ and let $Z = \underleftarrow{\lim}\{X_i/Y_i; \tilde{\pi}_{ij}\}$, where $\tilde{\pi}_{ij} : x_i + Y \mapsto \pi_{ij}(x_i) + Y_j$. Give X, Y, Z the usual cofinite topologies and give X/Y the cofinite topology induced from X.

Then Z is topologically isomorphic to X/Y.

Proof: Define $\varphi : X \to Z$ by $(x_i) \mapsto (x_i + Y_i)$. Then it is easy to check that φ is an algebra homorphism, and that Y is the kernel of φ.

We now show that φ is a surjection. Let $(x_i + Y_i) \in Z$. Since it is a vector subspace, Y_i is an algebraic subgroup of X_i, the group operation being addition, so that Y_i is closed when X_i has Zariski topology, whence $x_i + Y_i$ is a coset variety. Also, π_{ij} is an algebraic group morphism and induces the map $\tilde{\pi}_{ij}$. Since $\pi_{ij}(Y_i) \subseteq Y_j$, $\tilde{\pi}_{ij}$ is an affine map. In order to apply theorem

2.22, define $P = \varprojlim\{x_i + Y_i; \hat{\pi}_{ij}\}$, where each $\hat{\pi}_{ij}$ is the restriction of π_{ij} to $x_i + Y_i$. P is contained in X and each $\hat{\pi}_{ij}$ is well-defined, since $\hat{\pi}_{ij}(x_i + y_i) = \pi_{ij}(x_i) + \pi_{ij}(y_i) \in \pi_{ij}(x_i) + \pi_{ij}(Y_i) \subseteq \pi_{ij}(x_i) + Y_j = \tilde{\pi}_{ij}(x_i + Y_i) = x_j + Y_j$, since $(x_i + Y_i) \in Z$.

We now apply theorem 2.22 to see that P is non-empty, so there exists some $(h_i) \in P$, where $h_i \in x_i + Y_i$ and $\pi_{ij}(h_i) = h_j$. Therefore $(x_i + Y_i) = (h_i + Y_i) = \varphi((h_i))$, since $(h_i) \in X$. This shows that φ is a surjection.

Let θ be the map : $X/Y \to Z$ induced by φ, so that $(x_i) + Y \mapsto (x_i + Y_i)$. Then θ is an algebra isomorphism. To show that θ is continuous, let U be a closed vector subspace of Z. Then U has the form $\varprojlim\{U_i/Y_i\}$ for some vector subspaces U_i of X_i which contain Y_i, by corollary 2.14. But the inverse image under θ of $\varprojlim\{U_i/Y_i\}$ is $\varprojlim\{U_i\}/Y$, which is closed. It follows easily that θ is continuous. To show that θ is closed, let V/Y be a closed vector subspace of X/Y. Then V is closed in X, since the canonical map from X to X/Y is continuous, from definition 1.13. So by corollary 2.14, $V = \varprojlim\{V_i\}$ for some $V_i \subseteq X_i$, and where each V_i contains Y_i, since V contains Y. Thus θ maps V/Y to $\varprojlim\{V_i/Y_i\}$, which is closed in Z, by corollary 2.14 again. It is an easy step from this to see that θ is closed, and hence θ is the required topological isomorphism. □

Section 6 Some Further Properties of Cofinite Algebras

Proposition 2.25

A finite dimensional cofinite algebra is profinite.

Proof: Let A be a finite dimensional cofinite algebra. Then A has the affine topology, so is compact by proposition 2.2. □

Corollary 2.26

Let A be a cofinite algebra and let K be a finite dimensional vector subspace of A. Then K is closed in A.

Proof: K is cofinite in the relative topology, so is compact, by proposition 2.25. By corollary 1.10 we deduce that K is closed. □

Corollary 2.27

Let A be a residually finite algebra, $\{K_i : i \in I\}$ any finite residual system and K a finite dimensional vector subspace of A. Then $K = \bigcap_{i \in I} (K + K_i)$

Proof: Give A the cofinite topology determined by $\{K_i : i \in I\}$. Then $\overline{K} = \bigcap_{i \in I} (K + K_i)$ by proposition 1.6(i). But K is closed in A, by corollary 2.26; hence the result. □

Proposition 2.28

Let A be profinite algebra and suppose that H and K are closed vector subspaces of A. Then H + K is a closed vector subspace of A.

Proof:

Give A/K the cofinite topology induced from A. Then the map $\theta : A \to A/K$ given by $x \mapsto x + K$ is continuous.

Now $\theta(H) = (H + K)/K$. But H is closed, so $\theta(H)$ is closed, by proposition 1.15(ii), and so $\theta^{-1}(\theta(H))$ is closed, since θ is continuous. I.e. H + K is closed. □

Corollary 2.29

Let A be a profinite algebra, and let H, K be vector subspaces of A. Then $\overline{H + K} = \overline{H} + \overline{K}$.

Proof: That $\overline{H} + \overline{K} \subseteq \overline{H + K}$ is clear.

Conversely, $H + K \subseteq \overline{H} + \overline{K}$, and since $\overline{H} + \overline{K}$ is closed, from proposition 2.28, we can see that $\overline{H + K} \subseteq \overline{H} + \overline{K}$. □

Chapter 3 Profinite Completions

Section 1 Existence and Uniqueness

We aim to show that any cofinite algebra A can be embedded as a dense subset in a profinite algebra P, and that P is unique up to a topological isomorphism. P will be called the profinite completion of A, and we will see that P is topologically isomorphic to $\varprojlim\{A/K : K \in \mathcal{K}(A)\}$.

A cofinite algebra and its profinite completion have certain properties in common, and we may hope to use profinite completions as a tool for studying cofinite algebras.

Definition 3.1

Let A be a cofinite algebra. Then P is a profinite completion of A if P is a profinite algebra such that A embeds algebraically and topologically as a dense subset of P.

We first prove the existence of profinite completions:

Proposition 3.2

Any cofinite algebra has a profinite completion.

Proof: Let A be a cofinite algebra with $\mathcal{K}(A) = \{K_i : i \in I\}$, and let $P = \varprojlim\{A/K_i : i \in I\}$. Then P is profinite, by proposition 2.12, and the map $\varphi : A \to P$ given by $x \to (x + K_i)$ is a topological embedding, from the proof of theorem 2.17. Further, $\bar{A} = P$, by proposition 2.20, so that P is a profinite completion of A. □

We show that the profinite completion given above is in fact unique up to topological isomorphism.

Theorem 3.3

Let A be a cofinite algebra with a profinite completion P. Then P is topologically isomorphic to $\varprojlim\{A/K_i; \pi_{ij}: i \in I\}$ where $\{K_i: i \in I\} = \mathcal{K}(A)$ and where π_{ij} is the map $x + K_i \mapsto x + K_j$ for $j \leqslant i$.

Proof: Without loss of generality we may consider A to be a subset of P. We prove the result by showing that P is topologically isomorphic to $\varprojlim\{P/H_i\}$, where $\{H_i: i \in I\} = \mathcal{K}(P)$, which in turn is topologically isomorphic to $\varprojlim\{A/K_i\}$, which gives the required result.

Since A has the relative topology in P, $\mathcal{K}(A) = \{K_i = A \cap H_i: i \in I\}$. Now P is profinite, so is topologically isomorphic to $\varprojlim\{P/H_i; \rho_{ij}: i,j \in I\}$, by corollary 2.18, where for $j \leqslant i$, ρ_{ij} is the map $x + H_i \mapsto x + H_j$. Also, $\overline{A} = P$, so $P = A + H_i$ for each $i \in I$, by proposition 1.16. So $P/H_i = (A + H_i)/H_i$. Now for each i the map $\theta_i : (A + H_i)/H_i \to A/K_i$ given by $x + H_i \mapsto x + K_i$, is an algebra isomorphism, and is also a homeomorphism, since $(A + H_i)/H_i$ and A/K_i both have the affine topology, for every i. Also, the following diagrams commute:

$$\begin{array}{ccc} \frac{P}{H_i} & \xrightarrow{\rho_{ij}} & \frac{P}{H_j} \\ {\scriptstyle \theta_i}\downarrow & & \downarrow{\scriptstyle \theta_j} \\ \frac{A}{K_i} & \xrightarrow{\pi_{ij}} & \frac{A}{K_j} \end{array} \qquad \text{where} \qquad \begin{array}{ccc} p + H_i & \longrightarrow & p + H_j \\ \downarrow & & \downarrow \\ x_i + K_i & \longrightarrow & x_i + K_j = \\ & & x_j + K_j \end{array}$$

where, for $p \in P$, $p = x_i + h_i = x_j + h_j$ with $h_i \in H_i$, $h_j \in H_j$, x_i, $x_j \in A$.

Now $x_i - x_j \in H_j$, since $H_i \subsetneq H_j$, so $x_i - x_j \in A \cap H_j = K_j$. Thus $x_i + K_j = x_j + K_j$, confirming that the above diagrams commute.

So the map $\theta : \varprojlim\{P/H_i\} \to \varprojlim\{A/K_i\}$ given by $(p + H_i) \mapsto (x_i + K_i)$, where $p = x_i + h_i$, is a homeomorphism, by corollary 2.8. By proposition 2.9 . θ is an algebra isomorphism. □

Corollary 3.4

Let A be a cofinite algebra. Then any two profinite completions of A are topologically isomorphic. □

Corollary 3.5

Let A be a residually finite algebra. Then

(i) A is nilpotent if and only if each profinite completion of A is nilpotent, for any cofinite topology we may give A.

(ii) If A is a Lie algebra, then A is soluble if and only if each profinite completion of A is soluble, for any cofinite topology on A.

Proof: (i) If A is nilpotent, then $A^n = 0$ for some positive integer n, which implies that $(A/K)^n = 0$ for each $K \in \chi(A)$, and so $\varprojlim\{A/K\}^n \subsetneq \varprojlim\{(A/K)^n\} = 0$.

The converse is clear.

(ii) is proved similarly. □

Section 2 Extensions of Continuous Linear Maps to Profinite Completions

We show that a continuous algebra homomorphism of a cofinite algebra can be extended to a continuous algebra homomorphism of the profinite completion.

Theorem 3.6

Let A be a cofinite algebra with a dense subalgebra U. Let $\varphi : U \to W$ be a continuous algebra homomorphism from U to a profinite algebra W. Then

(i) There exists a continuous algebra homomorphism $\overline{\varphi} : A \to W$ extending φ.

(ii) $\overline{\varphi}(A) = \overline{\varphi}(\overline{U}) = \overline{\varphi(U)}$, when A is compact.

Proof: (i) (a) We first exhibit such a $\overline{\varphi}$. Consider $x \in A$. Since U is dense in A, $A = U + K_i$ for each $K_i \in \mathcal{K}(A) = \{K_i : i \in I\}$, by proposition 1.16. For each $i \in I$ there exist $v_i \in U$ and $k_i \in K_i$ such that $x = v_i + k_i$. Let $C_i = \varphi(v_i) + \bigcap_{M \in \mathcal{K}(W)} (\varphi(U \cap K_i) + M)$.

Define $S = \bigcap_{i \in I} C_i$. We want to show that this set has just one element, say $S = \{x'\}$. Then define $\overline{\varphi}(x) = x'$. We show that the C_i's have the finite intersection property, and so by the compactness of W, $\cap C_i$ is non-empty, and we then show from this that $\cap C_i$ is a singleton set.

1) Each C_i is clearly non-empty, and is closed, for

$$C_i = \bigcap_{M \in \mathcal{K}(W)} (\varphi(v_i) + \varphi(U \cap K_i) + M)$$

So C_i is the intersection of closed sets, hence is a closed set.

2) For each $r, i, j \in I$ such that $r \leqslant i$ and $r \leqslant j$ we show that $C_r \leqslant C_i \cap C_j$. Now $x \in v_i + K_i$, $x \in v_j + K_j$ and $x \in v_r + K_r$. Also, $K_r \leqslant K_i \cap K_j$. So $x - v_r \in K_r \subseteq K_i \cap K_j$. Therefore $v_i - v_r$ equals $(x - v_r) - (x - v_i)$, which lies in $K_r + K_i$, or K_i. Similarly $v_j - v_r \in K_j$. Thus $v_i - v_r \in K_i \cap U$ and $v_j - v_r \in K_j \cap U$, since all the v's are in U. So for all $M \in \mathcal{K}(W)$

$$\begin{aligned} \varphi(v_r) + \varphi(U \cap K_r) + M &= \varphi(v_r + (U \cap K_r)) + M \\ &\subseteq \varphi(v_r + (U \cap K_i)) + M \\ &= \varphi(v_i + (U \cap K_i)) + M \end{aligned}$$

Therefore $\bigcap_{M \in \mathcal{K}(W)} (\varphi(v_r) + \varphi(U \cap K_r) + M) \subseteq \bigcap_{M \in \mathcal{K}(W)} (\varphi(v_i) + \varphi(U \cap K_i) + M$ $= C_i$. So $C_r \leqslant C_i$. Similarly, $C_r \leqslant C_j$.

3) The set $\{C_i : i \in I\}$ is a family of closed non-empty subsets of W with the finite intersection property, by 2) above, so by the compactness of W, the intersection of the C_i's is non-empty.

4) We now show that $\bigcap_{i \in I} C_i$ is a singleton set. For $M \in \mathcal{K}(W)$, $\varphi^{-1}(M)$ is a closed ideal of U, since φ is a continuous algebra homomorphism. Now $U/\varphi^{-1}(M)$ is algebraically isomorphic to $U/\ker\varphi$ factored by $\varphi^{-1}(M)/\ker\varphi$, where $\ker\varphi$ denotes the kernel of φ, which in turn is isomorphic to $\varphi(U)/\varphi(\varphi^{-1}(M))$, i.e. to $(\varphi(U)+M)/M$, which is a vector subspace of W/M. But M has finite codimension in W, so $\varphi^{-1}(M)$ has finite codimension in U. Therefore $M \in \mathcal{K}(W)$ implies that $\varphi^{-1}(M) \in \mathcal{K}(U)$. So for each $M \in \mathcal{K}(W)$ we can find K' in $\mathcal{K}(U)$ such that $\varphi(K') \subseteq M$. K' has the form $U \cap K$ for some K in $\mathcal{K}(A)$, proposition 1.12.

Thus $S = \bigcap_{i \in I} \bigcap_{M \in \mathcal{K}(W)} (\varphi(v_i) + \varphi(U \cap K_i) + M)$

So $S = \bigcap_{M \in \mathcal{K}(W)} \bigcap_{i \in I} (\varphi(v_i) + \varphi(U \cap K_i) + M)$

But for each M we may use the axiom of choice to pick a $K_j \in \mathcal{K}(A)$ such that $\varphi(U \cap K_j) \subseteq M$, so that for each M,

$$\bigcap_{i \in I} (\varphi(v_i) + \varphi(U \cap K_i) + M) \subseteq \varphi(v_j) + M$$

Then $S \subseteq \bigcap_{M \in \mathcal{K}(W)} (\varphi(v_j) + M)$. But S is non-empty, by 3) above, so for any y in this set $S = y + \bigcap_{M \in \mathcal{K}(W)} M = \{y\}$.

So S is a singleton set, as claimed.

5) We now show that $\bar{\varphi}$ is well-defined i.e. that it does not depend on the choice of the expression $x = v_i + k_i$, for given x. Let x be an element of A and suppose that $x = v_i + k_i = u_i + h_i$ are two expressions for $x \in U + K_i$. Then $v_i - u_i = h_i - k_i \in U \cap K_i$ So $v_i + (U \cap K_i) = u_i + (U \cap K_i)$.

Thus $\bigcap_{\substack{i \in I \\ M \in \mathcal{K}(W)}} (\varphi(v_i) + \varphi(U \cap K_i) + M) = \bigcap_{\substack{i \in I \\ M \in \mathcal{K}(W)}} (\varphi(u_i + (U \cap K_i)) + M)$

$$= \bigcap_{i,M} (\varphi(u_i + (U \cap K_i)) + M)$$

So $\bar{\varphi}(x)$ is independent of the decomposition of x in A as a sum of elements from U and K_i, for each $i \in I$.

(b) We show that $\bar{\varphi}$ is an algebra homomorphism. Consider $x, y \in A$, and $a \in F$. Then for each i there exist $u_i, v_i \in U$, and $k_i, h_i \in K_i$ such that $x = v_i + k_i$ and $y = u_i + h_i$. Let $D_i = \bigcap_{M \in \mathcal{K}(W)} (\varphi(U \cap K_i) + M)$.

Now $\bar{\varphi}(x) = \bigcap_{i,M} (\varphi(v_i) + \varphi(U \cap K_i) + M)$, so $\bar{\varphi}(x) + D_i = \varphi(v_i) + D_i$ for each $i \in I$. Similarly, $\bar{\varphi}(y) + D_i = \varphi(u_i) + D_i$.

So for each i, $a\bar{\varphi}(x) + \bar{\varphi}(y) + D_i = a\varphi(v_i) + \varphi(u_i) + D_i$

$$= \varphi(av_i + u_i) + D_i.$$

Therefore $a\bar{\varphi}(x) + \bar{\varphi}(y) \in \varphi(av_i + u_i) + D_i$

i.e. $a\bar{\varphi}(x) + \bar{\varphi}(y) \in \bigcap_{i\in I} (\varphi(av_i + u_i) + D_i) = \{\bar{\varphi}(ax + y)\}$.

Thus $a\bar{\varphi}(x) + \bar{\varphi}(y) = \bar{\varphi}(ax + y)$ and $\bar{\varphi}$ is a linear map.

To check that $\bar{\varphi}$ preserves multiplication is straightforward.

(c) To see that $\bar{\varphi}$ extends φ, consider $x \in U$. Then $x = x + 0$ is a decomposition of x in $U + K_i$ for each K_i in $\mathcal{K}(A)$. So $\bar{\varphi}(x)$ equals $\bigcap_{i\in I} (\varphi(x) + D_i)$. But $0 \in \bigcap_{K_i, M} (\varphi(U \cap K_i) + M)$, so as $\bar{\varphi}(x)$ is single-valued, $\bar{\varphi}(x) = \varphi(x)$.

(d) We now show that $\bar{\varphi} : A \to W$ is continuous. Let $x + N$ be a closed subbasic set in W, so that $x \in W$ and $N \in \mathcal{J}(W)$, and let H be the inverse image under $\bar{\varphi}$ of N. If $y, y' \in \bar{\varphi}^{-1}(x + N)$, then $\bar{\varphi}(y - y') = \bar{\varphi}(y) - \bar{\varphi}(y') \in N$, which implies that $y - y' \in H$, so that $y + H = y' + H$. Thus $\bar{\varphi}^{-1}(x + N) = y + H$ for some $y \in A$. But by (a) of part 4), there exists K_j in $\mathcal{K}(A)$ such that $\varphi(U \cap K_j) \leqslant N$. We want to show that $K_j \subseteq H$, from which it will follow that $y + H$, and hence $\bar{\varphi}^{-1}(x + N)$, is closed in A, proving that $\bar{\varphi}$ is continuous. Consider $k \in K_j$. Then $k = 0 + k$ is a decomposition of k in $U + K_j$. Now $\bar{\varphi}(k) = \bigcap_{i\in I} (\varphi(v_i) + D_i)$

$$= z + \bigcap_{i,M} (\varphi(U \cap K_i) + M)$$

for some $z \in W$, since $\{\bar{\varphi}(k)\}$ is non-empty. So in particular, $z + D_j = \varphi(v_j) + D_j = 0 + D_j$, since $v_j = 0$. Therefore $z \in D_j$. Particularising again, $z \in \varphi(U \cap K_j) + N = N$. So $\bar{\varphi}(k)$ equals $z + \bigcap_{i,M} (\varphi(U \cap K_j) + M) \subseteq N$, and so $\bar{\varphi}(K_j) \subseteq N$. Thus $K_j \subset \bar{\varphi}^{-1}(N) = H$, as required. This completes the proof of (i).

(ii) Since $\bar{\varphi}$ is continuous, $\bar{\varphi}(A)$, which equals $\bar{\varphi}(\bar{U})$, is closed in W, by corollary 1.11, thus proving the first part.

Now $\bar{\varphi}(A)$ contains $\bar{\varphi}(U)$, which equals $\varphi(U)$. Therefore $\bar{\varphi}(A)$ contains $\overline{\varphi(U)}$.

Conversely, since $\bar{\varphi}$ is continuous, $\bar{\varphi}(\bar{X}) \subseteq \overline{\bar{\varphi}(X)}$ for all subsets X of A, so that $\bar{\varphi}(A) \subseteq \overline{\bar{\varphi}(U)} = \overline{\varphi(U)}$. Therefore $\bar{\varphi}(A) = \overline{\varphi(U)}$. □

Note that theorem 3.6 is analogous to theorem 2.1 in Hartley, [6], p196.

Remark

It could be that φ is not unique. For if $x \in A$, then $x = x_i + k_i$ for each $i \in I$, where $x_i \in U$, and intuitively $\bar{\varphi}(x)$ is a "limit" of $\{\bar{\varphi}(x_i) : i \in I\}$, which equals $\{\varphi(x_i) : i \in I\}$. However, A is not Hausdorff, so there may be more than one limit for this set. However, if $\bar{\varphi}$ is unique it may be possible to prove this in a way similar to that given in theorem 2.1 of Hartley, [6], page 198.

Corollary 3.7

Let A be a residually finite algebra and suppose that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are two cofinite topologies for A with $\mathfrak{J}_2 \subseteq \mathfrak{J}_1$. Let P_1 and P_2 be the profinite completions of A relative to $\mathfrak{J}_1$ and $\mathfrak{J}_2$ respectively. Let θ be the identity map on A.

Then θ can be extended to a continuous epimorphism $\bar{\theta} : P_1 \to P_2$.

Proof: The map θ maps $(A, \mathfrak{J}_1)$ to $(A, \mathfrak{J}_2)$, which is contained in P_2, and is given by $\theta(x) = x$. So we can consider θ to be the map $: (A, \mathfrak{J}_1) \to P_2$ given by $\theta(x) = x$. Then θ is clearly an algebra homomorphism, and is continuous, because $\mathfrak{J}_2 \subseteq \mathfrak{J}_1$. Since P_2 is

Chapter 4 Some Examples And Constructions

Because of corollary 2.18 we can find examples of profinite algebras either by taking inverse limits of finite dimensional algebras and giving them suitable topologies, or by finding residually finite algebras which can be given cofinite topologies relative to which they are compact.

A further method is to build up profinite algebras from known profinite algebras, by taking profinite completions of cofinite algebras , or by taking finite direct sums and certain tensor products.

Note that a vector space can be regarded as an algebra with trivial multiplication, and that any profinite algebra is also a profinite vector space if we disregard the multiplicative structure. Therefore we first consider profinite vector spaces.

Section 1 Profinite Vector Spaces

Notation

We shall denote by $Dr_{i\in I} V_i$ the (restricted) direct sum of the vector spaces (or algebras, as appropriate) V_i, and $Cr_{i\in I} V_i$ will denote the Cartesian sum of the V_i's.

In this section f will denote an arbitrary field (not necessarily algebraically closed, nor of characteristic zero).

The first result will provide a class of examples of both profinite algebras and of profinite vector spaces. The proof is effectively the same as that for the topological case (see Engelking, [5], Example 1, p67), but is given here for later

compact, θ extends to some $\bar{\theta} : P_1 \to P_2$. Now $\bar{\theta}(P_1) = \bar{\theta}(\bar{A})$, since A is dense in P_1, and $\bar{\theta}(\bar{A}) = \overline{\theta(A)} = P_2$, so that $\bar{\theta}$ is surjective. Also, $\bar{\theta}$ is continuous, by theorem 3.6. □

Corollary 3.8

Let A be a cofinite algebra, let B be a finite dimensional algebra with the affine topology and suppose that $\varphi : A \to B$ is a continuous algebra homomorphism. Suppose further that P is a profinite completion of A.

Then φ extends to a continuous algebra homomorphism $\bar{\varphi} : P \to B$.

Proof: This follows from proposition 2.25 and theorem 3.6. □

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reference.

Suppose that $A = Cr_{\lambda \in \Lambda} A_\lambda$, where each A_λ is a finite dimensional algebra. Let I be the set of finite subsets of Λ and partially order I by inclusion i.e. $i \leq j$ if and only if $i \subseteq j$. Then I is a directed set. For each $i \in I$ define $K_i = \{(x_\lambda) \in A: x_\lambda = 0$ for $\lambda \in i\}$. We will confuse K_i with $Cr_{\lambda \in \Lambda \setminus \{i\}} A_\lambda$.

It is clear that $\{K_i: i \in I\}$ is a finite residual system for A.

Proposition 4.1.

Let $A = Cr_{\lambda \in \Lambda} A_\lambda$, where each A_λ is a finite dimensional algebra. Give A the cofinite topology determined by the finite residual system $\{K_i: i \in I\}$ defined above.

Then A is profinite.

Proof: Let $P = \varprojlim\{A/K_i; \pi_{ij}: i,j \in I\}$, where for $j \leq i$, $\pi_{ij} : A/K_i \to A/K_j$ is defined by $x + K_i \mapsto x + K_j$, or, intuitively,

$$(\ldots.,0,x_{\lambda_1},0,\ldots,0,x_{\lambda_2},0,\ldots,0,x_{\lambda_n},0,\ldots) + K_i \text{ is mapped to}$$

$$(\ldots.,0,x_{\mu_1},0,\ldots,0,x_{\mu_m},0,\ldots.) + K_j, \text{ where } j = \{\mu_1,\ldots.,\mu_m\},$$

a subset of $i = \{\lambda_1,\ldots.,\lambda_n\}$.

Now define $\theta : P \to A$ by $(x_i + K_i) \mapsto (y_\lambda)_{\lambda \in \Lambda}$, where y_λ is the λ-coordinate of x_i whenever $\lambda \in i$. Now θ is well-defined, for if $\lambda \in i$ and $\lambda \in j$, then the λ-coordinate of x_i equals the λ-coordinate of x_j, since $\pi_{ij}(x_i + K_i) = x_j + K_j$, and the λ-coordinate is preserved.

Also, if $x_i + K_i = x_i' + K_i$, then $x_i - x_i' \in K_i$, so that the

λ-coordinate of $x_i - x_i'$ is zero, i.e. the λ-coordinate of x_i equals the λ-coordinate of x_i'. So y is uniquely determined, and hence θ is well-defined.

We now show that θ is a topological isomorphism, which together with proposition 2.12 will prove that A is profinite relative to the finite residual system $\{K_i : i \in I\}$, provided that P is given the usual cofinite topology. To prove that θ is an algebra homomorphism is straightforward.

To see that θ is injective, suppose that $\theta((x_i + K_i))$ equals $\theta((x_i' + K_i))$. Then $(y_\lambda) = (y_\lambda')$, so that $y_\lambda = y_\lambda'$ for each $\lambda \in \Lambda$. Therefore

$$x_i + K_i = (\ldots,0,y_{\lambda_1},0,\ldots,y_{\lambda_2},0,\ldots,y_{\lambda_n},0,\ldots) + K_i$$
$$= (\ldots,0,y_{\lambda_1}',0,\ldots,y_{\lambda_2}',0,\ldots,0,y_{\lambda_n}',0,\ldots) + K_i$$
$$= x_i' + K_i$$

This is true for each $i \in I$, so $(x_i + K_i) = (x_i' + K_i)$.

To show that θ is surjective, let (y_λ) be a typical element of A. Then for each $i \in I$, say $i = \{\lambda_1,\ldots,\lambda_n\}$, define

$$x_i + K_i = (\ldots,0,y_{\lambda_1},0,\ldots,0,y_{\lambda_2},0,\ldots,0,y_{\lambda_n},0,\ldots) + K_i$$

Then $\pi_{ij}(x_i + K_i) = x_j + K_j$, so that $(x_i + K_i) \in P$. Thus (y_λ) is the image of $(x_i + K_i)$ and θ is surjective, hence is an algebra isomorphism.

Finally, θ is a homeomorphism, from Engelking, [5], Example 1, p87. □

Remark 4.2

The topology induced on A by giving each A_λ the affine topology and giving A the Tychonoff topology is not necessarily a cofinite topology (see remark 2.15). However, suppose that $B = \varprojlim\{A_\lambda;\ \rho_{\lambda\mu}: \lambda, \mu \in \Lambda\}$ and $A = Cr_{\lambda\in\Lambda} A_\lambda$. Give A the cofinite topology determined by taking $\mathcal{K}(A) = \{K_i : i \in I\}$, using the notation of proposition 4.1, and give B the relative topology. Then B is a closed vector subspace of A. To see this we show firstly that the relative topology on $B, \mathfrak{T}_1$, is the same as the topology $\mathfrak{T}_2$ induced on B by the affine topologies on each A_λ. Now $\mathfrak{T}(A) = \{H \leqslant A: K_i \leqslant H \text{ for some } i \in I\}$. So the set $\mathcal{C}_1$, given by $\{B \cap (x + H): x \in A,\ H \in \mathfrak{T}(A)\}$ is a closed subbase for $\mathfrak{T}_1$. Also, $\mathcal{C}_2 = \{B \cap ((x_\mu + H_\mu) \times Cr_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda): x_\mu \in A_\mu,\ H_\mu$ is a vector subspace of $A_\mu,\ \mu \in \Lambda\}$ is a closed subbase for $\mathfrak{T}_2$, from the proof of proposition 2.12(ii).

Let $C = B \cap (x + H)$ be an element of $\mathcal{C}_1$. If C is empty then $C \in \mathcal{C}_2$, so suppose that C is non-empty. If $y \in C$, then C equals $y + (B \cap H)$. Now H contains some K_j, so $B \cap H$ contains $B \cap K_j$. But, again from the proof of proposition 2.12(ii), $B \cap K_j$ is $\mathfrak{T}_2$-closed in B. Thus $B \cap K_j \in \mathfrak{T}_2(B)$, so $B \cap H \in \mathfrak{T}_2(B)$, whence $y + (B \cap H)$ is $\mathfrak{T}_2$-closed. Therefore $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Conversely let $C = B \cap ((x + H_\mu) \times Cr_{\lambda\in\Lambda\setminus\{\mu\}} A_\lambda) \in \mathcal{C}_2$, where $x \in A_\mu$, and H_μ is a vector subspace of A_μ. Now $(x + H_\mu) \times Cr_{\lambda\neq\mu} A_\lambda$ equals $(y_\lambda) + Cr_{\lambda\in\Lambda} C_\lambda$, where $C_\lambda = A_\lambda$ for $\lambda \neq \mu$ and $C_\mu = H_\mu$, for some $(y_\lambda) \in A$, by part (b) of the proof of proposition 2.12(ii). But $\prod_{\lambda\in\Lambda} C_\lambda \in \mathfrak{T}(A)$, so if $C \in \mathcal{C}_2$ then $C \in \mathcal{C}_1$. Thus $\mathcal{C}_1 = \mathcal{C}_2$ and so $\mathfrak{T}_1 = \mathfrak{T}_2$.

Now A is compact, by proposition 4.1 and corollary 2.18, and B is compact in the $\mathcal{J}_2$-topology, and hence in the relative topology. So B is closed in A, by theorem 1.9.

With the notation of proposition 4.1, consider

$$A' = \mathrm{Dr}_{\lambda \in \Lambda} A_\lambda$$

$$= \{(x_\lambda) \in A: x_\lambda = 0 \text{ for all but finitely many } \lambda \in \Lambda\},$$

which is contained in A. Let $H_i = \{(x_\lambda) \in A': x_\lambda = 0 \text{ for all } \lambda \in i\}$. Then $\{H_i: i \in I\}$ is a finite residual system for A'.

Proposition 4.3

With the above notation, give A' the cofinite topology determined by $\{H_i: i \in I\}$. Then A is the profinite completion of A'.

Proof: Let $\theta : A' \to A$ be the identity embedding. By corollary 2.18 and proposition 4.1 A is profinite relative to the topology defined by $\{K_i: i \in I\}$. Also, θ is a topological embedding, since $H_i = A' \cap K_i$ for each $i \in I$. So it suffices to show that A' is dense in A, or, using proposition 1.16, that $A = A' + K_i$ for each $i \in I$. Take $x = (x_\lambda) \in A$ and fix $i \in I$, say $i = \{\lambda_1, \ldots, \lambda_n\}$. Let $y = (y_\lambda)$, where $y_\lambda = 0$ if $\lambda \notin i$ and $y_\lambda = x_\lambda$ for $\lambda \in i$. Then $x - y$ lies in K_i, since the λ_1-,.....,λ_n-coordinates of $x - y$ are all zero. Thus $x \in y + K_i$, which is a subset of $A' + K_i$, and hence $A = A' + K_i$. This is true for every i in I, so A' is dense in A, as required. □

Probably a well-known result is:

Lemma 4.4

Let $V = \mathrm{Dr}_{i\in I} V_i$, and let $V' = \mathrm{Cr}_{i\in I} V_i$, where each V_i is a finite dimensional vector space over the field f. Suppose that $|f| = \alpha$ and $|I| = \beta$, where β is an infinite cardinal number.

Then the dimension of V' equals the cardinality of V', which is α^β.

(So the dimension of V' is greater than the dimension of V).

Proof: We can split up each V_i into a sum of one-dimensional vector spaces without changing the size of the index set I, so we may assume that each V_i is one-dimensional.

We first deduce the number of elements in V. If J is a finite subset of I, then $|\sum_{i\in J} V_i| = \alpha^{|J|}$. By Zuckerman, [19], 6.9.6, p328, $|\mathcal{P}_n(I)| = |I|^n = \beta^n$ for $n \in \mathbb{N}$, the natural numbers, where $\mathcal{P}_n(I)$ is the set of subsets of I having cardinality n. So $|\{J \subseteq I: J \text{ is finite}\}| = \beta^0 + \beta^1 + \beta^2 + \ldots\ldots + \beta^n + \ldots\ldots$ where there are $\aleph_0$ (the smallest infinite cardinal) terms in the sum, so the sum is β. Now $|V| = |\bigcup_{\substack{J\subseteq I \\ |J|<\infty}} \sum_{i\in J} V_i|$

If α is infinite, then $|\sum_{i\in J} V_i| = \alpha^{|J|} = \alpha$. So $|V| = |\bigcup_{\lambda\in\Lambda} V_\lambda|$ where Λ is the set of finite subsets of I, and has cardinality β. Thus $|V| \leq |\Lambda|.|V_\lambda| = \alpha\beta$.

If α is finite, then $\sum_{i\in J} V_i$ has finite cardinality for any finite subset J of I, so that $|V| \leq |\Lambda|.\aleph_0 = \beta.\aleph_0 = \beta$.

So in either case, α finite or infinite, $|V| \leq \alpha\beta$.

Conversely it is clear that V has cardinality at least as great as both α and β, so that $|V| \geq \alpha\beta$. Therefore $|V| = \alpha\beta$.

Also, it is easy to see that $|V'| = \alpha^{\beta}$.

We now determine the dimension of V' in the two cases $\alpha \leq \beta$ and $\alpha > \beta$.

Case A: $\alpha \leq \beta$. Then $\alpha^{\beta} = 2^{\beta}$. Suppose that the dimension of V' is γ for some γ less than α^{β}. We derive from this a contradiction. V' is isomorphic to a direct sum of γ one-dimensional vector spaces over f, so $|V'| = \alpha\gamma$, as above. But α and γ are both less than 2^{β}, so $\alpha\gamma$ is less than $2^{\beta}.2^{\beta}$, which equals 2^{β}, by Zuckerman, [19], theorem 6.9.1(8), p321. So $|V'| < 2^{\beta}$. But $|V'| = \alpha^{\beta} = 2^{\beta}$ since $\alpha \leq \beta$, by Zuckerman, theorem 6.9.1(6), p321. This gives the required contradiction, so that the dimension of V' is greater than or equal to 2^{β}. However, dim V' $\leq |V'| = \alpha^{\beta} = 2^{\beta}$, and hence dim V' $= 2^{\beta} = \alpha^{\beta}$.

Case B: $\alpha > \beta$. Let $\{x_j \colon j \in J\}$ be distinct non-zero elements of f indexed by a set J having cardinality α. Now without loss of generality we may regard I as being well-ordered. Then we may define elements y_j in V' by taking $x_j^{\,n}$ to be the n^{th} coordinate of y_j when n is finite, and taking all other coordinates to be zero. Then $\{y_j \colon j \in J\}$ is a linearly independent set of vectors, since the Vandermonde determinant

$$\begin{vmatrix} 1 & x_1 & \cdots\cdots & x_1^{\,n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \vdots & & & \vdots \\ 1 & x_n & \cdots\cdots & x_n^{\,n} \end{vmatrix}$$

is non-zero.

Also, $\{y_j : j \in J\}$ has cardinality α, so $\dim V' \geq |\{y_j : j \in J\}| = \alpha$. But $\alpha^{\beta} = |V'| = \dim V' \times \alpha = \dim V'$, since the dimension of V' is greater than or equal to α, using Zuckerman, [19], theorem 6.9.i(6),p321.

Thus $\dim V' = \alpha^{\beta}$. □

Proposition 4.5

Let V be a vector space over the field f, where f has cardinality α. If $\dim V = \alpha^{\beta}$ for some infinite cardinal number β, then V can be given a topology relative to which V is profinite.

Proof: A vector space over a field f is determined uniquely (up to a vector space isomorphism) by its dimension, so V is isomorphic to a Cartesian sum of finite dimensional vector spaces. We may then give to this Cartesian sum a profinite topology as in proposition 4.1, and transfer this topology, via the isomorphism, to the vector space V, so that the isomorphism becomes a topological isomorphism, and V is profinite. □

Note

We do not say what the topology needed to make V into a profinite vector space is, since the proof of lemma 4.4 is not a constructive one, in that it does not provide us with an isomorphism between the Cartesian sum and a direct sum of equal dimension.

Remark 4.6

Proposition 4.5 shows that there exist direct sums of algebras which can be given a profinite topology; for example take $A = Dr_{i \in I} V_i$, where A has trivial multiplication, each V_i is a

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Thus $\dim V' = \alpha^\beta$. □

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Proposition 4.5 shows that there exist direct sums of algebras which can be given a profinite topology; for example take $A = Dr_{i \in I} V_i$, where A has trivial multiplication, each V_i is a one-dimensional vector space over f and I has cardinality $2^{|f|}$. Then A has dimension $2^{|f|}$, so can be given a profinite topology.

We now give an example of a vector space having different cofinite topologies with non-isomorphic profinite completions.

Example 4.7

Let $V = Cr_{i\in I} V_i$, where each V_i is a one-dimensional vector space, and suppose that V has dimension α, greater than $\aleph_0$. Then we may give V two cofinite topologies, such that the two profinite completions obtained have different dimensions.

Firstly, we may give V the cofinite topology defined in proposition 4.1, and V is then profinite, so is its own profinite completion, with dimension α.

Alternatively, there exist one-dimensional vector subspaces of V, $\{U_j: j \in J\}$ say, where J is an index set of cardinality α, such that V is isomorphic (as a vector space) to $Dr_{j\in J} U_j$. We may give this direct sum the cofinite topology defined in proposition 4.3, and then use this isomorphism to induce a corresponding cofinite topology on V. Now by proposition 4.3 $Dr_{j\in J} U_j$ has as its profinite completion $Cr_{j\in J} U_j$, and so this Cartesian sum is the profinite completion of V, and has dimension $|f|^{\alpha}$, which is greater than α.

Section 2 Profinite Algebras

Note that any finite dimensional algebra with the affine topology is a profinite algebra.

The next result shows that there exist Lie algebras of arbitrarily large dimension which cannot be given a profinite topology, because they cannot be given a cofinite topology (i.e. they are not residually finite).

Proposition 4.8

For any cardinal number α there exists a Lie algebra L of dimension α which cannot be embedded in a profinite Lie algebra.

Proof: Let H be a simple Lie algebra of dimension $\aleph_0$ and let K be a vector space of dimension α, so that K is a Lie algebra with trivial multiplication. Let $L = H \oplus K$. L has dimension α and clearly cannot be embedded in a residually finite Lie algebra. □

Corollary 4.9

For any cardinal number α there exists a Lie algebra which cannot be given a profinite topology. □

Proposition 4.10

If A is a profinite algebra and is simple, then A is finite dimensional.

Proof: Since A is profinite, A is residually finite, and the only possible finite residual system is $\{A,0\}$, so that 0 has finite codimension in A, i.e. A is finite dimensional. □

Example 4.11

Let $A = F[[t_1, \ldots\ldots, t_s]]$ be the formal power series in the commuting indeterminates $t_1, \ldots\ldots, t_s$ over the field F. Then A can be given a profinite topology.

To see this, let $K = (t_1) + \ldots\ldots + (t_s)$, where (t_r) denotes the principal ideal of A generated by t_r, for $r = 1, \ldots\ldots, s$. Define K_n to be K^n for each positive integer n. Then it is easy to check that $\{K_n : n \in \mathbb{N}\}$ is a finite residual system for A. Thus the desired topology on A is the cofinite topology determined by this finite residual system.

We can form the profinite completion $P = \varprojlim\{A/K_n; \pi_{nm} : n,m \in \mathbb{N}\}$, where $\pi_{nm} : A/K_n \to A/K_m$ is defined for $m \leqslant n$ and is the canonical projection. Let $\varphi : A \to P$ be given by $x \mapsto (x + K_n)$. Since P is a profinite completion of A the result follows if we show that φ is surjective.

Let $(x_n + K_n)$ be a typical element of P. Then $\pi_{nm}(x_n + K_n)$ equals $x_m + K_m$, so that $x_n - x_m \in K_m$ whenever $m \leqslant n$. That is, x_n and x_m have the same coordinates for degree less than $m + 1$. So let x be the limit of the sequence $x_1, x_2, \ldots\ldots, x_r, \ldots\ldots$ Then $\varphi(x) = (x_n + K_n)$, as required.

Thus A is a profinite commutative associative algebra.

The next two results show how we can obtain profinite algebras from known cases.

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Proposition 4.12

Let A, B be profinite algebras of the same kind and let C be the direct sum of A and B. Suppose $\mathcal{K}(A) = \{K_i : i \in I\}$ and $\mathcal{K}(B) = \{H_\alpha : \alpha \in \Omega\}$, and give C the cofinite topology determined by $\{K_i \oplus H_\alpha : (i,\alpha) \in I \times \Omega\}$, i.e. such that $\mathcal{K}(C) = \{K_i \oplus H_\alpha, H:$ H contains some $K_i \oplus H_\alpha$ and H is an ideal of C$\}$.

Then C is profinite.

Proof: Define the topological isomorphisms θ, θ' as in corollary 2.18 so that

$$\theta \;:\; A \to \varprojlim\{A/K_i;\ \pi_{ij} : i,j \in I\}$$

$$\theta' \;:\; B \to \varprojlim\{B/H_\alpha;\ \pi_{\alpha\beta} : \alpha,\beta \in \Omega\}$$

We shall denote elements of C by (x,y) where $x \in A$ and $y \in B$, and we shall also write $K \oplus H$ as (K,H) for ideals K of A and H of B. Let $\Lambda = I \times \Omega$ and partially order Λ by $\lambda = (i,\alpha) \leqslant (j,\beta)$ if and only if $i \leqslant j$ and $\alpha \leqslant \beta$. For $\mu \leqslant \lambda$ define $\pi_{\lambda\mu} : C/(K_i \oplus H_\alpha) \to C/(K_j \oplus H_\beta)$ by $(x,y) + (K_i,H_\alpha) \mapsto (x,y) + (K_j,H_\beta)$.

By corollary 2.18(ii) it suffices to show that the embedding $\varphi : C \to P = \varprojlim\{C/(K_i \oplus H_\alpha);\pi_{\lambda\mu} : \lambda,\mu \in \Lambda\}$ is surjective, where $(x,y) \mapsto ((x,y) + (K_i,H_\alpha))_{(i,\alpha)}$.

Let $((x_i,y_\alpha) + (K_i,H_\alpha))_{(i,\alpha)} \in P$. Then

$$\pi_{\lambda\mu} : (x_i,y_\alpha) + (K_i,H_\alpha) \mapsto (x_j,y_\beta) + (K_j,H_\beta).$$

Thus $(x_i,y_\alpha) + (K_j,H_\beta) = (x_j,y_\beta) + (K_j,H_\beta)$ and so $x_i + K_j = x_i + K_j$ and $y_\alpha + H_\beta = y_\beta + H_\beta$. From this we see that $(x_i + K_i) \in A$ and $(y_\alpha + H_\alpha) \in B$. Therefore $((x_i,y_\alpha) + (K_i,H_\alpha))_{(i,\alpha)}$ is the image under φ of $((x_i + K_i)_i,(y_\alpha + H_\alpha)_\alpha)$. □

Corollary 4.13

Let $A_1, \ldots, A_n$ be profinite algebras (of the same class) and let A be the direct sum of these algebras. If A is given the cofinite topology determined by the finite residual system $\{K_1 \oplus \ldots \oplus K_n : K_i \in \mathcal{K}(A_i)$ for $i = 1,2,\ldots,n\}$, then A is profinite. □

Note

There is no 4.14, nor 4.15

Pages 76 - 80 inclusive do not exist.

Section 3 An Alternative Construction of the Profinite Completion of a Cofinite Vector Space

We give a construction of the profinite completion of a cofinite vector space different to that given in theorem 3.3.

Construction 4.16

Let V be a cofinite vector space over the field F and let $\mathcal{J} = \{V_i : i \in I\}$ be the set of closed vector subspaces of V of finite codimension. Let $V' = \{f \in V^* : \ker f \supseteq \text{some } U \in \mathcal{J}\}$, where V^* is the dual space of V and $\ker f$ is the kernel of the map f. It is clear that $V' = \{f \in V^* : \ker f \in \mathcal{J}\}$.

Define $\overline{V}$ to be the dual space of V'. We shall show that, by giving $\overline{V}$ a suitable topology, $\overline{V}$ is the profinite completion of V.

(i) In order to define a cofinite topology on $\overline{V}$, consider any finite dimensional vector subspace U of V' and define

$$U^{\perp} = \{\theta \in \overline{V} : \theta(U) = 0\}$$

Let $\mathcal{P}$ be the set of finite dimensional vector subspaces of V' and let $\mathcal{R} = \{U^{\perp} : U \text{ is a finite dimensional vector subspace of } V'\}$. We claim that $\mathcal{R}$ is a finite residual system for $\overline{V}$.

(a) To show that the intersection of the elements of $\mathcal{R}$ is trivial, consider an element θ in this intersection. Then $\theta(U) = 0$ for all U in $\mathcal{P}$, so that $\theta(V') = 0$, since V' is the sum of its finite dimensional vector subspaces. Thus θ is the zero map, proving the claim.

(b) We now show that each $U^{\perp}$ from $\mathcal{R}$ has finite codimension in $\overline{V}$. Fix $U \in \mathcal{P}$ and consider the map $\psi : \overline{V}/U^{\perp} \to U^*$, where U^* is the dual space of U, given by $\theta + U \mapsto \theta_1$, where θ_1 denotes the

Corollary 4.13

Let $A_1, \ldots, A_n$ be profinite algebras (of the same class) and let A be the direct sum of these algebras. If A is given the cofinite topology determined by the finite residual system $\{K_1 \oplus \ldots \oplus K_n : K_i \in \mathcal{K}(A_i) \text{ for } i = 1,2,\ldots,n\}$, then A is profinite. □

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Define $\overline{V}$ to be the dual space of V'. We shall show that, by giving $\overline{V}$ a suitable topology, $\overline{V}$ is the profinite completion of V.

(i) In order to define a cofinite topology on $\overline{V}$, consider any finite dimensional vector subspace U of V' and define

$$U^{\perp} = \{\theta \in \overline{V} : \theta(U) = 0\}$$

Let $\mathcal{P}$ be the set of finite dimensional vector subspaces of V' and let $\mathcal{R} = \{U^{\perp} : U \text{ is a finite dimensional vector subspace of } V'\}$. We claim that $\mathcal{R}$ is a finite residual system for $\overline{V}$.

(a) To show that the intersection of the elements of $\mathcal{R}$ is trivial, consider an element θ in this intersection. Then $\theta(U) = 0$ for all U in $\mathcal{P}$, so that $\theta(V') = 0$, since V' is the sum of its finite dimensional vector subspaces. Thus θ is the zero map, proving the claim.

(b) We now show that each $U^{\perp}$ from $\mathcal{R}$ has finite codimension in $\overline{V}$. Fix $U \in \mathcal{P}$ and consider the map $\psi : \overline{V}/U^{\perp} \to U^*$, where U^* is the dual space of U, given by $\theta + U \mapsto \theta_1$, where θ_1 denotes the restriction of θ to U. Since U is finite dimensional, U* has the same dimension as U, so it suffices to show that ψ is an isomorphism. Since $\theta \in \overline{V}$, it maps V' to F so that θ_1 maps U to F. Further, if $\theta + U = \theta' + U$, then $(\theta - \theta')(U) = 0$, which implies that the restriction of θ to U equals the restriction of θ' to U; thus ψ is well-defined. That ψ is linear and injective is clear. To show that ψ is surjective, consider $\theta \in U^*$ and let W be a vector space complement to U in V', so that $V' = U \oplus W$. For $f \in V'$, $f = u + w$ for some $u \in U$ and $w \in W$. Define $\overline{\theta} \in \overline{V}$ by $\overline{\theta} : f \to \theta(U)$. Then $\overline{\theta}$ restricted to U is just θ so that θ is the image under ψ of $\overline{\theta} + U^{\perp}$; hence ψ is surjective, as claimed.

We have shown that ψ is an isomorphism (of vector spaces), from which we deduce that $U^{\perp}$ has finite codimension in $\overline{V}$.

(c) Consider $U_1^{\perp}, U_2^{\perp} \in \mathcal{R}$. We want to find some $U^{\perp} \in \mathcal{R}$ such that $U^{\perp} \leq U_1^{\perp} \cap U_2^{\perp}$. Take $U = U_1 + U_2$. Then U is finite dimensional. Also, if θ lies in $U^{\perp}$, then $\theta(U_1) = \theta(U_2) = 0$, implying that θ lies in $U_1^{\perp} \cap U_2^{\perp}$, so that $U^{\perp} \leq U_1^{\perp} \cap U_2^{\perp}$. Therefore $\mathcal{R}$ is a finite residual system for $\overline{V}$.

(ii) We may now give $\overline{V}$ the cofinite topology determined by the finite residual system $\mathcal{R}$, and show that $\overline{V}$ is profinite. It suffices to show that the map $\chi : \overline{V} \to P = \varprojlim\{\overline{V}/U^{\perp} : U \in \mathcal{P}\}$ is surjective, where $\theta \mapsto (\theta + U^{\perp})$. Index $\mathcal{R}$ by some set Ω, so that $\mathcal{R} = \{U_{\alpha}^{\perp} : \alpha \in \Omega\}$ and let $(\theta_{\alpha} + U_{\alpha}^{\perp})_{\alpha} \in P$. Then $\theta_{\alpha} + U_{\beta}^{\perp} = \theta_{\beta} + U_{\beta}^{\perp}$ for $\beta \leq \alpha$, so that $\theta_{\alpha} - \theta_{\beta} \in U_{\beta}^{\perp}$, i.e. $(\theta_{\alpha} - \theta_{\beta})(U_{\beta}) = 0$ whenever $\alpha \leq \beta$ ——(*)

Now for any $f \in V'$ define $U_{\alpha_f} = \langle f \rangle$, the one-dimensional vector space generated by f, where $\alpha_f \in \Omega$. Define $\theta : V' \to F$ by $\theta(f) = \theta_{\alpha_f}(f)$ for each $f \in V'$.

(a) We want to show firstly that θ is well-defined and lies in $\overline{V}$, for which it suffices to show that $f = g + h$ implies that $\theta_{\alpha_f}(f) = \theta_{\alpha_g}(g) + \theta_{\alpha_h}(h)$. So suppose that U_α contains $\langle g,h \rangle$. Then $U_{\alpha_f}, U_{\alpha_g}, U_{\alpha_h} \subseteq U_\alpha$, so that $\alpha_f, \alpha_g, \alpha_h \leq \alpha$. Thus $(\theta_\alpha - \theta_{\alpha_f})(U_{\alpha_f}) = 0$, by (*), and hence $\theta_\alpha(f) = \theta_{\alpha_f}(f)$. Similarly $\theta_\alpha(g) = \theta_{\alpha_g}(g)$ and $\theta_\alpha(h) = \theta_{\alpha_h}(h)$. We now see that $\theta_{\alpha_f}(f) = \theta_\alpha(f) = \theta_\alpha(g + h) = \theta_\alpha(g) + \theta_\alpha(h) = \theta_{\alpha_g}(g) + \theta_{\alpha_h}(h)$, as required.

(b) We now show that θ is the required map i.e. that $\chi(\theta) = (\theta_\alpha + U_\alpha^\perp)_\alpha$, for which it suffices to show that $\theta + U_\alpha^\perp = \theta_\alpha + U_\alpha^\perp$ for every $\alpha \in \Omega$. Fix $\alpha \in \Omega$ and let $f_1, \ldots\ldots, f_n$ be a basis for U_α. Then $\theta(f_r) = \theta_{\alpha_{f_r}}(f_r) = \theta_\alpha(f_r)$, from the above. So $(\theta - \theta_\alpha)(U_\alpha) = 0$ i.e. $\theta - \theta_\alpha \in U_\alpha^\perp$, so that $\theta + U_\alpha^\perp = \theta_\alpha + U_\alpha^\perp$. This is true for all α, so χ is surjective, as claimed.

We now show that $\overline{V}$ is a profinite completion of V. Consider the map $\varphi : V \to \overline{V}$ defined as follows: for $v \in V$ define the map $\varphi(v) : V' \to F$ by $f \mapsto f(v)$. We want to show that φ is a topological and algebraic embedding and that $\varphi(V)$ is dense in $\overline{V}$, from which it follows that $\overline{V}$ is the profinite completion of V.

(iii) To show that φ is an algebraic embedding, it is easy to check that φ is linear, and since $\varphi(v) \in \overline{V}$ for every $v \in V$ it is clear that φ is well-defined. Also, if $\varphi(v_1) = \varphi(v_2)$, then $f(v_1) = f(v_2)$ for every $f \in V'$, so that in particular, $1_V(v_1) = 1_V(v_2)$, where 1_V is the identity map on V; i.e. $v_1 = v_2$. Hence φ is a monomorphism.

(iv) To show that φ is a topological embedding, let $\mathfrak{I}_1$ be the subspace topology of $\varphi(V)$ in $\overline{V}$ and let $\mathfrak{I}_2$ be the topology of $\varphi(V)$ such that the map $\varphi' : V \to (\varphi(V), \mathfrak{I}_2)$ given by $v \mapsto \varphi(v)$ is a homeomorphism. Then φ is a topological embedding if $\mathfrak{I}_1 = \mathfrak{I}_2$, and to prove that $\mathfrak{I}_1 = \mathfrak{I}_2$ it suffices to show that for any vector subspace W of $\overline{V}$ of finite codimension, W is $\mathfrak{I}_1$-closed if and only if W is $\mathfrak{I}_2$-closed.

(a) Consider any $U^\perp \in \mathcal{R}$; we show that there exists $i \in I$ such that $\varphi(V) \cap U^\perp \supseteq \varphi(V_i)$, and then an application of C3 of definition 1.1 shows that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$.

Fix U and let $f_1, \ldots, f_n$ be a basis for U. Then there exist $i_1, \ldots, i_n \in I$ such that $f_j(V_{i_j}) = 0$ for each $j \in I_n$ because of the definition of V'. But there exists $i \in I$ such that $V_i \leq V_{i_j}$ for all $j \in I_r$, so that $f_j(V_i) \leq f_j(V_{i_j}) = 0$ for each $j \in I_n$. Thus for any $v \in V_i$ and any $f \in U$, $\varphi(v)(f) = f(v) = (\lambda_1 f_1 + \ldots + \lambda_n f_n)(v)$ for some $\lambda_1, \ldots, \lambda_n \in F$, and this equals $\lambda_1 f_1(v) + \ldots + \lambda_n f_n(v) = 0$. So $\varphi(V_i) \subseteq U^\perp$ and hence $\varphi(V_i) \subseteq \varphi(V) \cap U^\perp$, as required.

(b) To show that $\mathfrak{I}_2 \subseteq \mathfrak{I}_1$, we take $i \in I$ and show that there exists $U^\perp \in \mathcal{R}$ such that $\varphi(V) \cap U^\perp \subseteq \varphi(V_i)$, again using C3 of definition 1.1. Fix i and let $U = \{f \in V' : f(V_i) = 0\}$. Consider $\theta \in U^\perp \cap \varphi(V)$. Then $\theta(f) = 0$ for all $f \in U$, and $\theta = \varphi(v)$ for some

$v \in V$. So $\varphi(v)(f) = 0$, which implies that $f(v) = 0$ for every $f \in U$ and for some fixed $v \in V$ ---- (**)

But there exists $f' \in V^*$ such that $f'(x) = 0$ if and only if $x \in V_i$. Then $\ker f = V_i$, so that $f' \in V'$, and further $f' \in U$. So by (**), $f'(v) = 0$, and so $v \in V_i$. Therefore $\theta = \varphi(v) \in \varphi(V_i)$, so that $U^{\perp} \cap \varphi(V) \subseteq \varphi(V_i)$, as required.

(v) Finally we show that $\varphi(V)$ is dense in $\overline{V}$, for which it is sufficient to prove that $\overline{V} = \varphi(V) + U^{\perp}$ for every $U^{\perp} \in \mathcal{R}$, by proposition 1.16, i.e. to prove that for any $\theta \in \overline{V}$ there exist $v \in V$ and $u \in U^{\perp}$ such that $\theta = \varphi(v) + u$. But this is equivalent to showing that for some $v \in V$, $\theta - \varphi(v) \in U^{\perp}$, i.e. that the restriction of $\theta - \varphi(v)$ to U is the zero map, or that the restriction of θ to U equals the restriction of $\varphi(v)$ to U, for some $v \in V$.

Fix $U \in \mathcal{P}$. Then it is enough to show that every homomorphism : $U \to F$ has the form $f \mapsto f(v)$ for some $v \in V$. Define $U_1^{\perp} = \{v \in V: f(v) = 0$ for all $f \in U\}$. Then $V/U_1^{\perp}$ has finite dimension, since U is finite dimensional. Let $W = V/U_1^{\perp}$ and let $\sigma : U \to W^*$ be the map $f \to \sigma(f)$, where $\sigma(f) : V/U_1^{\perp} \to F$ is given by $v + U_1^{\perp} \mapsto f(v)$. It is easy to check that σ is a monomorphism, and to see that σ is surjective. Suppose that $\sigma(f)(v + U_1^{\perp}) = 0$ for some fixed $v \in V$ and for all $f \in U$. Then $f(v) = 0$ for all $f \in U$, so that $v \in U_1^{\perp}$, and thus $v + U_1^{\perp} = U_1^{\perp}$. Hence $\sigma(U)$ annihilates no non-zero vectors of W, so must be the whole dual space, i.e. $\sigma(U) = W^*$. So the dimension of U equals the dimension of W, and we may regard U and W as mutual dual spaces. Then W induces a dual space on U as follows: for $v \in V$, $v + U_1^{\perp} : U \to F$

is the map given by $f \to f(v)$, so that any element of U^* has the form $f \to f(v)$ for some $v \in V$, as required. □

Remark 4.17

This could provide a way of determining the dimension of the profinite completion of a cofinite vector space (and hence of a cofinite algebra). For suppose that, with the notation of construction 4.16, $\dim V' = \alpha$ for some cardinal number α. Then we can regard V' as a direct sum of α one-dimensional vector subspaces, so that $\overline{V}$, which equals V'^*, will have dimension $|F|^{\alpha}$, by lemma 4.4, since V'^* is isomorphic to a Cartesian sum of these one-dimensional vector spaces.

Thus if we could find the dimension of V' we would be able to find the dimension of $\overline{V}$, the profinite completion of V.

Theorem 4.18

A countable-dimensional algebra cannot be profinite.

Proof: Suppose that A is an infinite-dimensional profinite algebra. Consider only the vector space structure of A, and define a cofinite topology on A by taking as the closed vector subspaces of the vector space A the closed vector subspaces of the algebra A. It is clear that with this cofinite topology A is a profinite vector space. With the notation of construction 4.16, $\dim A' \geq \aleph_0$, the smallest infinite cardinal, so that $\dim \overline{A} \geq |F|^{\aleph_0} \geq 2^{\aleph_0} > \aleph_0$. Thus the profinite completion of A has dimension greater than $\aleph_0$, so for A to be profinite A must have dimension greater than $\aleph_0$. □

Corollary 4.19

Not all cofinite algebras are profinite. □

Corollary 4.20

Let A be a profinite algebra. Then A is finite dimensional or $\dim A \geq 2^{\aleph_0}$. □

Theorem 4.21

Let A be a cofinite algebra, with profinite completion P. Suppose that $|F| = \beta$ and $\dim A = \alpha$. Then $\dim P \leq 2^{\beta^{\alpha}}$.

Proof: With the notation of construction 4.16, $A' \subseteq A^*$, so we can regard $\bar{A}$ as a subset of A^{**}. Then it is sufficient to find the dimension of A^{**}, considering A only as a profinite vector space, as in the proof of theorem 4.18, since the profinite completion of A will be, when considered as a vector space, the profinite completion of the vector space A. Since A has dimension α there exist one-dimensional vector subspaces V_i of A such that $A = \mathrm{Dr}_{i\in I} V_i$ as vector spaces, and I has cardinality α. Then A^* is isomorphic to $\mathrm{Cr}_{i\in I} V_i$. So $\dim A^* = \beta^{\alpha}$, by lemma 4.4. Similarly $\dim A^{**} = \dim(A^*)^*$ $= \beta^{\beta^{\alpha}}$. But $\beta \leq \beta^{\alpha}$ so that $\beta^{\beta^{\alpha}} = 2^{\beta^{\alpha}}$. □

Remark 4.22

This is the best possible result in general, for let V be a residually finite algebra and give V the cofinite topology determined by taking $\mathfrak{J}(V)$ to be the set of all vector subspaces of finite codimension in V. Then with the notation of construction 4.16, $V' = V^*$ and $\overline{V} = V^{**}$ and $\overline{V}$ is the profinite completion of V, and achieves the bound given in theorem 4.21.

Chapter 5 Radicals of Cofinite Lie Algebras

From now on we shall concentrate on Lie algebras. In this chapter we introduce the concept of cofinitely soluble and cofinitely nilpotent Lie algebras, and generalise the definitions of the soluble and nilpotent radicals from finite dimensional Lie algebras to cofinite Lie algebras.

In view of theorem 2.17 we also consider inverse limits of finite dimensional Lie algebras.

Section 1 Cofinitely Soluble and Cofinitely Nilpotent Lie Algebras

Definition 5.1

Let L be a cofinite Lie algebra. Then L is cofinitely soluble if L/K is soluble for all $K \in \mathcal{X}(L)$.

L is cofinitely nilpotent if L/K is nilpotent for each $K \in \mathcal{X}(L)$.

Note 5.2

If X is a Lie subalgebra of the Lie algebra L, Then X is cofinitely soluble (respectively cofinitely nilpotent) in the relative topology if and only if $(X + K)/K$ is soluble (respectively nilpotent) for each $K \in \mathcal{X}(L)$, since $(X + K)/K$ is isomorphic to $X/(X \cap K)$.

The properties cofinitely soluble and cofinitely nilpotent are preserved by the closure operation:

Proposition 5.3

Let L be a cofinite Lie algebra and let H be a Lie subalgebra of L. Then H is cofinitely soluble (respectively cofinitely nilpotent) if and only if $\overline{H}$ is cofinitely soluble (respectively cofinitely nilpotent).

Proof: Suppose H is cofinitely soluble. Then for each $K \in \mathcal{K}$, $(H + K)/K$ is soluble. So for any $N \in \mathcal{K}$,

$$\frac{\overline{H} + N}{N} = \frac{\bigcap_{K \in \mathcal{K}} (H + K) + N}{N} \leq \frac{H + N}{N}$$

and this is soluble, and hence $\overline{H}$ is cofinitely soluble. The converse is obvious.

The cofinitely nilpotent case is proved in the same way. □

The cofinitely soluble and cofinitely nilpotent properties are also preserved when passing to the profinite completion:

Corollary 5.4

Let L be a cofinite Lie algebra. L is cofinitely soluble (respectively cofinitely nilpotent) if and only if its profinite completion is cofinitely soluble (respectively cofinitely nilpotent).

Further, if $\varphi : L \to P$ is the embedding of L into the profinite completion P, then for any subalgebra H of L, H is cofinitely soluble (respectively cofinitely nilpotent) if and only if $\overline{\varphi(H)}$ is cofinitely soluble (respectively cofinitely nilpotent).

Proof: This follows from proposition 5.3 and the fact that $\overline{\varphi(L)} = P$. □

Proposition 5.5

Suppose $L = \varprojlim\{L_i; \pi_{ij}\}$ and has the usual cofinite topology, where each L_i is a finite dimensional Lie algebra. Let $H = \varprojlim\{H_i; \rho_{ij}\}$ where ρ_{ij} is the restriction of π_{ij} to H_i, for $i \geq j$, be a closed subalgebra of L. Then H is cofinitely soluble (respectively cofinitely nilpotent) if and only if H_i is soluble (respectively nilpotent) for every i in I.

Proof: Suppose that each H_i is soluble. We shall find a finite residual system $\{K_i: i \in I\}$ of closed ideals of L cofinal in $\mathcal{X}(L)$ and show that each L/K_i is soluble, from which we infer that each L/K is soluble for $K \in \mathcal{X}(L)$. For each $j \in I$ let $p_j : L \to L_j$ be the map $(x_i) \to x_j$ and let K_j be the kernel of p_j. For each $i \in I$, $(H + K_i)/K_i$ is isomorphic to H_i, so is soluble. We can see, from part (d) of the proof of proposition 2.12(ii) for example, that $\{K_i: i \in I\}$ is cofinal in $\mathcal{X}(L)$. Consider $K \in \mathcal{X}$. There exists $j \in I$ such that $K_j \leq K$, so the map : $(H + K_j)/K_j \to (H + K)/K$ given by $h + K_j \mapsto h + K$ is an epimorphism. So the fact that $(H + K_j)/K_j$ is soluble implies that $(H + K)/K$ is soluble, and hence H is cofinitely soluble, from note 5.2.

Conversely, if H is cofinitely soluble then $(H + K_i)/K_i$ is soluble for each $i \in I$, which implies that every H_i is soluble.

The nilpotent case is proved similarly. □

Section 2 Radicals of a Cofinite Lie Algebra

We shall define the cofinitely soluble and cofinitely nilpotent radicals of a cofinite Lie algebra and show that they have properties similar to properties of the soluble and nilpotent radicals in finite dimensions.

Notation

Let L be a cofinite Lie algebra. We shall denote $\mathcal{K}(L)$ by $\{K_i : i \in I$ Also, for each $i \in I$, R_i/K_i will denote $\sigma(L/K_i)$, the soluble radical of the finite dimensional Lie algebra L/K_i.

Further, N_i/K_i will denote $\nu(L/K_i)$, the nilpotent radical of L/K_i.

Definition 5.6

Let L be a cofinite Lie algebra. Then

(i) $\sigma(L) = \bigcap_{i \in I} R_i$ is the cofinitely soluble radical of L

(ii) $\nu(L) = \bigcap_{i \in I} N_i$ is the cofinitely nilpotent radical of L.

Remark

These definitions reduce to the corresponding definitions for finite dimensional Lie algebras.

The following results show that $\sigma(L)$ and $\nu(L)$ have a number of expected properties.

Proposition 5.7

Let L be a cofinite Lie algebra. Then

(i) $\sigma(L)$ is the unique maximal cofinitely soluble ideal of L.

(ii) $\nu(L)$ is the unique maximal cofinitely nilpotent ideal of L.

Proof: That $\sigma(L)$ and $\nu(L)$ are ideals is clear.

(i) We show first that $\sigma(L)$ is cofinitely soluble. For $K_j \in \mathcal{K}(L)$, $\sigma(L)/(\sigma(L) \cap K_j) \cong (\sigma(L) + K_j)/K_j = ((\bigcap_{i\in I} R_i) + K_j)/K_j \subseteq \bigcap_{i\in I} (R_i + K_j)/K_j \subseteq R_j/K_j$, which is soluble. Thus $\sigma(L)/(\sigma(L) \cap K_j)$ is soluble, whence $\sigma(L)$ is cofinitely soluble.

We next show that $\sigma(L)$ is a maximal cofinitely soluble ideal of L. For suppose that $\sigma(L) \leqslant R$, where R is a cofinitely soluble ideal of L. Then $(R + K_i)/K_i$ is a soluble ideal of L/K_i for every $i \in I$. So $(R + K_i)/K_i \subseteq \sigma(L/K_i) = R_i/K_i$ for every i, and hence $R \subseteq \bigcap_{i\in I} R_i = \sigma(L)$. Thus $\sigma(L) = R$ and $\sigma(L)$ is maximal.

To show that $\sigma(L)$ is the unique maximal cofinitely soluble ideal of L, suppose that R' is also a maximal cofinitely soluble ideal of L. Then $\sigma(L) + R'$ is a cofinitely soluble ideal of L and hence $\sigma(L) = R'$, by maximality.

(ii) The proof for (L) is similar. □

Proposition 5.8

(i) If L is a cofinite Lie algebra, then $\sigma(L)$ and $\nu(L)$ are closed ideals of L.

(ii) If L is a profinite Lie algebra, then $\sigma(L)$ and $\nu(L)$ are profinite (in the relative topology).

Proof: (i) For each $i \in I$, R_i is a subalgebra of L containing K_i, so is closed, by C3. Therefore $\sigma(L)$, being the intersection of the R_i's, is closed. Similarly $\nu(L)$ is closed.

(ii) If L is compact, then $\sigma(L)$ is a closed ideal, by the above, so is compact by theorem 1.9 and hence is profinite. Similarly for $\nu(L)$. □

Proposition 5.9

Let L be a cofinite Lie algebra. Then

(i) $\nu(L) \subseteq \sigma(L)$.

(ii) $[L,\sigma(L)] \subseteq \nu(L)$.

(iii) $\sigma(L)^2$ is cofinitely nilpotent.

Proof: (i) For each $i \in I$, $N_i/K_i \subseteq R_i/K_i$, so that $N_i \subseteq R_i$ and hence $\nu(L) = \bigcap_{i\in I} N_i \subseteq \bigcap_{i\in I} R_i = \sigma(L)$.

(ii) $[L/K_i, (R + K_i)/K_i] \subseteq N_i/K_i$, from the finite dimensional theory, since $R + K_i \subseteq R_i$. So $[L,R] \subseteq N_i$ for every $i \in I$. Therefore $[L,R] \subseteq \bigcap_{i\in I} N_i = \nu(L)$, where $R = \sigma(L)$.

(iii) $\sigma(L)^2 \subseteq [L,\sigma(L)] \subseteq \nu(L)$, from part (ii) above. So $\sigma(L)^2$ is a subalgebra of a cofinitely nilpotent Lie algebra and hence is cofinitely nilpotent. □

Note

Let L be a cofinite Lie algebra and suppose that H is a subalgebra of L. Then $\sigma(H)$ is the cofinitely soluble radical of H obtained by giving H the relative topology i.e. if $Q_i/H_i = \sigma(H/(H \cap K_i)$ for each $i \in I$, then $\sigma(H) = \bigcap_{i \in I} Q_i$.

Recall that for a Lie algebra L, an ideal of L which is invariant under all derivations of L is called a characteristic ideal of L.

Proposition 5.10

Let L be a cofinite Lie algebra and let H be an ideal of L. Then
(i) $\sigma(H)$, $\nu(H)$ are ideals of L.
(ii) If H is closed in L, then $\sigma(H)$ and $\nu(H)$ are closed in L.

Proof: (i) Let $H_i = H \cap K_i$ and let $Q_i/H_i = \sigma(H/H_i)$ for each $i \in I$. Then H and H_i are ideals of L, so that H/H_i is an ideal of L/H_i. Also, $\sigma(H/H_i)$ is a characteristic ideal of H/H_i, so from Amayo and Stewart, [1], lemma 4.4, p13, $Q_i/H_i = \sigma(H/H_i)$ is an ideal of L/H_i, since Q_i/H_i is a characteristic ideal of H/H_i, which is an ideal of L/H_i. Thus Q_i is an ideal of L for every $i \in I$, and so $\sigma(H) = \bigcap_{i \in I} Q_i$ is an ideal of L.

Similarly $\nu(H)$ is an ideal of L.

(ii) By proposition 5.8, $\sigma(H)$ and $\nu(H)$ are closed in H, so if H is closed in L it follows that $\sigma(H)$ and $\nu(H)$ are closed in L. □

Corollary 5.11

Let L be a cofinite Lie algebra and let H be an ideal of L. Then

(i) $\sigma(H) = H \cap \sigma(L)$.

(ii) $\nu(H) = H \cap \nu(L)$.

Proof: (i) From proposition 5.10 $\sigma(H)$ is an ideal of L and so is a cofinitely soluble ideal of L. This implies that for every $i \in I$, $(\sigma(H) + K_i)/K_i$ is a soluble ideal of L/K_i and so $(\sigma(H) + K_i)/K_i$ is contained in R_i/K_i. Thus $\sigma(H) \subseteq R_i$ for every $i \in I$, from which it follows that $\sigma(H) \subseteq \bigcap_{i \in I} R_i = \sigma(L)$. Thus $\sigma(H) \subseteq H \cap \sigma(L)$.

Conversely, $H \cap \sigma(L)$ is a cofinitely soluble ideal of H, so by proposition 5.7, $H \cap \sigma(L) = \sigma(H)$.

(ii) This is proved in the same way as part (i). □

We now show that if $L = \lim\limits_{\leftarrow}\{L_i\}$ then $\sigma(L)$ is the inverse limit of the soluble radicals of the L_i's, and that the analogous result holds for (L).

Proposition 5.12

Let $L = \lim\limits_{\leftarrow}\{L_i;\ \pi_{ij}:\ i \in I\}$, where each L_i is a finite dimensional Lie algebra. Let ϱ_{ij} and σ_{ij} be the restrictions of π_{ij} to $\sigma(L_i)$ and

$\nu(L_i)$ respectively. Then

(i) $\sigma(L) = \varprojlim\{\sigma(L_i); \rho_{ij}\}$.

(ii) $\nu(L) = \varprojlim\{\nu(L_i); \sigma_{ij}\}$.

Proof: (i) Let $R = \sigma(L)$ and for each j in I define p_j and K_j as in proposition 5.5. Let $R' = \varprojlim\{\sigma(L_i)\}$. We show that $R = R'$. Now R is closed in L, by proposition 5.8, so there exist subalgebras H_i of L_i for each i such that $R = \varprojlim\{H_i\}$, and in view of proposition 2.10 we may consider the restriction of each π_{ij} to H_i to be surjective onto H_j. Now $(R + K_j)/K_j \cong p_j(R) = H_j$. Therefore each H_j is soluble. so that $H_j \leqslant \sigma(L_j)$. Thus $R \leqslant R'$.

Conversely, $(R' + K_i)/K_i = p_i(R') \leqslant \sigma(L_i)$, so is soluble. Therefore $R' \leqslant R_i$ for every $i \in I$ and so $R' \leqslant \bigcap_{i \in I} R_i = R$.

(ii) This is proved in the same way as part (i). □

Lemma 5.13

Let L be a cofinite Lie algebra and let $\pi_{ij} : L/K_i \to L/K_j$ be the canonical map : $x + K_i \mapsto x + K_j$ whenever $j \leqslant i$. Then

$$\pi_{ij}(R_i/K_i) = R_j/K_j \text{ .}$$

Proof: $\pi_{ij}(R_i/K_i) = (R_i + K_j)/K_j$, which is an ideal of L/K_j, since R_i is an ideal of L. But R_i/K_i is soluble, so $(R_i + K_j)/K_j$ is soluble. Therefore $\pi_{ij}(R_i/K_i) = (R_i + K_j)/K_j \subseteq \sigma(L/K_j) = R_j/K_j$.

Conversely, L/R_i is isomorphic to L/K_i factored by R_i/K_i, which is L/K_i factored by its soluble radical, and this is semisimple; hence L/R_i is semisimple. Now the map : $L/R_i \to L/(R_i + K_j)$ given by $x + R_i \mapsto x + (R_i + K_j)$ is a projection from a finite dimensional semisimple Lie algebra, so its image is semisimple; i.e. $L/(R_i + K_j)$ is semisimple. But this is isomorphic to L/K_j factored by $(R_i + K_j)/K_j$, which must therefore be semisimple. So $(R_i + K_j)/K_j$ contains $\sigma(L/K_j)$, which equals R_i/K_j. □

<u>Lemma 5.14</u>

Suppose $\varprojlim\{H_i; \rho_{ij}: i,j \in I\} = \varprojlim\{L_i; \pi_{ij}: i,j \in I\}$, where for each $i \in I$, H_i and L_i are finite dimensional Lie algebras, and H_i is a subalgebra of L_i with ρ_{ij} being the restriction of π_{ij} to H_i whenever $j \leq i$. Suppose that all of the π_{ij}'s are surjective.

Then $H_i = L_i$ for every $i \in I$.

<u>Proof:</u> Suppose there exists $j \in I$ such that $H_j \neq L_j$ i.e. such that $H_j < L_j$. We derive a contradiction to show that this is not possible. Consider $y \in L_j \setminus H_j$. Now there exists $(x_i) \in \varprojlim\{L_i\}$ such that $x_j = y$, by theorem 2.5(ii). Then $(x_i) \notin \varprojlim\{H_i\}$, which is the required contradiction. □

<u>Proposition 5.15</u>

Let L be a profinite Lie algebra. Then $R_i = \sigma(L) + K_i$ for every $i \in I$.

Proof: Since L is profinite the embedding $\varphi : L \to \varprojlim\{L/K_i; \pi_{ij}\}$ defined as in proposition 2.11 is a topological isomorphism. Let $R = \sigma(L)$. Now $\varphi(R) = \varphi(\sigma(L)) = \varprojlim\{R_i/K_i\}$, by proposition 5.12(i). But $\varphi(R) = \varprojlim\{(R + K_i)/K_i\}$, and $\pi_{ij} : R_i/K_i \to R_j/K_j$ is surjective, by lemma 5.14. Thus $R_i/K_i = (R + K_i)/K_i$ for all $i \in I$, by lemma 5.14, and hence $R_i = R + K_i$ for every i. □

Note

The corresponding result for the cofinitely nilpotent radical, i.e. that for each i, $N_i = \nu(L) + K_i$, is not true in general. For consider a finite dimensional soluble Lie algebra L which is not nilpotent and take $K_i = L^2$. Then $\nu(L/L^2) = L/L^2$ so that $N_i = L$. But $\nu(L) + K_i = \nu(L) + L^2 = \nu(L)$, and $\nu(L) \neq L$.

Proposition 5.16

Let L be a cofinite Lie algebra and let $\varphi : L \to \varprojlim\{L/K_i\} = P$ be the natural embedding defined in proposition 2.11. Then

(i) $\varphi(\sigma(L)) = \varphi(L) \cap \sigma(P)$.

(ii) $\varphi(\nu(L)) = \varphi(L) \cap \nu(P)$.

Proof: (i) The map φ is a topological and algebraic embedding, by corollary 2.18(i), and since $\sigma(L)$ is cofinitely soluble, $\varphi(\sigma(L))$ is cofinitely soluble. Thus by proposition 5.7(i), $\varphi(\sigma(L))$ contains $\varphi(L) \cap \sigma(P)$.

Conversely, by proposition 5.12, $\sigma(P)$ equals $\varprojlim\{R_i/K_i\}$, which contains $\varprojlim\{(\sigma(L) + K_i)/K_i\}$, and this in turn contains $\varphi(\sigma(L))$.

(ii) is proved similarly. □

Section 3 Pro-soluble and Pro-nilpotent Lie Algebras

In this section we define pro-soluble and pro-nilpotent Lie algebras analogously to the definitions of cofinitely soluble and cofinitely nilpotent Lie algebras (see definition 5.1) and give equivalent definitions for them.

Definition 5.17

A Lie algebra L is pro-soluble if L is profinite and L/K is soluble for every $k \in \mathcal{K}(L)$.

L is pro-nilpotent if L is profinite and L/K is nilpotent for every $K \in \mathcal{K}(L)$.

It is obvious that a pro-nilpotent Lie algebra is pro-soluble.

An alternative definition to 5.17 is indicated by the following (obvious) result:

Proposition 5.18

Let L be a profinite Lie algebra. Then

(i) L is pro-soluble if and only if $\sigma(L) = L$.

(ii) L is pro-nilpotent if and only if $\nu(L) = L$.

Proof: (i) Let $\mathcal{K}(L) = \{K_i : i \in I\}$ and let $\sigma(L/K_i) = R_i/K_i$. Suppose that L is pro-soluble. Then each L/K_i is soluble, so that $R_i/K_i = L/K_i$

for each $i \in I$. Therefore $R_i = L$ for all i and so $\sigma(L) = \bigcap_{i \in I} R_i = L$.

Conversely, suppose that $\sigma(L) = L$. Then $R_i = L$ for each $i \in I$ so that $\sigma(L/K_i) = R_i/K_i = L/K_i$ and it follows that each L/K_i is soluble.

(ii) is proved in the same way. □

Of course the same proof will show that a cofinite Lie algebra L is cofinitely soluble (respectively cofinitely nilpotent) if and only if $\sigma(L) = L$ (respectively $\nu(L) = L$).

Chapter 6 Pro-semisimple Lie Algebras

Section 1 Equivalent Forms of Pro-semisimple Lie Algebras

We Shall give the obvious definition of semisimple and pro-semisimple for cofinite and profinite Lie algebras. We shall show that a Lie algebra is pro-semisimple if and only if it is topologically isomorphic to an inverse limit of finite dimensional semisimple Lie algebras, and also that L is pro-semisimple if and only if L is topologically isomorphic to a Cartesian sum of finite dimensional simple Lie algebras.

Definition 6.1

A cofinite Lie algebra is semisimple if it has trivial cofinitely soluble radical.

A Lie algebra is pro-semisimple if it is profinite and semisimple.

Remark

It follows from the remark after definition 5.6 that this definition is consistent with the corresponding definition in finite dimensions.

We first show that a pro-semisimple Lie algebra can be regarded as an inverse limit of finite dimensional semisimple Lie algebras.

Theorem 6.2

Let L be a Lie algebra. L is pro-semisimple if and only if L is topologically isomorphic to an inverse limit of finite dimensional semisimple Lie algebras, this inverse limit having the usual cofinite topology.

Proof: Suppose that L is pro-semisimple. By corollary 2.18 the embedding $\varphi : L \to P = \varprojlim \{L/K_i\}$ is a topological isomorphism, where P has the usual cofinite topology, so $\sigma(P) = 0$, since $\sigma(L) = 0$. Hence it suffices to show that each L/K_i is semisimple. But by proposition 5.12(i), $\sigma(P) = \varprojlim \{\sigma(L/K_i)\}$. From lemma 5.13 and lemma 5.14 we see that $\sigma(L/K_i) = 0$ for every i, so that each L/K_i is semisimple.

Conversely, suppose that L is topologically isomorphic to $\varprojlim \{L_i ; \rho_{ij}\}$, where each L_i is a finite dimensional semisimple Lie algebra. Without loss of generality, we may assume equality. By proposition 2.10 we may assume also that the ρ_{ij}'s are surjective. Let K_j be the kernel of the projection map $p_j : L \to L_j$. Then each L/K_j, being isomorphic to L_j, is semisimple, and $\{K_i : i \in I\}$ is a finite residual system which is cofinal in $\mathcal{K}(L)$. As in the proof of proposition 5.5 we see that L/K is semisimple for each $K \in \mathcal{K}(L)$, so that L is pro-semisimple. □

Corollary 6.3

Let L be a pro-semisimple Lie algebra. Then L/K is semisimple for every $K \in \mathcal{X}(L)$. □

Corollary 6.4

If L is a profinite Lie algebra, then $L/\sigma(L)$ is pro-semisimple in the topology induced from L.

Proof: Let $R = \sigma(L)$. By proposition 5.8 R is closed in L, so using proposition 1.15 we see that L/R is profinite. Now $\mathcal{X}(L/R) = \{(K + R)/R : K \in \mathcal{X}(L)\}$ from definition 1.14, so it suffices to show that L/R factored by (K + R)/R is semisimple for all $K \in \mathcal{X}$. But this is isomorphic to L/(K + R), which in turn is isomorphic to L/K factored by (R + K)/K, which is semisimple, since $(R + K)/K = \sigma(L/K)$, by proposition 5.15. □

Recall that an ideally finite Lie algebra is a Lie algebra generated by finite dimensional ideals (see for example Stewart, [15], pp34-35).

In finite dimensions a semisimple Lie algebra is a direct sum of simple Lie algebras, and this result can be extended to the ideally finite case so that an ideally finite Lie algebra which is semisimple is a direct sum of finite dimensional ideals (see Stewart, [15], p44). The analogous result in the profinite case (theorem 6.7)

Corollary 6.3

Let L be a pro-semisimple Lie algebra. Then L/K is semisimple for every $K \in \mathcal{K}(L)$. □

Corollary 6.4

If L is a profinite Lie algebra, then $L/\sigma(L)$ is pro-semisimple in the topology induced from L.

Proof: Let $R = \sigma(L)$. By proposition 5.8 R is closed in L, so using proposition 1.15 we see that L/R is profinite. Now $\mathcal{K}(L/R) = \{(K + R)/R : K \in \mathcal{K}(L)\}$ from definition 1.14, so it suffices to show that L/R factored by $(K + R)/R$ is semisimple for all $K \in \mathcal{K}$. But this is isomorphic to $L/(K + R)$, which in turn is isomorphic to L/K factored by $(R + K)/K$, which is semisimple, since $(R + K)/K = \sigma(L/K)$, by proposition 5.15. □

Recall that an ideally finite Lie algebra is a Lie algebra generated by finite dimensional ideals (see for example Stewart, [15], pp34-35).

In finite dimensions a semisimple Lie algebra is a direct sum of simple Lie algebras, and this result can be extended to the ideally finite case so that an ideally finite Lie algebra which is semisimple is a direct sum of finite dimensional ideals (see Stewart, [15], p44). The analogous result in the profinite case (theorem 6.7) is that a pro-semisimple Lie algebra can be regarded as a Cartesian sum of finite dimensional simple Lie algebras. Before proving this we need the following lemma, which will also be useful later on:

Lemma 6.5

Let A be a profinite algebra, with $\mathcal{X}(A) = \{K_i : i \in I\}$. Suppose that for each $i \in I$ there exists a vector subspace H_i of A such that $K_i \leq H_i \leq A$ and such that $H_j + K_i = H_i$ whenever $i, j \in I$ and $j \geq i$. Let $H = \bigcap_{i \in I} H_i$.

Then $H + K_i = H_i$ for every $i \in I$.

Proof: Fix $i \in I$ and let $x \in H_i$. We show that $x \in H + K_i$, which will give the required result, since then $H_i \subseteq H + K_i$, and the converse is clear. For $j \geq i$, $K_j \leq K_i$, so we have $x \in H_j + K_i$. Then there exist $k \in H_j$ and $y \in K_i$ such that $x = k + y$, and then $y \in (x + H_j) \cap K_i$ i.e. $(x + H_j) \cap K_i$ is non-empty. But $x + H_j$ and K_i are both closed, so $(x + H_j) \cap K_i$ is closed. Also, $\{x + H_j : j \geq i\}$ has the finite intersection property, since $(x + H_{j_1}) \cap \ldots \cap (x + H_{j_n})$ contains $(x + K_{j_1}) \cap \ldots \cap (x + K_{j_n})$, which equals $x + \bigcap_{r=1}^{n} K_{j_r}$, which is non-empty. So, using the compactness of A,

$$\bigcap_{j \geq i} ((x + H_j) \cap K_i) \text{ is non-empty ------ (1)}$$

We shall show that $\bigcap_{j \geq i} H_j = \bigcap_{j \in I} H_j$ and shall use this together with (1) to see that $(x + H) \cap K_i$ is non-empty. Now $\bigcap_{j \in I} H_j$ is obviously contained in $\bigcap_{j\geq i} H_j$. Conversely, let $K_j \in \mathcal{K}(A)$ and let

$K_l = K_j \cap K_i$. Then $K_l \in \mathcal{K}(A)$ and $i \leq l$. Therefore $H_l \leq H_j$ ---(2)

So $\bigcap_{j\geq i} H_j \leq \bigcap_{\substack{j\geq i \\ K_l = K_j \cap K_i}} H_l \leq \bigcap_{j\in I} H_j$, by (2).

Thus $\bigcap_{j\geq i} H_j = \bigcap_{j\in I} H_j$, as claimed.

$$\begin{aligned} \text{So } \bigcap_{j\geq i} ((x + H_j) \cap K_i) &= \left(\bigcap_{j\geq i} (x + H_j)\right) \cap K_i \\ &= (x + \bigcap_{j\geq i} H_j) \cap K_i \\ &= (x + \bigcap_{j\in I} H_j) \cap K_i \\ &= (x + H) \cap K_i \end{aligned}$$

So by (1), $(x + H) \cap K_i$ is non-empty, from which it follows that $x \in H + K_i$, as required. □

Note that the same result, with $\mathcal{K}(A)$ replaced by $\mathcal{J}(A)$ can be proved in the same way.

For the proof of theorem 6.7 we extract the following corollary from lemma 6.5:

Corollary 6.6

With the notation of lemma 6.5, fix $i \in I$ and let $H' = \bigcap_{j\geq i} H_j$.

Then $H' + K_i = H_i$.

Proof: H is contained in H', so $H_i = H + K_i \subseteq H' + K_i$.

Conversely, H_j is contained in H_i for every $j \geqslant i$, since $H_j + K_i = H_i$. Then $H' = \bigcap_{j \geq i} H_j \subseteq H_i$, and so $H' + K_i \subseteq H_i$. □

Theorem 6.7

A Lie algebra L is pro-semisimple if and only if it is topologically isomorphic to a Cartesian sum of finite dimensional simple Lie algebras, where the topology on this sum is that described in proposition 4.1.

Further, these finite dimensional simple Lie algebras are the unique minimal ideals of L.

Proof: Suppose L is pro-semisimple, and fix $K \in \mathcal{X}(L)$. For any $H \in \mathcal{X}$ such that H is contained in K, L/H is semisimple, by corollary 6.3, so there exists a unique ideal U_H of L such that $L/H = K/H \oplus U_H/H$. Let $X_K = \bigcap_{\substack{H \leqslant K \\ H \in \mathcal{X}}} U_H$. We shall use corollary 6.6 in showing that $L = X_K \oplus K$.

If $H, H' \in \mathcal{X}$ and $H' \leqslant H$ with $\pi : L/H' \to L/H$ being the map $x + H' \mapsto x + H$, then $\pi(K/H') = K/H$. Then $\pi(U_{H'}/H') = U_H/H$, since L/H is finite dimensional and semisimple. But $\pi(U_{H'}/H') = (U_{H'} + H)/H$. Since H' is contained in H, $U_{H'} + H = U_H$. By corollary 6.5 we see that $X_K + H = U_H$. Then $L = U_H + K = X_K + H + K = X_K + K$, since $H \leqslant K$. We show that $L = U_H + K$ is a direct sum. Now $X_K \cap K \subseteq U_H \cap K$ if $H \leqslant K$, since X_K is the intersection of the U_H's, and $U_H \cap K$ is contained in

Proof: H is contained in H', so $H_i = H + K_i \subseteq H' + K_i$.

Conversely, H_j is contained in H_i for every $j \geq i$, since $H_j + K_i = H_i$. Then $H' = \bigcap_{j \geq i} H_j \subseteq H_i$, and so $H' + K_i \subseteq H_i$. □

Theorem 6.7

A Lie algebra L is pro-semisimple if and only if it is topologically isomorphic to a Cartesian sum of finite dimensional simple Lie algebras, where the topology on this sum is that described in proposition 4.1.

Further, these finite dimensional simple Lie algebras are the unique minimal ideals of L.

Proof: Suppose L is pro-semisimple, and fix $K \in \mathcal{K}(L)$. For any $H \in \mathcal{K}$ such that H is contained in K, L/H is semisimple, by corollary 6.3, so there exists a unique ideal U_H of L such that $L/H = K/H \oplus U_H/H$. Let $X_K = \bigcap_{\substack{H \leq K \\ H \in \mathcal{K}}} U_H$. We shall use corollary 6.6 in showing that $L = X_K \oplus K$.

If $H, H' \in \mathcal{K}$ and $H' \leq H$ with $\pi : L/H' \to L/H$ being the map $x + H' \mapsto x + H$, then $\pi(K/H') = K/H$. Then $\pi(U_{H'}/H') = U_H/H$, since L/H is finite dimensional and semisimple. But $\pi(U_{H'}/H') = (U_{H'} + H)/H$. Since H' is contained in H, $U_{H'} + H = U_H$. By corollary 6.5 we see that $X_K + H = U_H$. Then $L = U_H + K = X_K + H + K = X_K + K$, since $H \leq K$. We show that $L = U_H + K$ is a direct sum. Now $X_K \cap K \subseteq U_H \cap K$ if $H \leq K$, since X_K is the intersection of the U_H's, and $U_H \cap K$ is contained in

H since $K/H \cap U_H/H = 0$. Therefore $X_K \cap K \subseteq \bigcap_{H \leq K} H = 0$. Also, for any H contained in K, $[K,X_K] \subseteq [K,U_H] \subseteq H$.

So $[K,X_K] \subseteq \bigcap_{\substack{H \leq K \\ H \in \mathcal{K}}} H = 0$. Thus $L = X_K \oplus K$, and consequently X_K is finite dimensional and is uniquely defined. Also, X_K is isomorphic to L/K, so is semisimple, by corollary 6.3.

Let $X = \sum_{K \in \mathcal{K}} X_K$. We shall show that X is a direct sum of finite dimensional simple ideals of L, and that X has L as profinite completion, so that L is topologically isomorphic to the Cartesian sum of these finite dimensional simple ideals. Now X is ideally finite, and since each X_K is semisimple X is the direct sum of finite dimensional simple Lie algebras, and these are just the unique minimal ideals (see Stewart, [15], theorem 4.8, p44). So $X = \bigoplus_{\lambda \in \Lambda} Y_\lambda$, say, the direct sum of the minimal ideals Y_λ of L. Now for any $K \in \mathcal{K}$, $X + K = \sum_{H \in \mathcal{K}} X_H + K \supseteq X_K + K = L$, from above, so that by proposition 1.16 X is dense in L.

We therefore have that X is a cofinite Lie algebra which is a dense subalgebra of the profinite Lie algebra L, and hence L is the profinite completion of X. But by proposition 4.1, $Cr_{\lambda \in \Lambda} Y_\lambda$ is the profinite completion, so by corollary 3.4 L is topologically isomorphic to $Cr_{\lambda \in \Lambda} Y_\lambda$, where the Y_λ's are the minimal ideals of L.

Conversely, suppose that L is topologically isomorphic to $Cr_{\lambda \in \Lambda} Y_\lambda$, where each Y_λ is a finite dimensional simple Lie algebra.

Then as in proposition 4.1, L is topologically isomorphic to $\varprojlim\{L/K: K \in \mathcal{K}(L)\}$, where each L/K is finite dimensional and semisimple. Hence L is pro-semisimple, by proposition 6.2. □

Corollary 6.8

Let L be a cofinite semisimple Lie algebra, and suppose that L/K is semisimple for each $K \in \mathcal{K}(L)$. Then L embeds algebraically in a Cartesian sum P of finite dimensional simple Lie algebras.

If, further, P is given the cofinite topology defined in proposition 4.1, then this embedding is also a topological embedding.

Proof: The canonical embedding : $L \to \varprojlim\{L/K: K \in \mathcal{K}\}$, which is topologically isomorphic to P, will do. □

Proposition 6.9

Let L be a pro-semisimple Lie algebra. Then L is isomorphic (as a Lie algebra) to Inn(L), the Lie algebra of inner derivations of L.

Proof:
Follows from $S_1(L) = 0$. □

Note
There are no pages 111 – 118 inclusive.
6.10 – 6.17 do not exist.

Then as in proposition 4.1, L is topologically isomorphic to $\varprojlim\{L/K: K \in \mathcal{X}(L)\}$, where each L/K is finite dimensional and semisimple. Hence L is pro-semisimple, by proposition 6.2. □

Corollary 6.8

Let L be a cofinite semisimple Lie algebra, and suppose that L/K is semisimple for each $K \in \mathcal{X}(L)$. Then L embeds algebraically in a Cartesian sum P of finite dimensional simple Lie algebras.

If, further, P is given the cofinite topology defined in proposition 4.1, then this embedding is also a topological embedding.

Proof: The canonical embedding : $L \to \varprojlim\{L/K: K \in \mathcal{X}\}$, which is topologically isomorphic to P, will do. □

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Let L be a pro-semisimple Lie algebra. Then L is isomorphic (as a Lie algebra) to Inn(L), the Lie algebra of inner derivations of L.

Proof:
Follows from $\zeta_1(L) = 0$. □

Note
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Section 2 Some Further Properties of Pro-semisimple Lie Algebras

We show that in a pro-semisimple Lie algebra closed ideals are pro-semisimple, as are finite dimensional quotients. Also, any closed ideal is a direct summand, having its centraliser as its unique complement.

Lemma 6.18

Let L be a cofinite semisimple Lie algebra and let H be an ideal of L. Then H is semisimple (in the relative topology).

Proof: Let $R = \sigma(H)$. Then R is an ideal of L, by proposition 5.10. But for each $K \in \mathcal{K}$, $(R + K)/K$ is soluble, so is contained in $\sigma(L/K)$. Thus $R \leqslant \sigma(L) = 0$, so that $R = 0$ and H is semisimple. □

Proposition 6.19

Let L be a pro-semisimple Lie algebra and let H be a closed ideal of L. Then H is pro-semisimple in the relative topology.

Proof: The first part follows from lemma 6.18 and theorem 1.9. □

Remark 6.20

In the hypotheses of proposition 6.19 the condition that H be closed is necessary. For consider $L = Cr_{i \in I} L_i$ and $H = Dr_{i \in I} L_i$, where each L_i is a finite dimensional simple Lie algebra. Then L is pro-semisimple and H is an ideal of L, But H is not profinite.

Theorem 6.21

Let L be a pro-semisimple Lie algebra and suppose that H is a closed ideal of L. Then H is a direct summand of L.

Further, the complement of H in L is unique, and if $L = H + H'$, then H' is the centraliser of H in L and is pro-semisimple. (in the relative topology).

Proof: (i) We first obtain a complement of H in L. For each $i \in I$, L/K_i is semisimple, by corollary 6.3, so $L/K_i = (H + K_i)/K_i \oplus U_i/K_i$, where U_i/K_i is uniquely determined. Let π_{ij} be the canonical map $: L/K_i \to L/K_j$ whenever $j \leq i$. Then $\pi_{ij}((H + K_i)/K_i) = (H + K_j)/K_j$, and so as each L/K_i is semisimple, $\pi_{ij}(U_i/K_i) = U_j/K_j$. Hence $U_i + K_j = U_j$ whenever $j \leq i$.

Let $H' = \bigcap_{i \in I} U_i$. We show that this is a complement of H in L. By lemma 6.5 $U_i = H' + K_i$ for each $i \in I$. Therefore L/K_i equals $(H + K_i)/K_i \oplus (H' + K_i)/K_i$ for all i and so $(H + K_i) \cap (H' + K_i) \subseteq K_i$. Also, $L = \bigcap_{i \in I} (H + H' + K_i)$, which equals $H + H'$, by proposition 2.28. Now $H \cap H' = \bigcap_{i \in I} (H + K_i) \cap \bigcap_{i \in I} (H' + K_i) = \bigcap_{i \in I}((H + K_i) \cap (H' + K_i)) \subseteq \bigcap_{i \in I} K_i = 0$. Therefore $L = H \oplus H'$, and so H is a direct summand of L.

(ii) To see that H' is the centraliser of H in L, note firstly that $H' \subseteq C_L(H)$, since the sum $L = H + H'$ is direct. So consider x in $C_L(H)$. We show that $x \in H'$. Now $[x,H] = 0$, implying that $[x + K_i, H + K_i] \subseteq K_i$ and hence $[x + K_i, (H + K_i)/K_i] = 0$. I.e. $x + K_i$ is in the centraliser in L/K_i of $(H + K_i)/K_i$; but this is U_i/K_i. Hence $x \in U_i$ for each $i \in I$. Therefore $x \in \bigcap_{i \in I} U_i = H'$, as required.

(iii) Finally, H' is pro-semisimple, by proposition 6.19. □

Chapter 7 Levi Subalgebras of Profinite Lie Algebras

For a profinite Lie algebra L we shall define Levi subalgebras in a way that generalises from finite dimensions, and shall show that these are just the subalgebras Λ such that $(\Lambda + K)/K$ is a Levi subalgebra of L/K for each $K \in \mathcal{K}(L)$. We shall also show that they can be regarded, in a sense to be made clear, as inverse limits of finite dimensional Levi subalgebras. In chapter 10 we shall give a conjugacy theorem for Levi subalgebras of profinite Lie algebras.

Recall that a Levi subalgebra of a finite dimensional Lie algebra is a subalgebra that complements the soluble radical.

Notation

For an element x of the Lie algebra L, we shall denote by x^* the adjoint map : $L \to L$ given by $yx^* = [y,x]$.

We extract the following notation from Humphreys, [9], p82, for a finite dimensional Lie algebra L. An element x of L is strongly ad-nilpotent if there exists $y \in L$ and some non-zero eigenvalue λ of y^* such that x lies in the weight space L_λ in the weight space decomposition relative to y^*.

We shall denote by $\mathcal{N}(L)$ the set of strongly ad-nilpotent elements of L, and by $\mathcal{E}(L)$ the subgroup of Int(L) (the group generated by the inner automorphisms of L) generated by all $\exp x^*$ for $x \in \mathcal{N}(L)$.

Further Notation

Let L be a cofinite Lie algebra. Then $\mathcal{L}_i$ will denote the set of all Levi subalgebras of L/K_i, for each $i \in I$.

E_i will denote $\mathcal{E}(L/K_i)$ for each $i \in I$.

S_i will denote the set $\{\theta \in E_i : \theta(\Lambda) = \Lambda\}$ for some fixed Λ in $\mathcal{L}_i$ (we shall see later that for present purposes this choice of Λ is arbitrary).

φ will denote the embedding $: L \to P = \varprojlim\{L/K_i\}$ given by $x \mapsto (x + K_i)$.

Definition 7.1

Let L be a cofinite Lie algebra. Then Λ is a Levi subalgebra of L if Λ complements the cofinitely soluble radical of L, i.e. if $L = \Lambda \dotplus \sigma(L)$, and if Λ is closed.

We only concern ourselves here with Levi subalgebras of profinite Lie algebras. The following, 7.2 to 7.8, will effectively prove the existence of Levi subalgebras in a profinite Lie algebra. These results are proved in a different guise in Stewart, [16], but brief proofs are given here, as they will be used to indicate analogous proofs for Borel and Cartan subalgebras, and because the proof of theorem 7.12 is suggested by these results.

The idea is to show that each $\mathcal{L}_i$ is a coset variety and to find affine maps $\rho_{ij} : \mathcal{L}_i \to \mathcal{L}_j$ for $j \leq i$ and then apply theorem 2.22.

Lemma 7.2

For each $i \in I$, there is a bijection between the set of cosets E_i/S_i and $\mathcal{L}_i$.

Proof: The map $f_i : E_i/S_i \to \mathcal{L}_i$ given by $\theta S_i \to \theta(\Lambda)$ will serve as the required bijection. It is easy enough to show that f_i is injective, and that f_i is surjective follows from the fact that E_i is transitive on $\mathcal{L}_i$, by Stewart, [16], lemma 8.4, p208. □

Lemma 7.3

For each $i \in I$, E_i/S_i is a coset variety.

Proof: The sets E_i and S_i are algebraic groups, by Stewart, [16], p191, and S_i is an algebraic subgroup of E_i, so is Z-closed in E_i (i.e. is closed in the Zariski topology). □

Lemma 7.4

For fixed $i \in I$, a different choice of Λ in S_i gives the same coset variety structure on E_i/S_i, i.e. if S_i and S_i' are the stabilisers of Λ and Λ' respectively, then E_i/S_i and E_i/S_i' are homeomorphic when each has the $\mathcal{W}$-topology.

Proof: By Stewart, [16], lemma 8.4, p208, there exists $\theta \in E_i$ such that $\theta(\Lambda) = \Lambda'$. The map $\psi : E_i \to E_i$ given by $x \mapsto \theta x \theta^{-1}$ is a morphism of algebraic groups and induces the map $\tilde{\psi} : E_i/S_i \to E_i/S_i'$ given by $xS_i \mapsto \theta x \theta^{-1} S_i' = \psi(x)S_i'$. Now $\psi(S_i) \subseteq S_i'$, and

$\tilde{\psi}$ is a bijection; hence $\tilde{\psi}$ is a bijective affine map. By Stewart, [14] p189, affine maps between coset varieties having the $\mathcal{M}$-topology are continuous and closed. Hence $\tilde{\psi}$ is a homeomorphism. □

Remark 7.5

We may therefore give each $\mathcal{L}_i$ the coset variety structure of E_i/S_i, taking any convenient choice for Λ, or S_i, without affecting the topological structure of $\mathcal{L}_i$.

7.6

We now find affine maps $\rho_{ij} : \mathcal{L}_i \to \mathcal{L}_j$ for $j \leq i$. Fix j and i such that $j \leq i$. In order to find such maps we first find affine maps $\tilde{\psi}_{ij} : E_i/S_i \to E_j/S_j$, and using the bijections of lemma 7.2 we find a corresponding ρ_{ij} to fit our requirements. The canonical epimorphism $\pi_{ij} : L/K_i \to L/K_j$ induces the map $\psi_{ij} : E_i \to E_j$ given by $\exp x^* \mapsto \exp((\pi_{ij}(x))^*)$ for $x \in \mathcal{N}(L/K_i)$, and ψ_{ij} is surjective, by Humphreys, [9], p82. Further, ψ_{ij} is a morphism of algebraic groups. ψ_{ij} itself induces the map $\tilde{\psi}_{ij}$: $E_i/S_i \to E_j/S_j$ defined by $\tilde{\psi}_{ij}(xS_i) = \psi_{ij}(x)S_j$, where if S_i is the stabiliser of Λ we choose S_j to be the stabiliser of $\pi_{ij}(\Lambda)$. (Note that $\pi_{ij}(\Lambda) \in \mathcal{L}_j$, since epimorphisms of finite dimensional Lie algebras preserve Levi subalgebras)

Now $\psi_{ij}(S_i) \subseteq S_j$ and $\tilde{\psi}_{ij}$ is surjective, so $\tilde{\psi}_{ij}$ is a surjective affine map. We may therefore give $\mathcal{L}_i$ and $\mathcal{L}_j$ the coset variety structures of E_i/S_i and E_j/S_j respectively via the bijections f_i and f_j defined in lemma 7.2. Then $\tilde{\psi}_{ij}$ induces the map $\rho_{ij} : \mathcal{L}_i \to \mathcal{L}_j$ given by $\Lambda_i/K_i \mapsto \pi_{ij}(\Lambda_i/K_i) = (\Lambda_i + K_j)/K_j$, and the definition of ρ_{ij} is independent of the choice of S_i and S_j. The map ρ_{ij} is a surjective affine map, since the following diagram commutes:

$$\begin{array}{ccc} \dfrac{E_i}{S_i} & \xrightarrow{\ \tilde{\psi}_{ij}\ } & \dfrac{E_j}{S_j} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ \mathcal{L}_i & \xrightarrow{\ \rho_{ij}\ } & \mathcal{L}_j \end{array}$$

Thus we have found a suitable affine map $\rho_{ij} : \mathcal{L}_i \to \mathcal{L}_j$. □

7.7

The maps $\rho_{ij}: \mathcal{L}_i \to \mathcal{L}_j$ for $j \leq i$ are surjective affine maps on the coset varieties $\mathcal{L}_i, \mathcal{L}_j$, which have the (induced) $\mathcal{W}$-topologies. So by theorem 2.22, $\mathcal{L} = \varprojlim\{\mathcal{L}_i; \rho_{ij}\}$ is non-empty. □

We shall denote by $\mathfrak{L}$ the set $\{\varprojlim\{\Lambda_i; \sigma_{ij}\} : (\Lambda_i)_{i\in I} \in \mathcal{L}$ and σ_{ij} is the restriction of π_{ij} to $\Lambda_i\}$

Lemma 7.8

With the above notation, let $(\Lambda_i)_{i\in I} \in \mathcal{L}$. Then $\varprojlim\{\Lambda_i; \sigma_{ij}\}$ is a subalgebra of $\varprojlim\{L/K_i; \pi_{ij}\}$, where σ_{ij} is the restriction of π_{ij} to Λ_i.

Proof: By the definition of ρ_{ij}, $\pi_{ij}(\Lambda_i) = \Lambda_j$, and the maps σ_{ij} are surjective. Also, Λ_i is a subalgebra of L/K_i, for every $i \in I$, since $\Lambda_i \in \mathcal{L}_i$. So $\varprojlim\{\Lambda_i\} \leq \varprojlim\{L/K_i\}$, as claimed. □

Proposition 7.9

Let L be a cofinite Lie algebra with profinite completion $P = \varprojlim\{L/K_i\}$. If $\Lambda \in \hat{\mathcal{L}}$, then $P = \Lambda \dotplus \sigma(P)$.

Proof: Suppose $\Lambda = \varprojlim\{\Lambda_i/K_i\}$, where $\Lambda_i/K_i \in \mathcal{L}_i$ for each $i \in I$. Then $P = \varprojlim\{L/K_i\} = \varprojlim\{\Lambda_i/K_i \dotplus \sigma(L/K_i)\}$ by Levi's theorem, and this equals $\varprojlim\{\Lambda_i/K_i\} \dotplus \varprojlim\{\sigma(L/K_i)\}$, by proposition 2.23, which in turn equals $\Lambda \dotplus \sigma(P)$, by proposition 5.12(i). □

Theorem 7.10

Any profinite Lie algebra has Levi subalgebras.

Proof: Let L be a profinite Lie algebra and let P and $\hat{\mathcal{L}}$ be defined as above. Let M be an element of $\hat{\mathcal{L}}$. Then M is a Levi subalgebra of P, by proposition 7.9. Since φ is a topological isomorphism, $\varphi^{-1}(M)$ is a Levi subalgebra of L. □

The next theorem gives equivalent forms of Levi subalgebras in a profinite Lie algebra.

Lemma 7.11

Let L be a cofinite Lie algebra and let K be a subalgebra of L. Suppose that K is both cofinitely soluble and semisimple. Then $K = 0$.

Proof: Since K is cofinitely soluble, $\sigma(K/(K \cap K_i)) = K/(K \cap K_i)$, for each $i \in I$, and this implies that $\sigma(K) = \bigcap_{i \in I} K = K$. But as K is semisimple, $\sigma(K) = 0$, so that $K = 0$. □

Theorem 7.12

Let L be a profinite Lie algebra and let Λ be a subalgebra of L. Then the following are equivalent:

(i) $\varphi(\Lambda) \in \hat{\mathcal{L}}$

(ii) Λ is a Levi subalgebra of L

(iii) Λ is a closed maximal semisimple subalgebra of L.

Proof: (i) $\Rightarrow$ (ii): Suppose that $\varphi(\Lambda) \in \hat{\mathcal{L}}$. By proposition 7.9, $P = \varphi(\Lambda) \dot{+} \sigma(P)$. But φ is a topological isomorphism, so applying φ^{-1} to both sides of this equation we have $L = \Lambda \dot{+} \sigma(L)$ and so Λ is a Levi subalgebra of L.

(ii) $\Rightarrow$ (iii): Suppose that Λ is a Levi subalgebra of L. Let $R = \sigma(L)$. Then $L = \Lambda \dot{+} R$. By lemma 6.10, Λ is closed. Further, for each $i \in I$, $L/K_i = (\Lambda + K_i)/K_i \dot{+} (R + K_i)/K_i = (\Lambda + K_i)/K_i \dot{+} R_i/K_i$, by proposition 5.15. Thus $(\Lambda + K_i)/K_i$ is semisimple for each $i \in I$. Now, since Λ is closed, $\varphi(\Lambda) = \varprojlim\{(\Lambda + K_i)/K_i\}$, and so $\varphi(\Lambda)$ is semisimple, by theorem 6.2. As φ is a topological isomorphism Λ is also semisimple.

To prove that Λ is a maximal semisimple subalgebra of L, suppose that Λ' is a semisimple subalgebra of L containing Λ. Then $\Lambda' \cap R = 0$, by lemma 7.11, since $\Lambda' \cap R$ is both semisimple and cofinitely soluble. So $\Lambda' = \Lambda' \cap L = \Lambda' \cap (\Lambda + R)$, and this equals $\Lambda + (\Lambda' \cap R)$, by the modular law, which in turn equals Λ.

(iii) $\Rightarrow$(i): We use a proof similar to that of 7.2 to 7.8. Now Λ is closed in L, so is compact, and hence is pro-semisimple. By corollary 6.3 each $(\Lambda + K_i)/K_i$ is semisimple, so for each $i \in I$ let $\mathcal{F}_i$ be the set $\{\Lambda_i/K_i : (\Lambda + K_i)/K_i \leq \Lambda_i/K_i,\ \Lambda_i/K_i \in \mathcal{L}_i\}$. We shall show that $\mathcal{F} = \varprojlim\{\mathcal{F}_i\}$ is non-empty, and choosing an element $(\Lambda_i/K_i) \in \mathcal{F}$ we shall show that $\varphi(\Lambda) = \varprojlim\{\Lambda_i/K_i\}$, and so $\varphi(\Lambda) \in \hat{\mathcal{L}}$. Now each $\mathcal{F}_i$ is non-empty, since any semisimple subalgebra of a finite dimensional Lie algebra is contained in a Levi subalgebra. For each $i \in I$, consider the three groups of automorphisms:

$$G_i = \langle \exp(x^*) : x \in \nu(L/K_i) \rangle$$

$$C_i = C_{G_i}((\Lambda + K_i)/K_i)$$
$$= \{\theta \in G_i : (l + K_i)^\theta = l + K_i \text{ for all } l \in \Lambda + K_i\}$$

$$N_i = N_{C_i}(\Lambda_i/K_i)$$
$$= \{\theta \in C_i : (\Lambda_i/K_i)^\theta = \Lambda_i/K_i\} \text{ for some fixed } \Lambda_i/K_i \in \mathcal{F}_i$$

By $(l + K_i)^\theta$ and $(\Lambda_i/K_i)^\theta$ we mean the images under the automorphism θ of $l + K_i$ and Λ_i/K_i respectively.

Now G_i is an algebraic group, and by Stewart, [15], p31, C_i and N_i are algebraic subgroups of G_i. Then C_i/N_i is a coset variety for each $i \in I$. We show that C_i is transitive on the set of Levi subalgebras of L/K_i containing $(\Lambda + K_i)/K_i$, i.e. for $\Lambda_i'/K_i \in \mathcal{L}_i$, $\Lambda_i'/K_i \geq (\Lambda + K_i)/K_i$ if and only if there exists $\theta \in C_i$ such that

$$\Lambda_i'/K_i = (\Lambda_i/K_i)^\theta \text{ ------ } (*)$$

To prove this, suppose first that $\Lambda_i'/K_i = (\Lambda_i/K_i)^\theta$ for some $\theta \in C_i$. By the definition of C_i, θ acts trivially on $(\Lambda + K_i)/K_i$. So $\Lambda_i'/K_i = (\Lambda_i/K_i)^\theta \supseteq ((\Lambda + K_i)/K_i)^\theta = (\Lambda + K_i)/K_i$.

Conversely, suppose that Λ_i'/K_i contains $(\Lambda + K_i)/K_i$. Then there exists θ in G_i such that $(\Lambda_i/K_i) = \Lambda_i'/K_i$, since G_i is transitive on the Levi subalgebras of L/K_i, by Jacobson, [10], p92. We must show that this θ actually fixes $(\Lambda + K_i)/K_i$ pointwise. Consider $l \in (\Lambda + K_i)/K_i$ and $x \in N_i/K_i$. Then

$$l^{\exp x^*} = l(1 + x^* + \ldots\ldots + (x^*)^n/n!)$$

for some integer n dependent on x.

So $l^{\exp x^*} = l + [l,x] + \ldots\ldots + (1/n!)[l,_n x]$. But $x \in N_i/K_i$, which is contained in R_i/K_i, and so this implies that $[l,_m x]$ lies in R_i/K_i for each $m \in I_n$. Thus $l^{\exp x^*} = l + s$ for some $s \in R_i/K_i$ But θ is a product of automorphisms of the form $\exp y^*$ for $y \in N_i/K_i$, so, as $\exp x^*$ fixes R_i/K_i setwise we have $l^\theta = l + r$ for some $r \in R_i/K_i$. Now $l \in (\Lambda + K_i)/K_i \subsetneq \Lambda_i'/K_i$ and $l \in \Lambda_i/K_i$, so that $l^\theta \in (\Lambda_i/K_i)^\theta = \Lambda_i'/K_i$. Therefore $r = l^\theta - l \in \Lambda_i'/K_i \cap$

R_i/K_i, which is trivial, so that $l = l^\theta$ for all l in $(\Lambda + K_i)/K_i$. Thus $\theta \in C_i$, and this completes the proof of (*).

We now show that the map $f_i : C_i/N_i \to \mathcal{F}_i$ is a bijection, where f_i is given by $\theta N_i \mapsto (\Lambda_i/K_i)^\theta$. Now f_i is well-defined, since the fact that θ lies in C_i implies that $(\Lambda_i/K_i)^\theta \geq (\Lambda + K_i)/K_i$ by (*). Also, f_i is surjective, for if $\Lambda_i'/K_i \in \mathcal{F}_i$, then $\Lambda_i'/K_i \geq (\Lambda + K_i)/K_i$, by the definition of $\mathcal{F}_i$. So by (*) there exists $\theta \in C_i$ such that $\Lambda_i'/K_i = (\Lambda_i/K_i)^\theta$. Finally, f_i is injective, for if $(\Lambda_i/K_i)^\theta = (\Lambda_i/K_i)^\alpha$, then $(\Lambda_i/K_i)^{\theta\alpha^{-1}} = \Lambda_i/K_i$, which implies that $\theta\alpha^{-1} \in N_i$ and hence that $\theta N_i = \alpha N_i$. Thus f_i is a bijection, as claimed.

Whenever $j \leq i$, let ρ_{ij} be the map $: \mathcal{F}_i \to \mathcal{F}_j$ induced by π_{ij}, i.e. $\rho_{ij} : \Lambda_i/K_i \mapsto (\Lambda_i + K_j)/K_j$. Each ρ_{ij} is surjective. Let $\mathcal{F} = \varprojlim\{\mathcal{F}_i; \rho_{ij}\}$. Arguing as in 7.2 to 7.8 we may show that $\mathcal{F}$ is non-empty, so that there exists $(\Lambda_i/K_i) \in \mathcal{F}$ such that for each $i \in I$, $(\Lambda + K_i)/K_i \leq \Lambda_i/K_i$ and Λ_i/K_i is a Levi subalgebra of L/K_i and $\pi_{ij}(\Lambda_i/K_i) = \Lambda_j/K_j$ whenever $j \leq i$. Let $M = \bigcap_{i \in I} \Lambda_i$. Then M is a closed subalgebra of L. Thus $M + K_i = \Lambda_i$ for all $i \in I$, by lemma 6.5, so as Λ is contained in each Λ_i, we have $\Lambda \leq M$. But M is semisimple, for $\sigma(\varphi(M)) = \varprojlim\{\sigma((M + K_i)/K_i)\} = \varprojlim\{\sigma(\Lambda_i/K_i)\} = 0$, and so $\varphi(M)$, and hence M, is semisimple. By the maximality of Λ we have $\Lambda = M$ and $\varphi(\Lambda) = \varprojlim\{(\Lambda + K_i)/K_i\}$ $= \varprojlim\{\Lambda_i/K_i\}$. $\square$

Corollary 7.13

Let L be a profinite Lie algebra and let Λ be a closed subalgebra of L. Then Λ is a Levi subalgebra of L if and only if $(\Lambda + K_i)/K_i$ is a Levi subalgebra of L/K_i for each $i \in I$.

Proof: Λ is a Levi subalgebra of L if and only if $\varphi(\Lambda) \in \hat{\mathcal{L}}$, by theorem 7.12 i.e. if and only if $\varprojlim\{(\Lambda + K_i)/K_i\} \in \hat{\mathcal{L}}$, which is so if and only if $(\Lambda + K_i)/K_i \in \mathcal{L}_i$ for all $i \in I$. □

We finish this section with a generalisation of a well-known result in finite dimensions:

Proposition 7.14

Let L be a profinite Lie algebra and let H be a semisimple closed ideal of L.

Then H is a direct summand of L.

Proof: H is compact, by theorem 1.9, so is pro-semisimple and hence $H/(H \cap K_i)$ is semisimple for each $i \in I$, by corollary 6.3. But $H/(H \cap K_i)$ is isomorphic to $(H + K_i)/K_i$, and so each $(H + K_i)/K_i$ is semisimple. Replacing Λ by H in the proof of theorem 7.12, (iii) $\Rightarrow$ (i), we can show that there exists M = $\varprojlim\{\Lambda_i\}$ in $\varprojlim\{\mathcal{L}_i\}$ such that M contains $\varprojlim\{(H + K_i)/K_i\}$. M is a Levi subalgebra of L, by theorem 7.12.

The rest of the proof is effectively the same as the corresponding proof in finite dimensions. Let $K = \varphi^{-1}(M)$. Then K is a Levi subalgebra of L, so that $L = \Lambda \dot{+} \sigma(L)$. Now H is closed in L, so is closed in K and so is a direct summand of K, by theorem 6.21; say $K = H \oplus H'$. Then $L = K \dot{+} \sigma(L) = H + (H' \dot{+} \sigma(L))$, and as in the finite dimensional case this is a direct sum. □

Chapter 8 Borel Subalgebras of Profinite Lie Algebras

We extend the definition of Borel subalgebras from finite dimensions to cofinite and profinite Lie algebras and show that a number of properties of such subalgebras also generalise. The main result of this chapter (theorem 8.11) is the result that in a profinite Lie algebra a Borel subalgebra is equivalent to an inverse limit of finite dimensional Borel subalgebras, or that B is a Borel subalgebra of the profinite Lie algebra L if and only if $(B + K)/K$ is a Borel subalgebra of L/K for each K in $\mathcal{K}(L)$. In order to prove this theorem we first factor out the cofinitely soluble radical and prove the result for pro-semisimple Lie algebras, then extend to the general case.

In chapter 10 we shall show that the Borel subalgebras of profinite Lie algebras are conjugate.

Recall that in a finite dimensional Lie algebra a Borel subalgebra is a maximal soluble subalgebra. We can extend this definition to the cofinite case.

Definition 8.1

Let L be a cofinite Lie algebra. Then a subalgebra B of L is a Borel subalgebra if it is a maximal cofinitely soluble subalgebra of L.

The proof of existence is a straightforward application of Zorn's lemma:

Proposition 8.2

Let L be a cofinite Lie algebra. Then L contains Borel subalgebras.

Proof: Let $\mathcal{P}$ = {cofinitely soluble subalgebras of L}. Then $0 \in \mathcal{P}$, so that $\mathcal{P}$ is non-empty. Let $\mathcal{C}$ be an increasing chain in $\mathcal{P}$, and let $C = \bigcup_{B \in \mathcal{C}} B$. Then C is a subalgebra of L. We show that $C \in \mathcal{P}$, and so applying Zorn's lemma $\mathcal{P}$ has maximal elements, which are Borel subalgebras.

Now for any $B \in \mathcal{C}$ and for any $K \in \mathcal{K}$, $(B + K)/K$ is soluble, since each B is cofinitely soluble. Thus $(C + K)/K = ((\bigcup_{B \in \mathcal{C}} B) + K)/K$ which is contained in $\bigcup_{B \in \mathcal{C}} ((B + K)/K)$. But $\{(B + K)/K : B \in \mathcal{C}\}$ is a chain of finite dimensional soluble subalgebras of L/K, so must terminate after a finite number of terms, and hence $(C + K)/K$ is contained in $(B' + K)/K$ for some B' in $\mathcal{C}$. Thus $(C + K)/K$ is soluble and hence C is cofinitely soluble, whence $C \in \mathcal{P}$, as required. □

Proposition 8.3

Let L be a cofinite Lie algebra and let B be a Borel subalgebra of L. Then B is closed in L.

(Hence if L is profinite then B is compact)

Proof: By proposition 5.3 $\bar{B}$ is cofinitely soluble, so by the maximality of B, $B = \bar{B}$, so that B is closed. □

In a cofinite Lie algebra the Borel subalgebras contain the cofinitely soluble radical and are self-idealising:

Proposition 8.4

Let L be a cofinite Lie algebra and let B be a Borel subalgebra of L. Then $\sigma(L) \leq B$.

Proof: $\sigma(L)$ and B are both cofinitely soluble, so $B + \sigma(L)$ is cofinitely soluble. By maximality, $B = B + \sigma(L)$ and so $\sigma(L) \leq B$. □

Proposition 8.5

Let L be a cofinite Lie algebra and suppose that B is a Borel subalgebra of L. Then $B = I_L(B)$, the idealiser of B in L.

Proof: That $B \subseteq I_L(B)$ is clear.

Conversely, suppose that $x \in I_L(B)$ and let $C = B + Fx$, where Fx is the subalgebra of L generated by x. Then $C^{(1)} = B^{(1)} + [B,x] \subseteq B$. So $((C + K)/K)^{(1)} \subseteq (C^{(1)} + K)/K \subseteq (B + K)/K$. Hence $((C + K)/K)^{(1)}$ is soluble, and so C is cofinitely soluble.

By maximality $B = C$ and so $x \in B$. □

The following notation is similar to that used in chapter 7.

Notation

Let L be a profinite Lie algebra. For each $i \in I$ we shall denote by $\mathcal{B}_i$ the set of Borel subalgebras of L/K_i.

Let $\rho_{ij} : \mathcal{B}_i \to \mathcal{B}_j$ be the map $B_i/K_i \mapsto (B_i + K_j)/K_j$. Then $\mathcal{B}$ will denote $\varprojlim\{\mathcal{B}_i; \rho_{ij}\}$, and $\hat{\mathcal{B}}$ will denote $\{\varprojlim\{B_i; \sigma_{ij}\} : (B_i)_{i \in I} \in \mathcal{B}\}$.

The map $\varphi : L \to P = \varprojlim\{L/K_i\}$ denotes, as before, the canonical topological isomorphism.

Theorem 8.6

$\mathcal{B}$ is non-empty.

Proof: This is the same as the proof followed in 7.2 to 7.7 that $\mathcal{L}$ is non-empty, replacing $\mathcal{L}_i$ by $\mathcal{B}_i$ and $\mathcal{L}$ by $\mathcal{B}$.

For the proof of the main result of this chapter we shall need the following special case:

Theorem 8.7

Let L be a pro-semisimple Lie algebra, and let B be a subalgebra of L.

Then B is a Borel subalgebra of L if and only if $\varphi(B) \in \hat{\mathcal{B}}$, i.e. if and only if B is closed and $(B + K)/K$ is a Borel subalgebra of L/K for each $K \in \mathcal{K}(L)$.

Proof: (i) Suppose that B is a Borel subalgebra of L. That B is closed was proved in proposition 8.3, so it suffices to prove that for each $i \in I$, $(B + K_i)/K_i$ is a Borel subalgebra of L/K_i. Because of theorem 6.7 we may regard L as having the form $L = Cr_{\lambda \in \Lambda} L_\lambda$, where each L_λ is a finite dimensional simple Lie algebra, and where L has the profinite topology defined in proposition 4.1. We show firstly that $B = Cr_{\lambda \in \Lambda} B_\lambda$ for some set $\{B_\lambda : \lambda \in \Lambda\}$ such that for each $\lambda \in \Lambda$, B_λ is a Borel subalgebra of L_λ. For each $\mu \in \Lambda$, let $p_\mu : L \to L_\mu$ be the projection map and let $B_\mu = p_\mu(B)$, $K_\mu = \text{Ker } p_\mu$. Let $B' = Cr_{\lambda \in \Lambda} B_\lambda$. We show that $B = B'$, and then that each B_λ is a Borel subalgebra. Now $(B + K_\mu)/K_\mu$ is isomorphic to B_μ, and since each K_μ has the form $Cr_{\lambda \in \Lambda \setminus \{\mu\}} L_\lambda$, which is closed in L, it follows that as B is cofinitely soluble, then B_μ is soluble for each $\mu \in \Lambda$. Each K_i in $\mathcal{K}$ has the form $Cr_{\lambda \in \Lambda'} L_\lambda$ for some $\Lambda' \subseteq \Lambda$ such that $\Lambda \setminus \Lambda'$ is finite, by corollary 6.12, so that $(B' + K_i)/K_i$ is isomorphic to the direct sum of finitely many B_μ's and so is soluble, since each B_μ is soluble. Hence B' is cofinitely soluble. To see that B is contained in B', note that $B \subseteq p_\mu^{-1}(B_\mu)$ for each μ, so that $B \subseteq \bigcap_{\mu \in \Lambda} p_\mu^{-1}(B_\mu)$, which is just B'. Thus $B \subseteq B'$, and so by maximality, $B = B'$.

We now show that each B_λ is a Borel subalgebra of L_λ. Consider $\mu \in \Lambda$. There exists a Borel subalgebra B_μ', say, of L containing B_μ. Let $C = Cr_{\lambda \in \Lambda} B_\lambda'$, where $B_\lambda' = B_\lambda$ for $\lambda \neq \mu$. As in the case of B' above , C is cofinitely soluble and contains B, so that

$B = C$ by maximality, and hence $Cr_{\lambda \in \Lambda} B_\lambda' = Cr_{\lambda \in \Lambda} B_\lambda$, from which it follows that $B_\lambda' = B_\lambda$ for all $\lambda \in \Lambda$, and in particular $B_\mu' = B_\mu$. Thus each B_λ is a Borel subalgebra, as claimed.

To complete the proof of the first part of the theorem, note that for any $i \in I$, L/K_i is isomorphic to $L_{\lambda_1} \oplus \ldots \oplus L_{\lambda_n}$ for some subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ, and that $(B + K_i)/K_i$ corresponds under this isomorphism to $B_{\lambda_1} \oplus \ldots \oplus B_{\lambda_n}$. Since each of these B_λ's is a Borel subalgebra, the sum is a Borel subalgebra of $L_{\lambda_1} + \ldots + L_{\lambda_n}$ (see for example Stewart, [15], corollary 8.3, p65), and hence $(B + K_i)/K_i$ is a Borel subalgebra of L/K_i for every $i \in I$.

(ii) For the converse, suppose that $\varphi(B) \in \hat{\mathcal{B}}$. Now $\varphi(B)$ is closed in $\varprojlim\{L/K_i\}$, by corollary 2.14, and by corollary 2.18, B is closed in L. Since for each $i \in I, (B + K_i)/K_i \in \mathcal{B}_i$, B is cofinitely soluble, so it remains to prove maximality.

Suppose that $B \leqslant D$ and that D is cofinitely soluble. Then for each $i \in I$, $(B + K_i)/K_i \leqslant (D + K_i)/K_i$, and $(B + K_i)/K_i$ is a maximal soluble subalgebra of L/K_i, being a Borel subalgebra, and so $(B + K_i)/K_i = (D + K_i)/K_i$. Therefore $B + K_i = D + K_i$ for all $i \in I$, and so $D \leqslant \bigcap_{i \in I} (D + K_i) \leqslant \bigcap_{i \in I} (B + K_i) = B$. Therefore B is a Borel subalgebra of L. $\square$

$B = C$ by maximality, and hence $Cr_{\lambda \in \Lambda} B_\lambda' = Cr_{\lambda \in \Lambda} B_\lambda$, from which it follows that $B_\lambda' = B_\lambda$ for all $\lambda \in \Lambda$, and in particular $B_\mu' = B_\mu$. Thus each B_λ is a Borel subalgebra, as claimed.

To complete the proof of the first part of the theorem, note that for any $i \in I$, L/K_i is isomorphic to $L_{\lambda_1} \oplus \ldots\ldots \oplus L_{\lambda_n}$ for some subset $\{\lambda_1, \ldots\ldots, \lambda_n\}$ of Λ, and that $(B + K_i)/K_i$ corresponds under this isomorphism to $B_{\lambda_1} \oplus \ldots\ldots \oplus B_{\lambda_n}$. Since each of these B_λ's is a Borel subalgebra, the sum is a Borel subalgebra of $L_{\lambda_1} + \ldots\ldots + L_{\lambda_n}$ (see for example Stewart, [15], corollary 8.3, p65), and hence $(B + K_i)/K_i$ is a Borel subalgebra of L/K_i for every $i \in I$.

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Corollary 8.8

Let $L = Cr_{i\in I}L_i$, where each L_i is a finite dimensional simple Lie algebra. Give L the cofinite topology defined in proposition 4.1.

Then B is a Borel subalgebra of L if and only if $B = Cr_{i\in I}B_i$, where for each $i \in I$, B_i is a Borel subalgebra of L_i. □

We want to show that the conclusion of theorem 8.7 holds for any profinite Lie algebra. For the proof of this we shall require the following two lemmas:

Lemma 8.9

Let L be a profinite Lie algebra and let B be a subalgebra of L containing $\sigma(L)$. Give $L/\sigma(L)$ the cofinite topology induced from L.

Then B is cofinitely soluble if and only if $B/\sigma(L)$ is cofinitely soluble.

Proof: Let $R = \sigma(L)$. From definition 1.13, $\mathcal{K}(B/R) = \{(R + K_i)/R : i \in I\}$.

Suppose that B is cofinitely soluble. Now $(B + K_i)/R_i$ is isomorphic to $(B + K_i)/K_i$ factored by R_i/K_i, which is soluble,

Corollary 8.8

Let $L = Cr_{i \in I} L_i$, where each L_i is a finite dimensional simple Lie algebra. Give L the cofinite topology defined in proposition 4.1.

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Then B is cofinitely soluble if and only if $B/\sigma(L)$ is cofinitely soluble.

Proof: Let $R = \sigma(L)$. From definition 1.13, $\mathcal{K}(B/R) = \{(R + K_i)/R : i \in I\}$.

Suppose that B is cofinitely soluble. Now $(B + K_i)/R_i$ is isomorphic to $(B + K_i)/K_i$ factored by R_i/K_i, which is soluble, being the homomorphic image of $(B + K_i)/K_i$, which is soluble. Hence $(B + K_i)/R_i$ is soluble and so B/R is cofinitely soluble.

Conversely, if B/R is cofinitely soluble, then each $(B + K_i)/R_i$ is soluble, and then so is $(B + K_i)/K_i$ factored by R_i/K_i. Then $(B + K_i)/K_i$ is soluble, since $R_i/K_i = \sigma(L/K_i)$ and because the conclusion of the lemma holds for finite dimensional Lie algebras. Hence B is cofinitely soluble. □

Lemma 8.10

Let L be a profinite Lie algebra and suppose that B is a subalgebra of L containing $\sigma(L)$.

Then B is a Borel subalgebra of L if and only if $B/\sigma(L)$ is a Borel subalgebra of $L/\sigma(L)$.

Proof: Let $R = \sigma(L)$ and suppose that B is a Borel subalgebra of L. Because of lemma 8.9 it suffices to show that B/R is maximal as a cofinitely soluble subalgebra of L/R. So suppose that $B/R \leq B'/R$, where B'/R is cofinitely soluble. Then $B \leq B'$, and B' is cofinitely soluble, by lemma 8.9. Thus, by the maximality of B, $B = B'$ and so $B/R = B'/R$, showing that B/R is a Borel subalgebra of L.

Conversely, suppose that B/R is a Borel subalgebra of L/R. Again, it is sufficient to prove maximality of B as a cofinitely soluble subalgebra of L. If B' is a cofinitely soluble subalgebra of L containing B, then B'/R is cofinitely soluble and contains B/R, so that $B/R = B'/R$ and hence $B = B'$. □

Theorem 8.11

Let L be a profinite Lie algebra. Then B is a Borel subalgebra of L if and only if $\varphi(B) \in \hat{\mathcal{B}}$.

Proof: Suppose that B is a Borel subalgebra of L. Then B is closed, by proposition 8.3, so that $\varphi(B) = \varprojlim\{(B + K_i)/K_i\}$. We want to show that for each $i \in I$, $(B + K_i)/K_i \in \mathcal{B}_i$. By lemma 8.10 B/R is a Borel subalgebra of L/R, and together with theorem 8.7 this implies that each $(B + K_i)/R$ factored by R_i/R is a Borel subalgebra of L/R factored by R_i/R, since L/R is pro-semisimple, by corollary 6.4. This implies that $(B + K_i)/K_i$ factored by R_i/K_i is a Borel subalgebra of L/K_i factored by R_i/K_i, by the isomorphism theorems, and so $(B + K_i)/K_i$ is a Borel subalgebra of L/K_i, by lemma 8.10, as required.

The converse is proved in the same way as the corresponding case in theorem 8.7. □

Corollary 8.12

Profinite Lie algebras possess Borel subalgebras.

Proof: This is similar to the proof of theorem 7.10. □

Corollary 8.13

Let L be a profinite Lie algebra. A subalgebra B is a Borel subalgebra of L if and only if B is closed and each $(B + K_i)/K_i$ is a Borel subalgebra of L/K_i. □

Finally we show that in an infinite-dimensional profinite Lie algebra L the Borel subalgebras have the same dimension as L.

Proposition 8.14

Let L be a profinite Lie algebra and let B be a Borel subalgebra of L. If the dimension of L is infinite, then the dimension of B is the same as that of L.

Proof: By corollary 6.4 $L/\sigma(L)$ is pro-semisimple, so by theorem 6.7 we may consider $L/\sigma(L)$ to be a sum $Cr_{\lambda \in \Lambda} L_\lambda$, where each L_λ is a finite dimensional simple Lie algebra. Now $B/\sigma(L)$ is a Borel subalgebra of $L/\sigma(L)$, in the topology induced from L, by lemma 8.10, so $B/\sigma(L)$ has the form $Cr_{\lambda \in \Lambda} B_\lambda$, by corollary 8.8, where for each $\lambda \in \Lambda$, B_λ is a Borel subalgebra of L_λ.

There are two cases to consider:

Case 1: If Λ is infinite, then the dimension of $B/\sigma(L)$ equals $|\Lambda|$, which is the dimension of $L/\sigma(L)$. Therefore dim B = dim L.

Case 2: If Λ is finite, then $\sigma(L)$ has finite codimension in both L and B, so that dim B = dim $\sigma(L)$ = dim L. □

Chapter 9 Cartan Subalgebras of Profinite Lie Algebras

In this chapter we extend the definition of a Cartan subalgebra from finite dimensional Lie algebras to cofinite Lie algebras, and give an equivalent form for such a subalgebra. In the profinite case we also obtain a result analogous to theorem 8.11 i.e. that C is a Cartan subalgebra of the profinite Lie algebra L if and only if C is closed and $(C + K)/K$ is a Cartan subalgebra of L/K for each $K \in \mathcal{K}$.

Definition 9.1

Let L be a cofinite Lie algebra and let C be a subalgebra of L. C is a Cartan subalgebra of L if:

(a) C is cofinitely nilpotent

(b) If K is a closed subalgebra of L containing C and N is an ideal of K such that K/N is cofinitely nilpotent (in the topology induced from K), then $K = C + N$.

[Alternatively, in the language of formation theory, C is a cofinitely nilpotent projector.]

Note 9.2

This definition, along with those of Levi and Borel subalgebras (definitions 7.1 and 8.1) are consistent with the corresponding definitions in finite dimensions.

Remark 9.3

In definition 9.1 the topology on K/N is that obtained by giving K the relative topology, and then giving K/N the topology induced from K.

Then N will of necessity be a closed ideal of K, otherwise K/N could not be cofinitely nilpotent, since it would not be cofinite. To see this, note that $\mathcal{K}(K/N) = \{((K \cap K_i) + N)/N : K_i \in \mathcal{K}(L)\}$. But $\bigcap_{i \in I} ((K \cap K_i) + N)/N = 0$ if and only if $\bigcap_{i \in I} ((K \cap K_i) + N) = N$, and this holds if and only if N is closed, by proposition 1.6(i). □

Proposition 9.4

Let L be a cofinite Lie algebra and let C be a Cartan subalgebra of L. Then

(i) C is closed in L

(ii) C is a maximal cofinitely nilpotent subalgebra of L.

Proof: We shall use property (b) of the definition of a Cartan subalgebra. Suppose that H is a cofinitely nilpotent subalgebra of L containing C. Then $C \leq \overline{H}$ and $\overline{H}$ is cofinitely nilpotent, by proposition 5.3. Now 0 is a closed ideal of $\overline{H}$ and $\overline{H}/0$ is cofinitely nilpotent, so that by (b) of definition 9.1 $\overline{H} = C + 0 = C$. Hence C is closed, proving (i).

Further, $\overline{H} = C \leq H \leq \overline{H}$ so that C is maximal, proving (ii). □

Proposition 9.5

Let L be a cofinite Lie algebra and let C be a Cartan subalgebra of L. Then for each $K \in \mathcal{K}$, $(C + K)/K$ is a Cartan subalgebra of L/K.

Proof: Fix $K \in \mathcal{K}$. C is cofinitely nilpotent, so that $(C + K)/K$ is nilpotent. So it suffices to prove the projector property for $(C + K)/K$. Suppose that H/K is a subalgebra of L/K containing $(C + K)/K$ and that N/K an ideal of H/K such that H/K factored by N/K is nilpotent. Then H/N is nilpotent and H is a closed subalgebra of L, since H contains K, and contains C. Since C is a Cartan subalgebra of L, it follows from definition 9.1(b) that $H = C + N$, and then $H/K = (C + K)/K + N/K$. Thus $(C + K)/K$ is a Cartan subalgebra of L/K. □

To give an equivalent definition of a Cartan subalgebra of a cofinite Lie algebra we use the concept of quasiabnormality (for example, see Stonehewer, [18], p526).

Definition 9.6

Let L be a cofinite Lie algebra. A subalgebra H of L is quasiabnormal if all closed subalgebras of L containing H are self-idealising.

Theorem 9.7

Let L be a cofinite Lie algebra and let C be a subalgebra of L. Then C is a Cartan subalgebra of L if and only if each of the following conditions holds:

(i) C is cofinitely nilpotent

(ii) C is closed in L

(iii) C is quasiabnormal.

Proof: Suppose that C is a Cartan subalgebra of L. Then (i) follows from the definition, while (ii) follows from proposition 9.4(i). To prove (iii), let H be a closed subalgebra of L containing C, and consider $x \in I_L(H)$, the idealiser of H in L. Then $x + K \in I_{L/K}((H + K)/K)$ for all $K \in \mathcal{K}$. But $(C + K)/K$ is a Cartan subalgebra of L/K, by proposition 9.5, so that $(H + K)/K$ is self-idealising. Thus $x + K \in (H + K)/K$ for each $K \in \mathcal{K}$ and so $x \in \bigcap_{K \in \mathcal{K}} (H + K)$, which equals H, since H is closed. Therefore $H = I_L(H)$, and so C is quasiabnormal.

Conversely, suppose that conditions (i), (ii) and (iii) hold. We need only prove condition (b) of definition 9.1. Fix $K \in \mathcal{K}$. We show that $(C + K)/K$ is a Cartan subalgebra of L/K. $(C + K)/K$ is nilpotent, by (i), so it suffices to show that $(C + K)/K$ is self-idealising. Let $x + K \in I_{L/K}((C + K)/K)$. Then $[x + K, C + K] \subseteq C + K$ and so $x \in I_L(C + K)$. But $C + K$ is self-idealising, by (iii), so $x \in C + K$. Thus $x + K \in (C + K)/K$ and so $(C + K)/K$ is a Cartan subalgebra of L/K, as claimed.

Now suppose that H is a closed subalgebra of L containing C and that N is a closed ideal of H such that H/N is cofinitely nilpotent. We must prove that $H = C + N$. But for any $K \in \mathcal{K}$, $(H + K)/K$ is a subalgebra of L/K containing $(C + K)/K$, and $(N + K)/K$ is an ideal of $(H + K)/K$. Also, since H/N is cofinitely nilpotent, H/N factored by $((H \cap K) + N)/N$ is nilpotent, and so by the isomorphism theorems $H/((H \cap K) + N)$ is nilpotent. But $N + (H \cap K) = H \cap (N + K)$, by the modular law, so that $H/((H\cap(N+K))$ is nilpotent, and hence $(H + N + K)/(N + K)$ is nilpotent. Thus $(H + K)/K$ factored by $(N + K)/K$ is nilpotent. So we may use the fact that $(C + K)/K$ is a Cartan subalgebra of L/K to see that $(H + K)/K = (C + K)/K + (N + K)/K$, and then $H + K = C + N + K$. This is true for all $K \in \mathcal{K}$, so that

$$H = \overline{H} = \bigcap_{K\in\mathcal{K}} (H + K) = \bigcap_{K\in\mathcal{K}} (C + N + K).$$

But C is closed, by (ii), so $C + N$ is closed, by proposition 2.28, and then $H = C + N$, as required. $\square$

Proposition 9.8

Let L be a cofinite Lie algebra, let C be a Cartan subalgebra of L and let K be a closed subalgebra of L containing C.

Then C is a Cartan subalgebra of K.

<u>Proof:</u> Suppose that H is a closed subalgebra of K containing C and that N is a closed ideal of H such that H/N is cofinitely nilpotent. It suffices to show that H = C + N. Now H is a closed subalgebra of L containing C, and C is a Cartan subalgebra of L. Hence H = C + N. □

We now look at Cartan subalgebras of profinite Lie algebras, so by analogy with chapters 7 and 8 we introduce the following notation:

<u>Notation</u>

Let L be a cofinite Lie algebra. For each $i \in I$, $\mathcal{C}_i$ will denote the set of Cartan subalgebras of L/K_i, and for $j \leq i$, ρ_{ij} is the map $\rho_{ij} : \mathcal{C}_i \to \mathcal{C}_j$ given by $C_i/K_i \to (C_i + K_j)/K_j$.

We will denote $\varprojlim\{\mathcal{C}_i; \rho_{ij}\}$ by $\mathcal{C}$, and $\{\varprojlim\{\frac{C_i}{K_i}; \pi_{ij}|_{\frac{C_i}{K_i}}\}\}$ by $\hat{\mathcal{C}}$.

The map $\varphi : L \to \varprojlim\{L/K_i\}$ is the natural embedding.

The proof of the following is similar to that given for 7.1 to 7.7.

<u>Theorem 9.9</u>

The set $\mathcal{C}$ is non-empty. □

The next theorem is the analogue of corollary 7.3 and theorem 8.11.

Theorem 9.10

Let L be a profinite Lie algebra. Then C is a Cartan subalgebra of L if and only if $\varphi(C) \in \hat{\mathcal{C}}$.

Proof: Suppose that C is a Cartan subalgebra of L. Then C is closed, by proposition 9.4(i), so that $\varphi(C) = \varphi(\overline{C}) = \overline{\varphi(C)}$ $= \varprojlim\{(C + K)/K\}$. Also, $(C + K)/K$ is a Cartan subalgebra of L/K, so that $\varphi(C) \in \hat{\mathcal{C}}$.

Conversely, suppose that $\varphi(C) \in \hat{\mathcal{C}}$. $\varphi(C)$ is closed in $P = \varprojlim\{L/K_i\}$, so C is closed in L, as φ is a homeomorphism. Thus $(C + K)/K$ is a Cartan subalgebra of L/K for each $K \in \mathcal{K}$, so that each $(C + K)/K$ is nilpotent, and hence C is cofinitely nilpotent. In order to prove condition (b) of definition 9.1, suppose that H is a closed subalgebra of L containing C, and suppose that N is a closed ideal of H such that H/N is cofinitely nilpotent. Then $(H + K)/K$ is a subalgebra of L/K containing $(C + K)/K$, and $(N + K)/K$ is an ideal of $(H + K)/K$. Also, as in the proof of theorem 9.7, $(H + K)/K$ factored by $(N + K)/K$ is nilpotent, so that $(H + K)/K = (C + K)/K + (N + K)/K$. Hence $H + K = C + N + K$ for all $K \in \mathcal{K}$ and hence $H = C + N$, again as in the proof of theorem 9.7. Therefore C is a Cartan subalgebra of L. $\square$

This theorem gives the existence of Cartan subalgebras in profinite Lie algebras:

Corollary 9.11

Let L be a profinite Lie algebra. Then L possesses Cartan subalgebras.

Proof: This is the same as the corresponding proof for Levi subalgebras (theorem 7.10). □

We now briefly consider pro-semisimple Lie algebras.

Proposition 9.12

Let L be a pro-semisimple Lie algebra and suppose that C is a Cartan subalgebra of L. Then C is abelian.

Proof: By proposition 9.4, $(C + K)/K$ is a Cartan subalgebra of L/K, for each $K \in \mathcal{K}$. Also, each L/K is semisimple, by corollary 6.3. Thus $(C^2 + K)/K = ((C + K)/K)^2 = 0$, since Cartan subalgebras of finite dimensional semisimple Lie algebras are abelian. So $C^2 \subseteq K$ for all $K \in \mathcal{K}$ and so $C^2 = 0$. □

Proposition 9.13

Let L be a pro-semisimple Lie algebra and let C be a Cartan subalgebra of L. If L is infinite-dimensional, then C has the same dimension as L.

Proof: As in the corresponding proof for Borel subalgebras of

Corollary 9.11

Let L be a profinite Lie algebra. Then L possesses Cartan subalgebras.

Proof: This is the same as the corresponding proof for Levi subalgebras (theorem 7.10). $\square$

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Proposition 9.13

Let L be a pro-semisimple Lie algebra and let C be a Cartan subalgebra of L. If L is infinite-dimensional, then C has the same dimension as L.

Proof: As in the corresponding proof for Borel subalgebras of profinite Lie algebras, we may assume that $L = \mathrm{Cr}_{\lambda \in \Lambda} L_\lambda$ and that $C = \mathrm{Cr}_{\lambda \in \Lambda} C_\lambda$, where each L_λ is a finite dimensional simple Lie algebra , and where for each $\lambda \in \Lambda$, C_λ is a Cartan subalgebra of L_λ. Now Λ is infinite, otherwise L would be finite dimensional,and since each L_λ is simple, each C_λ is non-trivial. Hence $\dim C = |\Lambda| = \dim L$. □

Remark 9.14

Proposition 9.13 does not hold for profinite Lie algebras in general. For consider any infinite dimensional profinite vector space V, and let $\theta : V \to V$ be the identity map. Form the semi-direct sum $L = V \dot{+} \langle\theta\rangle$, making L into a Lie algebra by defining the Lie product as follows: for $u, v \in V$ and for $a \in F$, $[u,v] = 0$ and $[v, a\theta] = \theta(av) = av$. We give L a suitable profinite topology and then show that $\langle\theta\rangle$ is a Cartan subalgebra of L, providing the required counter-example.

Suppose $\mathcal{K}(V) = \{V_i : i \in I\}$. Then $\mathcal{K}(V)$ is a finite residual system for L. Give L the cofinite topology determined by $\mathcal{K}(V)$. The map $\varphi : V \to P = \varprojlim \{\frac{V}{V_i} ; \pi_{ij}\}$ is surjective, where $\pi_{ij} : u + V_i \mapsto u + V_j$, since V is profinite.

To show that L is profinite it suffices to show that the embedding $\psi : L \to Q = \varprojlim \{\frac{L}{V_i} ; \varrho_{ij}\}$ is surjective, where $\psi : x \mapsto (x + V_i)$ and where $\varrho_{ij} : x + V_i \mapsto x + V_j$. So consider $(x_i + V_i) \in Q$.

If $x_i + V_i = y_i + \lambda_i\theta + V_i$ for each i, where $y_i \in V$ and $\lambda_i \in F$, then $\varrho_{ij}(y_i + \lambda_i\theta + V_i) = y_j + \lambda_j\theta + V_j$

We now show that $\langle\theta\rangle$ is a Cartan subalgebra of L by using theorem 9.7. Conditions (i) and (ii) are satisfied because $\langle\theta\rangle$ is one-dimensional. To prove condition (iii) suppose that K is a subalgebra of L containing $\langle\theta\rangle$, and let x be an element of $I_L(K)$. Then $\theta \in K$, which implies that $[x,\theta] \in K$, i.e. that $x \in K$. So K is self-idealising. Hence $\langle\theta\rangle$ is a Cartan subalgebra of L, where L has infinite dimension and the dimension of $\langle\theta\rangle$ is finite i.e. one.

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Chapter 10 Conjugacy of Levi, Borel and Cartan Subalgebras of Profinite Lie Algebras

For a finite dimensional Lie algebra L the Borel subalgebras of L are conjugate, as are the Cartan subalgebras, under the group of automorphisms $\mathcal{E}(L)$ (see Humphreys, [9], p84), and Stewart has shown that the Levi subalgebras of L are also conjugate under this group ([16], p208). If L is a profinite Lie algebra, consider the inverse limit of the groups $\mathcal{E}(L/K_i)$. We shall show that this inverse limit may be regarded as a group of automorphisms of L, and that the classes of Levi, Borel and Cartan subalgebras of L are conjugate under this group.

Notation

Let L be a profinite Lie algebra and let E_i denote $\mathcal{E}(L/K_i)$ for each $i \in I$. For $j \leq i$, let $\rho_{ij} : E_i \to E_j$ denote the map $\exp(x + K_i)^* \mapsto \exp(x + K_j)^*$. Then $\mathcal{E}(L)$, or just $\mathcal{E}$, will denote $\lim\{E_i; \rho_{ij}: i,j \in I \text{ and } j \leq i\}$. ρ_{ij} is well-defined because of the lifting property of the groups $\mathcal{E}(L/K_i)$ i.e. if H, K are finite dimensional, then an epimorphism : H→K induces an epimorphism : $\mathcal{E}(H) \to \mathcal{E}(K)$ (see Humphreys, [9], p82).

Note that $\mathcal{E}$ is non-empty, as it contains the inverse limit of the identity elements of the E_i's.

Let $\hat{\mathcal{L}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}$ be defined as in chapters 7,8 and 9 respectively, i.e. the inverse limits of Levi, Borel and Cartan subalgebras of the finite dimensional factors by closed ideals.

Notation

Let $\mathfrak{X}$ be a set of subalgebras of the profinite Lie algebra L. We shall say that $\mathfrak{X}$ is $\mathcal{E}$-conjugate if for any two members X, Y of $\mathfrak{X}$ there is an element θ of $\mathcal{E}$ such that $\theta(X) = Y$.

For the next result we shall assume proposition 10.2

Proposition 10.1

The sets $\hat{\mathcal{L}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}$ are all $\mathcal{E}$-conjugate.

Proof: We prove the result for $\hat{\mathcal{L}}$ only, since the proofs for the remaining two cases are similar.

Consider $(M_i), (N_i) \in \hat{\mathcal{L}}$, and define $T_i = \{\theta_i \in E_i: \theta_i(M_i) = N_i\}$, $S_i = \{\varphi_i \in E_i: \varphi_i(M_i) = M_i\}$. We show that T_i is a coset variety by showing that $T_i = \theta_i S_i$ for some θ_i in T_i, and then find affine maps $\sigma_{ij} : T_i \to T_j$ for $j \leqslant i$. We may then apply theorem 2.22 to find an element of $\varprojlim\{T_i; \sigma_{ij}\}$ which will serve as the required map from (M_i) to (N_i).

For any $\theta_i \in T_i$ we claim that $T_i = \theta_i S_i$. For suppose θ_i is a fixed element of T_i. Then $\theta_i^{-1}\theta(M_i) = \theta_i^{-1}(N_i) = M_i$ for any $\theta \in T_i$. Hence $\theta_i^{-1}\theta \in S_i$, so that $\theta \in \theta_i S_i$, and $T_i \subseteq \theta_i S_i$.

Conversely, for $\theta \in S_i$, $\theta_i\theta \in \theta_i S_i$, so that $\theta_i\theta(M_i) = \theta_i(M_i) = N_i$, which implies that $\theta_i\theta \in T_i$, so that $\theta_i S_i \subseteq T_i$, proving the claim. Now for each $i \in I$, E_i is an algebraic group, S_i is an algebraic subgroup of E_i, so is closed in the $\mathcal{W}$-topology. Hence $\theta_i S_i$ is closed in the $\mathcal{W}$-topology, so that T_i is a coset variety. Fix i, $j \in I$ such that $j \leq i$. We want to find affine maps $\sigma_{ij} : T_i \to T_j$. Now $\pi_{ij} : L/K_i \to L/K_j$, the canonical map, induces the map $\rho_{ij} : E_i \to E_j$ given by $\exp x^* \mapsto \exp(\pi_{ij}(x)^*)$ and ρ_{ij} is a morphism of algebraic groups. ρ_{ij} in turn induces the map $\tilde{\rho}_{ij} : E_i/S_i \to E_j/S_j$ given by $\theta_i S_i \to \rho_{ij}(\theta_i)S_j$. Now S_i is the stabiliser of M_i, and S_j is the stabiliser of M_j, which equals $\pi_{ij}(M_i)$. Thus $\rho_{ij}(S_i) \subseteq S_j$. Choose θ_i, θ_j so that $\rho_{ij}(\theta_i) = \theta_j$. Let σ_{ij} be the restriction of ρ_{ij} to $\theta_i S_i = T_i$. The following diagram commutes

$$\begin{array}{ccc} M_i & \xrightarrow{\theta_i} & N_i \\ {\scriptstyle \pi_{ij}}\downarrow & & \downarrow{\scriptstyle \pi_{ij}} \\ M_j & \xrightarrow{\theta_j} & N_j \end{array}$$

and so $\sigma_{ij}(T_i) \subseteq T_j$, so that σ_{ij} is well-defined, and is an affine map. By theorem 2.22, $\varprojlim\{T_i; \sigma_{ij}\}$ is non-empty, so consider an element $\theta = (\theta_i)$ from this set. Then

$$\theta((M_i)) = (\theta_i(M_i)) = (N_i) \text{ and } \theta \in \mathcal{E}. \ \square$$

The sets $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$ for a profinite Lie algebra L correspond to the sets of Levi, Borel and Cartan subalgebras of L, so when we show that $\mathcal{E}$ corresponds to a group of automorphisms of L the required conjugacy result follows.

<u>Proposition 10.2</u>

Let L be a profinite Lie algebra. Then the group $\mathcal{E}$ embeds as a subgroup in Aut(L), the group of automorphisms of L.

<u>Proof:</u> Consider the profinite completion $P = \varprojlim\{L/K_i; \pi_{ij}\}$ of L, and the natural (surjective) embedding $\varphi : L \to P$. Σ acts on P, so we use the topological isomorphism φ to define the action of Σ on L. Consider $x \in L$ and $\theta = (\theta_i) \in \Sigma$. Define the action of θ on x to be $\theta(x) = \varphi^{-1}((\theta_i(x + K_i))_i)$. We need to check that θ is an automorphism of L. It is easy to see that θ is an injective Lie algebra homomorphism. θ is well-defined, for if $j \leq i$, then there exists $z \in L/K_i$ such that

$$\begin{aligned}\pi_{ij}(\theta_i(x + K_i)) &= \pi_{ij}(\exp(z^*)(y_i + K_i)) \\ &= \exp(\pi_{ij}(z)^*)(\pi_{ij}(x + K_i)) \\ &= \exp(\pi_{ij}(z)^*)(x + K_j) \\ &= \theta_j(x + K_j)\end{aligned}$$

So $(\theta_i(x + K_i)) \in P$, whence $\varphi^{-1}(\theta_i(x + K_i)) \in L$.

To show that θ is surjective it suffices to show that the map $(\theta_i) : P \to P$ is surjective. Let $x = (x_i) \in P$, and let $y_i = \theta_i^{-1}(x_i)$ for each $i \in I$. Then $\pi_{ij}(y_i) = \pi_{ij}(\theta_i^{-1}(x_i))$ $= \theta_j^{-1}(\pi_{ij}(x_i)) = \theta_j^{-1}(x_j) = y_j$. So $y = (y_i)$ is in P and then x is the image under (θ_i) of y. Hence θ is an automorphism of L, as required. □

Note 10.3

In future, therefore, we may either consider $\mathcal{E}(L)$ to be a subgroup of Aut(L) or a subgroup of Aut(P), where $P = \varprojlim \{L/K_i\}$, as appropriate.

Theorem 10.4

Let L be a profinite Lie algebra. Then

(i) The Levi subalgebras of L are $\mathcal{E}$-conjugate

(ii) The Borel subalgebras of L are $\mathcal{E}$-conjugate

(iii) The Cartan subalgebras of L are $\mathcal{E}$-conjugate.

Proof: This follows from proposition 10.1 and note 10.3. □

Chapter 11 Weight Spaces of Profinite L-Modules

Section 1 The Fitting Decomposition of a Profinite L-Module

For a finite dimensional nilpotent Lie algebra L consider a finite dimensional L-module M. For each $\lambda \in L^*$, the dual space of L, let $M_\lambda = \{m \in M$: for each $x \in L$, $m(x^* - \lambda(x))^n = 0$ for some n depending on $x\}$. λ is called a weight of M (or a weight of L on M), and M_λ is the corresponding weight space. Each M_λ is an L-submodule of M, and M is the direct sum of its weight spaces. The theory is best known when M is a semisimple Lie algebra and L is a Cartan subalgebra of M, and the action of L on M is given by Lie multiplication on the right. In this case the non-zero weights of M classify M. The above theory may be found in Humphreys, for example, [9].

In this chapter we consider the analogous decomposition in a profinite L-module, where L is a Lie algebra with certain finite dimensional nilpotent factors. Whereas in finite dimensions M is a direct sum of its weight spaces, a profinite L-module may be regarded as the Cartesian sum of its weight spaces.

Definition 11.1

Let L be a Lie algebra and let M be an L-module. Then M is a cofinite L-module if M is a cofinite vector space in which all M_i from $\mathcal{K}'(M)$ are L-submodules, for some set $\mathcal{K}'(M)$ cofinal in $\mathcal{K}(M)$.

A cofinite L-module is a profinite L-module if it is compact.

Notation

If M is a cofinite L-module we shall also write $\mathcal{K}(M)$ as $\{M_i : i \in I\}$.

Remark 11.2

We may regard each M/M_i as an L-module, under the action $(m + M_i)l = ml + M_i$.

Also, if C is an ideal of L such that $C \leq C_L(M/M_i)$, then we may regard each M/M_i as an L/C-module by defining the action $(m + M_i)(l + C) = ml + M_i$.

We shall obtain the weight spaces of a cofinite L-module by considering the weight spaces of its finite dimensional factors and taking a suitable intersection, as in the case of the radicals in chapter 5.

11.3

Let M be a cofinite L-module. Consider a set $\{C_i : i \in I\}$ such that

(i) $C_i \leq C_L(M/M_i)$ and C_i is an ideal of L, for all $i \in I$.

(ii) If $j \leq i$, then $C_j \geq C_i$.

(iii) L/C_i is finite dimensional and nilpotent for every i in I.

(When we particularise later on, M will be a Lie algebra and L will be a Cartan subalgebra of M, so that the L/C_i's defined above will correspond to Cartan subalgebras of the M/M_i's.)

We show that for any $\lambda \in L^*$, λ induces weights λ_i on each L/C_i-module M/M_i, and that conversely any weight λ_i of M/M_i can be induced in this way from some λ in L^*. We shall then define the weight space M_λ for the λ's which induce weights λ_i which have corresponding non-zero weight spaces.

Consider $\lambda \in L^*$, the dual space of L. For those $i \in I$ for which $C_i \leq \operatorname{Ker}\lambda$, let λ_i be the induced map in $(L/C_i)^*$ defined by $\lambda_i : l + C_i \mapsto \lambda(l)$. λ_i is well-defined, for if $l + C_i = h + C_i$, then $l - h \in C_i$ and then $\lambda(l - h) \in \lambda(C_i) \subseteq \lambda(\operatorname{Ker}\lambda) = 0$ so that $\lambda(l) = \lambda(h)$, i.e. $\lambda_i(l + C_i) = \lambda_i(h + C_i)$.

For each $i \in I$ define $(M/M_i)_{\lambda_i}$ to be zero if λ_i is not defined (i.e. if $C_i \nleq \operatorname{Ker}\lambda$), and to be the λ_i weight space of the L/C_i-module M/M_i otherwise. For any $i \in I$ we obtain all the weights of M/M_i in this way i.e. any weight λ_i of M/M_i is induced from some $\lambda \in L^*$. To see this, consider the weight λ_i in $(L/C_i)^*$ of M/M_i. Define $\lambda : L \to F$ by $\lambda : l \to \lambda_i(l + C_i)$. We show that λ is well-defined and that $C_i \subset \operatorname{Ker}\lambda$.

Since $(M/M_i)_{\lambda_i} \neq 0$ there exists $m \in M \setminus M_i$ such that $(m + M_i)((x + C_i)^* - \lambda_i(x + C_i))^n \subseteq M_i$ for each $x + C_i \in L/C_i$, and for some n. If $x + C_i = C_i$, then

$$(m + M_i)((x + C_i)^* - \lambda_i(x + C_i))^n$$
$$= (-\lambda_i(x + C_i)m + M_i)((x + C_i)^* - \lambda_i(x + C_i))^{n-1}$$
$$= (-\lambda_i(x + C_i))^n m + M_i, \text{ by induction.}$$

Then $(-\lambda_i(x + C_i))^n m + M_i = M_i$, whence $\lambda_i(x + C_i) = 0$, since $m \notin M_i$. Thus if $x \in C_i$, then $\lambda(x) = \lambda_i(C_i) = 0$ and so C_i is contained in Ker λ, as claimed.

Hence every $\lambda_i \in (L/C_i)^*$ is induced from some $\lambda \in L^*$. This λ is unique, for if λ and μ both induce λ_i, i.e. if $\lambda_i = \mu_i$, then for each $x \in L$, $\lambda(x) = \lambda_i(x + C_i) = \mu_i(x + C_i) = \mu(x)$, so that $\lambda = \mu$. □

Definition 11.4

With the above notation, fix $\lambda \in L^*$. Let $T_i/M_i = (M/M_i)_{\lambda_i}$ and let $M_\lambda = \bigcap_{i \in I} T_i$. Then M_λ will be called a weight space of M if it is non-zero, and λ is the corresponding weight.

We shall use the notation of 11.3 and definition 11.4 throughout this chapter.

The following result follows from the above definition.

Proposition 11.5

Let M be a cofinite L-module. Then

(i) The weight spaces of M are closed vector subspaces of M.

(ii) If λ and μ are distinct weights, then $M_\lambda \cap M_\mu = 0$.

Proof: (i) follows immediately from definition 11.4. To prove (ii), let T_i/M_i and S_i/M_i be the λ_i and μ_i weight spaces respectively of M/M_i. Then $M_\lambda \cap M_\mu = \bigcap_{i\in I} T_i \cap \bigcap_{i\in I} S_i = \bigcap_{i\in I} (T_i \cap S_i) \subseteq \bigcap_{i\in I} M_i = 0$. □

Proposition 11.6

Let M be a cofinite L-module, and let $\lambda \in L^*$ be a weight. Then with the above notation, $\pi_{ij}(T_i/M_i) = T_j/M_j$ whenever $j \leq i$.

Proof: Fix $i, j \in I$ with $j \leq i$. We show firstly that $\pi_{ij}(T_i/M_i) \subseteq T_j/M_j$. We may assume that $T_i/M_i \neq 0$, so that $C_i \subseteq \text{Ker}\,\lambda$. Let $m + M_i \in T_i/M_i$. Note that for any $x \in L$, $\lambda_i(x + C_i) = \lambda(x) = \lambda_j(x + C_j)$, since $C_j \leq C_i \leq \text{Ker}\,\lambda$. So $(m + M_i)((x + C_i)^* - \lambda_i(x + C_i))^n = 0$ for all x in L and for some n depending on x. Then $m(x^* - \lambda_i(x + C_i))^n \in M_i \subseteq M_j$, and so $(m + M_j)((x + C_j)^* - \lambda_j(x + C_j))^n = 0$. Hence $m + M_j$ lies in T_j/M_j, as required.

Now consider $m + M_j \in T_j/M_j$. We show that $\pi_{ij}(T_i/M_i)$

contains T_j/M_j. Now $m + M_i \in M/M_i$, so that $m + M_i$ equals $(m_{\lambda_1} + M_i) + \ldots\ldots + (m_{\lambda_n} + M_i)$, where $m_{\lambda_r} + M_i \in T_{r_i}/M_i$ and where M/M_i has the weight space decomposition $\bigoplus_{r=1}^{n} T_{r_i}/M_i$. Then

$$m + M_j = \sum_{r=1}^{n} \pi_{ij}(m_{\lambda_r} + M_i) = \sum_{r=1}^{n} (m_{\lambda_r} + M_j)$$

. But for each r, $m_{\lambda_r} + M_j \in T_{r_j}/M_j$. So as the weight space decomposition is a direct sum, say $M/M_j = \sum_{\lambda_j \in \Omega_j} (M/M_j)_{\lambda_j}$, where $\Omega_j = (L/C_j)^*$ and since $m + M_j \in T_j/M_j$ we must have $m_{\lambda_r} + M_j = M_j$ for $\lambda_r \neq \lambda_j$ and $m_{\lambda_r} + M_j = m + M_j$ for $\lambda_r = \lambda_j$.

So $\pi_{ij}(m_{\lambda_j} + M_i) = m_{\lambda_j} + M_j = m + M_j$, and so $m + M_j$ is the image under π_{ij} of an element of T_j/M_j, and hence π_{ij} is surjective. □

Remark 11.7

The definition of M_λ depends on the choice of $\{C_i : i \in I\}$. In what follows M_λ will often be assumed to be defined relative to some, possibly unspecified, set $\{C_i : i \in I\}$ satisfying (i), (ii) and (iii) of 11.3.

We now show that the weight spaces of a cofinite L-module are L-submodules.

Proposition 11.8

Let L be a Lie algebra and let M be a cofinite L-module. Then with the above notation, M_λ is an L-module for each λ in L*.

Proof: Fix $\lambda \in L^*$. By remark 11.2 we may regard each M/M_i as an L/C_i-module, and then by the corresponding theory in finite dimensions, each T_i/M_i is an L/C_i-module. So for any $l \in L$, $(T_i/M_i)(l + C_i) \subseteq T_i/M_i$, i.e. $T_i(l + C_i) \subseteq T_i$. But C_i centralises M/M_i, by (i) of 11.3, so that $MC_i \subseteq M_i \subseteq T_i$. Hence $T_i l \subseteq T_i$, and so $T_i L \subseteq T_i$. Hence each T_i is an L-module, so M_λ, being the intersection of such T_i's, is also an L-module. □

We have the following result for closed submodules, as in Stewart, [17], p79 :

Proposition 11.9

Let M be a cofinite L-module and let N be a closed submodule of M. Then $N_\lambda = M_\lambda \cap N$ for all $\lambda \in L^*$.

Proof: Fix $\lambda \in L^*$. Let $H_i/M_i = ((N + M_i)/M_i)_{\lambda_i}$ and let $N_i = H_i \cap N$ for each $i \in I$. Then

$N_i/M_i = (N + M_i)/M_i \cap T_i/M_i$ (see Stewart, [17], p79), so that $N_i = (N + M_i) \cap T_i$. So $N_\lambda = \bigcap_{i \in I} N_i$

$$= (\bigcap_{i \in I} T_i) \cap (\bigcap_{i \in I} (N + M_i))$$

$$= M_\lambda \cap \overline{N}. \quad \Box$$

We now show that a profinite L-module may be regarded as the Cartesian sum of its weight spaces:

Theorem 11.10

Let M be a profinite L-module with weight spaces M_λ, for $\lambda \in L^*$. Then M is isomorphic, as an L-module, to the L-module $Cr_{\lambda \in L^*} M_\lambda$.

Proof: Since the natural map $\varphi : M \to \varprojlim\{M/M_i\}$ is a vector space isomorphism, it is also an L-module isomorphism when $\varprojlim\{M/M_i\}$ is given an L-module structure by defining the L-action by $(m_i + M_i)l = (m_i l + M_i)$. So it suffices to show that there exists an L-module isomorphism $\theta : \varphi(M) \to Cr_{\lambda \in L^*} \varphi(M_\lambda)$. Now M/M_i is the direct sum of its weight spaces relative to L/C_i, so that any $m_i \in M/M_i$ can be expressed uniquely in the form $m_i = \sum_{\lambda_i \in \Omega_i} m_{\lambda_i}$, where $\Omega_i = (L/C_i)^*$. Define the map

$\theta : \varphi(M) \to Cr_{\lambda \in L^*} \varphi(M_\lambda)$ by $(m_i) \mapsto (y_\lambda)$, where for each $\lambda \in L^*$,

$y_\lambda = (m_{\lambda_i})$. To see that θ is well-defined, note that

$\pi_{ij}(\sum_{\lambda_i \in \Omega_i} m_{\lambda_i}) = \sum_{\lambda_j \in \Omega_j} m_{\lambda_j}$ and that $\pi_{ij}(m_{\lambda_i}) \in (M/M_j)_{\lambda_j}$. As the sum of the weight spaces is direct, $m_{\lambda_j} = \pi_{ij}(m_{\lambda_i})$, so that $y_\lambda \in \varphi(M_\lambda)$ and so θ is well-defined.

To see that θ is surjective, consider $(y_\lambda) \in Cr_{\lambda \in L^*}\varphi(M_\lambda)$. Then each y_λ lies in $\varphi(M_\lambda)$, so that y_λ has the form (m_{λ_i}), where for each $i \in I$ and $\lambda \in L^*$, $m_{\lambda_i} \in (M/M_i)_{\lambda_i}$. So $\pi_{ij}(m_{\lambda_i}) = m_{\lambda_j}$. Define m_i to be $\sum_{\lambda_i \in \Omega_i} m_{\lambda_i}$. Then $m_i \in M/M_i$. This sum is a finite one, since M/M_i has finitely many weights, being finite dimensional. Let $m = (m_i)$. Then $m \in \varphi(M)$, since m_i is in M/M_i for each i, and since $\pi_{ij}(m_i) = \pi_{ij}(\sum_{\lambda_i \in \Omega_i} m_{\lambda_i}) = \sum_{\lambda_j \in \Omega_j} m_{\lambda_j} = m_j$. Thus $\theta : m \to (y_\lambda)$ and θ is surjective.

It is easy to check that θ is linear and injective, and that $\theta(ml) = \theta(m)l$, and so θ is an L-module isomorphism. □

Definition 11.11

Let M be a profinite L-module. The expression $Cr_{\lambda \in L^*}M_\lambda$ of theorem 11.10 is called the Fitting decomposition of M.

We may consider the effect on the weight spaces of embedding a cofinite L-module into its profinite completion.

Proposition 11.12

Let M be a profinite L-module and let $\varphi : M \to \varprojlim\{M/M_i; \pi_{ij}\}$ be the natural isomorphism. Let M_λ be a weight space corresponding to the weight λ.

Then $\overline{\varphi(M_\lambda)} = \varprojlim\{T_i/M_i\}$.

Proof: From proposition 11.5, $T_j/M_j = \pi_{ij}(T_i/M_i) = (T_i + M_j)/M_j$. So whenever $j \leq i$, $T_j = T_i + M_j$. By lemma 6.5, $M_\lambda + M_i = T_i$ for every $i \in I$. Then $\overline{\varphi(M_\lambda)} = \varprojlim\{(M_\lambda + M_i)/M_i\} = \varprojlim\{T_i/M_i\}$. □

Thus the weight spaces of a profinite L-module may be regarded as the inverse limits of the corresponding weight spaces of the finite dimensional factors:

Corollary 11.13

With the above notation, suppose that M is a profinite L-module. Then $\varphi(M_\lambda) = \varprojlim\{T_i/M_i\}$.

Proof: This follows immediately from proposition 11.5(i) and proposition 11.12. □

Section 2 Weight Spaces of Cofinite Lie Algebras

For a cofinite Lie algebra L we may take a Cartan subalgebra C of L and consider L to be a cofinite C-module. We show that the weight space L_0 of L corresponding to the zero weight equals C, and that if L is pro-semisimple then the weight spaces corresponding to the non-zero weights are one-dimensional, as in the case when L is finite dimensional semisimple.

Remark 11.14

Let L be a cofinite Lie algebra, and let C be a Cartan subalgebra of L. Then L may be regarded as a C-module, where the C-action is merely the Lie multiplication, on the right, in L.

The set $\{C \cap K: K \in \mathcal{K}(L)\}$ satisfies (i), (ii) and (iii) of 11.3, and for $K \in \mathcal{K}(L)$ we can regard L/K as a $C/(C \cap K)$-module, defining the module multiplication by

$$(l + K)(c + (C \cap K)) = [l,c] + K$$

Equivalently, we may regard L/K as a $(C + K)/K$-module, where the module multiplication is given by

$$(l + K)(c + K) = [l,c] + K$$

Theorem 11.15

Let L be a profinite Lie algebra and let C be a Cartan subalgebra of L. Let $\{L_\lambda : \lambda \in C^*\}$ be the set of weight spaces of L regarded as a C-module, where L_0 is the weight space corresponding to the zero weight.

Then $L_0 = C$.

Proof: As in proposition 11.12, $L_0 + K_i = T_i$ for each $i \in I$. For $i \in I$, $(L_0 + K_i)/K_i = (L/K_i)_0 = (C + K_i)/K_i$ since L/K is a finite dimensional (C + K)/K-module and since (C + K)/K is a Cartan subalgebra of L/K, by proposition 9.5. Thus $L_0 + K = C + K$. So as L_0 and C are closed in L, by proposition 11.5 and proposition 9.4respectively, $L_0 = \overline{L_0} = \bigcap_{i \in I} (L_0 + K_i) = \bigcap_{i \in I} (C + K_i) = C$. □

The following products of weight spaces generalise from the corresponding results for finite dimensional Lie algebras.

Proposition 11.16

Let L be a cofinite Lie algebra and let C be a Cartan subalgebra of L. Consider L as a C-module and suppose that λ and μ are weights of L. Then

(i) $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$ if $\lambda + \mu$ is a weight

(ii) $L_{\lambda+\mu} = 0$ if $\lambda + \mu$ is not a weight

(iii) $[L_\lambda, L_\mu] = 0$ if $\lambda + \mu$ is not a weight.

Proof: (i) For each $i \in I$, let T_i^λ/K_i, T_i^μ/K_i and $T_i^{\lambda+\mu}/K_i$ be the weight spaces of L/K_i corresponding to the weights λ_i, μ_i and $\lambda_i + \mu_i$ respectively. Then $[T_i^\lambda/K_i, T_i^\mu/K_i]$ is contained in $T_i^{\lambda+\mu}/K_i$ if $\lambda_i + \mu_i$ is a weight, and equals zero otherwise. So $[T_i^\lambda, T_i^\mu] \subseteq T_i^{\lambda+\mu}$ for all $i \in I$. Hence

$$[L_\lambda, L_\mu] = [\bigcap_{i\in I} T_i^\lambda, \bigcap_{i\in I} T_i^\mu] \subseteq \bigcap_{i\in I} [T_i^\lambda, T_i^\mu] \subseteq \bigcap_{i\in I} T_i^{\lambda+\mu} = L_{\lambda+\mu}.$$

(ii) and (iii) follow similarly from the corresponding results for finite dimensional Lie algebras. □

We now consider weight spaces of a pro-semisimple Lie algebra.

Lemma 11.17

Let $A = \varprojlim\{A_i; \rho_{ij}\}$, where each A_i is an algebra having dimension one or zero.

Then A has dimension one or zero.

Proof: Let (x_i) and (y_i) be elements of A. Either the dimension of A is zero or we can assume that $(x_i) \neq 0$. Now $\rho_{ij}(x_i) = x_j$ and $\rho_{ij}(y_i) = y_j$. But for each $i \in I$ there exists $a_i \in F$ such that $y_i = a_i x_i$, since A_i has dimension one or zero. Now $a_j x_j = \rho_{ij}(a_i x_i) = a_i \rho_{ij}(x_i) = a_i x_j$. So for $x_j \neq 0$, $a_i = a_j$ when $j \leq i$. So $(y_i) = (a_i x_i) = a(x_i)$, where $a = a_i$ for all $i \in I$ such that $x_i \neq 0$. So if A is non-trivial, A has dimension one. □

Theorem 11.18

Let L be a pro-semisimple Lie algebra, and consider the C-module L for some Cartan subalgebra C of L.

For each non-zero weight λ in C*, L has dimension one.

Proof: Fix $\lambda \in C^*$. Let $\varphi : L \to \varprojlim\{L/K_i\}$ be the natural topological isomorphism. Now $\varphi(L_\lambda) = \varphi(\overline{L_\lambda})$ since L_λ is closed, by proposition 11.5, so $\varphi(L_\lambda) = \overline{\varphi(L_\lambda)} = \varprojlim\{T_i/K_i\}$, by proposition 11.12. But each T_i/K_i is one-dimensional, if $\lambda \neq 0$, since each L/K_i is finite dimensional semisimple. So by lemma 11.17, $\varphi(L_\lambda)$ has dimension one or zero. But φ is a topological isomorphism, so L_λ has dimension one or zero. However, L_λ is non-zero so must have dimension one. □

Chapter 12 Cofinitely Cleft Lie Algebras

The final two chapters follow closely the approach and methods used in Stewart, [17]. We want to obtain a Chevalley-Jordan decomposition for linear maps on a cofinite Lie algebra and define a cofinitely cleft Lie algebra accordingly, generalising some results from finite dimensions. In section 2 we look briefly at tori, and in section 3 at some properties of cofinitely cleft Lie algebras. Finally we consider cofinitely soluble, cofinitely cleft Lie algebras.

Section 1 Chevalley-Jordan Decomposition

Notation

Let L be a Lie algebra and let f be a linear map on L. We may regard L as an $\langle f \rangle$-module, where the module action is

$$xf = f(x).$$

If L is cofinite, and if $f(K_i) \subseteq K_i$ for each $K_i \in \mathcal{K}(L)$, then f induces linear maps on each L/K_i, which we shall denote by f_i, given by $x + K_i \mapsto f(x) + K_i$.

In the remaining chapters we shall assume that $f(K_i) \subseteq K_i$ for any linear map f on a cofinite Lie algera (or a cofinite vector space) L, for all $i \in I$.

Recall that if L is a finite dimensional Lie algebra and f is a linear map on L, then f is pure if there is a basis of eigenvectors of L, i.e. if f is diagonalisable. f is also referred to as semisimple or semi-regular in the literature.

Definition 12.1

Let L be a cofinite Lie algebra and suppose that f is a linear map on L. Then f is cofinitely pure if each f_i defined above is pure.

f is cofinitely nilpotent if each f_i is nilpotent.

An element x of L is cofinitely ad-pure if x* is cofinitely pure, and x is cofinitely ad-nilpotent if x* is cofinitely nilpotent.

If f is a cofinitely pure map on a cofinite vector space V, then the weight spaces of V consist of f-eigenvectors:

Proposition 12.2

Let f be a cofinitely pure linear map on the cofinite vector space V. Regarding V as an f -module, let $\{V_\lambda : \lambda \in \langle f \rangle^*\}$ be the set of weight spaces of V.

Then each V_λ consists of f-eigenvectors with eigenvalue λ $(= \lambda(f))$.

Proof: Fix $\lambda \in \langle f \rangle^*$, and consider $u \in V_\lambda$. Define T_i/K_i as in chapter 11, i.e. as the λ_i weight space of V/K_i. Then $u + K_i \in T_i/K_i$ for every $i \in I$, and the C_i's here are all 0, using the notation of chapter 11. Fix $i \in I$. Now f_i is pure, so $f_i(u + K_i) = \lambda_i(f_i).(u + K_i)$, since the result of the proposition holds in finite dimensions (see Stewart, [17], lemma 2.3, p80). Hence $f(u) + K_i = \lambda_i(f_i).u + K_i$. But $\lambda_i(f_i) = \lambda(f)$, as in 11.3. So $f(u) - \lambda(f).u \in K_i$. This holds for every $i \in I$, so that $f(u) = \lambda(f).u$. Hence u is an f-eigenvector belonging to the eigenvalue $\lambda(f)$. □

Proposition 12.3

Let A be a cofinite algebra and let f be a linear map on A. Consider A as an $\langle f \rangle$-module and suppose that H,K are $\langle f \rangle$-submodules of A. Suppose further that K is a closed ideal of A. Then

(i) If f is cofinitely pure on A, then f induces cofinitely pure maps on H and on A/K.

(ii) If f is cofinitely nilpotent on A, then f induces cofinitely nilpotent maps on H and on A/K.

<u>Proof:</u> We use the fact that the corresponding result holds for finite dimensional algebras (see Stewart, [17], corollary 2.4, p80).

(a) Let g be the restriction of f to H. Then for each $i \in I$, g_i is the map : $(H + K_i)/K_i \to (H + K_i)/K_i$ given by $x + K_i \mapsto g(x) + K_i = f(x) + K_i$. So g_i is the restriction of f_i to $(H + K_i)/K_i$. So if f is cofinitely pure, then f_i is pure for all $i \in I$, whence each g_i is pure, so that g is cofinitely pure.

Similarly, if f is cofinitely nilpotent then g is cofinitely nilpotent.

(b) Let h be the map on A/K given by $x + K \mapsto f(x) + K$. This is well-defined since K is an $\langle f\rangle$-module, and A/K is cofinite because K is closed in A, i.e. $h((K + K_i)/K) \subseteq (K + K_i)/K$, so that each $(K + K_i)/K$ is an $\langle h\rangle$-submodule of A/K.

If f is cofinitely pure, then each f_i is pure. Fix $i \in I$. The map h_i on A/K factored by $(K + K_i)/K$ given by

$$(x + K) + (K + K_i)/K \mapsto (f(x) + K) + (K + K_i)/K$$

is both the map induced from h and the map induced from f_i. So from the corresponding result in finite dimensions, h_i is pure, and so h is cofinitely pure.

The proof when f is cofinitely nilpotent is similar. □

Recall that for a finite dimensional algebra A a linear map f on A is cleft if f decomposes into a sum $f = p + n$, where p

is pure, n is nilpotent and p and n commute. This is also referred to as a splitting of f in the literature, and if f is cleft, f is also called splittable, or almost algebraic.

Definition 12.4

Let A be a cofinite algebra and let f be a linear map on A. Then f is cofinitely cleft if f decomposes into a sum $f = p + n$ such that p is cofinitely pure on A, n is cofinitely nilpotent on A and p and n commute.

Let L be a Lie algebra. Then $x \in L$ is cofinitely ad-cleft if x has decomposition $x = p + n$ such that p^* is cofinitely pure, n^* is cofinitely nilpotent and p, n commute.

L is cofinitely cleft if every x in L is cofinitely ad-cleft.

L is profinitely cleft if it is cofinitely cleft and compact.

We show that a cofinitely cleft map has a unique decomposition:

Proposition 12.5

Let A be a cofinite algebra, f a linear map on A and consider A as an $\langle f \rangle$-module. Suppose that f is cofinitely cleft with a decomposition $f = p + n$, where p is cofinitely pure, n is cofinitely nilpotent and $pn = np$.

Then p and n are uniquely defined.

Proof: Suppose that $f = p + n = p' + n'$ are two such decompositions.

Then $f_i = p_i + n_i = p'_i + n'_i$ for all $i \in I$. Let $g = p - p' = n' - n$. Then the induced maps are $g_i : A/K_i \to A/K_i$ given by $x + K_i \mapsto g(x) + K_i$. Fix $x \in A$. Now $g_i = p_i - p'_i$, so g_i is pure, since p_i and p'_i commute, from the finite dimensional theory. Similarly $g_i = n'_i - n_i$ is nilpotent, and hence g_i is the zero map. So $(p - p')(x) \in K_i$ for all $i \in I$ and so $p(x) = p'(x)$. This is true for all x in A, so $p = p'$ and $n = n'$. □

Corollary 12.6

Let L be a cofinite Lie algebra and consider $x \in L$. If x is cofinitely ad-cleft, then the decomposition of x into a sum of a cofinitely ad-pure and a cofinitely ad-nilpotent element which commute is unique modulo the centre of L.

Proof: Let Z be the centre of L. Suppose that $x = p + n = q + m$ are two such decompositions. Then $p^* + n^* = q^* + m^*$ and $p^* = q^*$, $n^* = m^*$ by proposition 12.5. So $(p - q)^*$ is the zero map, whence $p - q \in Z$. Similarly, $n - m \in Z$. □

Definition 12.7

With the notation of proposition 12.5, the decomposition $f = p + n$ is called the Chevalley-Jordan decomposition (or the cofinite cleaving) of f.

The next result follows from the corresponding result in finite dimensions:

Proposition 12.8

Let A be a cofinite algebra and let f be a linear map on A. Suppose that f has a cofinite cleaving $f = p + n$. Let U be a vector subspace of A. Then

(i) If $f(U) \subseteq U$, then $p(U), n(U) \subseteq \overline{U}$

(ii) If $f(U) = 0$, then $p(U) = n(U) = 0$.

Proof: (i) Suppose that $f(U) \subseteq U$. Then $f_i((U + K_i/K_i)$ is contained in $(U + K_i)/K_i$ for all $i \in I$. So for each $i \in I$, $p_i((U + K_i)/K_i) \subseteq (U + K_i)/K_i$, since A/K_i is finite dimensional. Therefore $p(U) \subseteq U + K_i$ for each $i \in I$, so that

$$p(U) \subseteq \bigcap_{i \in I} (U + K_i) = \overline{U}$$

Similarly, $n(U) \subseteq \overline{U}$.

(ii) This is proved similarly to part (i). □

Section 2 Tori of Cofinite Lie Algebras

We define a torus in a cofinite Lie algebra in the obvious way, and show that in a profinitely cleft Lie algebra the centraliser of a maximal torus is a Cartan subalgebra, and conversely that a Cartan subalgebra of a profinitely cleft Lie algebra is the centraliser of a unique maximal torus.

Recall that for a finite dimensional Lie algebra a torus is a subalgebra consisting of ad-pure elements.

Definition 12.9

Let L be a cofinite Lie algebra. Then T is a torus (or a toral subalgebra of L) if T is a subalgebra of L consisting of cofinitely ad-pure elements.

We give some simple properties of tori of cofinite Lie algebras.

Proposition 12.10

Let L be a cofinite Lie algebra. Then each torus is contained in a maximal torus.

Proof: We use a straightforward Zorn's lemma argument. Let T be a torus and let $\mathcal{P}$ be the set of tori containing T. $\mathcal{P}$ is non-empty since $T \in \mathcal{P}$, and it is easy to show that if $\mathcal{C}$ is a chain in $\mathcal{P}$, then $\bigcup_{T' \in \mathcal{C}} T'$ is in $\mathcal{P}$, so that $\mathcal{P}$ has maximal elements, i.e. there exist maximal tori containing T. □

Proposition 12.11

Let L be a cofinite Lie algebra and let T be a torus. Then $(T + K)/K$ is a torus in L/K for all $K \in \mathcal{K}(L)$.

Proof: Clear. □

Proposition 12.12

Let L be a cofinite Lie algebra and let T be a maximal torus in L. Then T is closed in L.

If, further, L is compact, then $\varphi(T)$ is an inverse limit of tori, where φ is the natural map $\varphi : L \to \varprojlim\{L/K_i\}$.

Proof: Consider $x \in \overline{T}$. Now $x + K \in (T + K)/K$ for each $K \in \mathcal{K}(L)$, so that $x + K$ is ad-pure. Therefore x is cofinitely ad-pure, and so $\overline{T}$ is a torus. But T is maximal, so $T = \overline{T}$ and hence T is closed.

To prove the second statement, note that $\varphi(T) = \varphi(\overline{T}) = \overline{\varphi(T)}$ $= \varprojlim\{(T + K_i)/K_i\}$ and the result follows from proposition 12.11. □

Remark 12.13

The above method of proof occurs frequently for various types of maximal subalgebras of a cofinite Lie algebra. If H and $\overline{H}$ share the property relative to which H is maximal, then $H = \overline{H}$ and so H is closed.

Proposition 12.14

Let L be a cofinite Lie algebra and let T be a torus in L. Then T is abelian.

Proof: Consider $x, y \in T$. Then for each $K \in \mathcal{K}(L)$,

$$[x + K, y + K] \in ((T + K)/K)^2 = 0$$

since $(T + K)/K$ is a torus in L/K, by proposition 12.11. Thus $[x, y] \in K$ for all $K \in \mathcal{K}$, so that $[x, y] = 0$. □

Corollary 12.15

Let L be a cofinite Lie algebra. If T is a torus in L, then T is a torus in T.(i.e. for $x \in T$, x is cofinitely ad-pure in its action on T).

Proof: T is abelian, so for $t \in T$, the restriction of t^* to T is the zero map. Therefore t^* is cofinitely pure on T.

Proposition 12.16

Let $L = \varprojlim\{L_i ; \rho_{ij}\}$, where each L_i is a finite dimensional Lie algebra, and let $T = \varprojlim\{T_i\}$ be a subalgebra of L, where

$\rho_{ij}(T_i) = T_j$. Give L the usual cofinite topology.

Then T is a torus in L if and only if T_i is a torus in L_i for every $i \in I$.

Proof: Suppose that T is a torus in L and fix $j \in I$. Consider $x_j \in T_j$. Then there exists $x = (y_i)$ in T such that $y_j = x_j$. Now x is cofinitely ad-pure in L, so that y_i is ad-pure in L_i for each $i \in I$, and in particular x_j is ad-pure. Thus T_j is a torus.

The converse is clear. □

We now use tori to obtain another expression for the Cartan subalgebras of a profinitely cleft Lie algebra (theorems 12.18 and 12.21).

We do not use all of the following lemma, but prove all of the parts for the sake of completeness.

Lemma 12.17

Let L be a cofinite Lie algebra, and let H be a vector subspace of L. Then

(i) $C_L(H)$ is a closed subalgebra of L

(ii) $C_L(\overline{H}) = C_L(H)$

(iii) If L is compact, then $\varphi(C_L(H)) = \varprojlim\{C_{L/K}((H + K)/K) : K \in \mathcal{K}(L)\}$

(iv) If H is closed in L, then $I_L(H)$ is closed in L

(v) If L is compact and H is closed, then $\varphi(I_L(H))$ equals

$\underleftarrow{\lim}\{I_{L/K}((H + K)/K): K \in \mathcal{K}(L)\}$.

Proof: (i) Let $C_K/K = C_{L/K}((H + K)/K)$ for each $K \in \mathcal{K}$ and let $C = \bigcap_{K \in \mathcal{K}} C_K$. We show that $C = C_L(H)$, whence $C_L(H)$ is closed. Consider $x \in C_L(H)$. Then $[x,H] = 0$, so that $[x + K, H + K] \subseteq K$ for all $K \in \mathcal{K}$. Therefore $x + K \in C_{L/K}((H + K)/K)$ for all K, and hence $x \in C_K$ for all K. Thus $x \in C$ and so $C_L(H) \subseteq C$.

Conversely, consider $y \in C$. Then $[y + K, H + K] \subseteq K$ for all $K \in \mathcal{K}$. Therefore $[y, H] \subseteq \bigcap_{K \in \mathcal{K}} K = 0$ and so $y \in C_L(H)$. Hence $C = C_{L(H)}$.

(ii) Now $\overline{H} + K = H + K$ for all $K \in \mathcal{K}$, so $C_K/K = C_{L/K}((H + K)/K) = C_{L/K}((\overline{H} + K)/K) = \overline{C}_K/K$, say. Then $C_K = \overline{C}_K$ for all $K \in \mathcal{K}$, and hence $C = \bigcap_{K \in \mathcal{K}} C_K = \bigcap_{K \in \mathcal{K}} \overline{C}_K = C_L(\overline{H})$.

(iii) It is clear that $\varphi(C_L(H)) \subseteq \underleftarrow{\lim}\{C_{L/K}((H + K)/K)\}$, so consider $(x_K + K)_K \in \underleftarrow{\lim}\{C_{L/K}((H + K)/K)\}$. Since L is compact, there exists $x \in L$ such that $x_K + K = x + K$ for all $K \in \mathcal{K}$, and $[x + K, H + K] = [x_K + K, H + K] \subseteq K$ for each K. Thus $[x, H] \subseteq K$ for every K and hence $[x, H] = 0$. So $x \in C_L(H)$, so that $(x_K + K) = (x + K) = \varphi(x) \in \varphi(C_L(H))$.

(iv) Consider $x \in I_L(H)$. Then $[x, H] \subseteq H$. So $[x + K, H + K] \subseteq H + K$ for all K. Therefore $x + K \in I_{L/K}((H + K)/K) = H_K/K$, say, for each K. Thus x lies in $\bigcap_{K \in \mathcal{K}} H_K$, which is closed. Hence $I_L(H) \subseteq \bigcap_{K \in \mathcal{K}} H_K$. It suffices now to prove equality.

Consider $y \in \bigcap_{K \in \mathcal{K}} H_K$. Then for each K, $[x + K, H + K] \subseteq H + K$.

Thus $[x, H] \subseteq \bigcap_{K \in \mathcal{K}} (H + K) = \overline{H} = H$. Therefore $x \in I_L(H)$ and so

$I_L(H) = \bigcap_{K \in \mathcal{K}} H_K$, from which it follows that $I_L(H)$ is closed.

(v) It suffices to show that $\varprojlim\{I_{L/K}((H + K)/K)\} \subseteq \varphi(I_L(H))$, since the converse is clear. Consider $(x_K + K) \in \varprojlim\{I_{L/K}((H + K)/K)\}$. Then since L is compact there exists x in L such that $(x_K + K) = (x + K)$. So $[x + K, H + K] \subseteq H + K$ for all K and hence $[x,H] \subseteq \bigcap_{K \in \mathcal{K}} (H + K) = \overline{H}$, whence $x \in I_L(H)$. Therefore

$$(x_K + K) = (x + K) = \varphi(x) \in \varphi(I_L(H)). \square$$

Theorem 12.18

Let L be a profinitely cleft Lie algebra and let T be a maximal torus in L.

Then $C_L(T)$ is a Cartan subalgebra of L.

Proof: We use theorem 9.7. Let $C = C_L(T)$.

(i) We show that for each $K \in \mathcal{K}$ and for each $c + K \in (C + K)/K$, $(c + K)^*$ is nilpotent on $(C + K)/K$ and then apply Engel's theorem to see that $(C + K)/K$ is nilpotent for every $K \in \mathcal{K}$.

Note that T is contained in C, since T is abelian, by proposition 12.14. Also, C is cofinitely cleft in L, for consider $c \in C$ and suppose that $c = p + n$ is a cofinite cleaving for c in L. Since $[c,T] = 0$, we have $[p,T] = [n,T] = 0$, by proposition 12.8(ii). Therefore $p,n \in C$ and so C is cofinitely cleft, as claimed. Now $T + \langle p \rangle$ is a torus in L, so by maximality $T = T + \langle p \rangle$ and hence $p \in T$. (Hence T is the set of cofinitely ad-pure elements of C.) So $[C,p] = 0$, since $[C,T] = 0$. Also, n^* is cofinitely ad-nilpotent on C, so that $c^* = p^* + n^*$ is cofinitely ad-nilpotent on C. Therefore $(c + K)^*$ is nilpotent on $(C + K)/K$ for all $K \in \mathcal{K}$ and for all $c \in C$. By Engel's theorem $(C + K)/K$ is nilpotent for every $K \in \mathcal{K}$, and so C is cofinitely nilpotent.

(ii) C is closed, by lemma 12.17(i).

(iii) Suppose that H is a closed subalgebra of L containing C, and suppose that $H \neq I_L(H)$. We obtain a contradiction, whence $H = I_L(H)$ and so C is quasiabnormal. Now there exists $x \in L \setminus H$ with $[x,H] \subseteq H$. Let $S = H \dotplus \langle x \rangle$. Then S is closed, by proposition 2.28. Now H is a closed ideal of S and both S and H can be regarded as T-modules, since T is contained in both S and H. Consider the Fitting decomposition of S and H as T-modules, so that, as T-modules, S is isomorphic to $Cr_{\lambda \in T^*} S_\lambda$ and H is isomorphic to $Cr_{\lambda \in T^*} H_\lambda$. Then, as vector spaces, S/H is isomorphic to $Cr_{\lambda \in T^*} S_\lambda / H_\lambda$. But H has codimension one in S, so there exists μ in T^* such that S_μ / H_μ has dimension one and $S_\lambda / H_\lambda = 0$ for all $\lambda \neq \mu$. So there exists $y \in S_\mu \setminus H_\mu \subseteq S$, and $y \notin H$, else y lies in $H \cap S_\mu$, which equals H_μ, by proposition 11.9. We show that $y \in C$, which is the required contradiction, since $C \subseteq H$.

Now for any $t \in T$, $\mu(t).y = [t,y] \in [H,y]$ since $t \in T \subseteq C \subseteq H$.

But $y \in S$ and H is an ideal of S, so that $[H,y] \subseteq H$, and so $\mu(t).y \in H$. If $\mu(t) \neq 0$, then $y \in H$, so $\mu(t) = 0$, and this is true for all t in T. Therefore $[T,y] = 0$ and so $y \in C$. □

Corollary 12.19

Let L be a profinitely cleft Lie algebra. Then any two maximal tori of L are Σ-conjugate.

Proof: Since the natural map $\varphi : L \to P = \varprojlim\{L/K_i\}$ is a topological isomorphism it suffices to show that any two maximal tori in P are Σ-conjugate, regarding Σ as a subgroup of Aut(P).

Let T, T' be two such tori and let $C = C_P(T)$, $C' = C_P(T')$. By proposition 10.1 there exists $\delta = (\delta_i)$ in Σ such that $\delta(C) = C'$. But from part (i) of the proof of theorem 12.18, T is the set of cofinitely ad-pure elements of C and T' is the set of cofinitely ad-pure elements of C'. We prove that δ preserves cofinitely ad-pure elements of P, from which it follows that $\delta(T) \subseteq T'$. Similarly $\delta^{-1}(T') \subseteq T$, so that $\delta(T) = T'$, the required result.

Consider $(x + K_i) \in P$ such that $(x + K_i)$ is cofinitely ad-pure. Then each $x + K_i$ is ad-pure in L/K_i, so that $\delta_i(x + K_i)$ is ad-pure in L/K_i, since δ_i is an auromorphism. Thus $\delta((x + K_i)) = (\delta_i(x + K_i))$ and so is cofinitely ad-pure in P, as claimed. □

Using the two previous results we may prove the following:

Proposition 12.20

Let L be a profinitely cleft Lie algebra, and let T be a maximal torus in L. Then for each $i \in I$, $(T + K_i)/K_i$ is a maximal torus in L/K_i.

Proof: Let $\mathcal{U}_i = \{T'/K_i : T'/K_i$ is a maximal torus containing $(T + K_i)/K_i\}$. Let $E_i = \mathcal{E}(L/K_i)$ and let $S_i = \{\theta \in E_i : \theta(T'/K_i) = T'/K_i\}$ for some fixed T'/K_i in $\mathcal{U}_i$. For each $i \in I$, consider the map $f_i : E_i/S_i \to \mathcal{U}_i$ given by $\theta S_i \mapsto \theta(T'/K_i)$. Now L is profinitely cleft, so it is clear that each L/K_i is cleft. By corollary 12.18 $\mathcal{U}_i$ is E_i-conjugate, so that f_i is surjective. It is easy to check that f_i is also injective. Arguing now as in 7.2 - 7.8 we may show that $\mathcal{U} = \varprojlim\{\mathcal{U}_i; \rho_{ij}\}$ is non-empty.

Consider $(T_i/K_i) \in \mathcal{U}$. Then $\varprojlim\{(T + K_i)/K_i\} \leq \varprojlim\{T_i/K_i\} \leq \varprojlim\{L/K_i\} = P$. Now T is a maximal torus in L, so $\varprojlim\{(T + K_i)/K_i\}$ is a maximal torus in P. But $\varprojlim\{T_i/K_i\}$ is a torus in P, by proposition 12.16, and contains $\varprojlim\{(T + K_i)/K_i\}$, so we must have equality.

Hence $(T + K_i)/K_i = T_i/K_i$ for each $i \in I$, since each projection map π_{ij} maps T_i/K_i onto T_j/K_j, using lemma 5.14, so that each $(T + K_i)/K_i$ is a maximal torus in L/K_i. □

We now prove the converse of theorem 12.18.

Theorem 12.21

Let L be a profinitely cleft Lie algebra and let C be a Cartan subalgebra of L. Then $C = C_L(T)$ for some maximal torus T, and T is uniquely determined.

Proof: We choose a maximal torus S in L and consider the Cartan subalgebra $C' = C_L(S)$. Then C and C' are conjugate under some δ in Σ. We then show that $\delta(S)$ is the required maximal torus.

Now 0 is trivially a torus, so by proposition 12.10 there exists a maximal torus, S, say, of L. Let $C' = C_L(S)$. Then C' is a Cartan subalgebra of L, by theorem 12.18, so by theorem 10.4(iii) there exists $\delta \in \Sigma$ such that $\delta(C') = C$. Let $T = \delta(S)$. As in the proof of corollary 12.19, δ preserves tori, so that T is a torus.

To show that T is a maximal torus, Suppose that T' is a maximal torus containing T (such tori exist, by proposition 12.10). Then $\delta^{-1}(T')$ is a torus containing S, so by the maximality of S, $\delta^{-1}(T') = S$, and then $T' = \delta(S) = T$.

We now show that $C = C_L(T)$. Consider $x \in C$. Then $\delta^{-1}(x) \in C'$, and for any $t \in T$, $\delta^{-1}(t) \in S$, so that $[\delta^{-1}(x), \delta^{-1}(t)] = 0$. Thus $\delta^{-1}([x,t]) = 0$ and hence $[x,t] = 0$. Therefore $x \in C_L(T)$ and so $C \leq C_L(T)$.

Conversely, consider $x \in C_L(T)$. Now $\delta^{-1}(x) \in C_L(S) = C'$, so

We now prove the converse of theorem 12.18.

Theorem 12.21

Let L be a profinitely cleft Lie algebra and let C be a Cartan subalgebra of L. Then $C = C_L(T)$ for some maximal torus T, and T is uniquely determined.

Proof: We choose a maximal torus S in L and consider the Cartan subalgebra $C' = C_L(S)$. Then C and C' are conjugate under some γ in ξ. We then show that $\gamma(S)$ is the required maximal torus.

Now 0 is trivially a torus, so by proposition 12.10 there exists a maximal torus, S, say, of L. Let $C' = C_L(S)$. Then C' is a Cartan subalgebra of L, by theorem 12.18, so by theorem 10.4(iii) there exists $\gamma \in \xi$ such that $\gamma(C') = C$. Let $T = \gamma(S)$. As in the proof of corollary 12.19, γ preserves tori, so that T is a torus.

To show that T is a maximal torus, Suppose that T' is a maximal torus containing T (such tori exist, by proposition 12.10). Then $\gamma^{-1}(T')$ is a torus containing S, so by the maximality of S, $\gamma^{-1}(T') = S$, and then $T' = \gamma(S) = T$.

We now show that $C = C_L(T)$. Consider $x \in C$. Then $\gamma^{-1}(x) \in C'$, and for any $t \in T$, $\gamma^{-1}(t) \in S$, so that $[\gamma^{-1}(x), \gamma^{-1}(t)] = 0$. Thus $\gamma^{-1}([x,t]) = 0$ and hence $[x,t] = 0$. Therefore $x \in C_L(T)$ and so $C \leqslant C_L(T)$.

Conversely, consider $x \in C_L(T)$. Now $\gamma^{-1}(x) \in C_L(S) = C'$, so

$x \in \delta(C') = C$ and hence $C_L(T) \leqslant C$.

To prove that T is unique, suppose also that $C = C_L(U)$ for some maximal torus U. Then $U \leqslant C$. By corollary 12.19 there exists $\delta' \in \Sigma$ such that $\delta'(T) = U$. Now T is the set of cofinitely ad-pure elements of C, so arguing as before, U consists of cofinitely ad-pure elements of C, so that $U \leqslant C$. By maximality, $T = U$ and so T is unique. □

Section 3 Profinitely Cleft Lie Algebras

We give necessary and sufficient conditions for a profinite Lie algebra to be profinitely cleft. Firstly that L is profinitely cleft if and only if each factor by elements of $\mathcal{K}(L)$ is cleft, and secondly L is profinitely cleft if and only if $\sigma(L)$ is profinitely cleft. It then follows that pro-semisimple Lie algebras are profinitely cleft. Finally we show that in a profinite Lie algebra L any Levi subalgebra Λ idealises a Cartan subalgebra of the cofinitely soluble radical, and that if L is profinitely cleft then Λ also idealises the corresponding maximal torus. This result will be useful in the next chapter when showing that a profinite Lie algebra may be embedded in a profinitely cleft Lie algebra.

Proposition 12.22

Let $L = \varprojlim\{L_i; \pi_{ij}\}$ where each L_i is a finite dimensional Lie algebra, and give L the usual cofinite topology. Then L is profinitely cleft if and only if L_i is cleft for every $i \in I$.

Proof: It is clear that if L is cofinitely cleft , then each L_i is cleft.

Conversely, suppose that each L_i is cleft. Consider $x = (x_i)$ in L, and for each $i \in I$ let $x_i = p_i + n_i$ be an ad-cleaving for x. We shall find elements z_i in the centre of each L_i such that

$(p_i + z_i)$, $(n_i - z_i) \in L$ and this will provide an ad-cleaving for x in L.

Now for $j \leq i$ $\pi_{ij}(x_i) = x_j$, so $\pi_{ij}(p_i + n_i) = p_j + n_j$. Hence $\pi_{ij}(p_i) - p_j = n_j - \pi_{ij}(n_i) \in \mathfrak{Z}_1(L_j)$, the centre of L_j. For each $i \in I$, let $Z_i = \mathfrak{Z}_1(L_i)$, and let $\mathcal{P}_i = \{p_i + z_i : z_i \in Z_i\} = p_i + Z_i$. Define $\rho_{ij} : \mathcal{P}_i \to \mathcal{P}_j$ to be the restriction of π_{ij} to $\mathcal{P}_i$. Then each ρ_{ij} is well-defined, since $\rho_{ij}(p_i + Z_i) = \pi_{ij}(p_i + Z_i) = p_j + Z_j$, as ad-pure elements are preserved by epimorphisms and since ad-cleavings are unique modulo the centre.

Now each L_i is an algebraic group under addition and Z_i is a closed subgroup. So each $\mathcal{P}_i$ is a coset variety and each ρ_{ij} is an affine map. By therem 2.22 therefore, $\mathcal{P} = \varprojlim\{\mathcal{P}_i ; \rho_{ij}\}$ is non-empty. Consider $(p_i + z_i) \in \mathcal{P}$. Then $\pi_{ij}(p_i + z_i) = \rho_{ij}(p_i + z_i) = p_j + z_j$ whenever $j \leq i$ and then also $\pi_{ij}(n_i - z_i) = n_j - z_j$. Since each p_i is ad-pure on L_i, $p_i + z_i$ is ad-pure for every i in I, so that $(p_i + z_i)$ is cofinitely ad-pure. Similarly $(n_i - z_i)$ is cofinitely ad-nilpotent.

So $x = (x_i) = (p_i + z_i) + (n_i - z_i)$ -------(*)

Now $(p_i + z_i)$ and $(n_i - z_i)$ lie in L, and $[(p_i + z_i),(n_i - z_i)] = ([p_i + z_i, n_i - z_i]) = ([p_i, n_i]) = 0$.

Hence (*) is a cofinite ad-cleaving of x in L. □

Corollary 12.23

Let L be a profinite Lie algebra. Then L is cofinitely cleft

$(p_i + z_i)$, $(n_i - z_i) \in L$ and this will provide an ad-cleaving for x in L.

Now for $j \leq i$ $\pi_{ij}(x_i) = x_j$, so $\pi_{ij}(p_i + n_i) = p_j + n_j$. Hence $\pi_{ij}(p_i) - p_j = n_j - \pi_{ij}(n_i) \in \mathfrak{Z}_1(L_j)$, the centre of L_j. For each $i \in I$, let $Z_i = \mathfrak{Z}_1(L_i)$, and let $\mathcal{P}_i = \{p_i + z_i : z_i \in Z_i\} = p_i + Z_i$. Define $\rho_{ij} : \mathcal{P}_i \to \mathcal{P}_j$ to be the restriction of π_{ij} to $\mathcal{P}_i$. Then each ρ_{ij} is well-defined, since $\rho_{ij}(p_i + Z_i) = \pi_{ij}(p_i + Z_i) = p_j + Z_j$, as ad-pure elements are preserved by epimorphisms and since ad-cleavings are unique modulo the centre.

Now each L_i is an algebraic group under addition and Z_i is a closed subgroup. So each $\mathcal{P}_i$ is a coset variety and each ρ_{ij} is an affine map. By therem 2.22 therefore, $\mathcal{P} = \varprojlim\{\mathcal{P}_i; \rho_{ij}\}$ is non-empty. Consider $(p_i + z_i) \in \mathcal{P}$. Then $\pi_{ij}(p_i + z_i) = \rho_{ij}(p_i + z_i) = p_j + z_j$ whenever $j \leq i$ and then also $\pi_{ij}(n_i - z_i) = n_j - z_j$. Since each p_i is ad-pure on L_i, $p_i + z_i$ is ad-pure for every i in I, so that $(p_i + z_i)$ is cofinitely ad-pure. Similarly $(n_i - z_i)$ is cofinitely ad-nilpotent.

So $x = (x_i) = (p_i + z_i) + (n_i - z_i)$ ------(*)

Now $(p_i + z_i)$ and $(n_i - z_i)$ lie in L, and $[(p_i + z_i),(n_i - z_i)] = ([p_i + z_i, n_i - z_i]) = ([p_i, n_i]) = 0$.

Hence (*) is a cofinite ad-cleaving of x in L. □

Corollary 12.23

Let L be a profinite Lie algebra. Then L is cofinitely cleft if and only if L/K is cleft for every $K \in \mathcal{K}(L)$.

Proof: This follows from corollary 2.18. $\square$

Proposition 12.24

Let L be a profinite Lie algebra. Then L is profinitely cleft if and only if $\sigma(L)$ is profinitely cleft.

Proof: It is sufficient to show that L is cofinitely cleft if and only if $\sigma(L)$ is cofinitely cleft.

By corollary 12.23 L is cofinitely cleft if and only if L/K is cleft for every $K \in \mathcal{K}$, and this is so if and only if $\sigma(L/K)$ is cleft for all $K \in \mathcal{K}$, since the conclusion of the proposition holds for finite dimensional Lie algebras (see Stewart, [17], lemma 6.5, p94). Finally this is equivalent to $\sigma(L)$ being cofinitely cleft, by proposition 12.22, since $\sigma(L)$ is topologically isomorphic to $\varprojlim\{\sigma(L/K): K \in \mathcal{K}\}$. $\square$

Recall that in finite dimensions a semisimple Lie algebra is cleft. The corresponding result holds for profinite Lie algebras:

Proposition 12.25

Pro-semisimple Lie algebras are profinitely cleft.

Proof: Let L be a pro-semisimple Lie algebra. Now $\sigma(L) = 0$, so $\sigma(L)$ is trivially profinitely cleft, so by proposition 12.24 L is profinitely cleft. □

Theorem 12.26

Let L be a profinite Lie algebra, let $R = \sigma(L)$ and let Λ be a Levi subalgebra of L. Then Λ idealises some Cartan subalgebra of R.

Further, if R is cofinitely cleft (i.e. if L is profinitely cleft), then Λ also idealises the maximal torus corresponding to this Cartan subalgebra, as given in theorem 12.21.

Proof: (i) We use the corresponding result from finite dimensions (see Stewart, [17], lemma 6.3, p93) on each factor $(\Lambda + K_j)/K_j$ and $\sigma(L/K_j)$ to find Cartan subalgebras of $\sigma(L/K_j)$ and piece them together, as in 7.2 - 7.7 to obtain a suitable Cartan subalgebra.

Fix $j \in I$. Then $(R + K_j)/K_j = \sigma(L/K_j) = R_j/K_j$ and $(\Lambda + K_j)/K_j$ is a Levi subalgebra of L/K_j, by corollary 7.13, so there exists a Cartan subalgebra C_j/K_j of R_j/K_j idealised by $(\Lambda + K_j)/K_j$. Let $\mathcal{D}_j$ be the set of Cartan subalgebras of R_j/K_j idealised by $(\Lambda + K_j)/K_j$. Also, for each $i \in I$, let $E_i = \mathcal{E}(\frac{L}{K_i})$, $S_i = N_{E_i}(\frac{\Lambda + K_i}{K_i})$, $T_i = \{\theta \in S_i : \theta(\frac{D_i}{K_i}) = \frac{D_i}{K_i}\}$ for some fixed element $\frac{D_i}{K_i} \in \mathcal{D}_i$. It can be shown that S_i is transitive on $\mathcal{D}_i$, and that elements of $\mathcal{D}_i$ are conjugate under S_i, so that $\mathcal{D}_i \leftrightarrow \frac{S_i}{T_i}$. Arguing as in 7.2 - 7.7 we may show that $\mathcal{D} = \varprojlim\{\mathcal{D}_i ; \rho_{ij}\}$ is non-empty, where for $j \leqslant i$, ρ_{ij} is induced by π_{ij}.

Choose an element $(D_i/K_i) \in \mathcal{D}$, and let $D = \varprojlim\{D_i/K_i; \pi_{ij}\}$. Now $D \leq \varphi(L) = \varprojlim\{L/K_i\}$ since $\varphi_{ij}(D_i/K_i) = D_j/K_j$. So D is a Cartan subalgebra of $\varphi(R)$. Let $C = \varphi^{-1}(D)$. It is easy to check that $\varphi([C,\Lambda]) \leq \varprojlim\{D_i/K_i\}$, so that $[C,\Lambda] \leq C$, and so the first part is proved.

(ii) Suppose that R is cofinitely cleft, and that the Cartan subalgebra C given above has the form $C = C_R(T)$ for some unique maximal torus T of R. Again we use the corresponding result from the finite dimensional case to show that Λ idealises T.

Now for each $i \in I$, $(C + K_i)/K_i$ is a Cartan subalgebra of R_i/K_i, and centralises $(T + K_i)/K_i$. Also, each $(T + K_i)/K_i$ is a maximal torus in R_i/K_i, by proposition 12.20, so that $(C + K_i)/K_i = C_{R_i/K_i}((T + K_i)/K_i)$. Now $(\Lambda + K_i)/K_i$ idealises $(T + K_i)/K_i$, so that $\varprojlim\{(\Lambda + K_i)/K_i\}$ idealises $\varprojlim\{(T + K_i)/K_i\}$ i.e. $\varphi(\Lambda)$ idealises $\varphi(T)$. Since φ is a topological isomorphism, we conclude that Λ idealises T. □

Choose an element $(D_i/K_i) \in \mathcal{D}$, and let $D = \varprojlim\{D_i/K_i; \pi_{ij}\}$. Now $D \leq \varphi(L) = \varprojlim\{L/K_i\}$ since $\rho_{ij}(D_i/K_i) = D_j/K_j$. So D is a Cartan subalgebra of $\varphi(R)$. Let $C = \varphi^{-1}(D)$. It is easy to check that $\varphi([C,\Lambda]) \leq \varprojlim\{D_i/K_i\}$, so that $[C,\Lambda] \leq C$, and so the first part is proved.

(ii) Suppose that R is cofinitely cleft, and that the Cartan subalgebra C given above has the form $C = C_R(T)$ for some unique maximal torus T of R. Again we use the corresponding result from the finite dimensional case to show that Λ idealises T.

Now for each $i \in I$, $(C + K_i)/K_i$ is a Cartan subalgebra of R_i/K_i, and centralises $(T + K_i)/K_i$. Also, each $(T + K_i)/K_i$ is a maximal torus in R_i/K_i, by proposition 12.20, so that $(C + K_i)/K_i = C_{R_i/K_i}((T + K_i)/K_i)$. Now $(\Lambda + K_i)/K_i$ idealises $(T + K_i)/K_i$, so that $\varprojlim\{(\Lambda + K_i)/K_i\}$ idealises $\varprojlim\{(T + K_i)/K_i\}$ i.e. $\varphi(\Lambda)$ idealises $\varphi(T)$. Since φ is a topological isomorphism, we conclude that Λ idealises T. □

Section 4 Cofinitely Cleft Cofinitely Soluble Lie Algebras

In this section we look briefly at cofinitely cleft cofinitely soluble Lie algebras, and show that if T is a maximal torus in a profinitely cleft pro-soluble Lie algebra L, then $L = \nu(L) + T$ and $T \cap \nu(L) = \zeta_1(L)$. From this we see that $L = \nu(L) \dot{+} T_0$ for some subalgebra T_0 of each such T. Also, for any Cartan subalgebra C of such an L, $L = \nu(L) + C$.

Notation

When a Lie algebra is referred to as cofinitely cleft and as cofinitely soluble, the topology concerned in each case will be the same.

We firstly obtain the centre of a cofinite Lie algebra L from the centres of the factors by elements of $\mathcal{K}$.

Lemma 12.27

Let L be a cofinite Lie algebra and for each $i \in I$ let Z_i/K_i be the centre of L/K_i.

Then the centre of L is a closed ideal of L and equals $\bigcap_{i \in I} Z_i$.

Proof: Consider x in Z, the centre of L. We show that x lies in every Z_i. For any $y \in L$, $[x + K_i, y + K_i] = [x,y] + K_i = K_i$ for every $i \in I$, so that $x + K_i \in Z_i/K_i$ for all i. Therefore x lies in $\bigcap_{i \in I} Z_i$ and so $\zeta_1(L) \subseteq \bigcap_{i \in I} Z_i$.

Conversely, consider $x \in \bigcap_{i \in I} Z_i$. Then for any $y \in L$, $[x + K_i, y + K_i] = K_i$, so $[x,y] \in K_i$ for every $i \in I$, and hence $[x,y] = 0$. Therefore $\zeta_1(L) = \bigcap_{i \in I} Z_i$.

$\zeta_1(L)$ is closed since each Z_i is closed. □

Proposition 12.28

Let L be a cofinitely cleft, cofinitely soluble Lie algebra and let T be a maximal torus of L. Then $\nu(L) \cap T = \zeta_1(L)$.

Proof: Consider $x \in \nu(L) \cap T$. For each $i \in I$, $x + K_i$ lies in $N_i/K_i \cap (T + K_i)/K_i$, and this is contained in $\zeta_1(L/K_i)$, which equals Z_i/K_i from the corresponding result in finite dimensions, since $(T + K_i)/K_i$ is a torus. So $x \in \bigcap_{i \in I} Z_i = \zeta_1(L)$.

Conversely, $\zeta_1(L/K_i) \subseteq N_i/K_i$ for all $i \in I$, so that $Z_i \subseteq N_i$ for every i. Thus $\zeta_1(L) = \bigcap_{i \in I} Z_i \subseteq \bigcap_{i \in I} N_i = \nu(L)$. Also, T is contained in $T + \zeta_1(L)$, so as T is a maximal torus, $T = T + \zeta_1(L)$, whence $\zeta_1(L) \subseteq T$. □

Lemma 12.29

Let L be a cofinitely soluble Lie algebra and consider $x \in L$.

x is cofinitely ad-nilpotent if and only if $x \in \nu(L)$.

Proof: We again use the corresponding result from finite dimensions (see Stewart, [17], lemma 5.2, p90). Now x is cofinitely ad-nilpotent if and only if $x + K_i$ is ad-nilpotent for every $i \in I$, and this is equivalent to $x + K_i \in N_i/K_i$ for every $i \in I$, which in turn holds if and only if $x \in \bigcap_{i \in I} N_i$, which equals $\nu(L)$. □

Proposition 12.30

Let L be a profinitely cleft, profinitely soluble Lie algebra and let T be a maximal torus in L.

Then $L = \nu(L) + T$.

Proof: We show first that $L = \nu(L) + C$, where $C = C_L(T)$, and then express a typical element of L as a sum of elements from $\nu(L)$ and from C. The Chevalley-Jordan decomposition in L of the element from C is then used to show that x lies in $\nu(L) + T$.

Now $L^2 \subseteq \nu(L)$, by proposition 5.9(iii), so that $L/\nu(L)$ is abelian and hence cofinitely nilpotent. Now C is a Cartan subalgebra of L, by theorem 12.18, and L is trivially a closed subalgebra of L containing C, with $\nu(L)$ a closed ideal of L such that $L/\nu(L)$ is cofinitely nilpotent. So by the definition of a Cartan subalgebra, $L = \nu(L) + C$.

Now consider $x \in L$. $x = m + c$ for some $m \in \nu(L)$ and some $c \in C$. Since L is cofinitely cleft, let $c = n + p$ be a cofinite ad-cleaving for c in L. But $n \in \nu(L)$, by lemma 12.29, so that $m + n \in \nu(L)$. Also, $p \in T$, so that $x = (m + n) + p \in \nu(L) + T$. Hence L is contained in $\nu(L) + T$, so that $L = \nu(L) + T$. □

Corollary 12.31

Let L be a profinitely cleft, profinitely soluble Lie algebra and let T be a maximal torus in L. If T_0 is any vector space complement to $\zeta_1(L)$ in T, then $L = \nu(L) \dot{+} T_0$ and every non-zero element of T_0 has a non-zero eigenvalue on L (i.e. for all non-zero t in T_0, $[x,t] = \lambda(t)x$ with $\lambda(t) \neq 0$).

Proof: T is abelian, so T_0 is a subalgebra of L. So $L = \nu(L) + T_0 + \zeta_1(L) = (\nu(L) + \zeta_1(L)) + T_0 = \nu(L) + T_0$. Also $T_0 \cap \nu(L) = T_0 \cap (T \cap \nu(L)) = T_0 \cap \zeta_1(L) = 0$. Thus $L = \nu(L) \dot{+} T_0$. Further, consider $t \in T_0$ and suppose, for the sake of a contradiction, that $\lambda(t) = 0$. Then $[x,t] = \lambda(t).x = 0$ for all $x \in L$. So $t \in \zeta_1(L)$ and hence $\zeta_1(L) \cap T_0 \neq 0$, a contradiction. □

Corollary 12.32

Let L be a profinitely cleft, profinitely soluble Lie algebra and let C be a Cartan subalgebra of L. Then $L = \nu(L) + C$.

Proof: By theorem 12.21, $C = C_L(T)$ for some maximal torus T. So as in the proof of proposition 12.30, $L = \ (L) + C$.

Chapter 13 Embedding In A Profinitely Cleft Lie Algebra

In this chapter we aim to show that a profinite Lie algebra embeds in a profinitely cleft Lie algebra.

Section 1 Cofinitely Semicleft Lie Algebras

In the ensuing sections we shall use the concept of semicleft. In the definition of a cofinite cleaving, if we relax the condition that the cofinitely pure and cofinitely nilpotent parts in the Chevalley-Jordan decomposition of a linear map need to commute, the result is a cofinite semicleaving. We show that a Lie algebra is cofinitely semicleft if and only if each L/K, for $K \in \mathcal{K}$, is semicleft, and that in the case of a pro-soluble Lie algebra the property of cofinitely semicleft is equivalent to the property of cofinitely cleft.

Recall that for a linear map f on a finite dimensional algebra A, f is semicleft if f can be expressed as a sum of linear maps, f = p + n, such that p is pure and n is nilpotent (see Stewart, [17], p89).

Definition 13.1

Let A be a cofinite algebra and let f be a linear map on A such that $f(K) \subseteq K$ for all $K \in \mathcal{K}$.

Then f is cofinitely semicleft if f can be expressed as a sum $f = p + n$, where p is cofinitely pure on A and where n is cofinitely nilpotent on A.

Definition 13.2

Let L be a cofinite Lie algebra. L is cofinitely semicleft if each $x \in L$ is cofinitely ad-semicleft, i.e. if for each $x \in L$, x^* is cofinitely semicleft.

Proposition 13.3

Let L be a profinite Lie algebra. Then L is cofinitely semicleft if and only if L/K is semicleft for each $K \in \mathcal{K}$.

Proof: This may be proved by a similar argument to that given in the proof of the analogous result for the cofinitely cleft case in proposition 12.22. $\square$

Theorem 13.4

Let L be a pro-soluble Lie algebra. L is cofinitely semicleft if and only if L is cofinitely cleft.

Proof: If L is cofinitely cleft it is clear that L is cofinitely semicleft.

For the converse we use the corresponding result from finite dimensions (see Mal'cev, [12], theorem 2, p233). If L is cofinitely semicleft then each L/K is semicleft, by proposition 13.3. Also, each L/K is soluble, so that each L/K is cleft, whence L is cofinitely cleft, by corollary 12.23. □

Section 2 Embedding A Profinite Lie Algebra In A Profinitely Cleft Lie Algebra

For a profinite Lie algebra L we shall find a profinitely cleft Lie algebra in which L embeds. For each $i \in I$ we take a subalgebra of the Lie algebra of derivations of L/K_i, and then take the inverse limit D of these. Then L factored by its centre embeds in D, so we use D to find a suitable Lie algebra in which L may embed.

Notation 13.5

For a cofinite Lie algebra L and for each $i \in I$ we shall denote by D_i the set of derivations of L/K_i which stabilise all ideals of L/K_i. (Note that each D_i is finite dimensional.)

For $j \leq i$, define maps $\rho_{ij} : D_i \to D_j$ as follows: ρ_{ij} maps θ_i to θ_j, where θ_j is defined by its action on L/K_j according to $\theta_j(x + K_j) = \pi_{ij}(\theta_i(x + K_i))$, where $\pi_{ij} : L/K_i \to L/K_j$ is the canonical map. So ρ_{ij} is defined so that the following diagram commutes

$$\begin{array}{ccc} \frac{L}{K_i} & \xrightarrow{\theta_i} & \frac{L}{K_i} \\ {\scriptstyle \pi_{ij}}\downarrow & & \downarrow{\scriptstyle \pi_{ij}} \\ \frac{L}{K_j} & \xrightarrow{\theta_j} & \frac{L}{K_j} \end{array}$$

We shall denote by D the inverse limit $\varprojlim\{D_i; \rho_{ij}\}$, and we give D the usual cofinite topology, so that D is profinite.

Remark 13.6

For $j \leq i$, ρ_{ij} is well-defined i.e. $\rho_{ij}(D_i) \leq D_j$. To see this, note firstly that θ_j is a derivation of L/K_j. Also, if H/K_j is an ideal of L/K_j, then $\theta_j(H/K_j) = \pi_{ij}(\theta_i(H/K_i)) \subseteq \pi_{ij}(H/K_i) = H/K_j$. Therefore $\theta_j \in D_j$, as required.

The next result paves the way for the embedding theorem.

Theorem 13.7

Let L be a cofinite Lie algebra and let D be as defined above. Let $\tau : L \to D$ be the map induced by the adjoint representation of L i.e. $x \mapsto ((x + K_i)^*)$. Then

(i) D is profinitely cleft

(ii) Ker $\tau = \mathfrak{Z}_1(L)$

If, further, L is compact, then

(iii) Im $\tau = \tau(L) = \varprojlim\{\mathrm{Inn}(L/K_i)\}$, where Im τ denotes the image of τ.

(iv) τ is continuous

(v) τ maps closed vector subspaces to closed vector subspaces.

(vi) If H is a closed ideal of L, then $\tau(H)$ is a closed ideal of D.

<u>Proof:</u> (i) D is compact because it is an inverse limit of finite dimensional Lie algebras, each one having the affine topology.

To show that D is cofinitely cleft it suffices to show that each D_i is cleft, by proposition 12.22. Fix $i \in I$ and consider $d \in D_i$. Since L/K_i is finite dimensional, d is L/K_i-cleft in $Der(L/K_i)$ - this is lemma B of Humphreys, [9], p18, i.e. there exist derivations p and n of L/K_i such that p and n commute and such that p is pure on L/K_i and n is nilpotent on L/K_i. Now p and n stabilise all ideals of L/K_i, by proposition 12.8(i), and hence $p, n \in D_i$. By Stewart, [17], corollary 4.2, p83, any subalgebra of the Lie algebra of linear maps of a finite dimensional vector space V which contains the pure and nilpotent parts of each of its elements (considered as linear maps on V) is cleft. Hence D_i is cleft.

(ii) An element x of L lies in the kernel of τ if and only if $(x + K_i)^* = 0$ for all $i \in I$, and this is equivalent to $[x,L] \subseteq K_i$ for all $i \in I$, which is equivalent to $x \in \zeta_1(L)$.

(iii) For $j \leq i$, ρ_{ij} maps $Inn(L/K_i)$ to $Inn(L/K_j)$ and is given by $(x + K_i)^* \mapsto (x + K_j)^*$. It is easy to check that each ρ_{ij} is a well-defined epimorphism, so we can form $N = \varprojlim\{Inn(L/K_i); \rho_{ij}\}$, a subalgebra of D. Now it is clear that $\tau(L) \leq N$, so it suffices to prove the converse.

Consider $((x_i + K_i)^*) \in N$. We want to find $x \in L$ such

Proof: (i) D is compact because it is an inverse limit of finite dimensional Lie algebras, each one having the affine topology.

To show that D is cofinitely cleft it suffices to show that each D_i is cleft, by proposition 12.22. Fix $i \in I$ and consider $d \in D_i$. Since L/K_i is finite dimensional, d is L/K_i-cleft in $Der(L/K_i)$ - this is lemma B of Humphreys, [9], p18, i.e. there exist derivations p and n of L/K_i such that p and n commute and such that p is pure on L/K_i and n is nilpotent on L/K_i. Now p and n stabilise all ideals of L/K_i, by proposition 12.8(i), and hence $p, n \in D_i$. By Stewart, [17] corollary 4.2, p83, any subalgebra of the Lie algebra of linear maps of a finite dimensional vector space V which contains the pure and nilpotent parts of each of its elements (considered as linear maps on V) is cleft. Hence D_i is cleft.

(ii) An element x of L lies in the kernel of τ if and only if $(x + K_i)^* = 0$ for all $i \in I$, and this is equivalent to $[x,L] \subseteq K_i$ for all $i \in I$, which is equivalent to $x \in \zeta_1(L)$.

(iii) For $j \leqslant i$, ρ_{ij} maps $Inn(L/K_i)$ to $Inn(L/K_j)$ and is given by $(x + K_i)^* \mapsto (x + K_j)^*$. It is easy to check that each ρ_{ij} is a well-defined epimorphism, so we can form $N = \varprojlim\{Inn(L/K_i); \rho_{ij}\}$, a subalgebra of D. Now it is clear that $\tau(L) \leqslant N$, so it suffices to prove the converse.

Consider $((x_i + K_i)^*) \in N$. We want to find $x \in L$ such that $((x + K_i)^*) = ((x_i + K_i)^*)$. We show that for each $i \in I$ there exists $z_i \in Z_i$ such that $(x_i + z_i + K_i) \in \varphi(L)$, where $Z_i/K_i = \zeta_1(L/K_i)$, and then show that there exists $x \in L$ such that $(x + K_i) = (x_i + z_i + K_i)$, and deduce that $(x + K_i)^* = (x_i + K_i)^*$ for all $i \in I$, so that $\tau(x) = ((x_i + K_i)^*)$ and then τ is surjective. For $j \leq i$, $\rho_{ij}((x_i + K_i)^*) = (x_j + K_j)^*$, so that $x_i - x_j \in Z_j$. Let $\mathcal{V}_i = \{x_i + z_i + K_i: z_i \in Z_i\} = (x_i + Z_i + K_i)/K_i$. Define $\sigma_{ij} : \mathcal{V}_i \to \mathcal{V}_j$ to be the restriction of π_{ij} to $\mathcal{V}_i$ whenever $j \leq i$. σ_{ij} is well-defined, since $\sigma_{ij}(x_i + z_i + K_i) = \pi_{ij}(x_i + z_i + K_i) = x_i + z_i + K_j = (x_i + K_j) + (z_i + K_j) = (x_j + K_j) + (z_i + K_j) \in \mathcal{V}_j$, since $Z_i \subsetneq Z_j$. We can check that each $\mathcal{V}_i$ is a coset variety and that each σ_{ij} is an affine map, and then by theorem 2.22 $\mathcal{V} = \varprojlim\{\mathcal{V}_i; \sigma_{ij}\}$ is non-empty. So there exists $(y_i) \in \mathcal{V}$ such that for each $i \in I$, $y_i = x_i + z_i + K_i$ for some $z_i \in Z_i$, and such that $\pi_{ij}(y_i) = y_j$. So $(x_i + z_i + K_i) \in \varphi(L)$ and hence there exists $x \in L$, by corollary 2.18, such that $(x + K_i) = (x_i + z_i + K_i)$. Then for each $i \in I$, $(x + K_i)^* = (x_i + z_i + K_i)^* = (x_i + K_i)^*$, as required.

(iv) Let J be a closed vector subspace of N having finite codimension in N. Since τ is linear and because the topology on N is a cofinite one it suffices to show that $\tau^{-1}(J)$ is closed in L. Now by corollary 2.14, $J = \varprojlim\{J_i\}$, where for each $i \in I$, $J_i = \{(x + K_i)^*: x \in H_i\}$ for some vector subspace

H_i of L. As in the proof of part (iii), if $((x_i + K_i)^*)$ lies in J there exists $x \in L$ such that $((x + K_i)^*) = ((x_i + K_i)^*)$ and $x \in \bigcap_{i\in I}(H_i + Z_i)$. So $\tau^{-1}(J) \subseteq \bigcap_{i\in I}(H_i + Z_i)$, and it is clear that $\tau(H_i + Z_i) \subseteq J$ for all $i \in I$, so that $\tau^{-1}(J) = \bigcap_{i\in I}(H_i + Z_i) = K$, say. But $K_i \leqslant Z_i \leqslant H_i + Z_i$ for every i, so that each $(H_i + Z_i)$ is closed in L, whence K is closed. Thus $\tau^{-1}(J)$ is closed, as required.

(v) This follows from part (iv) and corollary 1.11.

(vi) Suppose that H is a closed ideal of L. Consider $h \in H$ and $d = (d_i) \in D$. Then $[\tau(h),d] = [((h + K_i)^*),(d_i)] = ([(h + K_i)^*,d_i]) \in \varprojlim\{\mathrm{Inn}((H + K_i)/K_i)\}$. Now as in the proof of part (iii) we can show that $\tau(H) = \varprojlim\{\mathrm{Inn}((H + K_i)/K_i)\}$. Hence $[\tau(H),D] \subseteq \tau(H)$, so that $\tau(H)$ is an ideal of D. $\tau(H)$ is closed by part (v). $\square$

We now give the embedding theorem.

Theorem 13.8

Let L be a profinite Lie algebra. Then L embeds (algebraically and topologically) in a profinitely cleft Lie algebra.

Proof: Let τ be the map defined in theorem 13.7, let Λ be a Levi subalgebra of L and let $R = \sigma(L)$. We find a maximal torus

T of $\sigma(D)$ such that the split extension $X = R \dotplus (T + \tau(\Lambda))$ provides a suitable Lie algebra.

Now the restriction of τ to Λ is a Lie algebra isomorphism, since $\Lambda \cap \ker \tau = \Lambda \cap \mathfrak{Z}_1(L) = 0$, and is a homeomorphism, by parts (iv) and (v) of theorem 13.7. Thus $\tau(\Lambda)$ is semisimple and $\tau(\Lambda) \cap \sigma(D) = 0$, and $\tau(\Lambda)$ is compact, so is closed in D. D is cofinitely cleft by theorem 13.7(i), so by proposition 12.24 $\sigma(D)$ is cofinitely cleft. To obtain the torus T mentioned above, let $H = \sigma(D) \dotplus \tau(\Lambda)$. Then $\tau(\Lambda)$ is a Levi subalgebra of H and $\sigma(H) = \sigma(D)$, so that H is cofinitely cleft, again by proposition 12.24. Also, H is profinite, by proposition 2.28, since $\sigma(D)$ and $\tau(\Lambda)$ are both closed in H. Applying theorem 12.26 there exists a maximal torus T of $\sigma(H)$ idealised by $\tau(\Lambda)$, and such that $\tau(\Lambda) \cap T = 0$.

Now we can consider D, and hence T, to be an algebra of derivations on L, and by restriction, as an algebra of derivations on R. So we can form the split extension $X = R \dotplus (T + \tau(\Lambda))$. Give X the cofinite topology determined by taking $\mathfrak{I}(X) = \{K \dotplus H\colon K \in \mathfrak{I}(R),\ H \in \mathfrak{I}(T + \tau(\Lambda))\}$, which equals $\{(R \cap K) \dotplus ((T + \tau(\Lambda)) \cap H)\colon K \in \mathfrak{I}(L),\ H \in \mathfrak{I}(D)\}$. As in proposition 4.12 we may show that X is profinite with this topology. To see that L embeds in X, the map : $L \to R \dotplus \tau(\Lambda)$ given by $r + x \mapsto r + \tau(x)$ is both a Lie algebra isomorphism and a homeomorphism, since the restriction of τ to Λ is a homeomorphism. So L embeds, algebraically and topologically, in $X = (R \dotplus \tau(\Lambda)) \dotplus T$.

To complete the proof we show that X is cofinitely cleft, and to do this it suffices to show that $Y = R \dotplus T$ is cofinitely cleft, by proposition 12.24, since $Y = \sigma(X)$. Consider $x \in Y$. x has the form (r,u), where $r \in R$ and $u \in T$. T is maximal in $\sigma(D)$, so is closed, by proposition 12.12. Hence $T = \varprojlim\{T_i\}$ for some choice T_i of subalgebras of D_i, so $u = (u_i)$ with $u_i \in T_i$ for each $i \in I$.

For each $i \in I$, let $x_i = (r + (R \cap K_i), u_i)$. By proposition 12.30 there exists $n \in \nu(D)$ and $t = (t_i) \in T$ such that $r = n + t$. We may show, arguing as in the proof of theorem 6.7 in Stewart, [17], p96, that $x_i = (r + (R \cap K_i), u_i) = (r + (R \cap K_i), -t_i) + (0, u_i + t_i)$ is a semicleaving of x_i. So x is cofinitely ad-semicleft, and hence is cofinitely ad-cleft, by theorem 13.4, since $R \dotplus T$ is cofinitely soluble and since it is profinite, being the sum of closed subalgebras, by proposition 2.28. Hence Y is cofinitely cleft, as required. □

Note

Pages 211 - 221 do not exist

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