SUMMARY

In the first part of this thesis we prove that any (orientation- preserving)homeomorphism of a (orientable)connected sum of 3-manifolds can be written as a product g∘f where f preserves factors and g is a composition of loop homeomorphisms and permutations of factors.The method yields results about the higher homotopy groups of the space of automorphisms of a general 3-manifold.

In the second part we give a calculus of links to classify 4-manifolds similar to Kirby's calculus for 3-manifolds,using link pictures with certain identified links and corresponding allowable moves.We also consider a stable classification of 4-manifolds using such link pictures.

AUTOMORPHISMS OF 3-MANIFOLDS

AND

REPRESENTATIONS OF 4-MANIFOLDS

-by-

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A thesis submitted for the degree of Doctor of Philosophy at the University of Warwick, 1977

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Dedicated to TOIL

for

the sunshine I was not able to find in England

AUTOMORPHISMS OF 3-MANIFOLDS AND REPRESENTATIONS OF 4-MANIFOLDS

Introduction

This thesis is in two parts. In part I we shall consider the structure of the automorphism group of a connected sum of 3-manifolds and in part II apply some of the results of part I to obtain a simple link calculus to represent 4-manifolds.

Part I starts by reviewing Gluck's computation of the automorphism group of $S^1 \times S^2$ (the automorphism group is the group of isotopy classes of homeomorphisms) and extending the result to the "twisted" S^2-bundle over S^1, $S^1 \tilde{\times} S^2$. We then define three types of automorphisms of a connected sum of 3-manifolds, namely "generalised slides", "permutations" and "factor preserving automorphisms". "Generalised Slides" are the equivalent for a connected sum of handle slides in a handle-body – the point is that factors can be slid around as if they were handles.

Our first main theorem (theorem 3.1) proves that the three basic types of automorphisms generate the group of automorphisms (in fact, any automorphism is the composition $g \circ h$ where h preserves factors and g is a composition of slides and permutations). The method of proof is an adaptation of the familiar "ball push" method used by Milnor in [18]. (Orientation preserving automorphisms in the orientable case)

In order to obtain a canonical version of the decomposition given by the theorem we need to work with automorphisms fixed on a disc. Indeed we can apply the "Hatcher's method" [10] to make the process in theorem 3.2 completely canonical and deduce that there is a split epimorphism from the higher homotopy groups of the space of homeomorphisms fixing a disc to the direct product of the higher homotopy groups of the space of homeomorphisms fixing a disc of the factors.

Part I ends with a discussion of special cases. In some cases we can prove that the relative automorphism group (the group of automorphisms fixing a disc) coincides with the absolute one and in some cases it doesn't. We also consider standard fibrations [22] to analyse the higher homotopy groups of the absolute space of automorphisms in general and deduce some results on P^2-irreducible sufficiently large 3 manifolds.

Finally, we recover some results due to Laudenbach [17] on the automorphisms of a connected sum of S^2-bundles over S^1 and extend them to an arbitrary connected sum of such with P^2 - irreducible sufficiently large 3 manifolds.

In part II we use theorem 3.1 in the case of a connected sum of S^2 bundles over S^1 to prove that a 4-manifold is determined, up to homeomorphism, by its full 2-handles. Then we can use the Kirby link pictures [15] to give a calculus of links to classify orientable 4-manifolds. We then give a calculus for non-orientable 4 manifolds by allowing "twists" on passing through certain identified components of the link. We also consider the problem of stable classification of 4-manifolds (under connected sum with an S^2-bundle over S^2).

I would like to thank my supervisor Colin Rourke for his help and encouragement and also the Instituto Nacional de Investigação Científica, (Portugal), for its financial support.

I would also like to thank David Epstein for helpful comments on a previous version of part I of the thesis,in particular he suggested an improvement to theorem 2.2(from an isotopy classification of homeomorphisms to a homotopy classification)The proof of this included in the present version is due to Colin Rourke.

PART I

AUTOMORPHISMS OF 3-MANIFOLDS

1. PRELIMINAIRES

We assume the reader is familiar with basic works such as [21]. We work in the p.l. category. All manifolds are compact and connected and can be with or without boundary. The boundary of a manifold M will be denoted by ∂M, the interior by int M.

I^n denotes the n-cube i.e. the subset of points $(t_1,\ldots,t_n)$ of $\mathbb{R}^n$ with $0 \le t_i \le 1$ for $i=1,2,\ldots,n$.

We now give some definitions and results which will prove useful later on (this is by no means an exhaustive list but only a list of some probably not so well known results).

1.1 A <u>p.l. n-isotopy of M in Q</u> (M,Q are manifolds) is a p.l. embedding $F : M\times I^n \longrightarrow Q\times I^n$ which commutes with the projections onto the second factor.

So for any $t\in I^n$, there is an embedding $F_t : M \longrightarrow Q$ s.t F is given by $F(x,t) = (F_t(x),t)$ for all $x\in M$. We say F is a level preserving embedding.

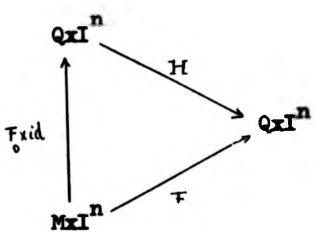
A 1-isotopy is just called an <u>isotopy</u>. Two embeddings are <u>n-isotopic</u> if there is an n-isotopy between them.

1.2 An <u>ambient pl n-isotopy</u> of Q is a level preserving p.l. homeomorphism $H : Q\times I^n \longrightarrow Q\times I^n$ s.t. $H_0 : Q \longrightarrow Q$ is the identity (0 is the origin in $\mathbb{R}^n$).

1.3 An n-isotopy F of M in Q is <u>fixed</u> on X if X is a subset of M and $F_{t/X} = F_{0/X}$ for all $t \in I^n$.

We say F has <u>support</u> in U if F fixes M-U. F is <u>mod X</u> or <u>relX</u> if it fixes X.

1.4 We say the ambient n-isotopy $H : Q \times I^n \longrightarrow Q \times I^n$ <u>covers</u> or <u>extends</u> $F : M \times I^n \longrightarrow Q \times I^n$ if the diagram



commutes, i.e. $F_t = H_t \circ F_0$, all $t \in I^n$

1.5 <u>Alexander's trick</u> [1] Any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.

1.6 (i) Let B^n, C^n be balls and h_0, h_1 homeomorphisms $B^n \longrightarrow C^n$ which agree on ∂B^n. Then h_0, h_1 are ambient isotopic mod ∂B^n.

(ii) If M is a manifold with compact boundary then any n-isotopy of ∂M extends to one of M with support in a collar of ∂M.

1.7 A p.l. n-isotopy F of M^m in Q^p is <u>allowable</u> if for some p.l. (m-1) sub-manifold N of ∂M, $F_t^{-1}(\partial Q) = N$ for all $t \in I^n$. N may be empty or it may be the whole of ∂M. A p.l. embedding $f : M^m \hookrightarrow Q^p$ is allowable if $f^{-1}(\partial Q)$ is a p.l. (m-1) sub-manifold of ∂M.

1.8 n-isotopy extension theorem [13]

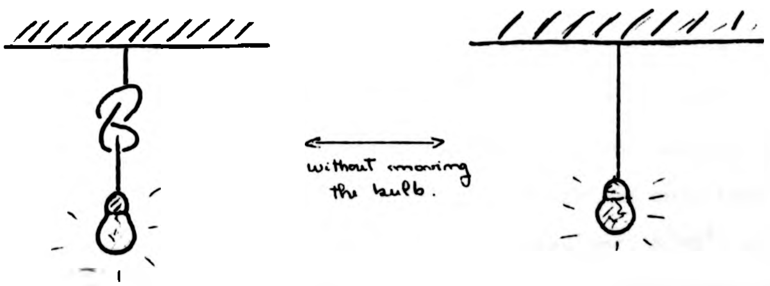
If $F : M\times I^n \longrightarrow Q\times I^n$ is an allowable n-isotopy (fixed on $Y\subset M$), M compact, and F is locally unknotted then there is an ambient n-isotopy of Q with compact support (fixed on Y) extending F.

1.9 Any two collars of ∂M in M are canonically ambient isotopic

1.10 Let $h_i : B^{q,n} \longrightarrow C^{q,n}$, i=1,2, be homeomorphisms between ball pairs that agree on $\partial B^{q,n}$. Then h_1 is ambient n-isotopic to h_2 rel $\partial B^{q,n}$.

1.11 Light bulb trick [8]

Let $*\in S^2$ be a base point in S^2. Then any tame arc properly embedded in $S^2\times I$ which connects $\{*\}\times\{0\}$ to $\{*\}\times\{1\}$ is ambient isotopic to $\{*\}\times I$ by an ambient isotopy keeping $S^2\times\{0,1\}$ fixed.



1.12 Aut M denotes the automorphism group of M i.e. the group of homeomorphisms of the manifold M factored out by the normal subgroup of those which are pl-isotopic to the identity.We will see later(1.14) that Aut M $= \pi_0$(PL(M)),where PL(M)is the space of pl-homeomorphisms of M.Elements of Aut M are called automorphisms.

If M is orientable we denote by Aut^+M the subgroup of AutM consisting of the isotopy classes of the orientation preserving homeomorphisms.

1.13 Connected sums and prime decompositions of 3-manifolds

We quote some results on 3-manifolds. For proofs and details see [11]. All manifolds are assumed to be compact and connected.

Let M_1, M_2 be n-manifolds, $B_i \subset \mathrm{int} M_i$ n-cells in M_i, $i=1,2$,. Let M be the space obtained from M_1, M_2 by removing the interior of B_i and identifying ∂B_1 with ∂B_2 by a homeomorphism which in the case M_1, M_2 oriented we require to be orientation **reversing** . M is called the connected sum of M_1, M_2 and denoted by $M_1 \# M_2$. Connected sum is a well defined associative and commutative operation in the category of oriented n-manifolds and orientation preserving homeomorphisms. For both oriented or non-oriented case **there** are at most two homeomorphism types for $M_1 \# M_2$ and only one, if one of M_1, M_2 admits a self-homeomorphism which fixes some point and reverses the orientation of a neighbourhood of this point [11].

A 3-manifold M is prime if $M = M_1 \# M_2$ implies one of M_1, M_2 is a 3-sphere. It is irreducible if each 2-sphere in its interior bounds a 3 cell. Clearly irreducible manifolds are prime. A prime manifold is either irreducible or a 2-sphere bundle over S^1 (There are only two 2-spheres bundles over S^1: the trivial or "untwisted" one, $S^1 \times S^2$, and the non-trivial or "twisted" one which we denote by $S^1 \tilde{\times} S^2$). Connected summing with $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ to a manifold M has the same effect as "adding a hollow handle" : choose two disjoint 3 cells in M, remove their interiors and match the resulting boundaries under an orientation reversing homeomorphism in the first case, an orientation preserving homeomorphism in the second case.

Let F be a surface in M (2 sided properly embedded) or in ∂M, F not being the 2-sphere. Then we say F is incompressible if $\mathrm{Ker}(\pi_1 F \longrightarrow \pi_1 M) = 0$. An irreducible manifold which is not a ball is sufficiently large if and only if there exists an incompressible surface in M.

A 3-manifold M is said to be P^2-irreducible if it is irreducible and contains no 2-sided projective plane.

Milnor showed that there exists a unique prime decomposition (under #) of oriented 3 manifolds up to homeomorphism and order of factors [18]. Proof of existence was given by Kneser [16]. In the non orientable case although the factorization always exists it is not unique since for any non orientable manifold M, $M \# S^1 \times S^2 \cong M \# S^1 \tilde{\times} S^2$ (for a proof see Hempel [11]). From this is follows that we can replace any factorization $M = M_1 \# \ldots \# M_n$ by one satisfying the following condition: if at least one of the M_i is $S^1 \times S^2$ then M is orientable. Such a factorization is called normal. Hempel [11] proved the uniqueness decomposition for normal factorizations. Putting together all these results we get the following:

THEOREM [11]

For any compact connected 3 manifold (closed or not*) there exists a finite normal decomposition into prime manifolds. Given any two normal factorizations $M_1 \# \ldots \# M_n = M_1' \# \ldots \# M_{n'}'$, then n=n' and, after possible reordering, M_i is homeomorphic to M_i' (orientation preserving homeomorphism in the oriented category).

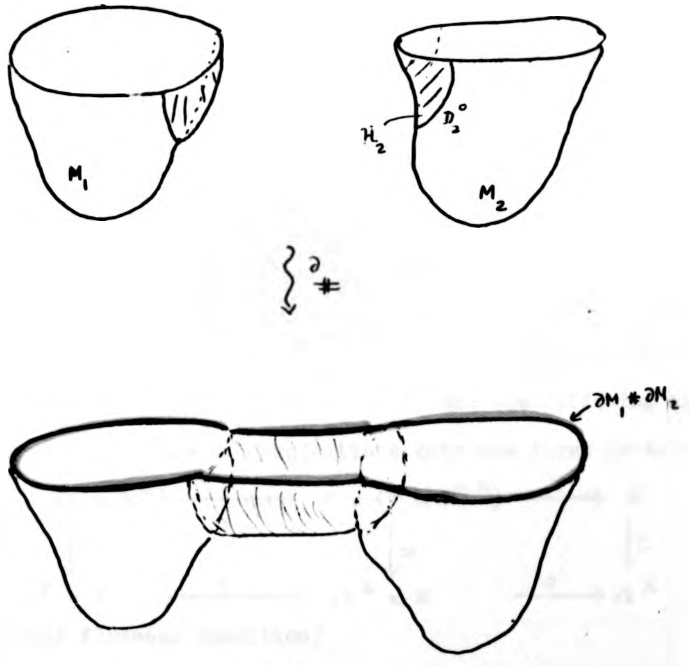
* If $\hat{M}$ denotes the manifold obtained from M by capping off each 2-sphere component of ∂M with a 3 cell B and $\hat{M} = M_1 \# \ldots \# M_q$ is a prime decomposition of M, then $M_1 \# \ldots \# M_q \# B^3 \# \ldots \# B^3$ is a prime decomposition of $\hat{M}$ where there are as many factors of B^3 as the number of 2-spheres in ∂M.

<u>The bounded connected sum</u>

If M_1, M_2 are two n manifolds with boundary the <u>bounded connected sum</u> $M_1 \overset{\partial}{\#} M_2$ is defined as follows:

Consider the p.l. half disc $H = D^{n-1} \times [0,1]$ where $D^{n-1} = [-1,1]^{n-1} \subset \mathbb{R}^{n-1}$ so that D^{n-1} is contained in ∂H^n. Find embeddings $(H_i^{\,n}, D_i^{\,n-1}) \hookrightarrow (M_i^{\,n}, \partial M_i)$ i=1,2 and define $(M_i^{\,0}, \partial M_i^{\,0})$, i=1,2, D^0, to be respectively $(\overline{M_i - H_i^{\,n}}, \overline{\partial M_i - D_i^{\,n-1}})$, i=1,2, $\overline{\partial H^n - D^{n-1}}$. Then $M_1 \overset{\partial}{\#} M_2$ is, by definition, $M_1^{\,0} \cup M_2^{\,0}$ with $D_1^{\,0} \subset \partial M_1$, identified with $D_2^{\,0} \subset \partial M_2^{\,0}$ by a homeomorphism which we require to be orientation reversing if both M_1 and M_2 are oriented.

$M_1 \overset{\partial}{\#} M_2$ is then a p.l n-manifold, orientable if both M_1, M_2 are, and $\partial(M_1 \overset{\partial}{\#} M_2) \cong \partial M_1 \# \partial M_2$.



$M_1 \overset{\partial}{\#} M_2$

1.14 (cf[22])

Let Y be a submanifold of the manifold M, Δ^k the standard k-simplex in $\mathbb{R}^k$.

(i) PL(M,Y) is defined to be the semi-simplicial complex whose k-simplices are p.l. homeomorphisms $\Delta^k \times M \longrightarrow \Delta^k \times M$ which commute with projection onto Δ^k and such that the restriction to $\Delta^k \times Y$ is the identity. It has the obvious boundary and degeneracy maps.

If $Y=\emptyset$ we write PL(M). If M is orientable SPL(M,Y) denotes the subcomplex of orientation preserving homeomorphisms. If 0 denotes the origin in $\mathbb{R}^k$, $PL(\mathbb{R}^k,0)$ is usually denoted by PL_k.

(ii) Let $i: Y^n \longrightarrow M^m$ be the inclusion map, Z a locally flat submanifold of Y. Y is also a locally flat submanifold of M.

$Emb_Z(Y^n,M^m)$ is defined to be the semi-simplicial complex whose k-simplices are embeddings $f: \Delta^k \times Y \longrightarrow \Delta^k \times M$ which commute with projection onto Δ^k and such that

(i) $f| \Delta^k \times Z = (id \times i)|(\Delta^k \times Z)$

(ii) $f^{-1}(\Delta^k \times \partial M) = \Delta^k \times i^{-1}(\partial M)$

(iii) given $(t,y) \in \Delta^k \times Y$ there is a closed neighbourhood U of t in Δ^k, a closed neighbourhood V of y in Y, and an embedding $\alpha : U \times V \times D^{m-n} \longrightarrow \Delta^k \times M$ s.t. the image of α is a closed neighbourhood of $f(t,y)$ in $\Delta^k \times M$ and the following diagram commutes, where π, π' are the projections onto the first factor:

$$\begin{array}{ccccc} U \times (V \times \{0\}) & \xrightarrow{\subset} & U \times (V \times D^{m-n}) & \xrightarrow{\pi} & U \\ \downarrow \subset & & \downarrow \alpha & & \downarrow \subset \\ \Delta^k \times Y & \xrightarrow{f} & \Delta^k \times M & \xrightarrow{\pi'} & \Delta^k \end{array}$$

(This is a local flatness condition)

$Emb_Z(Y,M)$ has the obvious boundary and degeneracy maps. If $Z=\emptyset$ we write Emb(Y,M). If M is oriented $SEmb_Z(Y,M)$ is the subcomplex of orientation preserving embeddings. PL(M,Y), SPL(M,Y), $Emb_Z(Y,M)$, $SEmb_Z(Y,M)$ are all Kan complexes, hence we can define their homotopy groups.

2. THE GROUP OF AUTOMORPHISMS OF A 2-SPHERE BUNDLE OVER S^1

(1) We consider first the trivial bundle $S^1 \times S^2$.

THEOREM 2.1

$$\mathrm{Aut}\, S^1 \times S^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad .$$

Proof

This was proved by Gluck [8]. We give a "geometric idea" of his proof:

Regard S^2 as the unit sphere in 3 space and S^1 as the space of complex numbers modulo 1.

A homeomorphism of $S^1 \times S^2$ induces an automorphism of $H_1(S^1 \times S^2;\mathbb{Z}) = \mathbb{Z}$ and an automorphism of $H_2(S^1 \times S^2;\, \mathbb{Z}) = \mathbb{Z}$ each of which depends only on the isotopy class of the homeomorphism. As $\mathbb{Z}_2$ is the group of automorphisms of $\mathbb{Z}$ we get a homomorphism ϕ : Aut $S^1 \times S^2 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $r : S^2 \longrightarrow S^2$ denote the antipodal map and $s : S^1 \longrightarrow S^1$ the complex conjugation. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the subgroup of Aut $S^1 \times S^2$ consisting of the isotopy classes of (id,id), (s,id), (s,r), (id,r). Let ρ be the isomorphism of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with this subgroup determined by the condition $\phi\rho = 1$ (1=id). Then ϕ splits. We will show that ker $\phi = \mathbb{Z}_2$ As a normal subgroup of order two is central and ϕ splits we get that Aut $S^1 \times S^2$ = Ker $\phi \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $f \in$ Ker ϕ.

(a) We first deform $f(\{0\} \times S^2)$ isotopically until $f/\{0\} \times S^2$ is the identity.

(1) By general position assume $f(\{0\} \times S^2)$ intersects $\{0\} \times S^2$ in a finite number of simple disjoint closed curves. We show how to isotope $f(\{0\} \times S^2)$ so as to reduce the number

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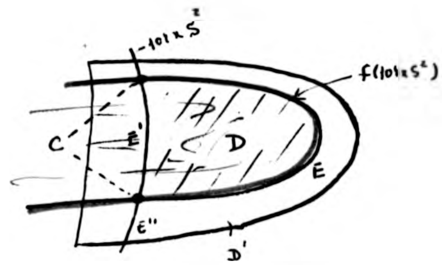
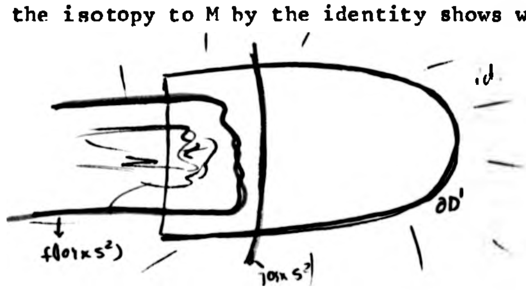
(1) By general position assume $f(\{0\}\times S^2)$ intersects $\{0\}\times S^2$ in a finite number of simple disjoint closed curves. We show how to isotope $f(\{0\}\times S^2)$ so as to reduce the number

of intersections. After a finite number of stages $f(\{0\}\times S^2)$ is disjoint from $\{0\}\times S^2$.

Let C be an innermost curve in $f(\{0\}\times S^2)$ in the intersection of $f(\{0\}\times S^2)$ with $\{0\}\times S^2$. Then C bounds a disc E in $f(\{0\}\times S^2)$ with no more intersection curves. C also bounds a disc E' in $\{0\}\times S^2$ s.t. $E'\cup_\partial E$ is a sphere which separates. Hence $E'\cup_\partial E$ bounds a 3 ball D in $S^1\times S^2$ as $S^1\times S^2$ is prime.



Let D' be a ball neighbourhood of D in $S^1\times S^2$. Then there exists an isotopy of D' which is the identity on the boundary that "pushes" $f(\{0\}\times S^2)$ across the ball D eliminating the intersection curve C (1.10). Extending the isotopy to M by the identity shows what we want.



Remark

This process of eliminating intersection curves is a special case of a general procedure to be used later on.

(2) As after stage (1) the two spheres are disjoint, we can isotope one into the other as now the region between them is an annulus (cut $S^1 \times S^2$ along one of the spheres to get a ball with 2 holes. Now the region between the two spheres becomes a regular neighbourhood of one of the holes hence an annulus by the regular neighbourhood collaring theorem. Gluing the holes back again doesn't affect it.)

(3) Now we have $f_{|\{0\}\times S^2}: \{0\}\times S^2 \to \{0\}\times S^2$, and as fεker φ is a degree one map, this restriction is isotopic to the identity.

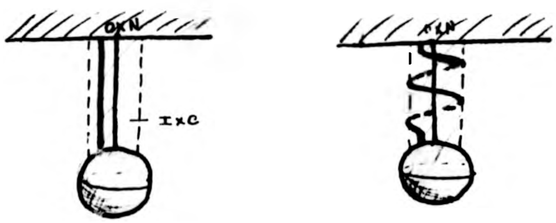
(b) We can now interpret f, by cutting $S^1 \times S^2$ along $\{0\}\times S^2$, as a map of $I\times S^2$ onto itself being the identity on boundary components.

Regard $I\times S^2$ as the space in between two 2-spheres in 3 space and denote by N the north pole.

We now deform, rel $\partial(I\times S^2)$, $f_{|(I\times\{N\})}$ till $f_{|(I\times\{N\})}$ is the identity:

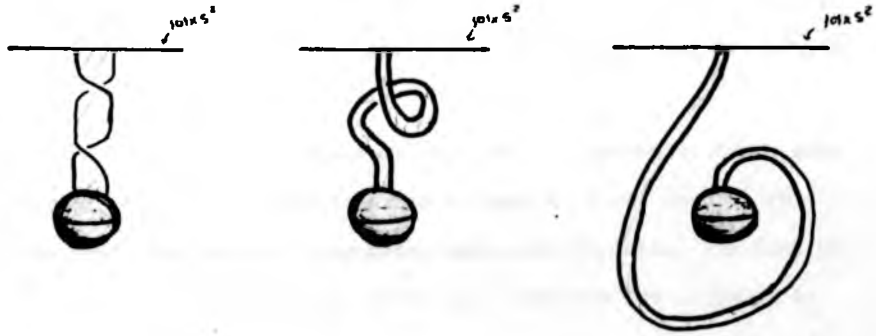
$f(I\times\{N\}$ is an arc from $\{0\}\times\{N\}$ to $\{1\}\times\{N\}$ that can have little knots. By the light bulb trick (1.11), we can unknot them, (the idea is to regard the central ball - see picture - as very small so that the arc and the ball can be regarded as a piece of string and then the knots can be slid off the end) and make $f(I\times\{N\}) = I\times\{N\}$ by an isotopy rel $\partial(I\times S^2)$. But as $f\epsilon \mathrm{Ker}\ \phi$, $f|I\times\{N\}$ must be orientation preserving, hence isotopic to the identity. As the isotopies used clearly extend to an isotopy of $S^1 \times S^2$ fixed on $\{0\} \times S^2$ we have the required result.

(c) Let C be a small circle on S^2 about the north pole N. The union of $I \times C$ with the two discs around respectively $\{0\} \times \{N\}$ and $\{1\} \times \{N\}$ bounded by C bounds a regular neighbourhood of $I \times \{N\}$ in $I \times S^2$. By the regular neighbourhood theorem we can then deform isotopically f by an isotopy rel $\partial(I \times S^2)$ s.t f takes $I \times C$ onto itself. But a homeomorphism of $I \times C$ onto itself is isotopic **rel ∂** to a standard n-tuple twist



(parametrize C by the angle β mod 2π. Then $f|I \times C$ is isotopic to one of the maps $f_n(t,\beta) = (t, \beta + 2\pi nt)$.

(d) We continue to deform f until $f|I \times C$ is either the identity on a standard 1-twist according to n is even or odd (see pictures).



(Pictures shows how to get rid of two twists. Then result follows after a finite number of stages.)

Again all the isotopies fix $\{0\}\times S^2 \cup \{1\}\times S^2$ hence extend to $S^1\times S^2$.

(e) Let τ denote the homeomorphism of $S^1\times S^2$ determined by $\tau(t,x) = (t,\phi_t(x))$ where ϕ_α denotes a rotation of S^2 about a diameter through the north and south poles, through an angle of $2\pi\alpha$ in some fixed direction. Let $K = \{0\}\times S^2 \cup \{1\}\times S^2 \cup I\times C$. Up to this point we have deformed f until f is the identity on $\{0\}\times S^2 \cup \{1\}\times S^2$ and is τ or the identity on $I\times C$. (Both τ,id Ker ϕ and are the identity on $\{0\}\times S^2 \cup \{1\}\times S^2$).

LEMMA 2.1

Let f and g be two homeomorphisms of $I\times S^2$ whose restrictions to the boundary are the identity and which agree on $I\times C$. Then f and g are isotopic rel $\partial(I\times S^2)$.

Proof

$f^{-1}g|K$ is the identity. $I\times S^2$ consists of two disjoint open 3-cells whose boundaries are non-singular and contained in K. But since $f^{-1}g$ cannot interchange these 3 cells, the restriction to each cell is a homeomorphism which is the identity on the boundary and hence isotopic to the identity rel $\partial(I\times S^2)$. Hence f and g are isotopic rel $\partial(I\times S^2)$. □

(f) It remains to prove that id,τ are not isotopic. Gluck uses Pontrjagin homotopic classification of maps of $S^1\times S^2$ onto S^2 to prove that they are not homotopic, hence not isotopic. In fact we will give a direct geometric proof that they are not homotopic in the next section. □

Two immediate corollaries of the theorem are the following:

COROLLARY 2.1

Two homeomorphisms of $S^1 \times S^2$ are isotopic iff they are homotopic.

COROLLARY 2.2

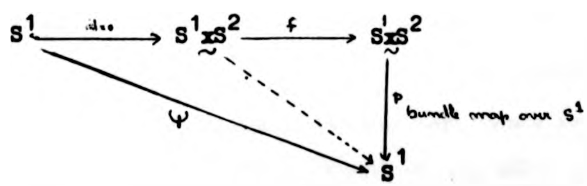
Any homeomorphism of $S^1 \times S^2$ extends to $S^1 \times B^3$.

(2) $S^1 \tilde{\times} S^2$

Consider $S^1 \tilde{\times} S^2$, the twisted S^2-bundle over S^1, as the space obtained from $[-1,1] \times S^2$ by identifying $\{-1\} \times S^2$ with $\{1\} \times S^2$ by an orientation reversing homeomorphism whose square is the identity (e.g. the antipodal map).

Any homeomorphism of $S^1 \tilde{\times} S^2$ induces an automorphism of $H_1(S^1 \tilde{\times} S^2; \mathbb{Z}) = \mathbb{Z}$ which depends only on the isotopy class of the homeomorphism. As the automorphism group of $\mathbb{Z}$ is $\mathbb{Z}_2$ we get a homomorphism ϕ : Aut $S^1 \tilde{\times} S^2 \longrightarrow \mathbb{Z}_2$. Let $\mathcal{H}$ be the subgroup of Aut $S^1 \tilde{\times} S^2$ generated by the isotopy classes of (id,id) and (s,id). $\mathcal{H} = \mathbb{Z}_2$

(i) To see that (s,id) is not isotopic to the identity consider the maps



If f = (s,id) Ψ has degree -1

f = (id,id) Ψ has degree 1

hence they are not isotopic.

(ii) To see that $(1,r)$ is isotopic to the identity lift the map to the universal cover $\mathbb{R}\times S^2$ to get a map $\widetilde{(1,r)} : \mathbb{R}\times S^2 \longrightarrow \mathbb{R}\times S^2$. If $\widetilde{id}$ denotes the lift of the identity obtained by choosing the same base point upstairs then $(\widetilde{id}) = \tau_1 \circ \widetilde{(1,r)}$ where $\tau_1 : \mathbb{R}\times S^2 \longrightarrow \mathbb{R}\times S^2$ is the covering translation $\tau_1(t,x) = (t+1,x)$. Hence $(1,r)$ is isotopic to the identity in $S^1 \tilde{\times} S^2$.

Now let ρ be the isomorphism of $\mathbb{Z}_2$ with ~~Rc~~ Aut $S^1 \tilde{\times} S^2$ determined by the condition $\phi\rho = 1$. Then ϕ splits. ϕ is onto. We show that Ker $\phi = \mathbb{Z}_2$ and hence that Aut $S^1 \tilde{\times} S^2 = \mathrm{Ker}\phi \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

<u>THEOREM 2.2</u>

<u>Aut $S^1 \tilde{\times} S^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.</u>

<u>Proof</u>:

It remains then to show that Ker $\phi = \mathbb{Z}_2$. The proof is essentially the same as in the orientable case and we only point out the differences.

Let h be a homeomorphism of $S^1 \tilde{\times} S^2$. h can be deformed isotopically until $h(\{0\}\times S^2) = \{0\}\times S^2$ as in the orientable case. Then $h/\{0\}\times S^2$ is either orientation preserving or orientation reversing. In the latter case apply the above to make it orientation preserving ($h/\{0\}\times S^2$ up to isotopy is either the identity or the antipodal map) hence isotopic to the identity. Then cutting $S^1 \tilde{\times} S^2$ along $\{0\}\times S^2$ we can think of h as a homeomorphism of $I\times S^2$ which is the identity on the boundary. Then, as before, h is either isotopic to the id or to τ in $S^1 \tilde{\times} S^2$.

<u>It remains to prove that τ, id are not isotopic:</u>

We prove that they are not homotopic, hence not isotopic.

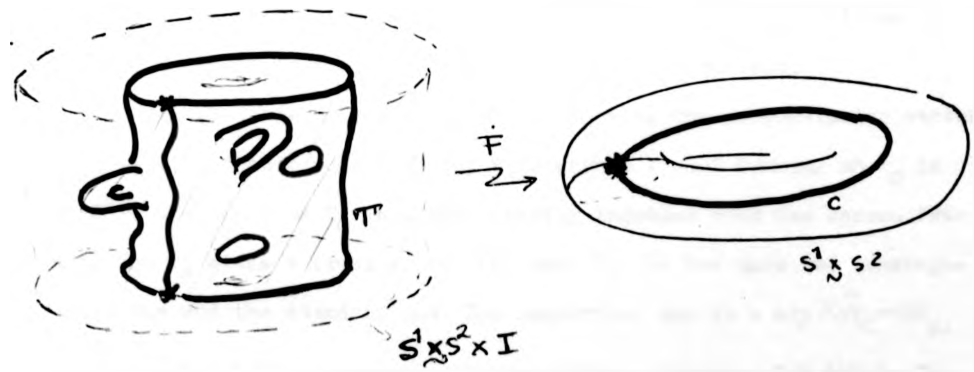
Let $q: S^2 \longrightarrow S^2$ be the reflection in a great circle through north and south poles. Think of $S^1\tilde{\times}S^2$ as obtained from $I \times S^2$ by identifying $\{0\}\times S^2$ with $\{1\}\times S^2$ by q. Let C be the circle which is the image of $I \times \{n\}$ in $S^1\tilde{\times}S^2$ (n denotes the north pole of S^2)and let * be the image of $\{0\}\times\{n\}$. Let τ be the self homeomorphism of $S^1\tilde{\times}S^2$ which rotates S^2 about the north and south poles once during I. Assume w.l.o.g. that the rotation is the identity near 0 and 1.

LEMMA 2.2

τ is not homotopic to the identity.

Proof:

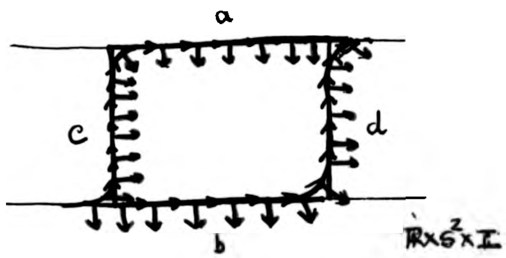
Suppose that τ and id were homotopic and w.l.o.g. assume that the homotopy is fixed near 0 and 1. Let $F: S^1\tilde{\times}S^2\times I \longrightarrow S^1\tilde{\times}S^2$ be the given homotopy, F_0=id, $F_1=\tau$. By the relative transversality theorem deform F to be transverse to C keeping fixed a neighbourhood of the levels 0 and 1. Let T be $F^{-1}(C)$ which is a surface in $S^1\tilde{\times}S^2\times I$ with two boundary components (the copies of C in levels 0 and 1).



Now deform F further until F/T is transverse to * in C and let C_0 be $F^{-1}(*)$ which is a system of circles and arcs on T(in fact one arc and n other circles).Let T_0 be T cut along C_0.Then T_0 is a surface with 2n+1 boundary components:one □ corresponding to the two copies of C and the arc,and others which come in pairs (Q_i,P_i) corresponding to the cir - cles which were cut.

Let Q be the universal cover of $S^1\tilde{\times}S^2\times I$.We can regard Q as a subset of $\mathbb{R}^4$ ($\mathbb{R}\times S^2\subset\mathbb{R}^3$ as a collar on S^2 and hence $\mathbb{R}\times S^2\times I\subset\mathbb{R}^3\times I\subset\mathbb{R}^4$) and hence $\tau(Q)$ (the tangent bundle of Q) has a standard framing.

T_0 can be lifted to a surface $\tilde{T}_0$ in Q(since the only obstruction to lifting lies in $\pi_1(S^1\tilde{\times}S^2\times I)\underset{F_*}{\approx}\pi_1(S^1\tilde{\times}S^2)$ and T_0 is the preimage of C cut at *, which is null homotopic). We now claim that $\tilde{T}_0$ can be framed so that the framing agrees with the following framing near □ :



first vector normal,second tangent- smooth at corners as shown.

and agrees up to an even number of twists with the outward-normal,tangent fra- ming near the other components(assume w.l.o.g. $n\neq 0$). This follows by an easy argument by induction on the genus of the surface.

C cut at * has a framing given by choosing two perpendicular vectors at n in S^2 and this pulls back by F to give a normal framing on T_0 in $S^1\tilde{\times}S^2\times I$ and hence on $\tilde{T}_0$ in Q.This framing together with the chosen fra- ming on $\tilde{T}_0$ gives a framing of $\tau(Q)$ near T_0 .We now have two framings- -this one and the standard one. The comparison map is a map $\lambda:\tilde{T}_0\to SO_4$. We claim that $\lambda/\partial\tilde{T}_0$ represents the non trivial element of $H_1(SO_4)=\mathbb{Z}_2$ (and this is impossible since $\lambda/\tilde{T}_0$ is then a homology of this to zero).

Step 1 <u>$\lambda/\square$ represents the non zero element.</u>

Since we have assumed F and τ to be the identity near 0 and 1, everything is standard near the corners of $\square$ and so we can think of $\square$ as made of four pieces and measure the contribuitions separately:

λ/a gives one twist on the normal framing by definition of τ and therefore represents 1 in $\mathbb{Z}_2$(note that $H_1(SO_4)$ is generated by the image $\pi_1(SO_2)$= circle group).

λ/b gives no twist .

λ/c and λ/d are related by covering translation(expansion of $\mathbb{R}^4$ plus a reflection) which differ by multiplication by a constant element of SO_4,hence they give the same element in H_1.

<u>Step 2</u> : <u>$\lambda/Q_1 = \lambda/P_1$</u>

As in the proof of step 1 if we had chosen the normal-tangent framing near Q_1 and P_1 then λ/Q_1 and λ/P_1 would give the same element of H_1,but the actual framing chosen differs from the normal-tangent framing by an even number of twists, hence they still represent the same element of $H_1(SO_4)$.

Hence $\lambda/\partial\tilde{T}_0$ represents the non zero element of $H_1(SO_4)$.Contradiction. $\square.\square$

As in the orientable case the next two corollaries follow immediatly. $S^1\tilde{\times}B^3$ denotes the twisted 3-disc bundle over S^1.

<u>COROLLARY 2.3</u>

<u>Any homeomorphism of $S^1\tilde{\times}S^2$ extends to $S^1\tilde{\times}B^3$.</u>

<u>COROLLARY 2.4</u>

<u>Two homeomorphisms of $S^1\tilde{\times}S^2$ are homotopic if and only if they are isotopic.</u>

3. HOMEOMORPHISMS OF 3-MANIFOLDS

I. Let M be a closed 3 manifold $P_1 \# \dots \# P_s \# P_{s+1} \# \dots \# P_n$, the normal factorization [11] of M into prime manifolds where $P_1 \dots P_s$ are irreducible, $P_{s+1} \dots P_n$ are 2 sphere bundles over S^1. Our aim is to study the group of homeomorphisms of M. For simplicity we consider M closed although the techniques used are true for M with boundary with minor changes. So, unless otherwise stated, manifold will mean closed manifold.

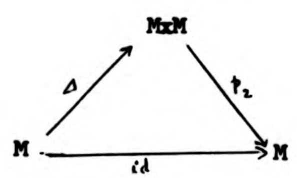
We first need some preliminaries lemmas:

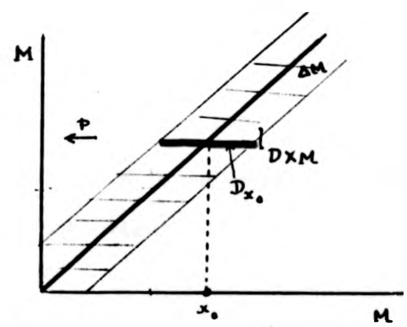
LEMMA 3.1

A parallelization of an orientable 3 manifold determines a choice of a standard disc neighbourhood of each point of M.

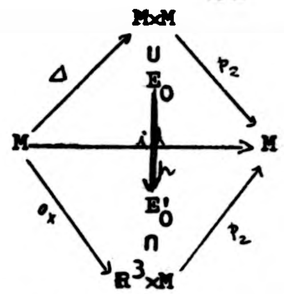
Proof

The tangent microbundle of M is defined to be the microbundle





where Δ denotes the diagonal map [19].
Thus the "fibre" over a point $x_0 \in M$ is the set of pairs (y, x_0) where y ranges over an arbitrary neighbourhood of x_0 in M. It is known that for orientable 3 manifolds the tangent microbundle is trivial.
i.e. there exist neighbourhoods E_0 of $\Delta(M)$ in M×M and E_0' of $\{0\} \times M$ in $\mathbb{R}^3 \times M$ and a homeomorphisms h from E_0 to E_0' making the following diagram commutative



We can then choose $E'_0=D^3\times M$ for a small enough disc D^3 neighbourhood of the origin in $\mathbb{R}^3$. Thus the parallelization(i.e.the choice of a particular homeomorphism h) determines a choice of a disc for each point of M.For each point x_0 in M the fibre in D×M is a disc $D_{x_0}\cong D$. $p_1(D_{x_0})$ is the required disc neighbourhood of x_0 in M(p_1 is the projection on the first factor). □

We will say that a choice of something is <u>canonical</u> (e.g. a choice of a homeomorphism satisfying certain conditions,a choice of a disc,etc) if the space of choices is contractible i.e. a choice is defined up to an isotopy which in turn is defined up to an isotopy, which is in turn defined up to an isotopy and so on.

A unique choice is canonical.

Suppose given an parallelization of M if M is orientable, a parallelization of the orientable double cover otherwise.

<u>COROLLARY 3.1</u>

<u>Given an arc in M and an orientation on one end,there exists a canonical extension to an isotopy of embeddings of a 3-disc in M and a homotopy rel ends between any two such arcs extends canonically to a homotopy rel ends between the two isotopies of embeddings.</u>

<u>Proof:</u>

If M is orientable, define $\bar{\alpha}: D\times I \longrightarrow M\times I$ by $\bar{\alpha}(D\times\{t\})=(D_{\alpha(t)},t)$ where

α:I ⟶ M is the arc. If M is non orientable,lift the arc to the orientable double cover. choosing a lift of one end according to the orientation given.We can then as before ,choose in a continuous way a disc for each point of the arc.Then project into M.Second part of the corollary is similar. □

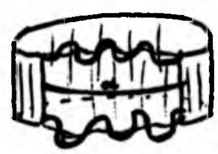
We remark that if the arc is an orientation preserving loop we will end up with the same disc we have started off with,since we will end up in the same sheet upstairs, hence the lift is a loop.

LEMMA 3.2

There are only two choices of extensions of an arc to an isotopy of embeddings of a disc in M up to isotopy rel ends:the canonical one defined by corollary 3.1 and the twisted one,(given an orientation of one end).

Proof:

Let f,h::DxI ⟶ MxI be any two such isotopies,$f_0 = h_0$,$f_1 = h_1$. f(DxI) and h(DxI) determine two normal bundles of an arc α in MxI (trivial, of course).But as ends are fixed,we think of any two such choices as two trivialisations of a normal bundle of S^1 in MxI (4manifold). As such trivializations are classified by $\pi_1(O_3)=Z_2$ the lemma follows.



LEMMA 3.3

Given an isotopy of embeddings of a disc D in M starting with the inclusion ,there is a canonical extension to an ambient isotopy H_t of M .Hence the final homeomorphism* H_1 ,which is well defined up to isotopy rel D, is canonical.

*If $H: M\times I^n \longrightarrow M\times I^n$ is an n-isotopy of M, then $H_1: M\times\{1\} \longrightarrow M\times\{1\}$ is called the final homeomorphism (here $I^n=[0,1]^n$, $1\in I^n$ is the point $(1,\dots,1)$).

<u>Proof</u>:

Let $h_t: D \longrightarrow M$ be the isotopy of embeddings. By the isotopy extension theorem there is an extension to an isotopy of $M, H_t: M \rightarrow M$ such that $H_0=id$. This proves the first part of the lemma.

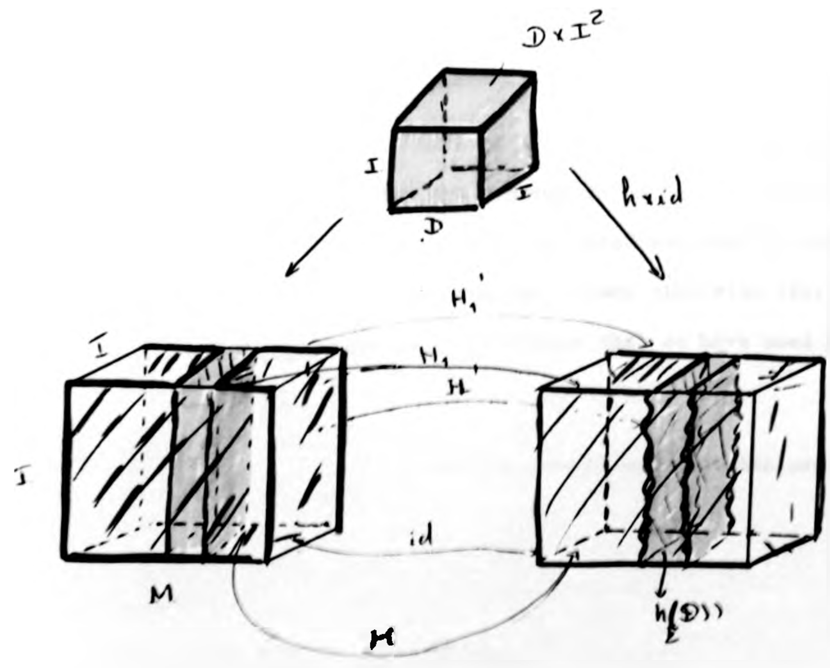
We now show that such an extension is canonical:

Suppose then that $H'=(H_t')_{t\in I}$, is another extension of $h_t, H'_0=id$. Define a map $g^{(2)}$ from $M\times I\times\{0\}\cup M\times I\times\{1\} \cup M\times\{0\}\times I \cup D\times I^2$ into $M\times I^2$ by mapping

$M\times I\times\{0\}$	into $M\times I\times\{0\}$	by H
$M\times I\times\{1\}$	into $M\times I\times\{1\}$	by H'
$M\times\{0\}\times I$	into $M\times\{0\}\times I$	by id
$D\times I^2$	into $D\times I^2$	by $h\times id$

$(h=(h_t)_{t\in I}, H=(H_t)_{t\in I})$

This is possible since both H and H' are extensions of h.



By the 2-isotopy extension theorem $g^{(2)}$ extends to a 2- isotopy $G^{(2)}$ of M with $G_0^{(2)}$=id. $G^{(2)}/ M\times\{1\}\times I : M\times\{1\}\times I \longrightarrow M\times\{1\}\times I$ gives an isotopy rel D between H_1 and H_1' .Hence H_1 is well defined up to isotopy rel D.

$G^{(2)}/ M\times I\times\{s\}$ defines an isotopy $G_s^{(2)}$ between the two extensions H and H', through extensions (i.e. each $G_s^{(2)}$ is an extension of h)

It remains thus to show that any two such isotopies $G^{(2)}$ and $G^{(2)'}$ between the extensions H and H' are isotopic through such isotopies and so on. Define a map $g^{(3)}$ from $M\times I^2\times\{0\} \cup M\times I^2\times\{1\} \cup M\times\{0\}\times I \cup D\times I^3$ by mapping

$M\times I^2\times\{0\}$	into $M\times I^2\times\{0\}$	by $G^{(2)}$
$M\times I^2\times\{1\}$	into $M\times I^2\times\{1\}$	by $G^{(2)'}$
$M\times\{0\}\times I$	into $M\times\{0\}\times I$	by id
$D\times I^3$	into $D\times I^3$	by h×id

This is possible since both $G^{(2)}$ and $G^{(2)'}$ agree on $D\times I^3$.
Again by the 2-isotopy extension theorem $g^{(3)}$ extends to a 3- isotopy $G^{(3)}$ of M with $G_0^{(3)}$=id. $G_s^{(3)} = G^{(3)}/ M\times I^2\times s$ gives an isotopy between the two isotopies $G^{(2)}$ and $G^{(2)'}$ through such isotopies.

We then carry on in the same way. □

We suppose given parallelizations of the factors or double covers of the factors , according as they are orientable or not,so that discs ends of handles or discs used to form the connected sum are the standard ones in the sense of lemma 3.1. From now on ,unless otherwise stated, whenever we talk about connected sums we assume that we have used standard discs.

We now described certain types of homeomorphisms that can occur in a 3- manifold.

(1) __The generalised slides__

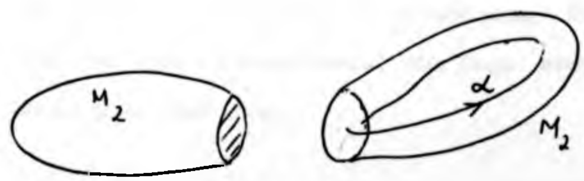
We consider two cases although they are essentially the same(in fact the second one is a particular case of the first one).

(i) Suppose $M=M_1\# M_2$ and D ($i_k: D\longrightarrow M_k$,k=1,2) is the embedded disc used to form the connected sum, i.e. $M=\overline{M_1-i_1(D)}\cup\overline{M_2-i_2(D)}$ (if no confusion arises we will often write $\overline{M_k-i_k(D)}$ as $\overline{M_k-D}$,k=1,2). Slide the disc in one of the factors M_1, say, along an arc α .This determines an isotopy of embeddings h_t of a disc in M_1.

Let H_t be an extension of the isotopy to M_1. For each $t\in[0,1]$ the map f_t defined by H_t in $\overline{M_1-h_t(D)}$ and by id on $\overline{M_2-i_2(D)}$ defines a homeomorphism between M and $\overline{M_1-h_t(D)}\cup_{h_t}\overline{M_2-i_2(D)}$ where $\partial(\overline{M_1-h_t(D)})$ is identified with $\partial(\overline{M_2-i_2(D)})$ by h_t. This me ns ,for instance in the orientable case, if $r:D\to D$ is the standard orientation reversing homeomorphism such that $i_1(D)$ is identified with $i_2(D)$ by $i_2\circ r\circ i_1^{-1}/i_1(D)$ then $h_t(D)$ is identified with $i_2(D)$ by $i_2\circ r\circ h_t^{-1}/h_t(D)$.

Denote by $M_1\#_{h_t}M_2$ (or $M_1\#_{h_t}M_2$) the homeomorphic copy of M obtained in this way.For each t , $f_t: M\longrightarrow M_1\#_{h_t}M_2$ is called an __arc homeomorphism__ . f_t is well defined up to isotopy by lemma 3.3 and depends only on the homotopy class of $\alpha/[0,t]$ rel ends and on an element of $\mathbb{Z}_2$ (corollary 3.1 and lemma 3.2)

In general f_t is not a self homeomorphism of M,but if α is an __orientable loop__ then as h_t starts and ends with the inclusion , f_1 is a self homeomorphism of M. We then say that f_1 is a __loop homeomorphism__.



Also note that if $M' = M\#N$, a loop homeomorphism on M extends to a loop homeomorphism on M' by the identity on N, as we can always assume the homeomorphism on M to be the identity on the disc where we form the connected sum with N. Same considerations for arc homeomorphisms.

If the extensions of the arc or loop to an isotopy of embeddings are the canonical ones we shall call an arc or loop homeomorphism, respectively, a <u>partial</u> or <u>generalised slide</u>, and we shall sometimes refer to the partial or generalised slides as the standard arc or loop homeomorphisms (also we shall sometimes omit the word generalised).

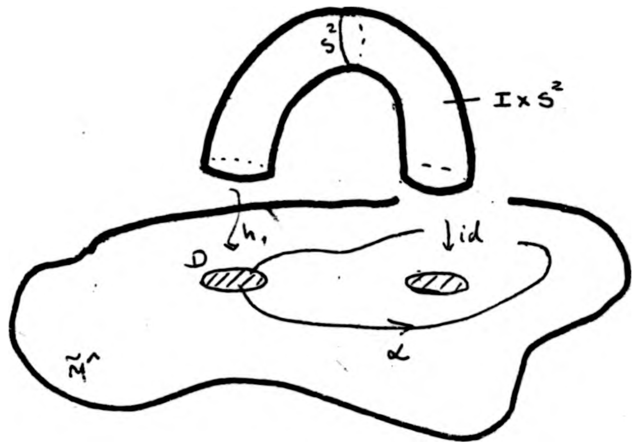
Note that once given the (orientable) loop or the arc, a generalised or partial slide is canonical by lemma 3.3.

<u>Remark:</u>

If α is a non orientable loop at the end of the isotopy h_t, D will have its orientation reversed. We can suppose, w.l.o.g., that the map $h_1^{-1} \circ i_1$ is the standard orientation reversing homeomorphism r. Then if M_2 admited a self homeomorphism which is r on D, R, say, we could glue R and H_1 together along ∂D to get a self-homeomorphism of M. However this will not be needed in what follows.

(ii) The second case is obtained by sliding one end of the hollow handle around a orientable loop α away from the other end of the handle (we sometimes say handle instead of hollow handle). Precisely, if $\tilde{M}$ denotes $\overline{M - I\times S^2}$ and $\hat{\tilde{M}}$, $\tilde{M}$ with the two sphere components capped off with 3 balls, is a loop in $\hat{\tilde{M}}$ starting at the centre point of one of those 3 balls and not intersecting the other. As before there is an isotopy H_t of $\hat{\tilde{M}}$ that drags the disc D corresponding to that end (see picture) around the loop starting and ending with the identity on that disc.

Note that we can avoid the possibility of one end of the handle kicking the other by choosing small enough standard discs i.e. by choosing a suitable trivialisation of the tangent microbundle.



Final homeomorphism H_1 is well defined up to isotopy rel D and there are at most two choices as before. As the loop only meets the hollow handle at one end we can assume that H_t is the identity on the other disc corresponding to the other end. Then f_1 defined by H_1 on $\tilde{M}$ and the identity on $I \times S^2$ defines a self homeomorphism of M which in the case the choice of the isotopy of embeddings of the disc in $\tilde{M}^\wedge$ is the canonical one we call a generalised slide. Note that there isn't a self homeomorphism of $I \times S^2$ which is the identity on one end and r on the other, hence we couldn't get a self homeomorphism if we slid around a non orientable loop. We could though obtain a partial slide. (Similarly to case (i) we have partial slides and given the loop or arc the generalised or partial slides are canonical).

If α goes around another handle then f_1 is a (hollow) handle slide. Again as everything can be assumed to be the identity outside a compact set a slide in a factor of M extends by the identity to a slide on M.

Some remarks

(1) In case (ii) we are only interested in slides around orientable loops. If α were non orientable we would get a homeomorphism between $M = M_1 \# S^1 \tilde{\times} S^2$ and $M = M_1 \# S^1 \times S^2$.

(2) In the orientable case all slides are orientation preserving homeomorphisms.

(3) Any two different choices of homeomorphisms obtained by sliding one end of a handle around a loop differ by the homeomorphism τ defined in section 2.

(2) Homeomorphisms preserving factors

These are the homeomorphisms that when restricted to each factor define a self homeomorphism of the factor (In fact the restriction defines a homeomorphism of $\overline{P_i - 3 \text{ ball}}$ which extends conewise to P_i).

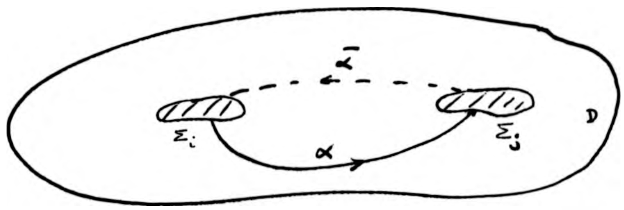
If in a decomposition $P_1 \# \ldots \# P_n$, $P_i \cong P_j$, $i \neq j$, and the homeomorphism sends $\tilde{P}_i$ to $\tilde{P}_j$ ($\tilde{P}_i$ denotes P_i with the interior of a 3 cell removed), $\tilde{P}_j$ to $\tilde{P}_i$ and $\tilde{P}_k$ to $\tilde{P}_k$ for $k \neq i,j$ we say the homeomorphism sends factor to factor. Same thing for any composition of these.

(3) Permutations of factors

Suppose M is given as obtained by attaching the different factors to disjoint cells in S^3. Suppose furthermore that there are some repeated factors in the decomposition. We consider homeomorphisms which interchange such factors.

Permutations are generated by homeomorphisms which interchange only two factors and can be defined as follows:

Suppose $P_i \cong P_j$, $i \neq j$ and let Σ_i, Σ_j be the respective separating spheres. Denote by M' the manifold before connecting P_i, P_j. Σ_i, Σ_j bound 3 balls B_i, B_j in M'. Let α be a path between them in the sphere part of M' intersecting **the balls only in** Σ_i, Σ_j in its end points. Consider a ball D regular neighbourhood of $B_i \cup B_j \cup \alpha$ in the sphere part of M'.



Then there is an isotopy s_t of M' which is the identity outside D s.t. s_1 interchanges the two balls. $s_1/M' - (\text{int } B_i \cup \text{int } B_j)$ extends to a homeomorphism which interchanges the two factors. If the arc homeomorphism thus determined (B_i is slid along α and B_j along $\bar{\alpha}$ ($= \alpha^{-1}$, see picture) is the **standard** one we say the homeomorphism thus determined is a <u>permutation.</u> **Permutations are well defined up to isotopy.**

We now show that these three types of homeomorphisms generate the group of homeomorphisms of the manifold:

THEOREM 3.1

Any (orientation preserving) homeomorphism of a(orientable) 3-manifold M can be obtained ,up to isotopy,as a composition of the following homeomorphisms:

(a) Homeomorphisms preserving factors;

(b) Permutations of factors;

(c) Generalised slides.

Proof:

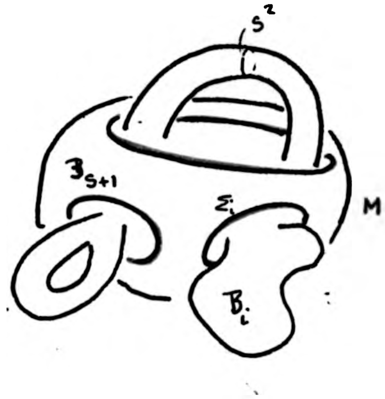
We give here the proof for the orientable case.For the non orientable case the proof will follow as a corollary of a relative version of the theorem(Theorem 3.2), where we consider homeomorphisms fixed on a disc-any homeomorphism of a non orientable manifold can be assumed,up to isotopy the identity on a disc.

Thus ,from now on ,unless otherwise stated,homeomorphism will mean orientation preserving homeomorphism,manifold will mean orientable manifold.

We shall show that given any homeomorphism f of M then we can find another homeomorphism h (not necessarily unique) suct that h∘f sends factors to factors and hence,composing with some permutations we shall get a homeomorphism preserving factors.As h is obtained,up to isotopy, by a composition of homeomorphisms of types (a) and (c) this will prove the theorem.

Choosing separating spheres $\Sigma_1 \dots \Sigma_n$ ($M = P_1 \# \dots \# P_n$) so that each Σ_j divides M into two parts one of which homeomorphic to $\overline{P_j - \text{3-ball}}$, which we denote by $\tilde{P}_j$ (i.e. attach the manifolds P_j to n disjoint standard discs in S^3). Regard the P_i, $i \geq s+1$ as(hollow) handles h_i attached.Denote by $S^2_{s+1} \dots S^2_n$ the belt spheres of these handles and let S be $\{\Sigma_1 \dots \Sigma_s, S_{s+1} \dots S_n\}$

Cutting M along S we obtain a non connected manifold with (s+1) components B_i where $B_i \tilde{=} P_i$ for $i \leq s$ and B_{s+1} is a 3-sphere with 2n-s holes(consider B_i with boundary).Let T be one of the separating spheres .Then f(T) is a sphere that separates.By general position and transversality $f(T) \cap S$ consists of a finite number of simple closed curves.



<u>The idea of the proof is as follows:</u>

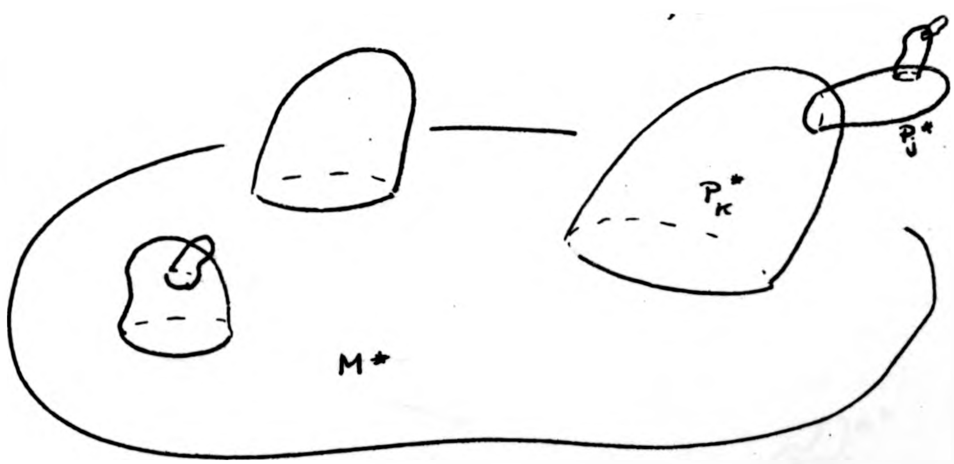
If $f(T) \cap S \neq \emptyset$ we first show how to reduce the number of intersections. By induction,after a finite number of steps $f(T) \cap S = \emptyset$ .The next step will be to make the separating spheres go to separating spheres.In both cases we deform M (and hence f) by a series of isotopies,partial and generalised slides.As partial slides are not self-homeomorphisms of M,in general,after each step we shall probably have moved to a homeomorphic copy of M,M" say. After having the separating spheres back in separating spheres we are in M again.

The fact that during the process we move to a homeomorphic copy of M will be irrelevant in the end as we shall end up with the same copy of M and show later that those partial slides can be put together to give generalised slides. Then the theorem will follow.

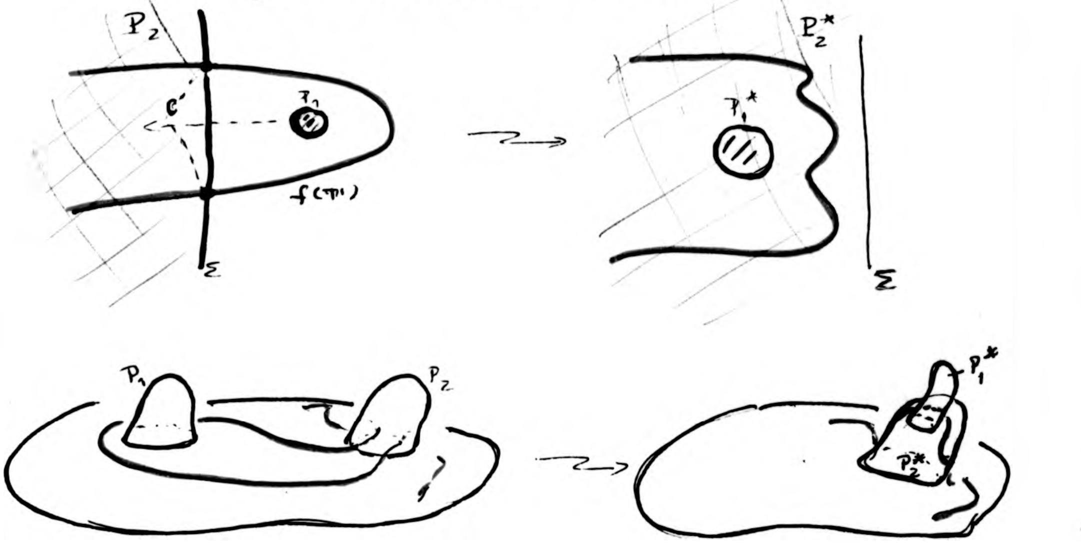
Some remarks:

As already quoted during the process we move to a homeomorphic copy of M obtained by inductively gluing the factors P_i choosing discs lying either in S^3 or inside one of the components already glued in. We call such a copy M* (bearing in mind that it is not always the same copy. We will often say "...by a homeomorphism of M*..." meaning "...by a homeomorphism of M* into another copy of M...". This will have the advantage of simplifying considerably the notation) and write P^*_i for the factor corresponding to P_i (notice that P^*_i might be glued to a disc inside P^*_j.-i.e. P^*_j is not, in general, the image of P_j under the homeomorphism $M \rightarrow M^*$).

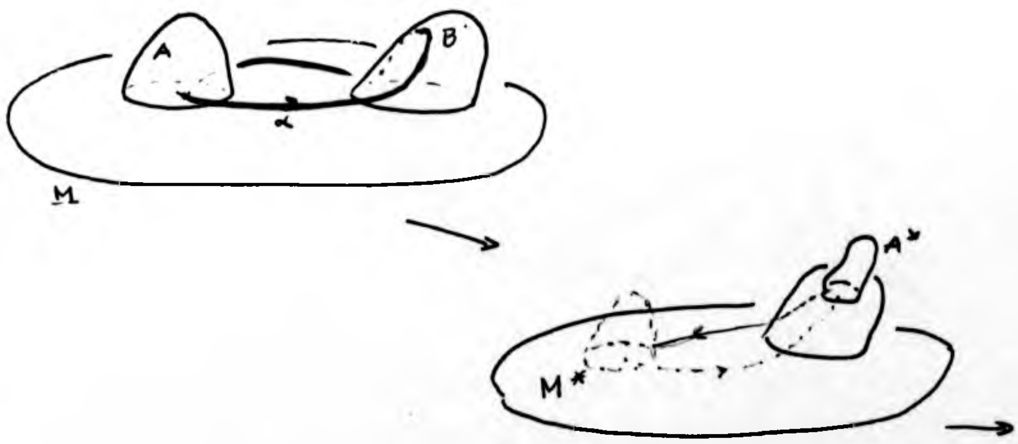
Here is a typical picture of M*:

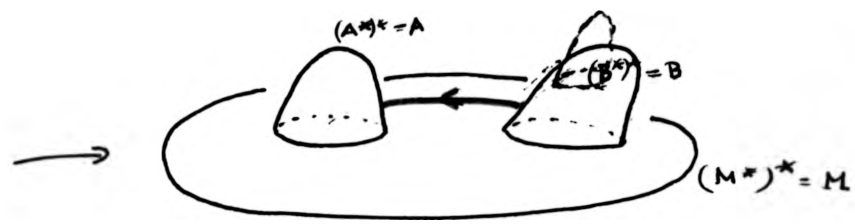


We want this, as the basic process will be to eliminate intersection curves of $f(T) \cap S$ and a typical situation will be ,for instance, the one pictured below where to eliminate an intersection the factor P_1 will have to be moved into P_2.(the other situations are similar).



For example,suppose M=A#B and A is moved along a loop that goes through B.Then the picture for the process will be:

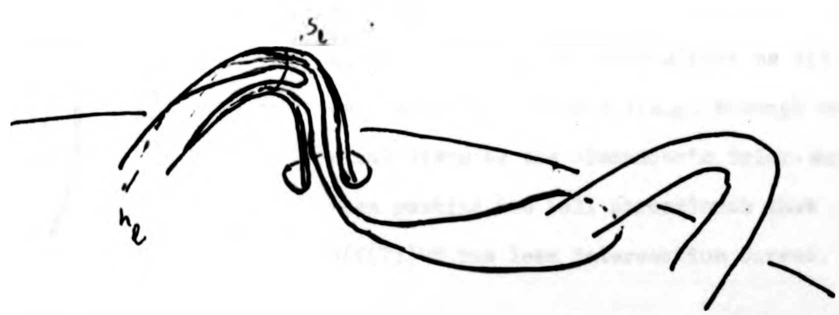


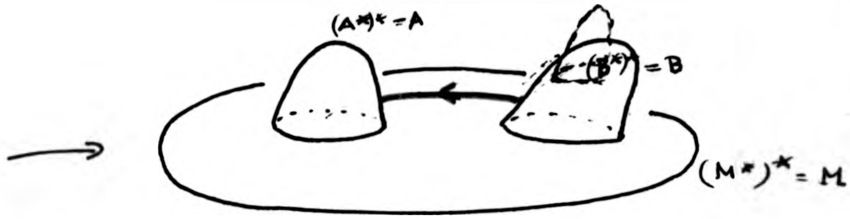


Finally as all the homeomorphisms and isotopies can be assumed to keep the manifold fixed outside some compact set ,we can always suppose, if necessary,that one doesn't destroy the effect of the previous ones- this will become clear in the proof.It will always be assumed that what should be kept fixed will be,although sometimes it is not explicitly mentioned.Also,for simplicity,we will stll call f the deformed homeomorphism.

(1) Among the intersections of f(T) with S choose an innermost curve C in f(T).Then C bounds a 2 cell E,say, in f(T) containing no more intersection curves.$E\subset B_j$, some j.

(a) Suppose first $C\subset S^2_\ell$,some ℓ. Then j=s+1.





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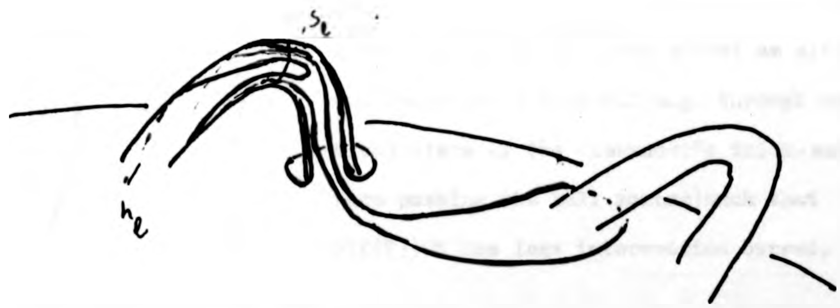
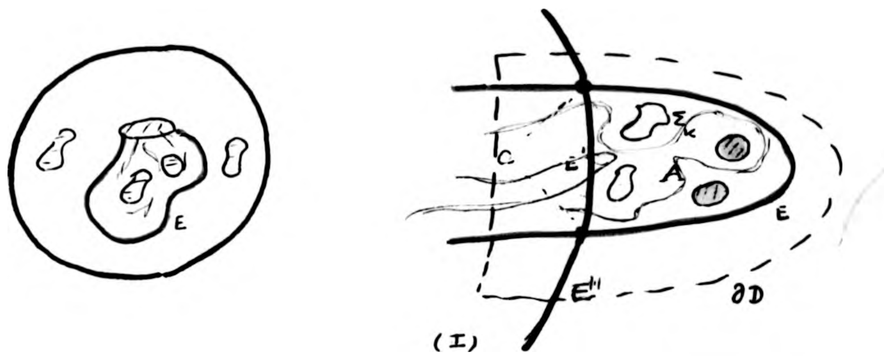
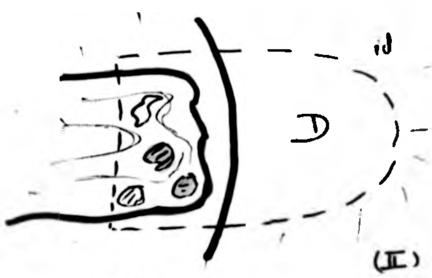


C divides S_ℓ into two 2 discs E', E''. Both $E''\cup_\partial E$, $E'\cup_\partial E$ are 2 spheres By the Schoenflies theorem both of them divide B_{s+1} into 3 balls with possibly some balls removed. Choose E'' or E' s.t. the component of $B_{s+i} - E''\cup_\partial E$ (or $B_{s+i} - E'\cup_\partial E$) bounded by $E''\cup_\partial E$ ($E''\cup_\partial E$) does not contain the "other hole "corresponding to h_ℓ. Suppose we have chosen E' and denote by A the component satisfying the required condition.

A is a 3 cell with possibly some balls removed

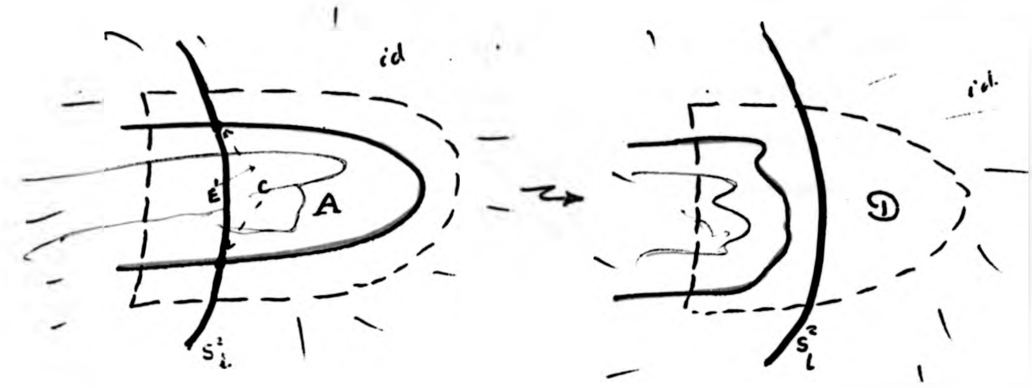


Let $\hat{B}_{s+1}$ denote B_{s+1} with these holes capped off with 3 cells. Then there is an isotopy q_t in $\hat{B}_{s+1}$ (1.10) which takes picture (I) to picture (II) and is the identity outside a 3 ball D neighbourhood of A. Then $q_1/\hat{B}_{s+1}$ extends to a composition of partial slides Q,say,(as it has the same effect as sliding one factor at a time-w.l.o.g. through canonical discs by the Alexander's trick- and then pushing the ball across)such that $Q(f(T))\cap S$ has less intersection curves.



Remark

In the special case where A is a 3 cell, after an isotopy identity outside a 3 ball neighbourhood of A we replace f(T) by another sphere with less intersection curves with S^2_ℓ. (c.f. Theorem 2.1.).

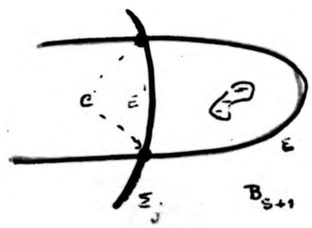


(b) <u>If $C \subset \Sigma_j$ $j \leq s$ we proceed as follows:</u>

Let E',E" be the 2 cells in Σ_j bounded by C. We consider two cases

(i) <u>$E \subset B_{s+1}$</u>

Then $E' \underset{\partial}{\cup} E$, say, bounds a 3 ball in B_{s+1} with possibly some points removed. As before by partial slides



we reduce the number of intersection curves.

(ii) <u>$E \subset B_m$ $m \leq s$</u>

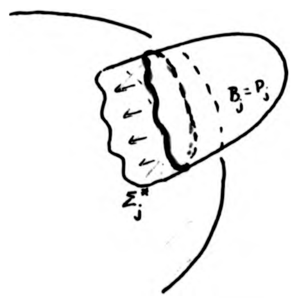
As $B_m \cong \tilde{P}_m$ where P_m is an irreducible manifold one of the 2-spheres $E'' \underset{\partial}{\cup} E$, $E' \underset{\partial}{\cup} E$ bounds a 3-ball with possibly some holes. Then proceed as before to get $f(T) \cap \Sigma_j$ with less intersection curves.

(2) Thus after a finite number of stages f(T) lies in int B_j^*, some j (recall we still denote by f the resulting homeomorphism $M \longrightarrow M^*$). If $T=\Sigma_k$, say, by choice of separating spheres f(T) divides M^* into two components one of which is homeomorphic to $\tilde{P}_k$.

We now try to make f(T) = T.

(i) <u>$f(T) \subset \text{int } B_j^*$, $j \leq s$</u>

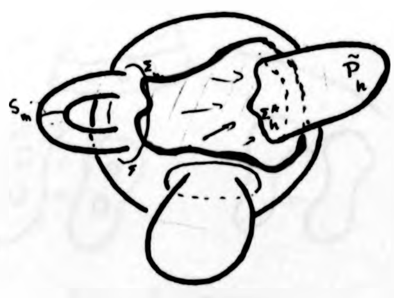
Then $T = \Sigma_k$ for $k \leq s$. It follows from the fact that P_j is prime and irreducible that f(T) divides $\tilde{P}_j^*$ into two components one of them homeomorphic to $\tilde{P}_j$ and the other a collar on $\partial \tilde{P}_j^* = \Sigma_j^*$.



Hence we can isotope f(T) so as to coincide with Σ_j^*.

(We can assume the isotopy fixed outside a collar of Σ_j^* in $\overline{M-\tilde{P}_j^*}$).

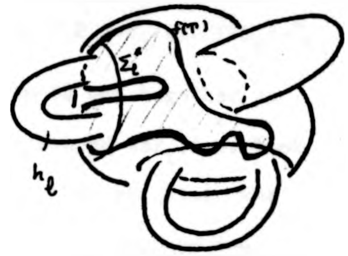
(ii) <u>$f(T) \subset \text{int } B_{s+1}$ and $T=\Sigma_j$, $j \leq s$</u>



By the Schoenflies theorem f(T) divides B_{s+1}^* into two components each of which is a 3 ball with holes and one of them has only one hole on it i.e. it is an annulus. Then if Σ_h^* is the boundary of that hole we isotope f so as to make $f(T) = \Sigma_h^*$ (Hence $P_h \cong P_j$). We can assume the isotopy fixed outside ∂B_{s+1}^*.

(iii) $f(T) \subset \text{int } B^*_{s+1}$ and $T = \Sigma_j$, $j > s$

As $T = \Sigma_j, j>s$, one of the components bounded by $f(T)$ in B^*_{s+1} is a ball with two holes corresponding to the handle h_ℓ say (Then $P_\ell \cong P_j$). Then after a partial slide $f(T) = \Sigma^*_\ell$.



At this stage we have $f: M \longrightarrow M^*$ where $M^* = \# P^*_i$ and $f(\tilde{P}_i) = \tilde{P}^*_j$, $f(\Sigma_i) = \Sigma^*_j$. By further partial slides, we can also assume $f(B_{s+1}) = B^*_{s+1}$:

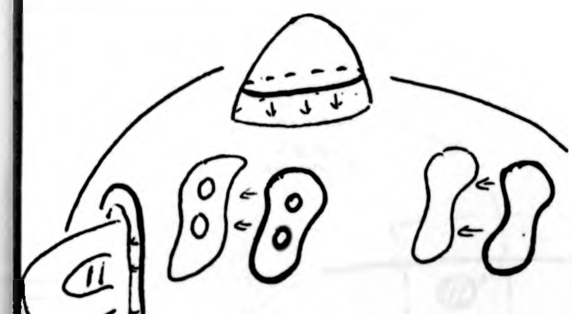
(i.e. we make the holes go to the corresponding holes)

Both B_{s+1}, B^*_{s+1} are 3 spheres with holes and $\Sigma^*_j \subset B_k$ some k. (This follows from the proof). If $k \leq s$ we make, as before, $\Sigma_k = \Sigma^*_j$ by an isotopy.

Now $\hat{B}_{s+1} \cong \hat{B}^*_{s+1}$ where $\wedge$ denotes the manifold after capping off its sphere components.

Then by a composition of partial slides we can make Σ^*_j go to Σ_j (making the holes corresponding to the handles go to the corresponding holes in case $j>s$).

The cases to consider are shown in the picture. (This follows from the proof).

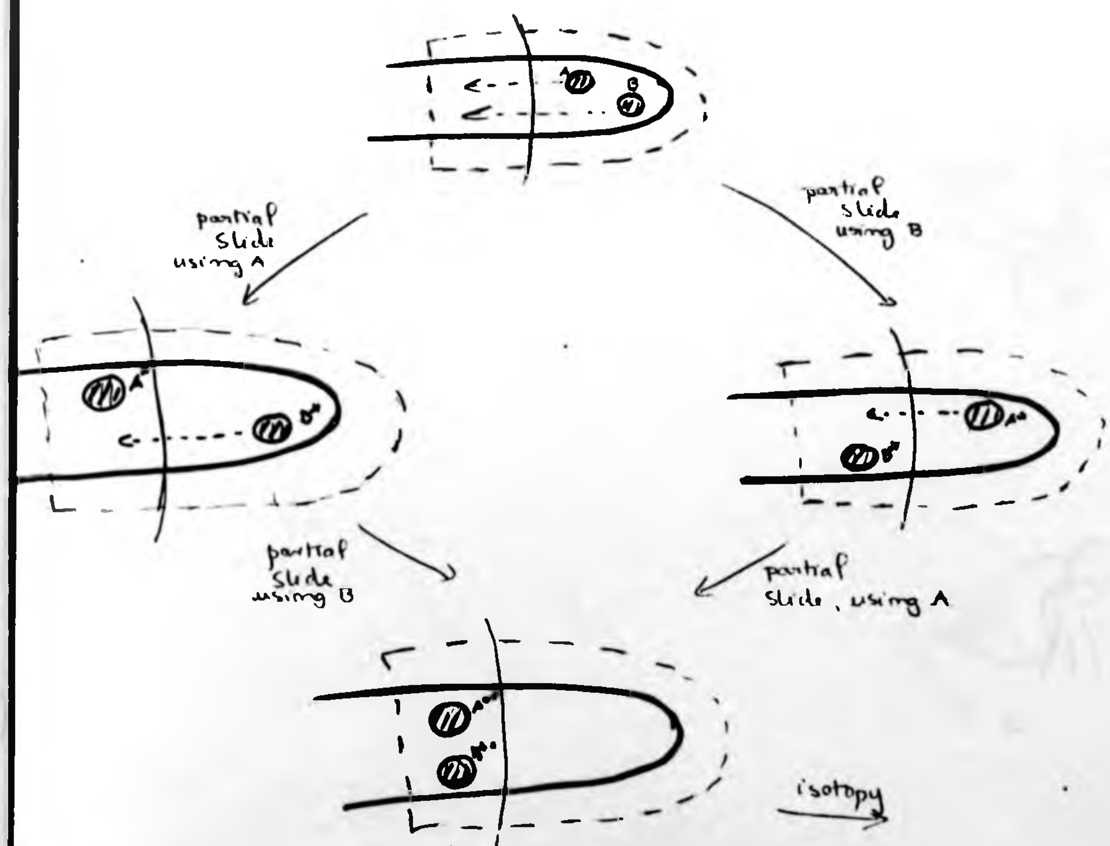


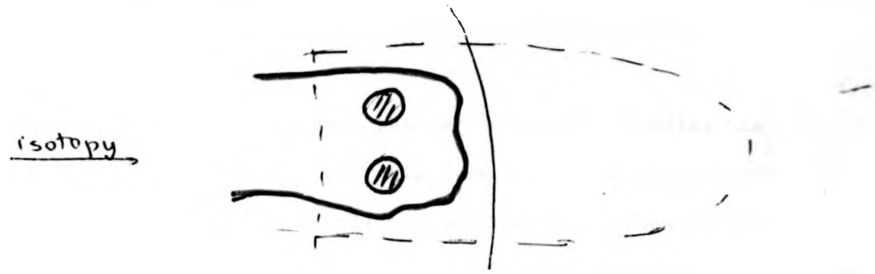
Cases to consider

Finally after a finite number of stages f will send $\tilde{P}_j$ to $\tilde{P}_i$ and Σ_j to Σ_i ,i.e. we are in M again.Then composing with a suitable permutation the homeomorphism will preserve factors, i.e. $f(\tilde{P}_i)=\tilde{P}_i$ and $f(\Sigma_i)=\Sigma_i$.

It remains to prove that all partial slides can be put together to give generalised slides(as in the final stage Σ_i is mapped to Σ_i)i.e. that we can "permute" the partial slides in such a way so that we obtain the same effect as if we slid one factor at a time around a loop.Once the separating spheres are already disjoint or coincident with their images,clearly the order in which we make them coincide is irrelevant as each homeomorphism can be assumed to be the identity on either the other separating spheres or their images.There are similar considerations for the permutations.So let's look closely at stage(1)

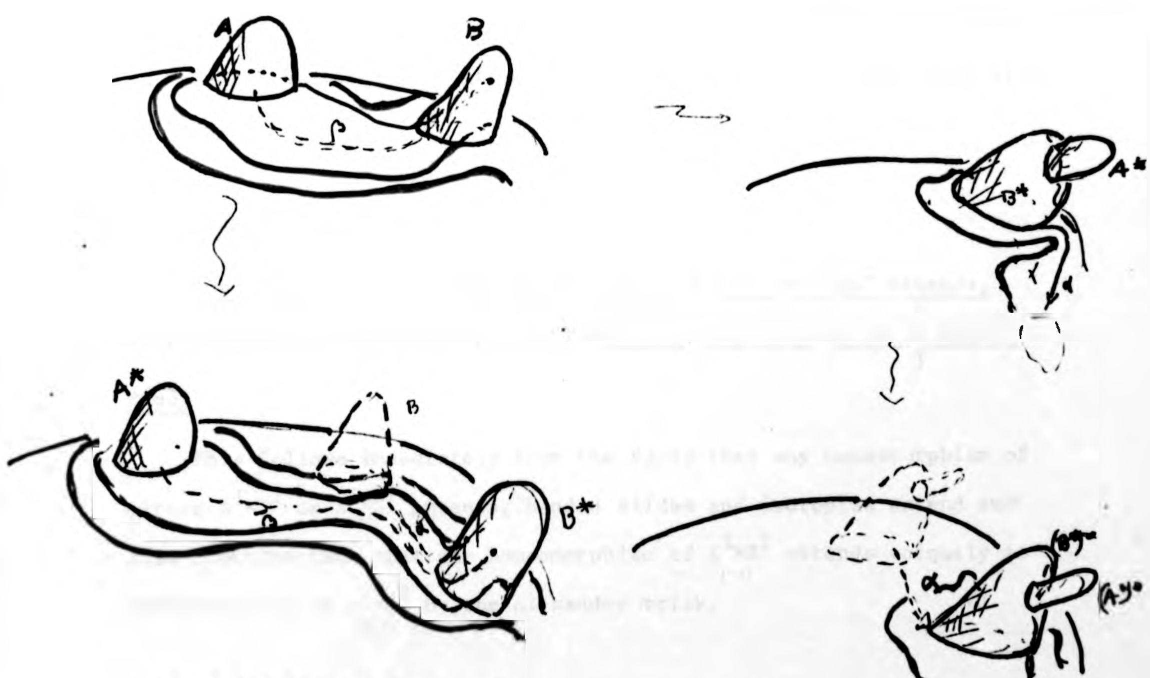
We have already quoted that the effect of such stage is a composition of partial slides.





We could have slid one factor at a time. As a partial slide can be assumed to be the identity outside the track of the factor ,the order in which we slide is irrelevant(see the pictures above).

Now consider one factor A,say.Again if the next stage is the identity on A we can interchange the two stages.If not it is because A has been moved into another factor B,say, and the next stage moves B (or it has been moved out of B.Proof is the same) But this clearly has the same effect as moving B first and then A into B(see pictures below).



Hence the theorem is proved. □

Remarks

(a) In the non orientable case, the non orientable handles must be slid, in the end, around an orientable loop as we end up with the same factorization (for if we slide a non orientable handle over an orientation reversing handle the handle will become orientable and $S^1\tilde{\times}S^2 \not\cong S^1\times S^2$) although in intermediate stages we will probably have to use different factorizations.

(b) It follows from the last remark that <u>we don't need to consider normal factorizations.</u> If we start with a certain factorization we will end up with the same factorization by choice of separating spheres : orientable handles must go to orientable handles and non orientable handles to non orientable handles.

(c) Also as already observed in the beginning the theorem is true for the bounded case. If for instance we consider homeomorphisms fixed on the boundary the theorem will follow with the "obvious" changes using the uniqueness of normal factorizations for 3 manifolds with boundary.

COROLLARY 3.2

<u>Any homeomorphism of $\#_i S^1\times S^2$, $\#_i S^1\tilde{\times}S^2$ or $\#_i S^1\times S^2 \#_j S^1\tilde{\times}S^2$ extends, respectively, to a homeomorphism of $\#^{\partial}_i S^1\times B^3$, $\#_i S^1\tilde{\times}B^3$ or $\#^{\partial}_i S^1\times B^3 \#^{\partial}_j S^1\tilde{\times}B^3$.</u>

Proof

This follows immediately from the facts that any homeomorphism of either $S^1\times S^2$ or $S^1\tilde{\times}S^2$ extends, handle slides and isotopies extend and also from the fact that any homeomorphism of $S^1\times_{(\sim)}S^2$ extends uniquely to a homeomorphism of $S^1\times_{(\sim)}S^2$ by the Alexander trick. □

This result has also been obtained by Laudenbach [17]. In fact we shall recover some more general results of Laudenbach later on.

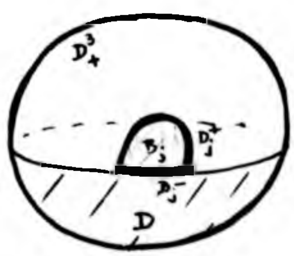
II. The Case of homeomorphisms fixed on a disc

We now recast the theorem in a canonical form. We do this by working relative to a disc.

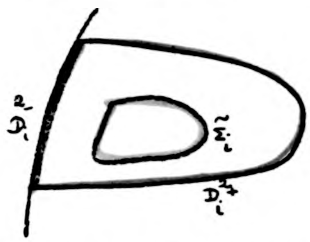
Let Aut(M,D) denote the group of isotopy classes of homeomorphisms of M which are the identity on a fixed disc, $D \hookrightarrow \text{int } M$, and also the isotopies are fixed on D. Call an isotopy mod D a D-isotopy and denote by $\sim_D$ the D-isotopy equivalence relation, $[\ ]_D$ the D-isotopy classes. Similarly define S(M,D), P(M,D) respectively the D-isotopy classes of generalised slides and permutations (whenever possible).

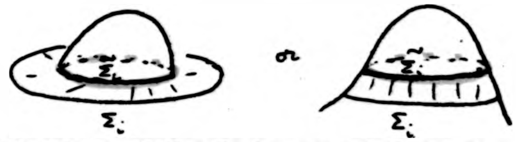
Let $M = \#_n P_i$ where, as before, for $i \leq s$ P_i are irreducible and for $i > s$ P_i is a 2 sphere bundle over S^1. $\tilde{P}_i$ will denote the closure of P_i-3disc. Given $D \hookrightarrow M$ choose separating spheres Σ_i satisfying the following conditions ($\#_n P_i$ denotes the connected sum of n prime manifolds P_i):

Consider S^3 as the union of two 3-balls D^3_+, D^3_- ($D^3_- = D$) along their common boundary. On D^3_+ choose n-disjoint 3 balls B_j with boundary $\partial B_j = D_j^{2+} \cup D_j^{2-}$ where D_j^{2-} is a 2-disc in ∂D^3_- and $D_j^+ \subset D^3_+$. Then for $j > s$ form the connected sum using these balls i.e. $\partial B_j = \Sigma_j$.



Choose a parallel separating sphere $\tilde{\Sigma}_i \subset B_i$ (i.e. the region between $\tilde{\Sigma}_i$ and $\Sigma_i = \overset{2}{D}_i^{\,} \cup \overset{2}{D}{}_i^{-}$ is an annulus in B_i) and for $i \leq s$, attach P_i by $\tilde{\Sigma}_i$ (Clearly we can also think of P_i attached by Σ_i)



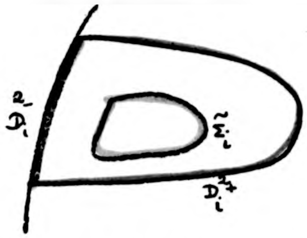


Suppose also that we choose parallelizations of the factors or double covers of the factors, according as they are orientable or not, so that discs ends of handles or discs bounded by $\tilde{\Sigma}_i$ in both the sphere or in the factors are the canonical ones in the sense of Lemma 3.1.

Let $\underset{i}{X}(\mathrm{Aut}\ P_i, D)$ denote the D-isotopy classes of homeomorphisms of M obtained by **gluing** up isotopy classes of homeomorphisms of factors which fix a disc (and isotopies fixing a disc). For each factor $\mathrm{Aut}(P_i, D)$ means B_i - isotopy classes of homeomorphisms of P_i fixing B_i. (For $i \leq s$ think of P_i attached by the Σ_i). A typical element is denoted by $\underset{i}{\#} f_i$ where $f_i \in \mathrm{Aut}(P_i, D)$. The notation $\underset{i}{X}(\mathrm{Aut}\ P_i, D)$ is justified as this group is clearly isomorphic to the direct product of the $(\mathrm{Aut}\ P_i, D)$.

We now show that theorem 3.1 generalises to this case and that it can even be strengthened as we will be able to find a well defined D-isotopy class $[g]_D$ where g is a composition of slides and permutations for every D-isotopy class $[f]_D \in \mathrm{Aut}(M, D)$ s.t. $g \circ f$ preserves factors. This will allow us to compute $\mathrm{Aut}(M, D)$ in terms of $P(M, D)$ $S(M, D)$ and $\underset{i}{X}(\mathrm{Aut}\ P_i, D)$.

Choose a parallel separating sphere $\tilde{\Sigma}_i \subset B_i$ (i.e. the region between $\tilde{\Sigma}_i$ and $\Sigma_i = D_i^{2+} \cup D_i^{2-}$ is an annulus in B_i) and for $i \leq s$, attach P_i by $\tilde{\Sigma}_i$ (Clearly we can also think of P_i attached by Σ_i)



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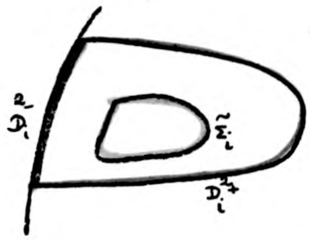

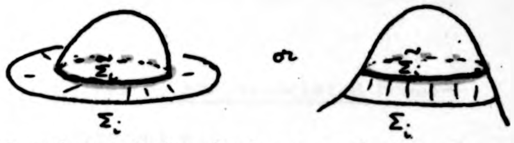

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THEOREM 3.2

(1) For any homeomorphism f of M which is the identity on a fixed disc D there exists a well defined $[g]_D \in Aut(M,D)$, where g is a composition of generalised slides and permutations, such that $g \circ f$ preserves factors.

(2) Moreover if f_1 is isotopic to f_2 rel D, the associated homeomorphisms given by (1) are isotopic rel D.

Proof;

(a) First part of the proof is essentially the same as in theorem 3.1 but instead of trying to make the sphere $f(\Sigma_i)$ disjoint from the S_j's and Σ_i's (i≤s) we try to make f (int $\overset{2+}{D_i}$) disjoint from $S_j \tilde{\Sigma}_i$ always making sure that D^3_- (D^3_-=D) remains fixed.

Again, for simplicity, we suppose M is orientable.

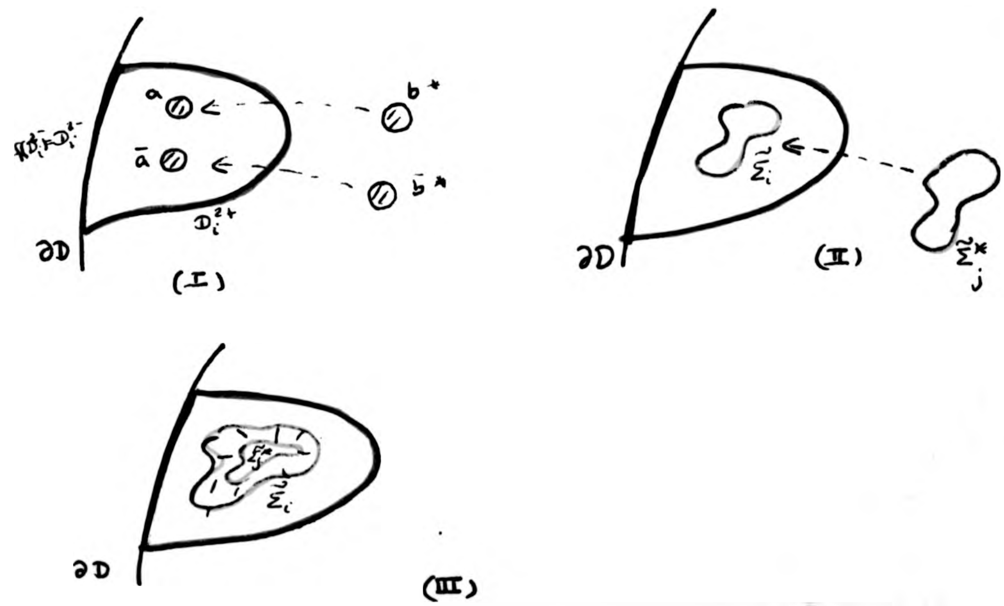
It follows from the Alexander's trick, lemma 3.3 and corollary 3.1 that the homeomorphism which eliminates an innermost curve is well defined up to isotopy. In fact we can suppose it is even canonical, as each factor can be pushed along a collar line, through canonical discs, of a collar structure of the 3-ball B (see picture)

Note that a partial slide is a standard arc homeomorphism. We also point out that an innermost curve C bounds a unique disc in $f(\overset{2}{D_i})$ and a unique disc in $\tilde{P}_j$ in the irreducible case. In the sphere bundle cases only one of the discs E',E" is such that the component bounded by $E' \cup_\partial E$ or $E'' \cup_\partial E$ does not contain the other hole (cf. theorem 3.1).

<u>This uniqueness is crucial for the proof of statement (2) of the theorem.</u>

(b) Having already made $f(\text{int}D_i^{2+})$ disjoint from the belt spheres S_j of the handles and from the separating spheres Σ_i of the irreducible factors, we now need to make $f(D_i^{2+})=D_i^{2+}$.

At this stage we are in a homeomorphic copy of M denoted by M* (of theorem 3.1). As before, we make $\tilde{\Sigma}_i^*$ coincide with $\tilde{\Sigma}_j$, some i,j, and holes corresponding to the hollow handles in M* go to corresponding holes in M. We have to consider the following cases pictured below (the fact that these are the only cases follows from the proof).



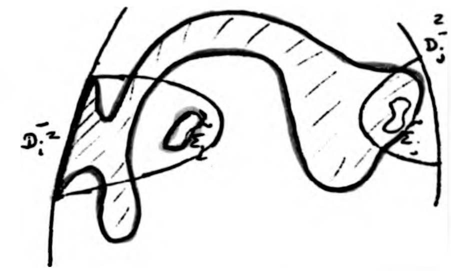
Case I shows discs a,ā which are ends of a handle and ends b*,b* of a probably different handle in M* . We make them coincide by a partial slide. Cases II and III show a separating sphere $\tilde{\Sigma}_i$ of an irreducible factor and the two possible positons of a probably different separating sphere, $\tilde{\Sigma}_j^*$, in M^*. In case II we use a partial slide, in case III an isotopy (the region between $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_j^*$ is a collar).

Again all the choices are unique up to isotopy rel D (for the last case use (1.9); also note that the slides are along arcs in a simply connected space so any two choices of arcs are homotopic).

After all these moves we are in M again. We are now ready to make $f(D_i^{2+})$ coincide with D_i^{2+}. <u>We consider two cases</u> each of which is divided into three subcases.

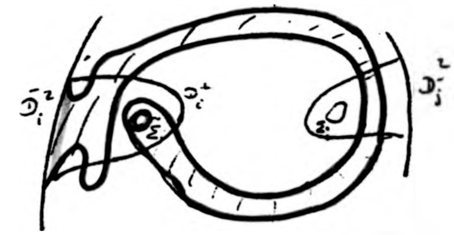
(a) <u>$i \leq s$</u>

(II)



(I)

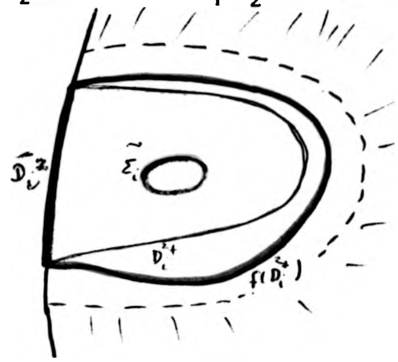
(III)



Let B denote the closure of D_+^3 with both the interiors of the two balls corresponding to the hollow handles and the interiors of the balls bounded by the $\tilde{\Sigma}_j$ ($j \leq s$) removed. B is a sphere with some points removed (thus $\Pi_1(B) = 0$). $f(\text{int}\, D_i^{2+}) \subset \text{int} B$.

Again we can assume(rel D),possibly after a permutation,(case in picture II) and partial slides that $f(\text{int}\ D_i^{2+}) \cap \text{int}\ D_i^{2+}$ is empty.Choices and homeomorphisms are again unique up to isotopy.

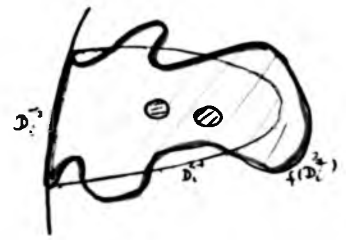
Let $\tilde{B}$ denote B with the 2 sphere $\tilde{\Sigma}_i$ capped off with a 3 ball. Then D_i^+, $f(D_i^{2+})$ bound (resp) 3 balls G_1, G_2 with either $G_1 \subset G_2$ or $G_2 \subset G_1$. Also $\partial G_1 \cap \partial G_2 = D_i^{2-}$. Then as a regular neighbourhood of $G_1 \cup G_2$ is a 3 ball, there exists an isotopy of $\tilde{B}$ rel $\partial\tilde{B}$ taking $f(D_i^{2+})$ to D_i^{2+} which is the identity outside that ball and on D_-^3. By a further isotopy, we can also make



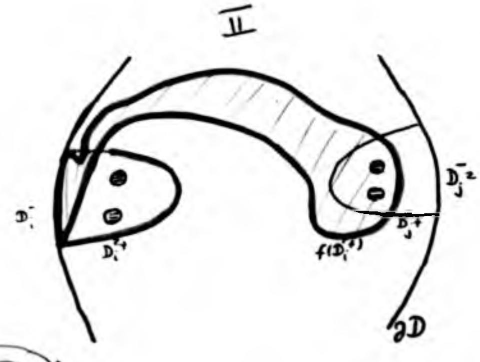
$\tilde{\Sigma}_i$ go canonically to $\vec{\Sigma}_i$. This extends to a homeomorphism of M and again choices are unique up to isotopy. **In fact it is canonical by the Alexander's trick.**

f/D_i^{2+} an orientation preserving self-homeomorphism which is the identity on the boundary hence isotopic to the identity (Alexander's trick).

(b) i>s



(I)



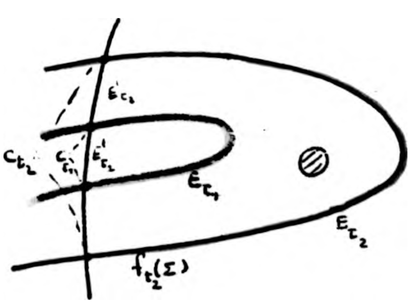
(II)

Again we have to consider two subcases (see pictures). After a permutation we are reduced to subcase I. The only difference is that now $\tilde{B}$ is B with the two holes corresponding to the hollow handles capped off with balls. $\Pi_1(\tilde{B}) = \Pi_1(B) = 0$ hence any two choices of arcs are homotopic, **thus the homeomorphisms are unique up to isotopy.**

Thus, in the end, we will have a homeomorphism fixed on D which will preserve factors. Also as on $\partial(\overline{D^3_+ - \bigcup_i B_i}) = S^2$, f is the identity, by a further D-isotopy we make f to be the identity on $\overline{S^3 - \bigcup B_i}$ (Alexander's trick). The proof of statement (1) is now complete as g is well defined by all the uniqueness referred relative to choices of homeomorphisms (g is the composition of slides and permutations obtained through the proof s.t. $g \circ f$ preserves factors.) Clearly it also follows that $[g]_D$ is well defined.

For the proof of statement (2) we refer the reader to the proof of Proposition 1 of [10]. The method of proof follows immediatly for k=1, as all our choices are unique up to isotopy.The choice of standard arc homemorphisms is crucial here to get a coherent choice all along the isotopy.We remark that in the non orientable case,as we are working rel D, the factors must be slid, in the end , along orientable loops,and also that in lemma 1 of [10] we need to choose the intervals I_i small enough so that "nearby curves behave similarly" as we want to eliminate intersections in a standard way for all $t \in I_i$.

For example:



The situation showed in the picture is not allowed for $t_1, t_2 \in I_i$, some i,since $E'_{t_2} \cup E_{t_2}$ bounds a disc with a hole in it while $E'_{t_1} \cup E_{t_1}$ doesn't,and hence in the first case we should have to use first an arc homeomorphism or a partial slide which it would not be the case in the second one.

Clearly we can always assume this.Hence the theorem is proved.

□

We now give some corollaries to the theorem.

COROLLARY 3.3

There is a well defined map $\phi : \mathrm{Aut}(M,D) \longrightarrow \underset{i}{X}\ \mathrm{Aut}(P_i,D)$ defined by $\phi[f]_D = [g\circ f]_D$. ϕ is onto and splits by the inclusion.

Proof

ϕ is well defined by the theorem. The facts that ϕ is onto and splits by the inclusion are clear. □

Unfortunately ϕ is not a homomorphism in general: take for instance, $M = \#_2 S^1\times S^2$, f_1 a permutation, $f_2 = a_1 \# a_2$, where a_i, $i=1,2$, are homeomorphisms of $\mathrm{Aut}(S^1\times S^2,D)$ with $[a_1]_D \neq [a_2]_D$. Then

$$\phi[f_1]_D = [id]_D \# [id]_D$$

$$\phi[f_2]_D = [a_1]_D \# [a_2]_D$$

and $$\phi[f_2\circ f_1] = [a_2]_D \# [a_1]_D \neq (\phi[f_2]_D)\circ\phi([f_1]_D) = [a_1]_D \# [a_2]_D$$

But ϕ as a map of pointed sets (basepoint being the identity in both cases) ϕ has a kernel K. It follows from the proof of theorem 3.2 that K is the subgroup of $\mathrm{Aut}(M,D)$ generated by the generalised slides and permutations.

LEMMA 3.4

K is the semi-direct product of $P(M,D)$ and $S(M,D)$.

Proof

(1) $S(M,D)$ is a normal subgroup of K : It is enough to prove that the composition (permutation) slide $(\mathrm{permutation})^{-1}$ is a (probably different slide).

If we denote by U_{ik} the slide obtained by sliding either the factor

P_i, if P_i is irreducible, or the end of the "handle" of P_i, if P_i is a 2-sphere bundle over S^1, along the factor P_k, $k \neq i$, and by $P_{j\ell}$, $j \neq \ell$ the permutation interchanging P_j with P_ℓ (if possible) then it follows from the definitions that

(1) $P_{ik}U_{ik} = U_{ki}P_{ik}$

(2) $P_{ij}U_{ik} = U_{jk}P_{ij}$

(3) $P_{ij}U_{ki} = U_{kj}P_{ij}$

(4) $P_{i\ell}U_{jk} = U_{jk}P_{i\ell}$ $i \neq j \neq k \neq \ell$

proving (1).

(2) We therefore have a split exact sequence

$$0 \longrightarrow S(M,D) \xrightarrow{i_1} K \underset{i_2}{\overset{p}{\rightleftarrows}} P(M,D) \longrightarrow 0.$$

where i_1, i_2 are the inclusions and p is the quotient map

$$K \xrightarrow{p} K/_{S(M,D)} \cong P(M,D)$$

i.e. K is the semi-direct product of $P(M,D)$, $S(M,D)$. (Note that $P(M,D)$ is not a normal subgroup in general. For example take $M = \#_2 S^1 \times S^2$ and let $P_{1,2}$ be the permutation that interchanges the two factors and h the homeomorphism obtained by sliding the first handle over the second. Then $h^{-1}P_{1,2}h$ is not a permutation. Look at map induced on Π_1. □

Remarks

(1) We cannot do the same with the sequence of maps

$$0 \longrightarrow K \longrightarrow \mathrm{Aut}(M,D) \xrightarrow{\phi} \underset{i}{X}\mathrm{Aut}(P_i,D) \longrightarrow 0$$

as neither K nor $\underset{i}{X}\mathrm{Aut}(P_i,D)$ are normal subgroups (in general) of $\mathrm{Aut}(M,D)$. K is not normal from the above. To see $\underset{i}{X}\mathrm{Aut}(P_i,D)$ is not always a normal subgroup, consider $M = \#_2 S^1 \times S^2$, h a homeomorphism which is the identity on one factor and on the other interchanges the two ends of the handle. Then if g is a generalised slide which slides the second factor along the first one, $g^{-1} \circ h \circ g \notin \times_i \mathrm{Aut}(P_i, D)$ -- look at the map induced on π_1 (π_1 is not abelian). □

(2) If $M = \#_i P_i$ where all P_i are irreducible then $\times_i \mathrm{Aut}(P_i, D)$ is a normal subgroup of $\mathrm{Aut}(\#_i P_i, D)$ and then $\mathrm{Aut}(\#_i P_i, D)$ is the semi-direct product of $\times_i \mathrm{Aut}(P_i, D)$ and K, i.e.

$$0 \longrightarrow \times_i \mathrm{Aut}(P_i, D) \underset{i}{\overset{\phi}{\rightleftarrows}} \mathrm{Aut}(\#P_i, D) \underset{i}{\overset{p}{\rightleftarrows}} K \longrightarrow 0$$

is a split exact sequence. (i is the inclusion, p the projection $\mathrm{Aut}(\#_i P_i, D) \longrightarrow \mathrm{Aut}(\#_i P_i, D) / \times_i \mathrm{Aut}(P_i, D)$). □

From theorem 3.2 we can also get some information about the higher homotopy groups of $PL(M,D)$. The deformation described gives a well defined map $\phi_k : \pi_k\, PL(M,D) \longrightarrow \times_i \pi_k\, PL(P_i, D)$ (in fact for k=0 we have ϕ of corollary 3.3) defined by $\phi_k([f]_D) = [g \circ f]_D$ ($\times_i \pi_k PL(P_i, D)$ means the obvious thing).

THEOREM 3.3

$\phi_k : \pi_k PL(M,D) \longrightarrow \times_i \pi_k PL(P_i, D)$ is a split epimorphism for $k \geq 1$

Proof:

The process, described in the theorem 3.1, of taking a homeomorphism of M and possibly composing it with slides and permutations till it preserves factors is, actually, canonical when the homeomorphism is isotopic to the identity as non-trivial permutations cannot be involved. We can thus appeal to Proposition 1 of [10] to show that the deformation described induces an epimorphism $\phi_k : \pi_k\, PL(M,D) \longrightarrow \times_i \pi_k\, PL(P_i, D)$. Clearly ϕ_k splits by the inclusion.

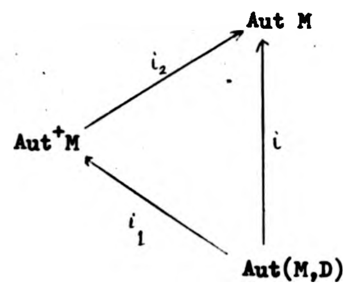
□.

Remark:

The deformation fails to be an isotopy of one space into another because of slides of the type described in pictures(III) of pages 41 and 42.

III Comparing Aut(M,D) with Aut M

There is a natural map Aut(M,D) $\longrightarrow$ Aut(M) which, in the orientable case, factors through two other maps induced by inclusions



By the disc theorem i_1 is onto in the orientable case, i is onto in the non orientable case. A natural question is to ask if, these homomorphisms are into i.e. if $Aut^+(M) = Aut(M,D)$, in the orientable case, or if $Aut\ M = Aut(M,D)$ in the non orientable case.

We first consider the particular cases where M is a 2 sphere bundle over S^1 and show that the answer is affirmative. Then we consider the general case and reduce the problem to one involving exact sequences of two fibrations.

(1) M is a 2-sphere bundle over S^1

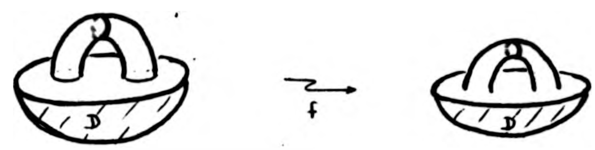
THEOREM 3.4

$$Aut\ (S^1 \times S^2, D) = Aut^+(S^1 \times S^2).$$

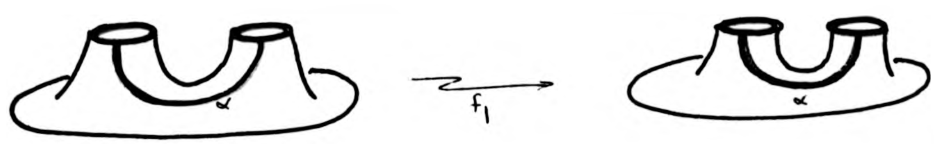
Proof:

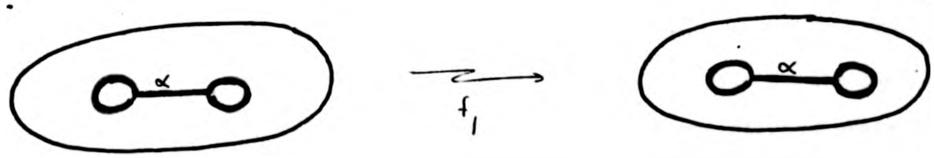
$i_1: Aut(S^1 \times S^2, D) \longrightarrow Aut^+(S^1 \times S^2)$ is onto by the disc theorem. We show $Ker\ i_1 = \{id\}_D$.

Let $[f]_D \in Aut(S^1 \times S^2, D)$ s.t $f \sim id$ (~ mean isotopic to)



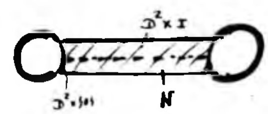
By the same sort of arguments as in theorem 3.2 we can assume rel D, that f sends the belt sphere to the belt sphere. As the inclusion $\overline{S^1 \times S^2 - D} \hookrightarrow S^1 \times S^2$ induces an isomorphism on H_1 and H_2 and $f \sim id$, $f/$belt sphere : $S^2 \longrightarrow S^2$ is a degree one map hence isotopic to the id. Extend the isotopy to a D-isotopy of $S^1 \times S^2$. Then cutting $S^1 \times S^2$ along the belt sphere and along D we can think of f as a map from a 3-disc B with two holes into itself being the identity on boundary components.



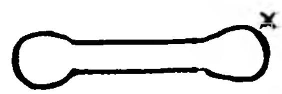


Now as in Gluck's proof (Theorem 2.1) we can assume, rel ∂B, that f is the identity on an arc α between the two holes (see pictures above) and in the neighbourhood of that arc it is τ or id.

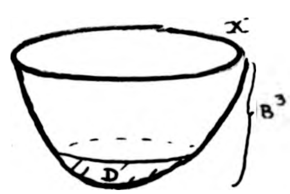
Denote by N a regular neighbourhood of α in B. N is homeomorphic to $D_1^2 \times I$ and we can assume that N only meets ∂B in $D_1^2 \times \{0\} \cup D_1^2 \times \{1\}$ where D_1^2 a disc in the (previous) belt sphere.



Let X be the 2-sphere obtained by removing these 2-discs $D_1^2 \times \{0\}$, $D_1^2 \times \{1\}$ from the two sphere components of ∂B different from ∂D, and adding $\partial D_1^2 \times I$ along their common boundary (see picture).

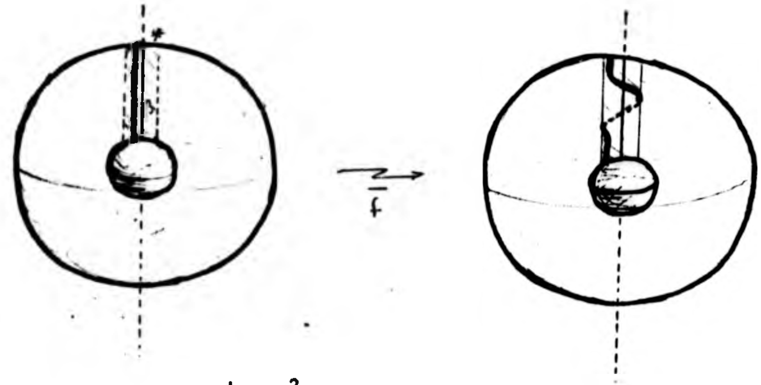


f/X extends uniquely up to isotopy to the 3 ball B^3 bounded by X on $S^1 \times S^2$.



By the disc theorem there is an isotopy rel ∂B^3 s.t the extension is the identity on D i.e. the isotopy class of f in $Aut^+ S^1 \times S^2$ is determined by f/X. But by hypothesis f is isotopic to the identity. Hence f/X = id, up to isotopy. (note that f/X is orientation preserving).

Thus, up to D-isotopy, f defines a map $\bar{f} : I \times S^2 \longrightarrow I \times S^2$ which is the identity on the boundary, $\bar{f}$ being orientation preserving.

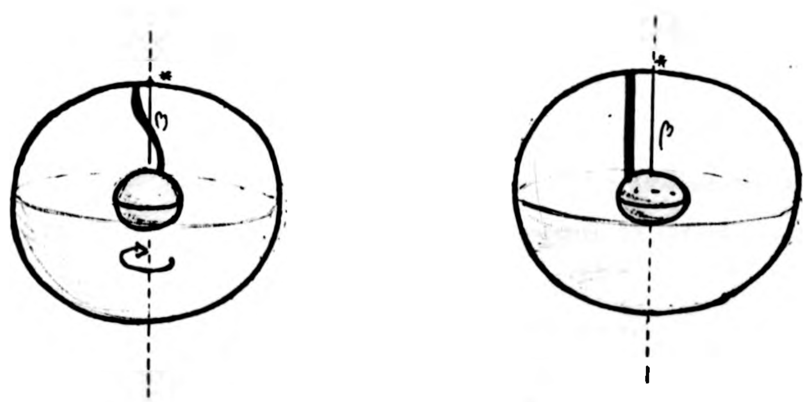


But by Gluck [8] $Aut^+(I\times S^2,\partial) = \mathbb{Z}_2$ and f is either τ or id in a neighbourhood of an arc $\beta = I\times\{*\}$ $(*\in S^2)$.

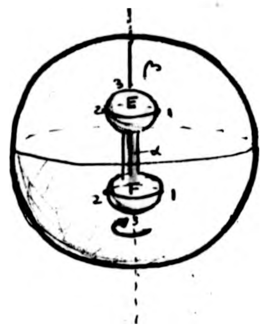
If it is the identity we are done because extending the isotopies trivially to $S^1\times S^2$ defines a D-isotopy of f to the identity.

If not, we know that f restricted to a neighbourhood of the arc β twists a parallel curve once around it and its isotopy class is completely determined by this fact.

Now rotating the inside sphere (see pictures) once around its axis (through *) has the effect of undoing the twist.



It is clear that we can always choose α,β s.t. we can think of $\overline{S^1\times S^2 - D}$ as the space obtained by identifying the two spheres E,F as shown in the following picture.



Numbers in the inside sphere show how to identify E,F to get $\overline{S^1 \times S^2 - D}$. As the rotation inside is compatible with the identifications and the isotopy in the identity on ∂D^3, the isotopy of $I \times S^2$, rel ∂, defines an isotopy of $S^1 \times S^2$, rel D.

Hence $f \underset{D}{\sim} id$ as required. This i_1 is an isomorphism and

$$Aut(S^1 \times S^2, D) = Aut^+(S^1 \times S^2).$$ □

THEOREM 3.5

$$Aut(S^1 \underset{\sim}{\times} S^2, D) = Aut(S^1 \underset{\sim}{\times} S^2).$$

Proof:

As already said it remains to prove that Ker i = $\{id_D\}$. Let $[f]_D \in Aut(S^1 \underset{\sim}{\times} S^2, D)$ s.t. f~id. Again as in the previous theorems we can assume, rel D, that f sends the belt sphere to the belt sphere. By homology arguments f/belt sphere : $S^2 \longrightarrow S^2$ is isotopic to the identity. But in $S^1 \underset{\sim}{\times} S^2$ the antipodal map on $\{0\} \times S^2$ is isotopic to the identity on $\{0\} \times S^2$. That is not the case in $\sim_D$ for if we cut $S^1 \underset{\sim}{\times} S^2$ along S^2 we would get a homeomorphism of $I \times S^2$ into itself which was orientation reversing and the identity on a disc, which is **absurd** as $I \times S^2$ in orientable. Hence f/belt sphere is D-isotopic to the identity.

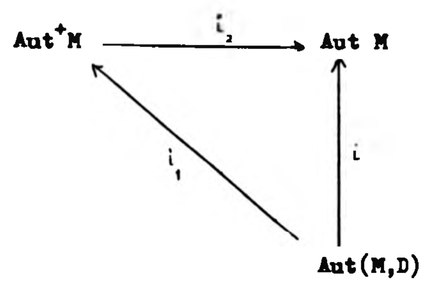
Remaining of the proof is an in the orientable case.

(2) The general case

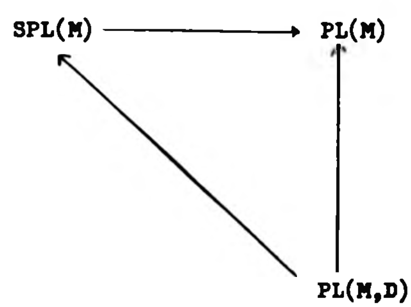
As already mentioned the problem will be reduced to a problem of exact sequences of certain fibrations. We consider separately the orientable and the non orientable cases.

(a) Orientable case

Let M be an orientable 3 manifold, D a 3 disc in int M. We have already considered the commutative diagram



where i, i_1, i_2 are the natural maps and i_1 is onto by the disc theorem. Corresponding diagram without considering isotopies is



as

$$\text{Aut } M = \pi_0(PL(M))$$

$$\text{Aut}(M,D) = \pi_0(PL(M,D))$$

$$\text{Aut}^+M = \pi_0(SPL(M,D))$$

The natural map $SPL(M) \longrightarrow S\,Emb(D,M)$ defined by restriction is a fibration (cf[22]) with fibre $PL(M,D)$. Hence we get a long exact sequence of homotopy groups:

$$\dots \xrightarrow{\partial} \pi_n PL(M,D) \longrightarrow \pi_n SPL(M) \dashrightarrow \pi_n(S\,Emb(D,M)) \longrightarrow \dots \quad (1)$$

ending up with

$$\longrightarrow \pi_1\; S\,Emb(D,M) \xrightarrow{\partial} Aut(M,D) \xrightarrow{i_*} Aut^+M \longrightarrow 0$$

as $\pi_0(S\,Emb\,(D,M)) = 0$ by the disc theorem. We also have another fibration (cf [22])

$$SPL_3 \overset{j}{\hookrightarrow} SEmb(D,M) \xrightarrow{f} M \quad (2)$$

coming from the fibration

$$S\,Emb_*(D,M) \longrightarrow S\,Emb(D,M) \xrightarrow{\bar{f}} Emb(*,M)$$

where $\bar{f}$ is the map that associates to an embedding of D in M the image of the centre point of the disc and using the facts that $M \simeq Emb(*,M)$ and as M is a 3-manifold, $S\,Emb_*(D,M) \simeq S\,Emb_*(D,\mathbb{R}^3) \simeq SPL_3$.

From the fibration we get a long exact sequence of homotopy groups

$$\dots \longrightarrow \pi_n SPL_3 \xrightarrow{j_*} \pi_n S\,Emb(D,M) \xrightarrow{f_*} \pi_n M \longrightarrow \dots$$

Actually we can show more as lemma 3.1 says: A parallelization of M^3 gives a section $s : M \longrightarrow SEmb(D,M)$ of the fibration $SEmb(D,M) \xrightarrow{f} M$.

Thus, for all n, the sequence

$$0 \longrightarrow \pi_n\; SPL_3 \xrightarrow{j_*} \pi_n\; SEmb(D,M) \underset{s_*}{\overset{f_*}{\rightleftarrows}} \pi_n M \longrightarrow 0 \quad (3)$$

is exact and splits. Therefore

$$\pi_n\; SEmb(D,M) = \pi_n SPL_3 \times \pi_n M \quad (4)$$

Replacing in (1) we get:

$$\cdots \longrightarrow \pi_n(M)\times\pi_n SPL_3 \longrightarrow \pi_{n-1}PL(M,D) \longrightarrow \pi_{n-1}(SPL(M)) \longrightarrow \cdots \qquad (5)$$

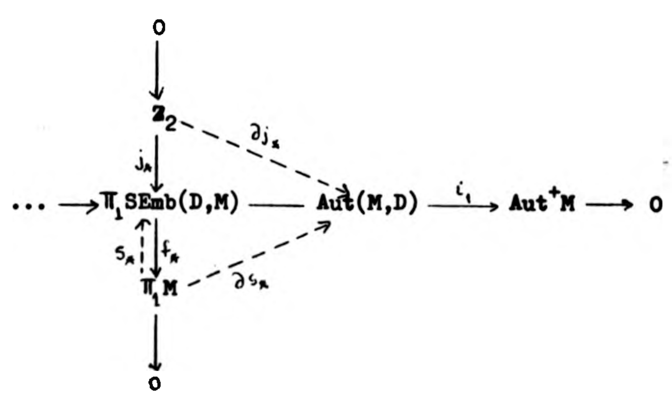
In particular for n=1 as $\pi_1 SPL_3 = \mathbb{Z}_2$ ($\pi_i SPL_3 = \pi_i SO_3$ for $i\leq 3$), we have

$$\cdots \longrightarrow \pi_1 M\times\mathbb{Z}_2 \xrightarrow{\partial} Aut(M,D) \xrightarrow{i_1} Aut^+ M \longrightarrow 0 \qquad (6)$$

We already know that i_1 is onto. Our problem was to know Ker i_1. From this sequence Ker i_1 = im ∂ by exactness. We look at im ∂.

A geometric interpretation of ∂

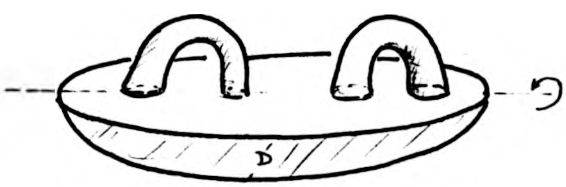
A loop in S Emb(D,M) is an isotopy of embeddings h of a disc in M starting and ending with the inclusion. Extend the isotopy to an isotopy H_t of M. Then $\partial([h]) = [H_1]_D$. From lemmas 3.2, 3.3, ∂ is well defined and depends only on the homotopy class of the loop and on an element of $\mathbb{Z}_2$ which is precisely what says the following diagram obtained by combining sequences (1) and (3).



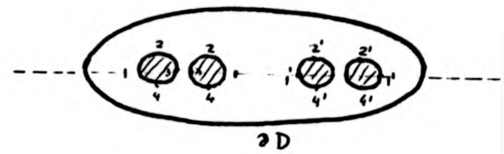
∂ is completely determined by ∂s_* and ∂j_*. The first one is the canonical automorphism determined by sliding a disc around a loop, the latter is the one determined by rotating the disc once around its axis.

Some particular cases

1. For $M = \#_n S^1\times S^2$, im $\partial_* = 0$



(I)



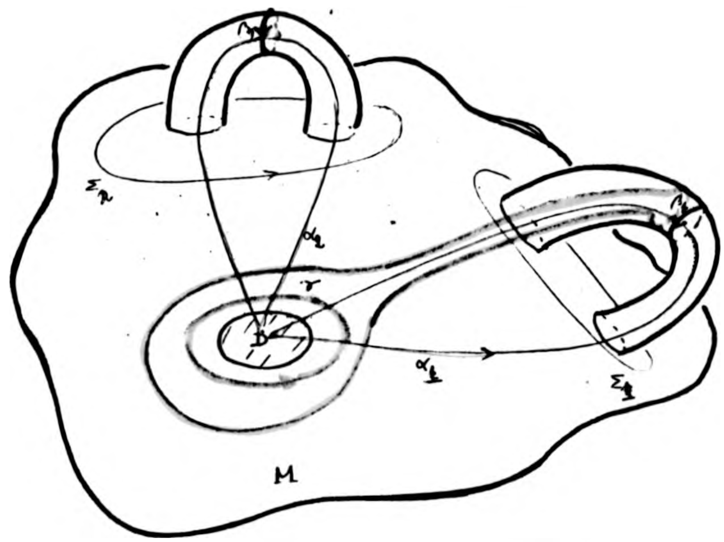
(II)

Consider $\#_n S^1\times S^2$ as obtained by attaching the handles to S^3 ($=D^3_+ \cup_\partial D^3_-$) ($D^3_- = D$) along the symmetry axis of D (see picture I). We can also think of $\#_n S^1\times S^2$ as in picture II where the boundary of the holes are to be identified as shown. But now we can easily see that the rotation of the disc once around its axis is compatible with the identifications hence it extends to an ambient isotopy of $\#_n S^1\times S^2$. Thus $\mathrm{im}\,\partial j_* = 0$ and $\mathrm{im}\,\partial = \mathrm{im}\,\partial_*$

2. For $M = \#_n S^1\times S^2$, $n>1$, im $\partial_* \neq 0$

Let $\tilde M$ denote M with the interior of D removed. $\pi_1 M$ is a free product of n copies of $\mathbb{Z}$ and is generated by the loops that go once around each handle. Assume the base point is some point in D, for instance its centre point, and denote by $\alpha_1,\dots,\alpha_n$ the generators of π_1.

Generators for $\pi_2\tilde M$ are the belt spheres of the handles $\beta_1\dots\beta_n$ and a sphere γ parallel to ∂D in $\tilde M$ (we confound homotopy classes with their representatives if no confusion arises). If $\Sigma_1,\dots,\Sigma_n$ denote the homotopy classes of the separating spheres, γ is clearly isotopic in $\tilde M$ to $\Sigma_1+\dots+\Sigma_n$.

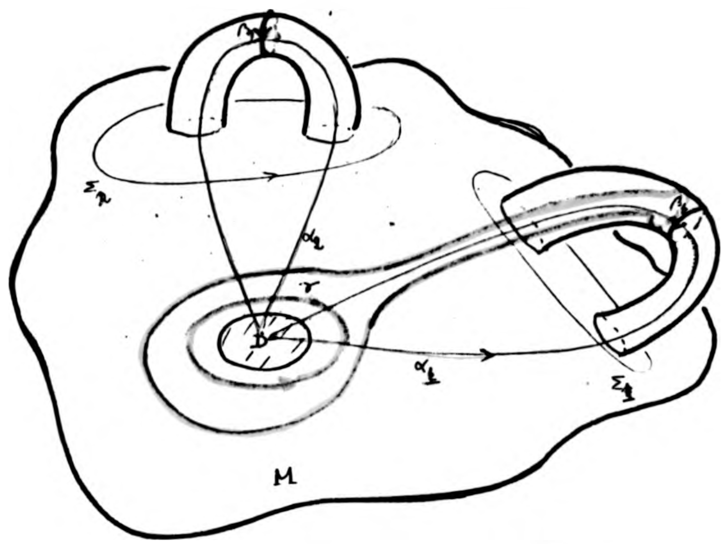


Note that each Σ_i is homologous to zero in $\tilde{M}$ but <u>not isotopic</u> to zero. Similarly for γ.

Now suppose we have dragged D once around α_1, say. Let $H_1^{\alpha_1}$ be the corresponding element in im ∂s_*. $H_1^{\alpha_1}(\beta_1)$ is clearly isotopic (rel D) to $\beta_1 + \gamma$ (see picture). But $\beta_1 + \gamma$ is not isotopic to γ (as $\gamma \not\sim 0$) in $\tilde{M}$, hence $H_1^{\alpha_1}$ cannot be isotopic to the id, rel D. Thus im $\partial s_* \neq 0$ as required. □

Remark

Note that the proof doesn't follow for n=1 (a fact already known as we proved that $Aut(S^1 \times S^2, D) = Aut^+(S^1 \times S^2)$ as $\beta_1 + \gamma \sim \beta_1$.



Note that each Σ_i is homologous to zero in $\tilde{M}$ but not isotopic to zero. Similarly for γ.

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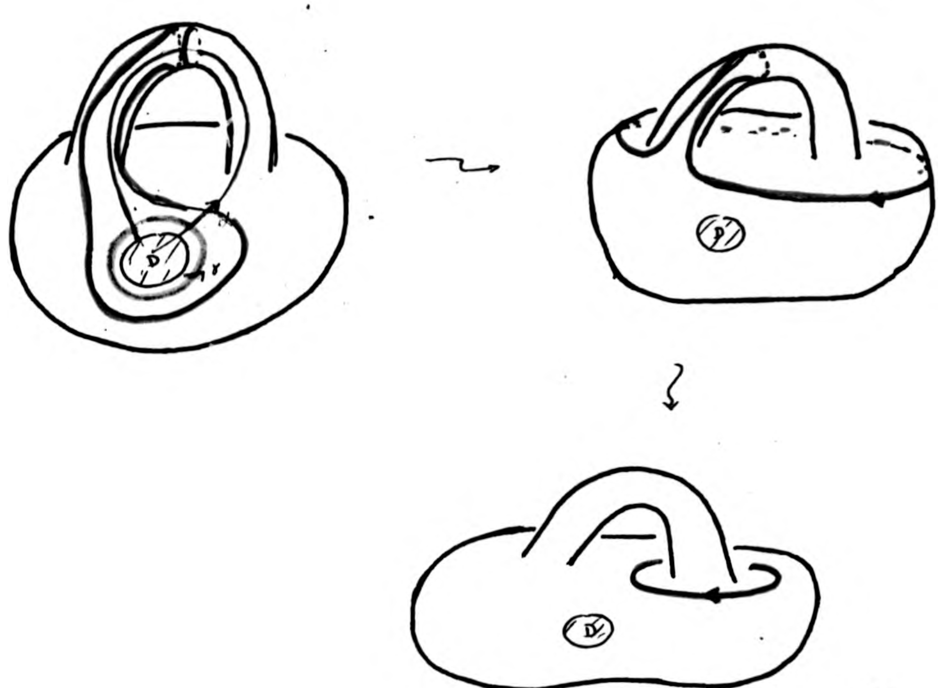
□

Remark

Note that the proof doesn't follow for n=1 (a fact already known as we proved that $Aut(S^1 \times S^2, D) = Aut^+(S^1 \times S^2)$ as $\beta_1 + \gamma \sim \beta_1$.



□

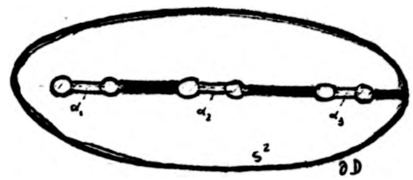
From 1, 2. we get

THEOREM 3.6

$\mathrm{Aut}(\#_n S^1\times S^2, D) \neq \mathrm{Aut}^+(\#_n S^1\times S^2)$, $n>1$.

3. It follows from Theorem 3.2 that a homeomorphism in $\times_n \mathrm{Aut}(S^1\times S^2, D)$ which we denote by $\phi_1\# \dots \#\phi_n$ is D-isotopic to the identity iff ϕ_i is isotopic to the identity, $i=1,\dots,n$.

Consider homeomorphisms of this form with $\phi_i = \varepsilon$ or id. Is such a homeomorphism isotopic to the id? As they induce the id on homology we can think of them as homeomorphism of a disc with holes being the identity on boundary components (cf th 3.4). Run arcs α_i between the corresponding holes and as in theorem 3.4 the homeomorphism in the



neighbourhood of the arcs is τ or id (in the D-isotopy class). Then make (as in th 3.4) the homeomorphism to be the identity on the shaded region (we are still in the D-isotopy class which is now completely determined as what is left in a ball - i.e. the D-isotopy class of the homeomorphism is completely determined by the restriction of the homeomorphism to the regular neighbourhoods of the arcs α_i)

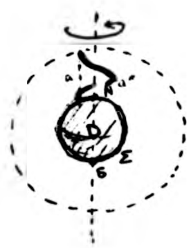
If f is such a homeomorphism with at least one $\phi_i = \tau$ (i.e. $[f]_D \neq [id]_D$) then as clearly $[f]_D \notin$ im ∂s_*, <u>f is not isotopic to the id.</u>

6. <u>THEOREM 3.7</u>

<u>For M orientable ·irreducible sufficiently large Aut(M,D) $\neq$ Aut^+M.</u>

<u>Proof</u>

We show that $\partial j_* \neq 0$. Let D be the disc, ϕ the automorphism obtained



by rotating the disc once around its axis. Let $\Sigma = \partial D$ and let $U = \Sigma\times[0,1]$ be a collar on $\Sigma(\Sigma=\Sigma\times\{0\})$. Now the effect on U of the rotation is to rotate $\{*\}\times[0,1]$ where $*$ is any point in Σ different from the poles, once around the axis keeping end points fixed, i.e. we get τ (cf theorem 2.1) on the collar. But Hendriks [12] showed that for P^2-irreducible sufficiently large manifolds (orientable or not) this is never D-isotopic to the id. Hence $\partial j_* \neq 0$. □

<u>Remarks</u>

(1) We shall be able to prove Hendriks result quoted above later on.

(2) Hendriks calls τ <u>a rotation parallel to a sphere</u> (Σ in this case).

(b) The non orientable case

If M is non orientable in order to study the map $Aut(M,D) \xrightarrow{i} Aut\ M$ we consider, similarly to the orientable case, two fibrations, but in this case it makes no sense to talk about orientation preserving homeomorphisms. Corresponding fibrations are:

$$PL(M,D) \hookrightarrow PLM \longrightarrow Emb(D,M) \qquad (7)$$

$$PL_3 \xrightarrow{j} Emb(D,M) \xrightarrow{f} M \qquad (8)$$

given long exact sequences of homotopy groups:

$$\dots \longrightarrow \pi_n(Emb(D,M)) \xrightarrow{\partial} \pi_{n-1}(PL(M,D)) \xrightarrow{i_*} \pi_{n-1}(PL(M)) \longrightarrow \dots \qquad (9)$$

$$\dots \longrightarrow \pi_n PL_3 \xrightarrow{j_*} \pi_n(Emb(D,M)) \xrightarrow{f_*} \pi_n M \longrightarrow \dots \qquad (10)$$

Consider the **first sequence:**

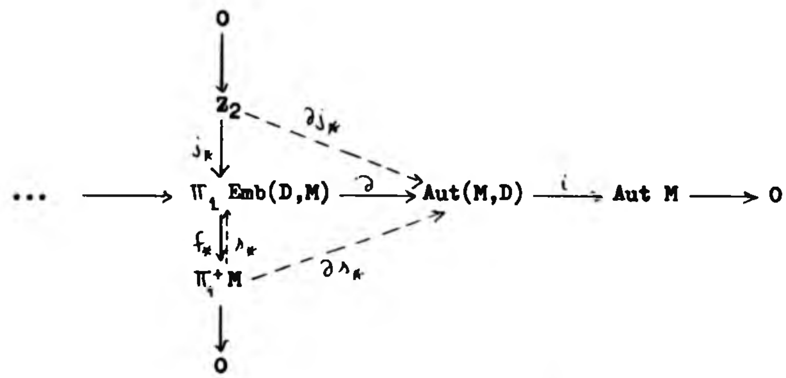
As $\pi_0(Emb(D,M)) = 0$ by the disc theorem and from the fact that M is non orientable we get:

$$\dots \longrightarrow \pi_1 Emb(D,M) \xrightarrow{\partial} Aut(M,D) \xrightarrow{i} Aut\ M \longrightarrow 0 \qquad (11)$$

For the second sequence as $\pi_0 PL_3 = \mathbb{Z}_2$ we have

$$\dots \longrightarrow \mathbb{Z}_2 \xrightarrow{j_*} \pi_1(Emb(D,M)) \xrightarrow{f_*} \pi_1 M \xrightarrow{\delta} \mathbb{Z}_2 \longrightarrow 0 \qquad (12)$$

where δ is the orientation homomorphism. Let $\pi_1{}^+M = Ker\ \delta$. $\pi_1{}^+M$ is the subgroup of $\pi_1 M$ of homotopy classes of orientable loops. Then $f_* : \pi_1(Emb(D,M)) \longrightarrow \pi_1{}^+M$ **is onto and splits by corollary 3.1. Thus we get the following commutative diagram:**

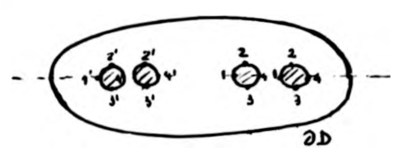


Keri = im ∂ is completely determined by im ∂j_*, im ∂s_*

Particular cases

1. For $M = \#_n S^1 \tilde{\times} S^2$ im $\partial j_* = 0$

   Proof is essentially the same as in orientable case



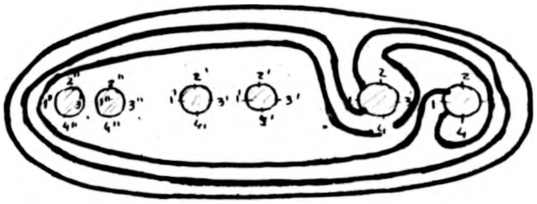
Boundary of the holes are identified as shown. The identification is compatible with the rotation of the disc once around its axis. □

2. For $M = \#_n S^1 \tilde{\times} S^2$, $n>1$, im $\partial s_* \neq 0$

   Proof is a little bit different as the α_i are non orientable.

   For simplicity we use $2\alpha_1$ instead.

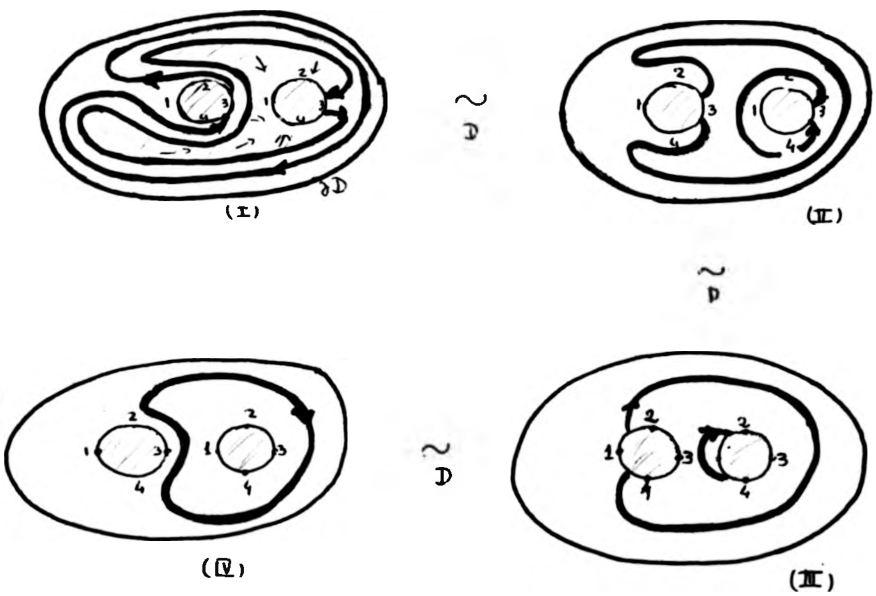
   Using the same notation as in the orientable case we see that $H_1^{2\alpha_1}(\beta_1) \sim \beta_1 + 2\gamma$ which is not D-isotopic to β_1 for $n>1$. □



∂D

<u>Remark</u>

For <u>n=1</u> this is not the case



□

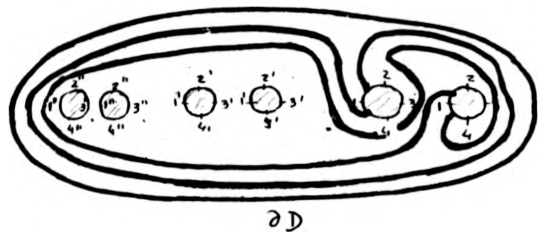
Hence we get

<u>THEOREM 3.8</u>

$$\mathrm{Aut}(\#_n S^1 \tilde{\times} S^2, D) \neq \mathrm{Aut}(\#_n S^1 \tilde{\times} S^2) \quad n>1.$$

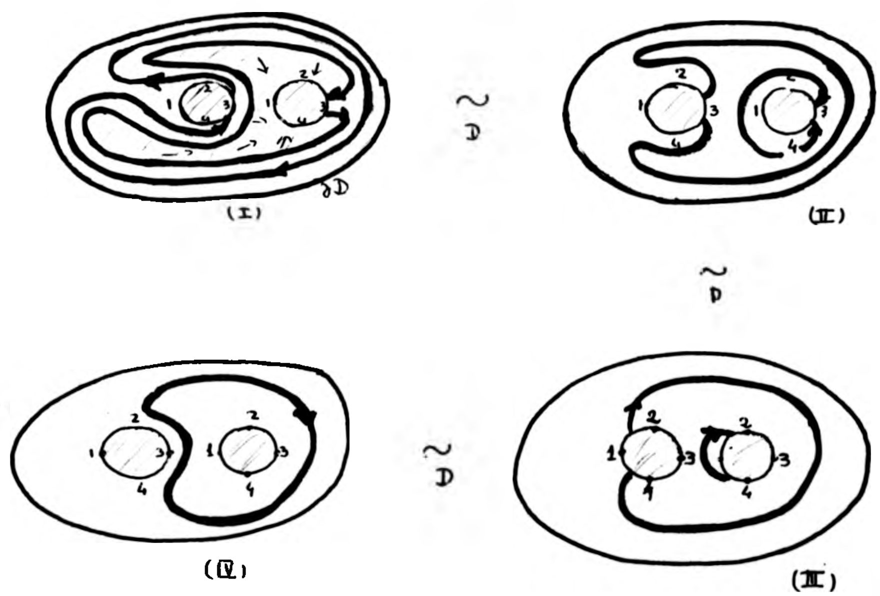
---

3. Again from Hendriks [12] we get the following.



Remark

For n=1 this is not the case



□

Hence we get

THEOREM 3.8

$$\mathrm{Aut}(\#_n S^1 \tilde{\times} S^2, D) \neq \mathrm{Aut}(\#_n S^1 \tilde{\times} S^2) \quad n>1.$$

3. Again from Hendriks [12] we get the following:

THEOREM 3.9

For any non orientable P^2-irreducible sufficiently large 3 manifold M, $\mathrm{Aut}(M,D) \neq \mathrm{Aut}(M)$.

□

IV Some results for P^2-irreducible sufficiently large 3-manifolds

Let M be a P^2-irreducible sufficiently large 3-manifold. Hatcher showed [10] that if $\partial M=\emptyset$, $\pi_k PL(M)=0$ for $k\geq 2$, and if $\partial M\neq\emptyset$, $\pi_k PL(M,D)=0$ for $k\geq 1$. Also as M is a $K(\pi,1)$, $\pi=\pi_1 M$, we have for $k\geq 2$, $\pi_k M=0$.

Suppose for simplicity that M is closed and consider the exact sequence

$$\ldots \longrightarrow \pi_n M \oplus \pi_n PL_3 \xrightarrow{\partial} \pi_{n-1} PL(M,D) \longrightarrow \pi_{n-1} PL(M) \quad \ldots \quad (n>1)$$

(π_n is abelian for $n>1$, hence we have direct sums).

For manifolds with boundary consider homeomorphisms keeping the boundary fixed. Also consider $n>1$ so that orientation has no effect on the sequence.

Then from the above we get

$$\pi_{n+1} PL_3 \cong \pi_n PL(M,D) \quad \text{for} \quad n\geq 2 \qquad (13)$$

We now try to calculate $\pi_n PL(M,D)$ for $n<2$. For simplicity of notation we consider M orientable and without boundary (see remark above). The results follow for the other cases with minor changes.

Denote by $G(M)$ the (simplicial) space of homotopy equivalences of M. $G(M,x_0)$ is the subspace of the ones that fix a point x_0 in M. If M has boundary then $G(M,\partial M)$ denotes the space of homotopy equivalences of M which restrict to the identity on ∂M. Since M is a $K(\pi,1)$ ($\pi=\pi_1 M$) it is an easy consequence of obstruction theory to determine the homotopy type of $G(M)$, $G(M,x_0)$ at least when $\partial M=\emptyset$. One finds that

$$\pi_0 G(M,x_0) = \text{Aut}\,\pi_1(M,x_0)$$

(we write $\pi_1 M$ for $\pi_1(M,x_0)$).

$$\begin{aligned}\pi_0 G(M) &= \text{Aut}\,\pi_1 M \,/\text{inner automorphisms of } \pi_1 M\\ &= \text{Out}\,\pi_1 M \text{, the outer automorphism group of } \pi_1 M.\end{aligned}$$

$$\pi_1 G(M) = \text{centre of } \pi_1 M.$$

and the higher homotopy groups vanish.

Hatcher [10] proved that the inclusion $i: PL(M) \longrightarrow G(M)$ (or $PL(M,\partial M) \longrightarrow G(M,\partial M)$) is a homotopy equivalence. It follows then that $i': PL(M,x_0) \longrightarrow G(M,x_0)$ is also a homotopy equivalence.

(Consider the commutative diagram

$$\begin{array}{ccccc} G(M,x_0) & \longrightarrow & G(M) & \xrightarrow{f} & M \\ \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle id} \\ PL(M,x_0) & \longrightarrow & PL(M) & \xrightarrow{f} & M \end{array}$$

where i, i' are the inclusions, f is the restriction to a point x_0 in M and the horizontal lines are fibrations. Then $i'_* : \pi_k\, PL(M,x_0) \longrightarrow \pi_k\, G(M,x_0)$ is an isomorphism by the five lemma).

Replacing in the results above we get

$$\begin{aligned} \pi_0\, PL(M,x_0) &= \text{Aut}\, \pi_1 M \\ \pi_0\, PL(M) &= \text{Out}\, \pi_1 M \\ \pi_1 PL(M) &= \text{centre of}\, \pi_1 M \end{aligned}$$

Now consider again the fibration

$$PL(M,x_0) \longrightarrow PL(M) \xrightarrow{f} M$$

We get a long exact sequence

$$\ldots \underset{\substack{\shortparallel \\ 0}}{\pi_2 M} \longrightarrow \pi_1 PL(M,x_0) \longrightarrow \pi_1 PL(M) \xrightarrow{f_*} \pi_1 M \longrightarrow \pi_0 PL(M,x_0) \longrightarrow \pi_0 PL(M) \longrightarrow \pi_0 M \qquad (14)$$

which we can write as

$$\ldots\ 0 \longrightarrow \pi_1 PL(M,x_0) \longrightarrow \text{centre of}\, \pi_1 M \longrightarrow \pi_1 M \longrightarrow \text{Aut}\, \pi_1 M \xrightarrow{p} \text{Out}\, \pi_1 M \longrightarrow \pi_0 M \qquad (15)$$

The map centre of $\pi_1 M \longrightarrow \pi_1 M$ is, in fact, the inclusion. This follows from the definitions of f and of the isomorphism centre $\pi_1 M \cong \pi_1 PL(M)$. Also $p: \text{Aut}\, \pi_1 M \longrightarrow \text{Out}\, \pi_1 M$ is onto as it is the quotient map. Again this follows from the definitions of the isomorphisms $\pi_0 PL(M,x_0) = \text{Aut}\, \pi_1 M$ and $\pi_0 PL(M) = \text{Out}\, \pi_1 M$.

Hence we get

$$\pi_1 PL(M,x_0) = 0$$

$$0 \longrightarrow \text{centre of } \pi_1 M \longrightarrow \pi_1 M \longrightarrow \text{Aut } \pi_1 M \longrightarrow \text{Out}\,\pi_1 M \longrightarrow 0 \quad \text{is exact.} \tag{16}$$

Also as for $k\geq 2$, $\pi_k M = 0$

$$\pi_k PL(M,x_o) = \pi_k PL(M) = 0 \tag{17}$$

We also have another fibration

$$PL(M,D) \longrightarrow PL(M,x_0) \xrightarrow{f} \text{Emb}_{x_0}(D,M) \tag{18}$$

giving long exact sequence of homotopy groups:

$$\cdots \pi_{k+1} PL_3 \longrightarrow \pi_k PL(M,D) \longrightarrow \pi_k PL(M,x_0) \longrightarrow \pi_k PL_3 \longrightarrow \cdots \tag{19}$$

as $\text{Emb}_{x_o}(D,M) \simeq PL_3$ (cf [22]) .Hence for $k<2$ we get

$$\longrightarrow \underset{0}{\pi_2 PL_3} \longrightarrow \pi_1 PL(M,D) \longrightarrow \underset{0}{\pi_1 PL(M,x_0)} \longrightarrow \underset{Z_2}{\pi_1 PL_3} \longrightarrow \pi_0 PL(M,D) \longrightarrow \pi_0 PL(M,x_0) \xrightarrow{w} \underset{Z_2}{\pi_0 PL_3}$$

Thus,

$$\pi_1 PL(M,D) = 0$$

$$0 \longrightarrow Z_2 \xrightarrow{j} \pi_0 PL(M,D) \xrightarrow{p} \pi_0 SPL(M,x_0) \longrightarrow 0 \quad \text{is exact} \tag{20}$$

where $\pi_0 SPL(M,x_o) = \text{Ker } w$ (cf remark (1) on pag 58).

As $\pi_0 PL(M,x_0) \simeq \text{Aut}\,\pi_1 M$ we identify Ker w with a subgroup of $\text{Aut}\,\pi_1 M$ which we shall note by $\text{Aut}^+\pi_1 M$ .In this particular case (i.e. M closed orientable) $\text{Aut}^+\pi_1 M = \pi_0 SPL(M,x_0)$ (21).

Summarizing all the results we get for M closed ,orientable, P^2-irreducible sufficiently large:

$$0 \longrightarrow Z_2 \longrightarrow \pi_0 PL(M,D) \longrightarrow \text{Aut}^+\pi_1 M \longrightarrow 0 \quad \text{is exact} \tag{22}$$

and as $\pi_2 PL_3 = 0 = \pi_1 PL(M,D)$,

$$\pi_n PL(M,D) = \pi_{n+1} PL_3 \quad \text{for } n\geq 1$$

The following table compares the homotopy groups of $PL(M,D)$, $PL(M,x_0)$ and $PL(M)$:

	PL(M,D)	PL(M,x₀)	PL(M)
π_0	$0 \to Z_2 \to \pi_0 PL(M,D) \to Aut^+\pi_1 M \to 0$ is exact	$Aut\pi_1 M$	$Out\pi_1 M$
π_1	0	0	Centre of $\pi_1 M$
π_2	Z	0	0
π_3	$\pi_4 PL_3$	0	0
⋮	⋮	⋮	⋮
π_n	$\pi_{n+1} PL_3$	0	0
⋮	⋮	⋮	⋮

<u>Remarks:</u>

(1) For M non-orientable the only difference is that (21) does not hold (and in other places replace SPL by PL). $Aut^+\pi_1 M$ will consist of the automorphisms which respect the orientation homomorphism.

(2) For M with boundary we consider homeomorphisms fixed on the boundary and as $\pi_n PL(M,\partial M) = 0$ for $n\geq 1$, we get

$$\pi_n PL(M,D\cup\partial M) = \pi_{n+1} PL_3 \quad \text{for } n\geq 1$$

$$0 \longrightarrow Z_2 \longrightarrow \pi_0 PL(M,D\cup\partial M) \longrightarrow \pi_0\, PL(M,x_0\cup\partial M) \longrightarrow 0 \quad \text{is exact.}$$

V – Comparing Aut(M,D) with Aut π_1(M)

Laudenbach [17] has shown that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_0 SPL(\#_p S^1\times S^2,\ x_0) \xrightarrow{\pi} Aut(*_p \mathbb{Z}) \longrightarrow 0$$

We first show that this result follows from theorem 3.2 and then give a generalization for connected sums of $S^1\times S^2$'s with P^2-irreducible sufficiently large 3 manifolds.

For simplicity we consider M closed orientable and try to compare Aut(M,D) with Aut π_1M. (In the non orientable case we have to consider only the subgroup of Aut π_1M of automorphisms which respect the orientation homomorphism. We then obtain the same results).

Remark

$Aut^+(\#_p S^1\times S^2,\ x_0) = Aut(\#_p S^1\times S^2, D)$ (cf sequence (6) and the result obtained for $\#_p S^1\times S^2$ – i.e. $\partial j_* = 0$).

(i) The automorphism group of a free product

We start by looking at Aut π_1M. As $M = \#_n P_i$ where all P_i are prime. $\pi_1 M$ is a free product $G = A_1 * \ldots * A_m * A_{m+1} * \ldots * A_n$ where each A_i is irreducible and $A_1 \ldots A_m$ are infinite cyclic.

This group has been studied by Fuchs-Rabinovitch [6] [7]. The particular case where all the groups are infinite cyclic has been studied by Nielsen [20]. They give a system of generators and relations for the group.

Let $a_i^{(k)} \in A_i$, $i=1,\ldots,n$, denote a typical element. Let $A_1 = \{a_1\} \ldots A_m = \{a_m\}$ so that $a_i^{(k)} = a_i^k$ for $i \le m$.

<u>AutG is generated as follows:</u>

(a) $\emptyset_i$ the automorphism group of each A_i – we have automorphisms ϕ_i s.t $\phi_i a_i^{(k)} = \overline{\phi}_i a_i^{(k)}$, $\phi_i a_j^{(k)} = a_j^{(k)}$ $j \neq i$, $\overline{\phi_i} \in \emptyset_i$.

(b) For each ordered pair (i,j) $i \neq j$, $j > m$, $1 \leq i,j \leq n$, we have a group of automorphisms isomorphic to A_i given by conjugation of A_j by A_i, fixing the other factors. If

$$a_i^{(k)} \in A_i \qquad \alpha_{ij}^{(k)} a_\ell^{(m)} = a_\ell^{(m)} \quad (\ell \neq j)$$

$$\alpha_{ij}^{(k)} a_j^{(m)} = a_i^{(k)-1} a_j^{(m)} a_i^{(k)}$$

(c) For $i \neq j$, $j \leq m$, $1 \leq i \leq n$, we have the automorphisms

$$\beta_{ij}^{(k)} = \begin{cases} \beta_{ij}^{(k)} a_\ell^{(p)} = a_\ell^{(p)} & \ell \neq j \\ \beta_{ij}^{(k)} a_j^{(p)} = a_j^{(p)} a_i^{(k)} & \end{cases}$$

$$\gamma_{ij}^{(k)} = \begin{cases} \gamma_{ij}^{(k)} a_\ell^{(p)} = a_\ell^{(p)}, & \ell \neq j \\ \gamma_{ij}^{(k)} a_j^{(p)} = a_i^{(k)} a_j^{(p)} & \end{cases}$$

(d) Split the indices $1,2 \ldots n$ into blocks $I_1, I_2, \ldots, I_t$ where for all $i \in I_j$, the A_i are isomorphic (For example $I_1 = \{1 \ldots m\}$ corresponds to the infinite cyclic factors). Then we have the symmetric group on each block as group of automorphisms: If $A_i \cong A_j$ then we have automorphism ω_{ij} defined by

$$\omega_{ij} a_i^{(k)} = a_j^{(k)}$$

$$\omega_{ij} a_j^{(k)} = a_i^{(k)}$$

$$\omega_{ij} a_\ell^{(m)} = a_\ell^{(m)} \quad (\ell \neq i,j)$$

(we suppose that the isomorphism $A_i \longrightarrow A_j$ is given by $a_i^{(k)} \longleftrightarrow a_j^{(k)}$).

The group is thus generated by <u>permutations (d), proper automorphisms of the components</u> (a), <u>elementary conjugations</u> (b) and <u>Nielsen</u>

<u>transformations</u> (c).

For a set of defining relations see [7]. For $i \leq m$ the group of automorphisms ϕ_i are known. They are groups of order 2, generated by elements σ_i which take a_i into a_i^{-1}.

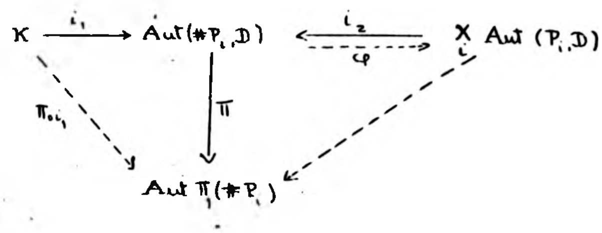
(ii) Consider the homomorphism

$$\mathrm{Aut}(M,D) \xrightarrow{\pi} \mathrm{Aut}\ \pi_1 M$$

that associates to a homeomorphism the induced automorphism on π_1. (Centre point of D is the base point). We now show that in certain cases π is onto i.e. that we can realise all the automorphisms of π_1 by automorphisms fixing a disc.

We first remark that the automorphisms (b) (c) (d) correspond respectively to generalised slides when the whole factor is slid along a curve in another factor, generalised slides when the end of a handle is slid along a curve in another factor and permutations of factors (all homeomorphisms can be assumed to be fixed on a disc). Hence to prove π is onto we only have to see if the automorphism groups of the fundamental groups of the factors correspond to homeomorphisms in the manifold. We also know by theorem 3.2 that the homeomorphisms of M are generated by generalised slides, permutations and homeomorphisms of the factors. Hence the problem will be to see if the automorphisms of the fundamental group of the factors can be realised by homeomorphisms of the factors. We shall show that is the case if M is $\#_n S^1 \times S^2$, $n \geq 1$, and that the kernel is a direct sum of $\mathbb{Z}_2$'s (one for each factor) and then give a generalisation for a $\#_i P_i$ or $\#_i P_i \#_n S^1 \times S^2$ where P_i is irreducible sufficiently large.

In general we have a diagram



where K is the semi-direct product of P(M,D) and S(M,D), M = # P_i, i_1, i_2 are the inclusions.

$\pi \circ i_1$ is 1-1. This follows by looking at the relations [8] (all the relations between slides and permutations can be realised geometrically). Hence as together K and $\underset{i}{X}$(Aut P_i,D) generate Aut(#P_i,D) Ker π = Ker $\pi \circ i_2$

We now consider some particular cases:

(1) $M = S^1 \times S^2$

$\pi_1(S^1 \times S^2) = \mathbb{Z}$ and Aut $\pi_1(S^1 \times S^2)$ is generated by taking the generator x to x^{-1}. This is realised by the homeomorphism that interchanges the two ends of the handle (see picture). x can be represented by a loop that goes once around the handle.

Hence π is onto and Aut$(S^1 \times S^2, D) = \mathbb{Z}_2 \oplus$ Aut $\pi_1(S^1 \times S^2)$ as π splits (cf th. 2.1.).

(2) $\underline{M = \#_p S^1\times S^2}$

Again π is onto as we only have to see that the automorphisms defined by

$$\begin{cases} x_k \longmapsto x_k^{-1} \\ x_\ell \longmapsto x_\ell & \ell \neq k \end{cases}$$

are realised by homeomorphisms where $x_1 \ldots x_p$ are the generators of $*_p\, \mathbb{Z}$, (Each one can be represented by the loop that goes once around a handle), and as in case (1) they correspond to interchange the two ends of a handle.

We then have

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Ker}\ \pi & \longrightarrow & \mathrm{Aut}(\#_p S^1\times S^2, D) & \xrightarrow{\pi} & \mathrm{Aut}(*_p\ \mathbb{Z}) \longrightarrow 0 \\ & & \uparrow i_2 & & \\ & & X_p(Z_2\times \mathrm{Aut}\ \pi_1(S^1\times S^2)) & & \end{array}$$

$\mathrm{Ker}\ \pi = \mathrm{Ker}(\pi . i_2)$. But $\mathrm{Aut}\ \pi_1(S^1\times S^2)$ maps injectively into $\mathrm{Aut}(*_p\ \mathbb{Z})$. The $\mathbb{Z}_2$ part corresponds to a rotation parallel to the belt sphere of the handle (τ) which induces the identity on π_1. Hence we have a $\mathbb{Z}_2$ coming from each factor i.e. $\mathrm{Ker}\ \pi = \bigoplus_p \mathbb{Z}_2$. Thus we recover <u>Laudenbach's exact sequence</u>

$$0 \longrightarrow \bigoplus_p Z_2 \longrightarrow \mathrm{Aut}(\#_p S^1\times S^2, D) \xrightarrow{\pi} \mathrm{Aut}(*_p \mathbb{Z}) \longrightarrow 0$$

□

(3) <u>The general case</u>

For an arbitrary 3-manifold $M = \# P_i$ we do not know if $\mathrm{Aut}(M,D)$ maps onto $\mathrm{Aut}\ \pi_1 M$. We have a diagram:

$$\begin{array}{ccccc} K & \xrightarrow{i_1} & \mathrm{Aut}(\#_n P_i, D) & \xleftarrow{i_2} & \times_n \mathrm{Aut}(P_i, D) \\ & & \downarrow \pi & & \\ & & \mathrm{Aut}\,\pi_1(\#_n P_i) & & \end{array}$$

As before $\mathrm{Ker}\,\pi = \mathrm{Ker}\,\pi \circ i_2$ and from (i) and remarks in (ii) we can prove by a similar method to (2) the following:

<u>THEOREM 3.10</u>

Let $M = \#P_i$ and for each P_i let $K_{P_i} \longrightarrow \mathrm{Aut}(P_i,D) \longrightarrow Q_{P_i}$ be the short exact sequence where $Q_{P_i} = \mathrm{im}\,\pi, K_{P_i} = \mathrm{Ker}\,\pi$ $(\pi : \mathrm{Aut}(P_i,D) \longrightarrow \mathrm{Aut}\,\pi_1 P_i)$. Then there is a short exact sequence

$$\times_n K_{P_i} \longrightarrow \mathrm{Aut}(\#_n P_i, D) \longrightarrow G$$

where G is the subgroup of $\mathrm{Aut}\,\pi_1 M$ generated by (b),(c),(d) and by Q_{P_i} in factors. □

We now consider certain cases where we know K_{P_i}, Q_{P_i}.

(4) <u>M is a P^2-irreducible sufficiently large 3-manifold</u>

We already know that we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(M,D) \xrightarrow{\pi} \mathrm{Aut}^+\pi_1 M \longrightarrow 0$$

(5) <u>$M = \#_i P_i$ where all P_i are P^2-irreducible sufficiently large 3-manifolds</u>

π maps onto the subgroup G of $\mathrm{Aut}\,\pi_1(\#P_i)$ generated by (b),(c),(d) and $\mathrm{Aut}^+\pi_1 P_i$ on factors by (3) and (4).

$\times_i K_{P_i} = \times_i \mathbb{Z}_2$ (or $\oplus_i \mathbb{Z}_2$ as $\mathbb{Z}_2$ is abelian and normal in $\times_i \mathbb{Z}_2$). The $\mathbb{Z}_2$'s parts correspond to rotations parallel to the separating spheres which clearly induce the identity on $\mathrm{Aut}\,\pi_1(\#_i P_i)$.

Hence we get the exact sequence

$$0 \longrightarrow \oplus_i \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\#_i P_i, D) \xrightarrow{\pi} G \longrightarrow 0$$

(6) From the results above we get:

THEOREM 3.11

If $M = \#_i P_i$, where P_i is either a irreducible sufficiently large closed orientable 3-manifold or $S^1 \times S^2$, then the sequence

$$0 \longrightarrow \bigoplus_i Z_2 \longrightarrow \mathrm{Aut}(\#_i P_i, D) \xrightarrow{\pi} G \longrightarrow 0$$

is exact where G is the subgroup of Aut $\pi_1 M$ generated by (b),(c),(d), $\mathrm{Aut}^+ \pi_1 P_i$ if P_i is P^2-irreducible sufficiently large and Aut $\pi_1(S^1 \times S^2) = \mathbb{Z}_2$ on factors. Each $\mathbb{Z}_2$-factor in the kernel of π comes either from a rotation parallel to the separating sphere of an irreducible factor or from a rotation parallel to the belt sphere of a handle.

□

PART II

REPRESENTATIONS OF 4-MANIFOLDS

Unless otherwise stated, all manifolds are assumed to be closed [redacted] and connected.

1. STABLE CLASSIFICATION OF 4-MANIFOLDS

Let $S^2 \underset{\sim}{\times} S^2$ denote the non trivial S^2- bundle over S^2.

DEFINITION 1.1

We say that two 4-manifolds M_1, M_2 are stably equivalent if

$$M_1 \# t_1(S^2 \times S^2) \# s_1(S^2 \underset{\sim}{\times} S^2) \cong M_2 \# t_2(S^2 \times S^2) \# s_2(S^2 \underset{\sim}{\times} S^2)$$

for some t_i, $s_i \geq 0$, $i = 1,2$.

Denote by $\underset{s}{\sim}$ the stable equivalence relation. It follows immediately from Van Kampen's theorem and from the fact that $\pi_1(S^2 \times S^2) = \pi_1(S^2 \underset{\sim}{\times} S^2) = 0$ that a necessary condition for M_1, M_2 to be stably equivalent is that $\pi_1 M_1 = \pi_1 M_2$.

(a) The orientable case

We consider first oriented manifolds.

For any CW X^n with fundamental group π there is a natural inclusion $X \hookrightarrow K(\pi,1)$ for we can build up a model for $K(\pi,1)$ by attaching cells to X to kill higher homotopy groups. Denote by $\eta(X)$ the class in $\tilde{H}_n(\pi) = \tilde{H}_n(K(\pi,1))$ defined by that inclusion. In particular, if X is an n-manifold that inclusion defines a natural class $\lceil X \rceil_s$ in $\Omega_n(\pi) = \Omega_n(K(\pi,1))$ the oriented n-bordism group of $K(\pi,1)$.

We want to relate $\Omega_4(\pi)$ with stable equivalence classes of oriented 4-manifolds with fundamental group π. We need some preliminary results.

LEMMA 1.1

If X, Y are oriented homotopy equivalent n-circuits with same orientation homomorphism and [redacted] fundamental group π then $\eta(X) = \eta(Y)$ in $\tilde{H}_n(\pi)$.

Proof.

As X,Y are n-circuits, $H_n(X) = \mathbb{Z}$ generated by the homology class of X,[X], $\tilde{H}_n(Y) = \mathbb{Z}$ generated by [Y]. Furthermore, if X,Y are homotopy equivalent and if $f : X \to Y$, $g : Y \to X$ are such that $fg \simeq id_Y$, $gf \simeq id_X$ then

$$f_*[X] = r[Y] \quad , r \in \mathbb{Z}$$

$$g_*f_*[X] = g_*(r[Y]) = r.g_*[Y]$$

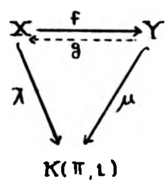
$$= id_*[X] = [X]$$

hence, $r = \pm 1$. Similarly, $g_*[Y] = \pm[X]$.

Thus, if $f : X \to Y$ is a homotopy equivalence we can assume, w.l.o.g., that $f_*[X] = [Y]$ (i.e. the homotopy equivalence commutes with orientation homomorphisms

We say that X,Y have same orientation homomorphism.

Then the diagram



.where, λ,μ are the natural inclusions, commutes up to homotopy and hence

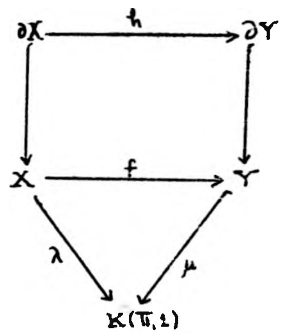
$$\eta(X) = \lambda_*[X] = (\mu f)_*[X] = \mu_*f_*[X] = \mu_*[Y] = \eta(Y)$$ as required. □

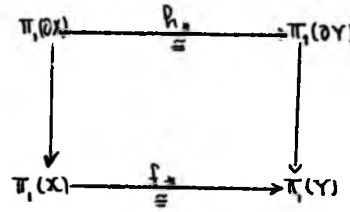
COROLLARY 1.1.

If X,Y are homotopy equivalent n-manifolds with homeomorphic boundaries, same orientation homomorphism and the homotopy equivalence extends an orientation preserving homeomorphism of the boundaries, then $\eta(X \cup_\partial (-Y)) = 0$ in $\tilde{H}_n(\pi)$ where $\pi = \pi_1 X = \pi_1 Y$.

Proof.

Let f be the homotopy equivalence extending h, an orientation preserving homeomorphism of the boundaries. We then have commutative diagrams





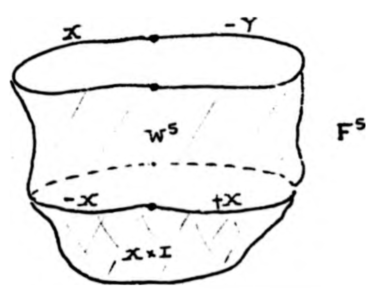
μ, λ as before.

Identify $K(\pi_1(X),1)$ with $K(\pi_1(Y),1)$ (up to homotopy) using f_* and glue up the two natural maps together to obtain a map

$$\rho : X \cup_\partial (-Y) \longrightarrow K(\pi,1)$$

ρ determines $\eta(X \cup_\partial (-Y))$ in $\tilde{H}_n(\pi)$. We'll prove $\eta(X \cup_\partial (-Y)) = 0$ in $\tilde{H}_n(\pi)$.

As $X \cup_\partial (-Y) \simeq X \cup_\partial (-X) \cong 2X \cong \partial(X\times I)$, by lemma 1, $\eta(X \cup_\partial (-Y)) = \eta(\partial(X\times I))$. Hence there is a 5-cycle W^5 with $\partial W^5 = (X \cup_\partial (-Y)) \cup \partial(X\times I)$ and a map $W^5 \to K(\pi,1)$ extending the map on the boundary.



Form $F^5 = W^5 \cup_\partial X\times I$. Then $\partial F^5 = X \cup_\partial (-Y)$ and as it follows from Van Kampen's theorem that there is a map $F^5 \to K(\pi,1)$ extending ρ, $\eta(X \cup_\partial (-Y)) = 0$ in $\tilde{H}_n(\pi)$ as required. □

When $\eta(X) = \eta(Y)$ in $\tilde{H}_n(\pi)$ we say X,Y are homologous over π.

We now consider the case of 4-manifolds.

THEOREM 1.1

If M_1, M_2 are oriented 4-manifolds with [redacted] fundamental group π, then they are stably equivalent iff they define the same class in $\Omega_4(\pi)$.

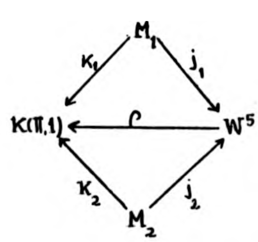
Remarks : (1) by the 'class defined by M' we mean the natural class defined above.

(2) If M_1, M_2 define the same class in $\Omega_4(\pi)$ we say they are bordant over π.

(3) The proof of the theorem will use arguments from [5].

Proof:

(a) Suppose first that $[M_1]_s = [M_2]_s$ in $\Omega_4(\pi)$. Then there exists a a 5-manifold W^5 with boundary $\partial W^5 = M_1 \cup M_2$ (disjoint union) and a map $W^5 \xrightarrow{\rho} K(\pi,1)$ extending the natural maps $M_i \xrightarrow{K_i} K(\pi,1)$ $i = 1,2$. We then have a commutative diagram



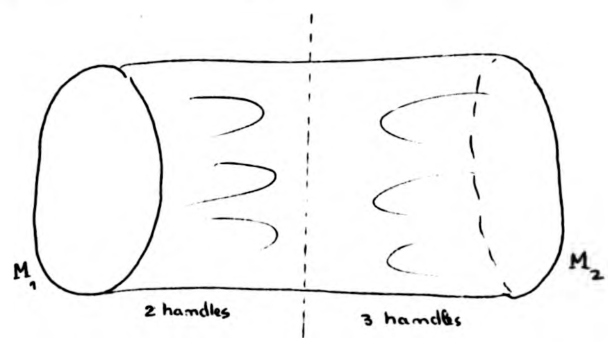
where j_1, j_2 are the inclusions.

Then $\rho_* : \pi_1 W^5 \longrightarrow \pi = \pi_1 M_1 = \pi_1 M_2$ splits $j_{i*} : \pi_1 M_i \longrightarrow \pi_1 W^5$ $i = 1,2$.

Let $s_i : \pi_1 W^5 \longrightarrow \pi_1 M_i$ be the split map.

Perform surgery on W^5 to make j_{1*}, j_{2*} isomorphisms (i.e. kill the normal subgroup Ker s_1 = Ker s_2 of $\pi_1 W^5$. The normal subgroup in question is the normal closure of a finite number of elements) to get a new cobordism, W'^5 between M_1, M_2 with $\pi_1 W'^5 = \pi$. Consider a handle decomposition of W'^5. By connectedness assume there are no 0,5 handles and cancel 1,4 handles using the fact that $\pi_1(W', M_i) = 0$ [21]. We can thus assume that W'^5 has only 2,3 handles. As $\pi_1 M_1 \to \pi_1 W'$ is an isomorphism it follows that the 2-handles must be attached by null homotopic curves in M_1 and, as dim $M_1 = 4$, the effect of adding these handles to M_1 is to change it to

$\bar{M} = M_1 \# t_1(S^2\times S^2) \# s_1(S^2 \tilde{\times} S^2)$ some $t_1, s_1 \geq 0$.



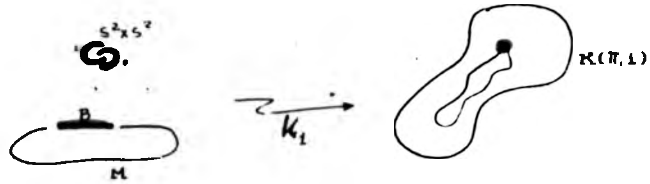
As 3-handles are dual 2-handles, $\bar{M}$ can also be obtained from M_2 by attaching 2-handles by null homotopic curves. Hence

$$\bar{M} = M_1 \# t_1(S^2\times S^2) \# s_2(S^2 \tilde{\times} S^2) \cong M_2 \# t_2(S^2\times S^2) \# s_2(S^2 \tilde{\times} S^2)$$

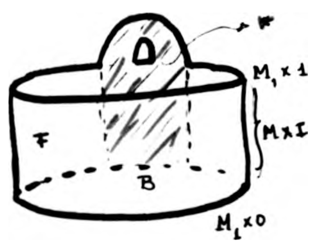
some $s_i, t_i \geq 0$, as required.

(b) For the converse, it is enough to show that adding $S^2\times S^2$ or $S^2 \tilde{\times} S^2$ to M_1, say, doesn't change its cobordism class.

We can assume w.l.o.g. that $M_1 \xrightarrow{K_1} K(\pi,1)$ maps a ball B to the base point $*$ in $K(\pi,1)$. We use this ball to form the connected sum and map the whole of $S^2 \times S^2$ (or $S^2 \tilde{\times} S^2$) to $*$



This defines a map $\bar{K}_1 : M_1 \# S^2 \underset{(\sim)}{\times} S^2 \longrightarrow K(\pi,1)$ ($S^2 \underset{(\sim)}{\times} S^2$ means either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ as convenient). It remains thus to show that $\ulcorner M_1, K_1 \urcorner = \ulcorner M_1 \# S^2 \underset{(\sim)}{\times} S^2, \bar{K}_1 \urcorner$ in $\Omega_4(\pi)$. We construct a cobordism R^5 between M_1 and $M_1 \# S^2 \underset{(\sim)}{\times} S^2$ and a map $F : R^5 \longrightarrow K(\pi,1)$ s.t. $F/M_1 = K_1$, $F/M_1 \# S^2 \underset{(\sim)}{\times} S^2 = \bar{K}_1$.



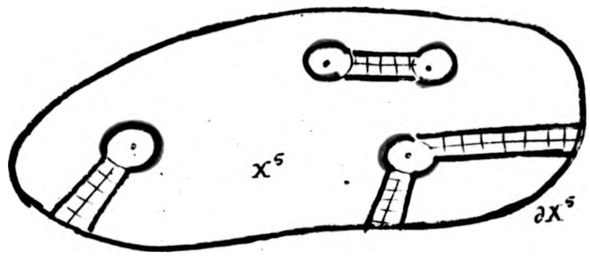
Let $R = M_1 \times I \cup$ 2-handle, attached to B

Then $\partial R = -M_1 \times 0 \cup M_1 \# S^2 \underset{(\sim)}{\times} S^2$. Map all the region $B \times I \underset{\partial}{\cup} (S^2 \underset{(\sim)}{\times} B^3 - 4 \text{ ball})$ to $*$.

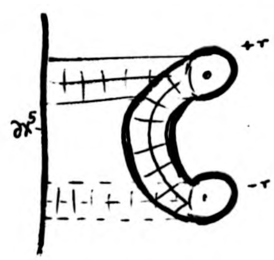
This extends clearly to the required F. $\square$

If M_1, M_2 are oriented closed 4-manifolds with [redacted] fundamental group π, [redacted] and homologous over π we can resolve the homology to have only a finite number of points of singularities (as oriented bordism groups Ω_1, Ω_2, Ω_3 are all zero) each of which has for a link an orientable 4-manifold. As $\Omega_4 = \mathbb{Z}$ is detected by the index and index $\mathbb{C}P^2 = 1$, we can resolve the singularities by adding some copies of $\mathbb{C}P^2$ or $-\mathbb{C}P^2$ to the links :

Denote by X the homology and by M_0, M with a ball removed.



Then cutting out tubes from the links to $M_1 \subset \partial X^5$ and replacing them by $(\mathbb{C}P_2)_0 \times I$ or $(-\mathbb{C}P_2)_0 \times I$, we can kill the obstruction index and resolve the homology to a 5-manifold. Also, every time we have links with opposite sign, instead of piping them to the boundary, we can pipe them together away from the boundary. Clearly π_1 is not affected and the 5 manifold then obtained



gives a cobordism of $M \# \pm \mathbb{C}P^2$ to M_2 over π

THEOREM 1.2

If M_1, M_2 are homotopy equivalent oriented closed 4-manifolds (with same orientation homomorphism) then they are stably equivalent.

Remark This result was suggested by Kirby, without proof, in a private communication.

Proof.

Let $\pi = \pi_1 M_1 = \pi_1 M_2$. By lemma 1.1 $\eta(M_1) = \eta(M_2)$ in $H_4(\pi)$($\tilde{H}_4(\pi)$)Hence by the above, $M_1 \# \pm \mathbb{C}P^2$ is bordant to M_2 over π. But as M_1, M_2 are homotopy equivalent, index M_1 = index M_2 and so we must have an equal number of $\mathbb{C}P^2$'s and $-\mathbb{C}P^2$'s. Then we can resolve all the singularities away from the boundary of the homology to get a cobordism over π between M_1, M_2 as required. □

COROLLARY 1.2

Two homotopy equivalent oriented 4-manifolds with homeomorphic boundaries same orientation homomorphism and the homotopy equivalence extending an orientation preserving homeomorphism on the boundaries, are stably equivalent.

Proof.

Same as above using corollary 1.1 instead of lemma 1.1. □

THEOREM 1.3

For any group π

$$\Omega_4(\pi) = Z \oplus H_4(\pi)$$

Proof.

(If π is, for instance, any finitely presented group, then there is a 4-manifold with fundamental group π.)

It follows from what we have said before that there is an exact sequence of abelian groups (note that we don't need to have M^4 with fundamental group π but M^4 and a map $M^4 \longrightarrow K(\pi,1)$).

$$Z \xrightarrow{\ i\ } \Omega_4(\pi) \xrightarrow{\ j\ } H_4(\pi)$$

where $\underline{j}$ is the natural map and $\underline{i}$ is defined by sending 1 to the class of

$\mathbb{C}P^2$ with the trivial map to $K(\pi,1)$. As index $\mathbb{C}P^2 = 1$ and $\mathbb{C}P^2$ doesn't bound $i(1)$ is non-zero and therefore im $i = \mathbb{Z}$. i splits by the index map. Hence i is onto. Also j is onto as there are no obstructions to resolve a 4-homology to a 4-manifold (an algebraic argument can be found in Conner and Floyd "Differentiable periodic maps" Springer, Berlin 1964). Hence we have in fact a split exact sequence

$$0 \longrightarrow \mathbb{Z} \underset{\leftarrow - - -}{\overset{i}{\longrightarrow}} \Omega_4(\pi) \overset{j}{\longrightarrow} H_4(\pi) \longrightarrow 0$$

Thus

$$\Omega_4(\pi) = \mathbb{Z} \oplus H_4(\pi) \qquad \text{as required.}$$

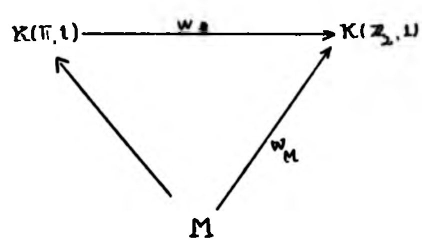
□

In particular, for any oriented 4-manifold M with $\pi_1 = \pi$ its stable equivalence class is determined by the index and the natural class $\eta(M)$ in $H_4(\pi)$.

(b) The non-orientable case

In this case we have to work in the category of spaces over $K(\mathbb{Z}_2,1)$ and consider bordism and homology with "twisted coefficients" which can be defined as follows:

Let M be an n-manifold. $w_M : \pi_1 M \longrightarrow \mathbb{Z}_2$ its orientation homomorphism. Considering then the diagram



where $\pi = \pi_1 M$, and defining two such diagrams to be bordant or homologous respectively by the "obvious things" we get resp. the bordism groups

$\Omega_n^t(\pi)$ or the homology groups $H_n^t(\pi)$ with twisted coefficients. Geometrically, these groups can be interpreted as bordism or homology classes, resp, of singular n-circuits in $K(\pi,1)$ where both the circuits, bordisms and the homologies are locally orientable and the orientation homomorphisms commute with w_*.

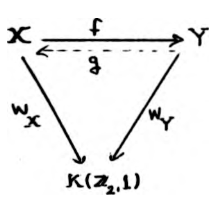
Denote again by $\eta(M)$ the natural class in $H_n^t(\pi,M)$. The corresponding results in this case are :

LEMMA 1.1'

If X,Y are non-orientable homotopy equivalent over $\mathbb{Z}_2$ n-circuits with [redacted] fundamental group π and same orientation homomorphism then $\eta(X) = \eta(Y)$ in $H_n^t(\pi)$.

Proof.

Let $f : X \longrightarrow Y$ be a homotopy equivalence over $\mathbb{Z}_2$. This means we have



a homotopy commutative diagram where g is the homotopy inverse and also the homotopies are homotopies over $K(\mathbb{Z}_2,1)$.

In particular, X and Y are homotopy equivalent in the usual sense. Then the same proof as in the oriented case $(H_n(X;K(\mathbb{Z}_2,1)) = \mathbb{Z})$ working in the category of spaces over $K(\mathbb{Z}_2,1)$ gives the result. $\square$

COROLLARY 1.1'

If X,Y are non-orientable homotopy equivalent (over $\mathbb{Z}_2$) n-manifolds with homeomorphic boundaries same orientation homomorphisms and the homotopy equivalence extends a homeomorphism of the boundaries, then

$\eta(X \cup_\partial Y) = 0$ in $H_n^t(\pi)$ where $\pi = \pi_1 X = \pi_1 Y$.

Proof.

As before with the obvious changes. □

Case of non-orientable 4-manifolds

For non-orientable 4-manifolds there is a bordism invariant - the reduction mod 2 of the Euler characteristic denoted by χ_2.

$$\text{As } \chi(M) = \dim H_0(M;\mathbb{Z}_2) - \dim H_1(M;\mathbb{Z}_2) + \dim H_2(M;\mathbb{Z}_2) - \dim H_3(M;\mathbb{Z}_2) + \dim H_4(M;\mathbb{Z}_2)$$

where $H_i(M;\mathbb{Z}_2)$ is considered as a vector space over $\mathbb{Z}_2$ it follows from Duality that $\chi(M) = \dim H_2(M;\mathbb{Z}_2) \bmod 2$.

THEOREM 1.1'

If M_1, M_2 are non-orientable 4-manifolds with fundamental group π and same orientation homomorphisms then they are stably equivalent iff they define the same class in $\Omega_4^t(\pi)$.

Proof.

As in the orientable case but working in the category of spaces over $K(\mathbb{Z}_2, 1)$.

THEOREM 1.2'

If M_1, M_2 are homotopy equivalent over $\mathbb{Z}_2$, non-orientable closed 4-manifolds with same orientation homomorphisms then they are stably equivalent.

$\eta(X \cup_\partial Y) = 0$ in $H_n^t(\pi)$ where $\pi = \pi_1 X = \pi_1 Y$.

Proof.

As before with the obvious changes. □

Case of non-orientable 4-manifolds.

For non-orientable 4-manifolds there is a bordism invariant - the reduction mod 2 of the Euler characteristic denoted by χ_2.

$$\text{As } \chi(M) = \dim H_0(M;\mathbb{Z}_2) - \dim H_1(M;\mathbb{Z}_2) + \dim H_2(M;\mathbb{Z}_2) - \dim H_3(M;\mathbb{Z}_2) + \dim H_4(M;\mathbb{Z}_2)$$

where $H_i(M;\mathbb{Z}_2)$ is considered as a vector space over $\mathbb{Z}_2$ it follows from Duality that $\chi(M) = \dim H_2(M;\mathbb{Z}_2) \bmod 2$.

THEOREM 1.1'

If M_1, M_2 are non-orientable 4-manifolds with fundamental group π and same orientation homomorphisms then they are stably equivalent iff they define the same class in $\Omega_4^t(\pi)$.

Proof.

As in the orientable case but working in the category of spaces over $K(\mathbb{Z}_2,1)$.

THEOREM 1.2'

If M_1, M_2 are homotopy equivalent over $\mathbb{Z}_2$, non-orientable closed 4-manifolds with same orientation homomorphisms then they are stably equivalent.

Proof.

Let $\pi = \pi_1 M_1 = \pi_1 M_2$. By lemma 1.1' $\eta(M_1) = \eta(M_2)$ in $H_4^t(\pi)$. As the classes are all locally orientable and the singularities appearing in a homology are all local, we can resolve a homology between M_1, M_2 at the expenses of introducing some $\pm\mathbb{C}P^2$'s as before. But as for M non-orientable, $M \# \mathbb{C}P^2 \cong M \# (-\mathbb{C}P^2)$ (slide $\mathbb{C}P^2$ along a non-orientable curve) we get after resolving the homology, that either $M_1 \# \mathbb{C}P^2$ is bordant to M_2 over π or M_1, M_2 are bordant over π. (If there are an even number of singularities we pipe them together in pairs away from the boundary. If not, we pipe all except one away from the boundary).

As $M_1 \simeq M_2$, dim $H_2(M_1)$ = dim $H_2(M_2)$, hence $\chi_2(M_1) = \chi_2(M_2)$. But as $\chi_2(M_1 \# \mathbb{C}P^2) = \chi_2(M_1) + 1 \neq \chi_2(M_2)$, $M_1 \# \mathbb{C}P^2$, M_2 cannot be bordant over π. Hence M_1, M_2 are bordant over π as required. □

Similarly we get :

COROLLARY 1.2'

Two homotopy equivalent over $\mathbb{Z}_2$ non-orientable 4-manifolds with homeomorphic boundaries, same orientation homomorphisms and the homotopy equivalence extending a homeomorphism on the boundaries are stably equivalent.

THEOREM 1.3'

For any group π and for every non-trivial homomorphism $\pi \longrightarrow \mathbb{Z}_2$ we have

$$\Omega_4^t(\pi) = \mathbb{Z}_2 \oplus H_4^t(\pi).$$

Proof.

It follows from the discussion above that there is an exact sequence

$$\mathbb{Z}_2 \xrightarrow{i} \Omega_4^t(\pi) \xrightarrow{j} H_4^t(\pi)$$

defined similarly to the orientable case, i(1) is the class determined by $\mathbb{C}P^2$. Hence $\operatorname{im} i = \mathbb{Z}_2$. Also, as before, j is onto and as we have an exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i} \Omega_4^t(\pi) \xrightarrow{j} H_4^t(\pi) \longrightarrow 0$$

As mod 2 reduction of the Euler characteristic defines a split map

$$\Omega_4^t(\pi) \longrightarrow \mathbb{Z}_2$$
$$M \longrightarrow \frac{X(M)}{2}$$

we get $$\Omega_4^t(\pi) = \mathbb{Z}_2 \oplus H_4^t(\pi)$$ as required. □

This result together with theorem 1.1' prove the following:

Stable equivalence classes of non-orientable 4-manifolds with fundamental group π are in 1-1 correspondence with the elements of $\Omega_4^t(\pi)$ and are determined by the reduction mod 2 of the Euler characteristic and by a "natural" 4-dimensional homology class with twisted coefficients.

SOME REMARKS AND EXAMPLES.

(1) We need at most one $S^2 \tilde{\times} S^2$ in the stable equivalence as
$S^2 \times S^2 \# S^2 \tilde{\times} S^2 \cong S^2 \tilde{\times} S^2 \# S^2 \tilde{\times} S^2$

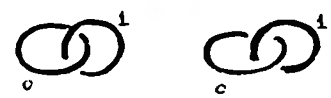
Proof : we use the language of [15]. To a framed link in S^3 we can associate a 1-connected 4-manifold with boundary by attaching 2 handles on the boundary of B^4 along the framed link. Components of the framed link represent 2 spheres corresponding to the second homology classes of the 4-manifold. Hence there is a 1-1 correspondence between framed links in S^3 and 1-connected 4-manifolds with boundary which admit handle decompositions with only handles of index 2. We also say that the link represents the boundary of the 4-manifold. In cases where the boundary is S^3, capping off with a 4-ball we may also say that the link represents the closed 4-manifold thus obtained. Both $S^2 \times S^2$ and $S^2 \underset{\sim}{\times} S^2$ are such cases and their link pictures are

$S^2 \times S^2$

$S^2 \underset{\sim}{\times} S^2$

A link picture for $S^2 \times S^2 \# S^2 \underset{\sim}{\times} S^2$ is then given by

; a link picture for $S^2 \underset{\sim}{\times} S^2 \# S^2 \underset{\sim}{\times} S^2$ is



We now prove that they are equivalent by Kirby's band moves - band moves correspond in the 4-manifold to 2-handle slides and hence the 4-manifold is not affected, up to homeomorphism.

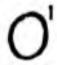
Another proof can be found in [24].

(2) Cappell and Shaneson found a homotopy $\mathbb{R}P^4$, HP^4, s.t. there is no $t',t > 0$ s.t. $HP^4 \# t(S^2\times S^2) = \mathbb{R}P^4 \# t'(S^2\times S^2)$. We know in fact from theorem 2' and from the first remark that for some t,t'

$$HP^4 \# t(S^2\times S^2) \# S^2\tilde{\times}S^2 \cong \mathbb{R}P^4 \# t'(S^2\times S^2) \# S^2\tilde{\times}S^2.$$

(3) <u>If M,N are homologous over π and index M = index N + k, k $\geq$ 0 then N, M # k($-\mathbb{C}P^2$) are bordant over π.</u>

<u>Proof:</u>

We first show that $\mathbb{C}P^2 \# (-\mathbb{C}P^2) \# (-\mathbb{C}P^2) \cong S^2\tilde{\times}S^2 \# (-\mathbb{C}P^2) \cong S^2\times S^2 \# (-\mathbb{C}P^2)$, using link pictures (another proof in [24]). A link picture for $\mathbb{C}P^2$ is given by 

($-\mathbb{C}P^2$ with -1)(

also represents S^3).

$\mathbb{C}P^2 \# (-\mathbb{C}P^2) \# (-\mathbb{C}P^2)$ = 

$S^2\tilde{\times}S^2 \# (-\mathbb{C}P^2)$ = 

= 

= 

$S^2\times S^2 \# (-\mathbb{C}P^2)$ = 

Now the result follows immediately. $\square$

(4) We give some examples where stable equivalence classes are determined by the index for oriented manifolds.

(a) <u>$\pi_1 = 0$</u> $\Longrightarrow$ $\Omega_4(\pi) = Z$ as $H_4(\pi) = 0$.

This is, in fact, Wall's result [23] that any two simply connected oriented 4-manifolds with same quadratic form (and hence the same index) are stably equivalent. Wall proves also that no $S^2 \tilde{\times} S^2$ are needed. For simply connected oriented 4 manifolds to have isomorphic quadratic forms is equivalent to say that they are homotopy equivalent. Wall showed then that they are, in fact, h-cobordant and it follows from this fact that there are no $S^2 \tilde{\times} S^2$ in the stable equivalence.

(b) For $\underline{\pi_1 = \mathbb{Z}}$ or $\underline{\pi_1 = \mathbb{Z}_p}$ as $\underline{H_4(\pi) = 0 \text{ in both cases}}$, we have $\underline{\Omega_4(\pi) = \mathbb{Z}}$. For $\pi_1 = \mathbb{Z}$, $K(\pi,1) = S^1$ and $H_4(S^1) = 0$. For $\pi_1 = \mathbb{Z}_p$, homology is periodic of period 2 and hence $H_i(\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$ $i > 0 \quad p > 1$ [11].

Then (a), (b) and theorem 3 give the following result:

<u>Two oriented closed 4 manifolds with $\pi_1 = 0$, $\mathbb{Z}$ or $\mathbb{Z}_p$ are stably equivalent iff they have the same index.</u>

2. A LINK REPRESENTATION FOR 4-MANIFOLDS.

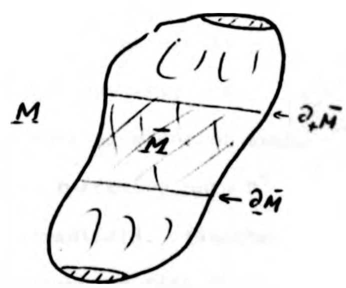
As already quoted there is a 1-1 correspondence between framed links in S^3 and 4 manifolds with boundary which admit a given nice handle decomposition with only handles of index 2. In this section we try to generalise this result to arbitrary 4 manifolds and give a 'link representation' of any closed 4 manifolds. We then define a series of "allowable moves" in the link picture that will enable us to see when two different link pictures represent the same 4 manifold. Finally, we consider the stable case.

We assume the reader is familiar with [15] and refer to it for definitions and details. We will deal with the orientable and non-orientable cases separately.

The orientable case

1. Let M^4 be a closed oriented 4 manifold with a given nice handle decomposition $\mathcal{H}_1$ which, w.l.o.g. : we can assume to have only one 0 and one 4 handle (Recall all our manifolds are connected.) If we remove from M^4 the 0,1,3,4 handles we obtain a cobordism $\bar{M}$ between a connected sum of i copies of $S^1\times S^2$ ($i \geq 0$ - In the case $i = 0$, $\#_0 S^1\times S^2$ denotes S^3, by convention) where i is the number of 1-handles of $\mathcal{H}_1$, and a connected sum of j copies of $S^1\times S^2$ ($j \geq 0$) where j is the number of 3-handles in $\mathcal{H}_1$. The cobordism $\bar{M}$ has then only 2 handles $\partial\bar{M} = \partial_+\bar{M} \cup \partial_-\bar{M}$ where $\partial_+\bar{M} = \#_j S^1\times S^2$ $\partial_-\bar{M} = \#_i S^1\times S^2$.

Conversely given $\bar{M}$ i.e. given only the full 2-handles (a full handle is the cobordism associated to the attaching of the handle) we can recover M uniquely up to homeomorphism:



LEMMA 2.1

For any orientable closed 4 manifold given only the cobordism formed by the full 2-handles we can recover the manifold uniquely up to (orientation preserving) homeomorphism.

Proof.

The union of (0,1) handles (respect. union of 3,4 handles) is homeomorphic to $\overset{\partial}{\underset{i}{\#}} S^1\times B^3$ (resp. $\overset{\partial}{\underset{j}{\#}} S^1\times B^3$). To prove the lemma it is enough to show that if we give $\overset{\partial}{\underset{i}{\#}} S^1\times B^3$ and $\overset{\partial}{\underset{j}{\#}} S^1\times B^3$ back again by two different homeomorphisms on their boundaries we get the same manifold up to homeomorphism. But this follows immediately from the fact that any homeomorphism on $\underset{k}{\#} S^1\times S^2$ extends to a homeomorphism of $\overset{\partial}{\underset{k}{\#}} S^1\times B^3$. $\square$

Remark : We will see later that a sort of converse holds.

2. A link picture for M^4 with a given handle decomposition.

(a) We suppose, w.l.o.g., given an oriented closed 4 manifold M^4 with a nice handle decomposition $\mathcal{H}_1$ with only one 0 and 4 handles.

Suppose given a certain link picture of a 1-connected 4-manifold with boundary. Components of the framed link represent 2-spheres corresponding to the second homology classes of the 4 manifold. Surgering the 2-spheres corresponding to an unknotted circle with 0-framing, corresponds to trading a 2-handle in for a 1-handle, hence changing the 4 manifold (but not the boundary). We can then think of representing a 1-handle ($\cong S^1\times B^3$) by putting a dot on such a circle; this means that we first attach a 2 handle with 0-framing onto B^4 along the unknot and get $B^2\times S^2$, then surger S^2 from this manifold. (Another way of picturing this is by pushing the interior of the spanning disc D^2 of the unknot in S^3 into B^4, and by removing an open tubular neighbourhood of D^2 from B^4. For instance, a connected sum of $S^1 \times B^3$'s can be represented by disjoint unknotted dotted circles in S^3 (disjoint means separated in S^3 by embedded 2-spheres), meaning that first we attach 2-handles along the curves with 0-framing to get $\#_k S^2 \times B^2$ then we surger the 2 spheres. Clearly we can either surger one at a time or all at the same time as we can always do one surgery so that it doesn't affect the others.

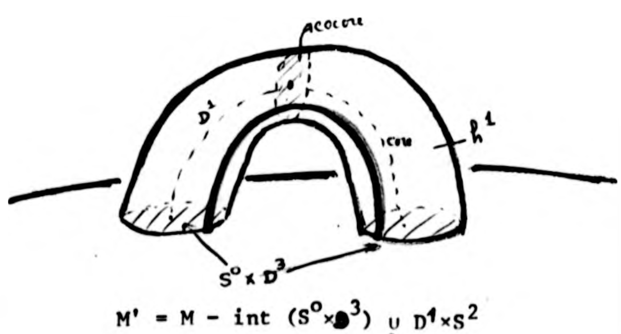
Hence, given M, $\mathcal{H}_1$, we consider only the 0,1,2 handles to get a manifold M_1 with $\partial M_1 = \#_j S^1 \times S^2$ (which by lemma 2.1 determines M uniquely). Then we trade all 1-handles for 2-handles and get another 4-manifold with boundary ∂M_1, with the property of being 1-connected. Following Kirby, we'll then have a framed link representation of it where disjoint unknotted circles with 0-framing will appear. Surger the 2-spheres corresponding to those unknotted circles trading the handles back again - In the picture put a dot on such circles. Then we get a "link picture" L_1 of M_1 and hence of M, associated to the handle decomposition $\mathcal{H}_1$. This link picture determines M uniquely up to homeomorphism by lemma 2.1. Converse is not true: for two different handle decomposition give different link pictures. We call such a link a special framed link. From now on every time we talk about link pictures we mean special framed link pictures. We can also consider the undotted curves as a framed link in $\#_k S^1 \times S^2$. By a framed link L in a closed [redacted] 3 manifold M we mean a finite collection of disjoint PL embeddings of $S^1 \times D^2$. Any image of $S^1 \times \{0\}$ will be called a component of L and the associated image $S^1 \times \{x\}$, $x \neq 0$, a parallel curve. Framings are determined by the parallel curves. (If $H_1(M) = 0$ then the parallel curves are determined by linking numbers). Hence, in our case, when having a numbered link this will tell us how to attach the 2-handles to S^3 after trading the 1-handles into 2-handles and then trading them back again to get the original manifold. In this sense, the effects on framings obtained by Kirby moves on undotted circles are as in [15]. Otherwise they are determined by the effect on parallel curves.

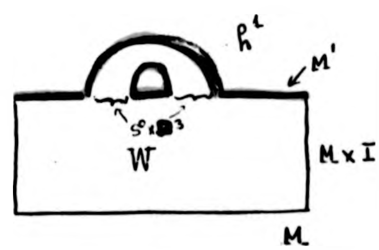
Also think of the link pictures as either representing the closed 4-manifold or the boundary of the manifold obtained by removing the 3,4 handles.

We now take a closer look at the trading process which will prove useful later on.

(b) Trading a 1-handl_ for a 2-handle in an orientable 4-manifold.

Let M^3 be an oriented 3-manifold. Remove two disjoint 3 balls from M^3 and identify the resulting boundaries by an orientation reversing homeomorphism. The resulting manifold M' is said to be obtained from M^3 by adding an orientable 1-handle or by surgering an S^o.





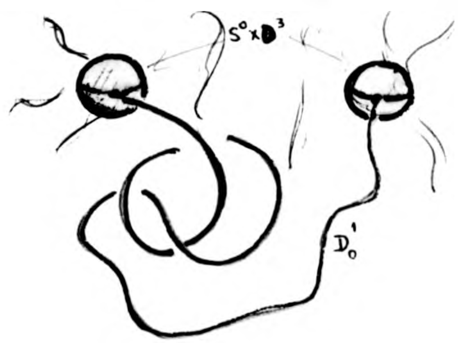
$$M' = M - \text{int}\,(S^o \times D^3) \cup_\partial D^1 \times S^2$$

Corresponding to this surgery there is an elementary cobordism W with only one handle:

Form $M\times I$. Instead of removing $S^0\times D^3$ from $M\times I$ we glue a 4 ball $D^1\times D^3$ to $M\times I$. $D^1\times D^3$ has boundary $S^0\times D^3 \underset{\partial}{\cup} D^1\times S^2$. We then glue $S^0\times D^3$ in $\partial(D^1\times D^3)$ to $S^0\times D^3$ in $M\times I$ to obtain a 4-manifold W whose boundary is the disjoint union $M \cup M'$ where M is identified with $M\times 0$.

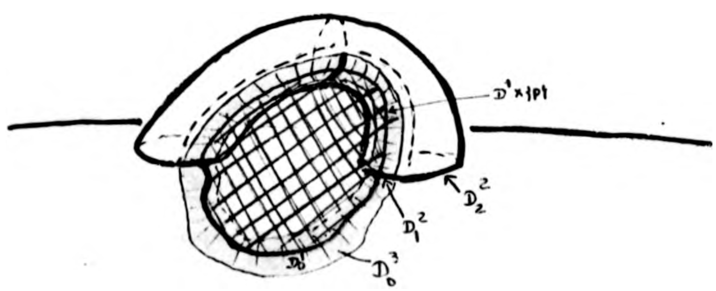
We now show how to replace this cobordism by another cobordism between M,M' with only a 2-handle :

$$S^0\times \{p\} \subset S^0\times S^2 = \partial(S^0\times D^3) \text{ bounds an arc } D^1_0 \text{ in } M.$$



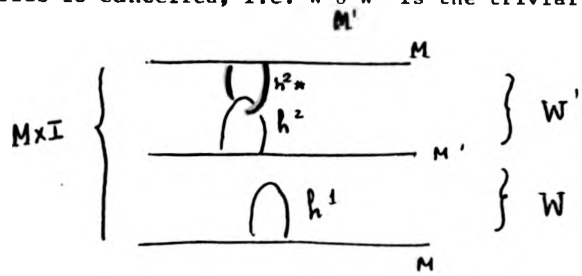
The arc may wander around in the 3-manifold.

The attaching sphere of the 1-handle h^1. $S^0\times S^2$, can be expressed as the union $S^0\times D^2_1 \underset{\partial}{\cup} S^0\times D^2_2$ (see picture)

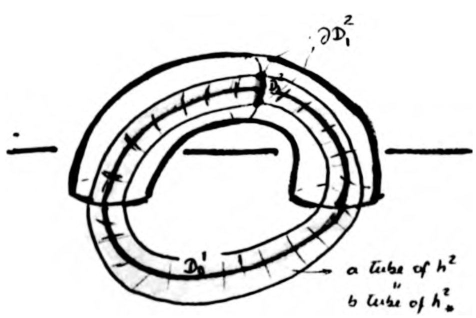


The cells $D^1\times\{p\}$ and D_0^1 joined along their boundaries form a sphere S^1 in M'. By orientability and by the regular neighbourhood theorem, S^1 has a regular neighbourhood of the form $D^1\times D_1^2 \underset{\partial}{\cup} D_0^3$ where $D^1\times D_1^2$ is a neighbourhood of $D^1\times\{p\}$ in $D^1\times S^2$, D_0^3 is a neighbourhood of D_0^1 in $M-\mathrm{int}(S^0\times D^3)$.

Now if we perform surgery on the 1-sphere S^1, thus obtained, we recapture M. This is because the associated cobordism W' has a 2-handle h^2 attached by this S^1 and as h^2, h^1 are then complementary handles, the effect of doing these two surgeries is cancelled, i.e. $W \cup W'$ is the trivial cobordism $M\times I$.



Considering the situation dually, M' is obtained from M by a 1-surgery and W' gives a cobordism between M,M' with only a 2 handle h^2_* (the dual handle to h^2 in W').

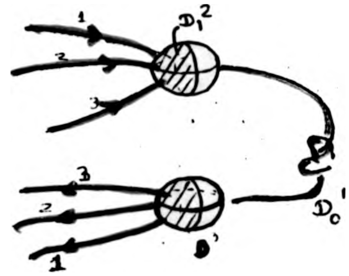


The shaded region in the picture can be considered as either the attaching tube of h^2 or the belt tube of the dual handle h^2_*.

Thus M' is obtained from M by surgery along the curve ∂D_1^2 which is an unknotted circle (with 0-framing, if $M = S^3$).

Note that we have changed the cobordism. We say we have traded a 1-handle by a 2-handle (represented by that unknotted circle).

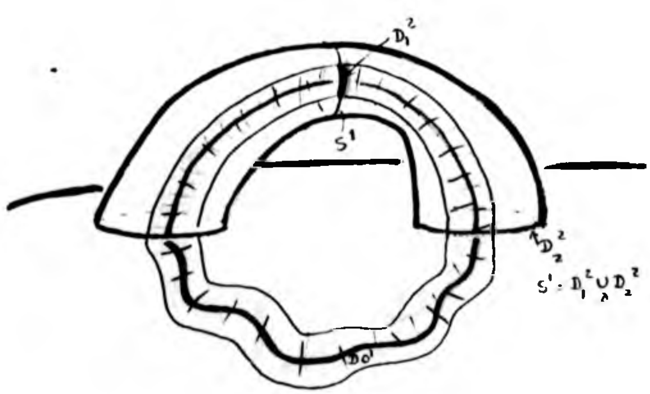
For simplicity we consider the case when $M = S^3$ and then try to see the effect of trading on the picture of M'.

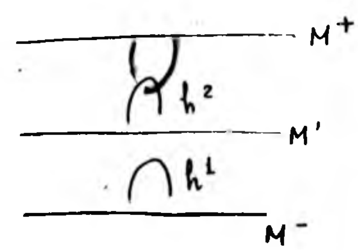


The two balls are to be removed and their boundaries identified by an orientation reversing homeomorphism

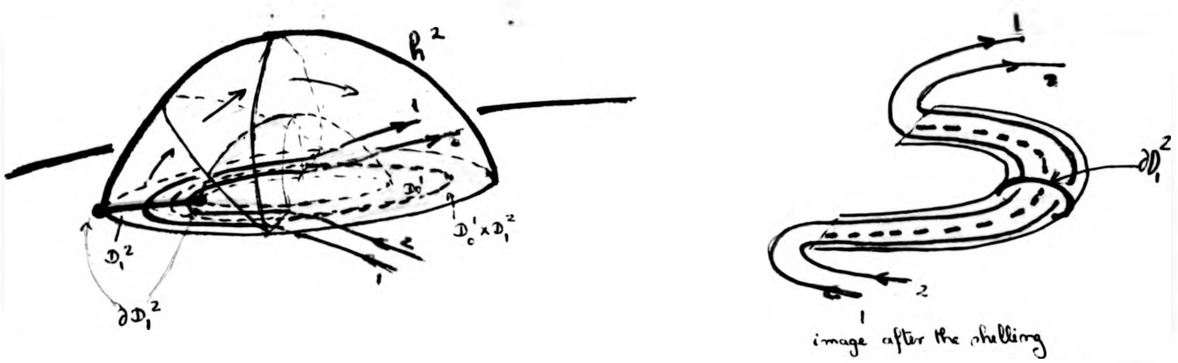
We first note that all the curves attaching spheres of the 2-handles that pass through the 1-handle can be assumed to pierce only D_1^2. We can also assume that the orientation reversing homeomorphism that identifies the boundaries of the balls maps D_1^2 to D_1^2 (for instance, the reflection through the equatorial plane of D^3 - see picture).

Then when replacing this 1-handle by a 2-handle the curves that pass through the handle are completed along the path D_0^1 and ringed by a small curve labelled 0.

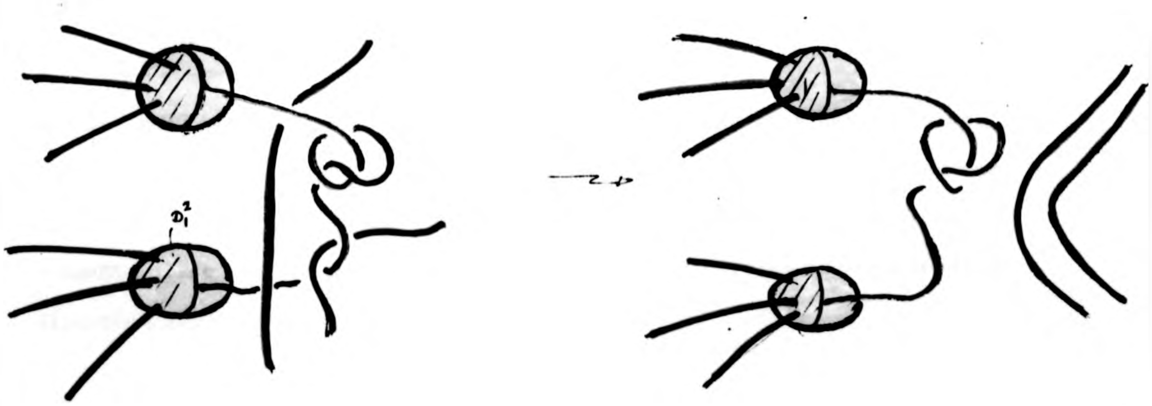




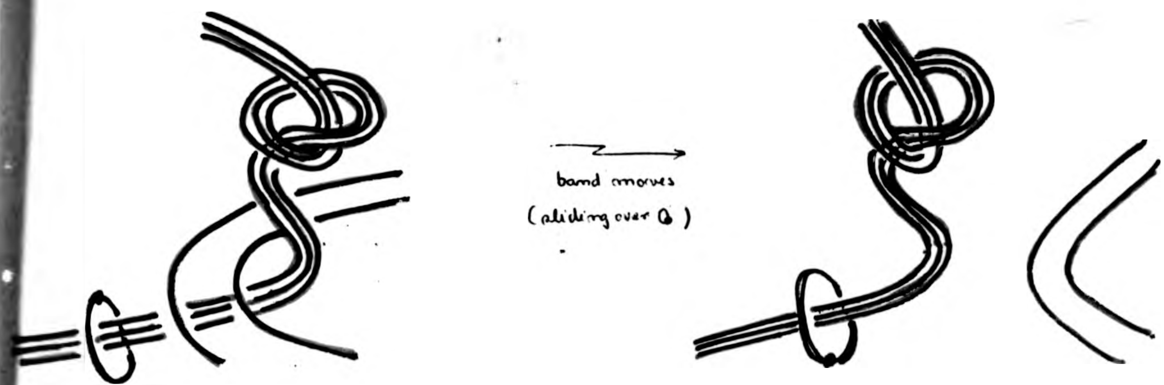
A homomorphism between $M^+ = M^- \cup h^1 \cup h^2$ and M^- is obtained by shelling first h^1 from $D^1 \times D^2_2$ (D^1 core of 1-handle) to $D^1 \times D^2_1$ and then shelling h^2 onto $D^1_0 \times D^2_1$. (See pictures.)



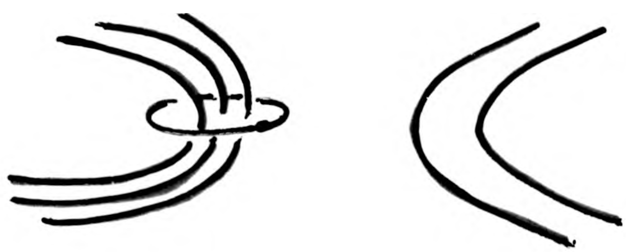
Choice of arc D^1_0 determines the trading but, no matter which choice, the end result is always the same as we can suppose that D^1_0 doesn't have little knots and is unlinked from other curves attaching spheres of 2-handles by sliding around $\{pt\} \times S^2 \subset S^0 \times S^2$ (this is in fact the reason why sliding over an unknotted curve labelled 0 removes the linking and knotting as any such curve introduces a 2-sphere).



in the link picture:



Also by band moves using the circle labelled 0 introduced we unknot and unlink the other curves to get the following pictures



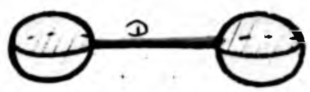
Remark :for the general case when $M \neq S^3$ see [5]. The proof is essentially the same : only for the unknotting a certain curve has to represent 0 in π_1)

(c) 1-handle slides and slides of dotted curves.

We have seen, so far, how a 1-handle can be represented by a dotted curve coming from the belt sphere of a complementary 2-handle. We now try to see the effect of a 1-handle slide on the dotted curves and we will show that a 1-handle slide corresponds to a slide of the dotted circles in the opposite direction with a change of sign.

(c_1) First we see that a 1-handle slide in two complementary pairs has the same effect as a 2-handle slide.

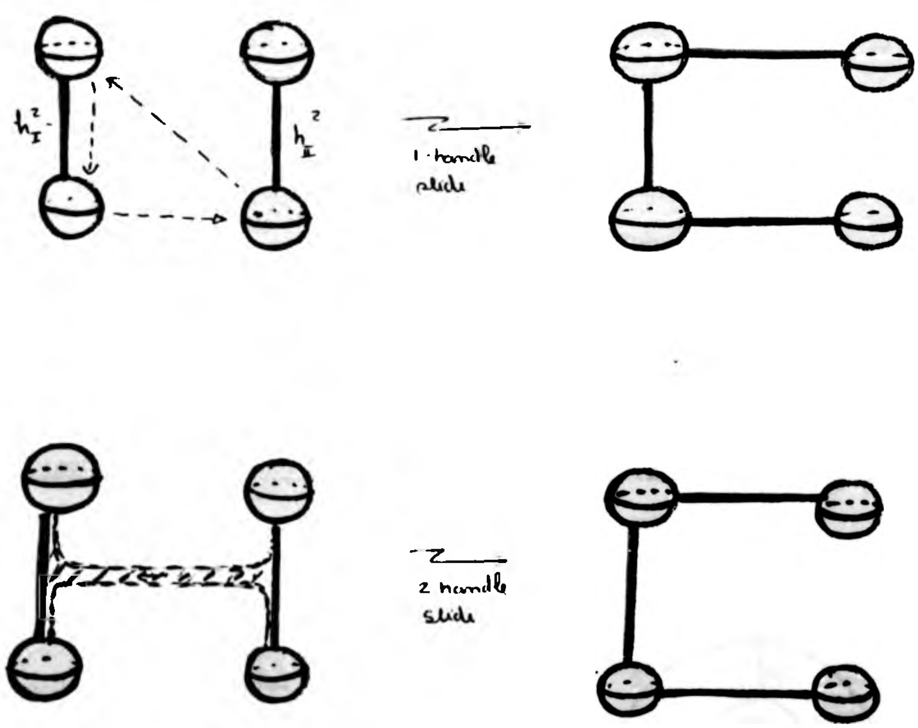
[A 1-handle with a cancelling 2-handle can pictured as in the following picture:



the 2-balls are to be removed and their boundaries identified. The arc D becomes a circle, attaching sphere of the 2-handle.

Of course, this is a simplified picture, the arc D may wander around and have little knots. The same comment for the pictures that follow. However, as this doesn't affect the proof we picture the simpliest case.].

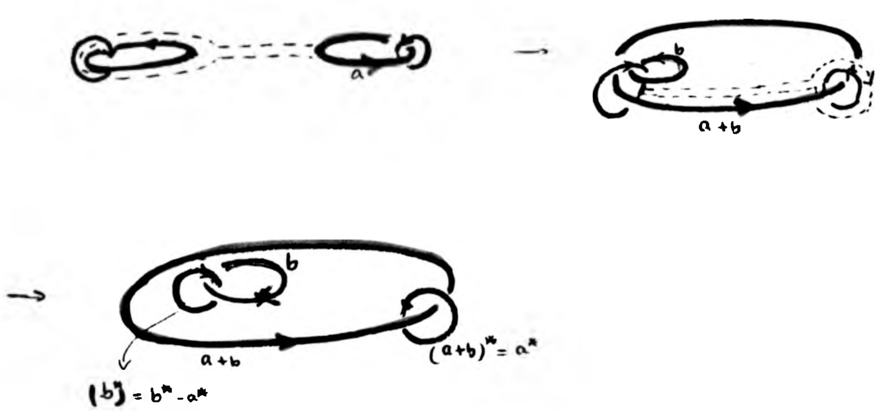
Proof. (cf [5])



(c_2) <u>Next we see how the dual circles of the 2-handles are slid in the opposite direction.</u>

If the 2-handle is represented by its attaching sphere a the dual circle a*, the attaching sphere of the dual handle, (i.e. our dotted curve) is represented by a simple curve a* linking a only once

Then the effect of the slide on the dual circles can be pictured as follows.



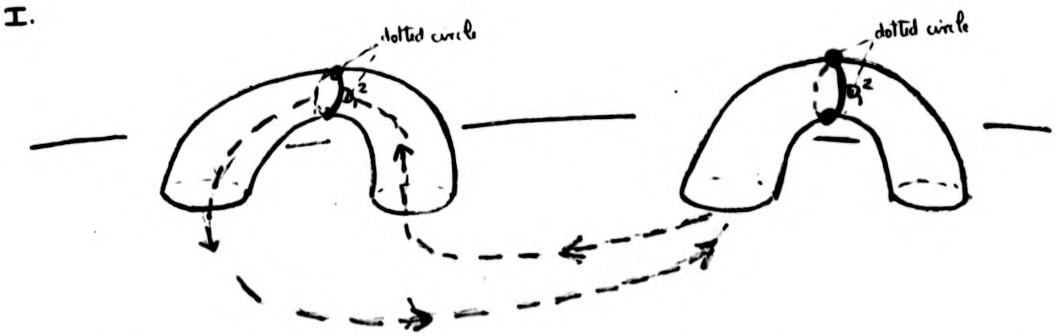
Dual circles after the slide are $(a+b)^* = a^*$

$(b)^* = b^* - a^*$ as required.

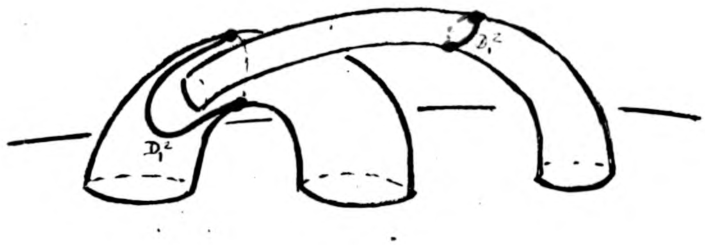
$(\ldots)^*$ means image of $(\ldots)$ after the slide.

(c_3) <u>Another picture of what happens without considering the complementary 2-handles in the following:</u>

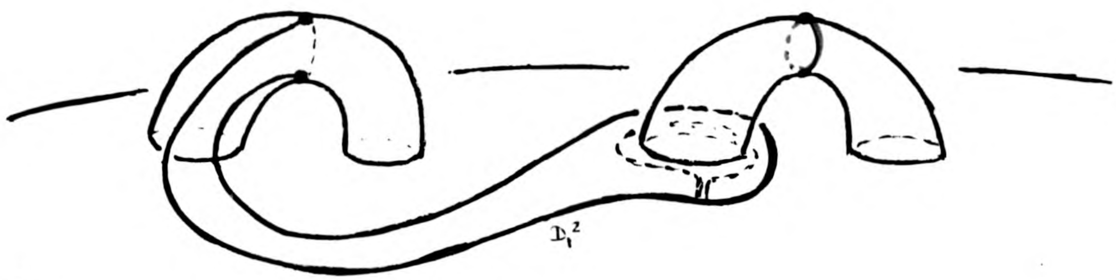
I.



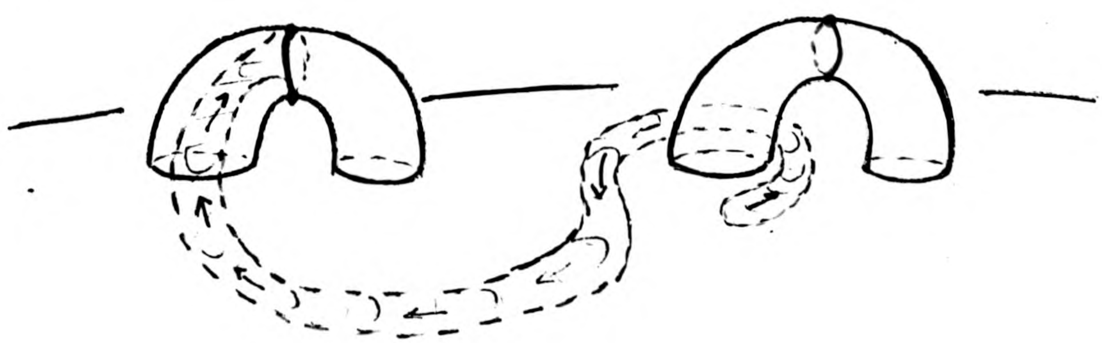
II.



III.



IV.



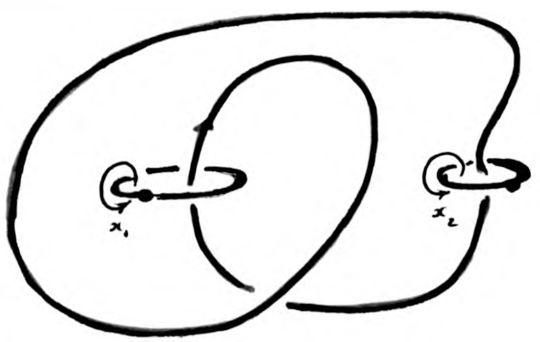
<u>Remark</u> : as the dimensions are wrong, pictures can be misleading. It actually shows the slides of the disc <u>bounded by</u> the dotted circle.

(d) Link pictures and presentations of the fundamental group.

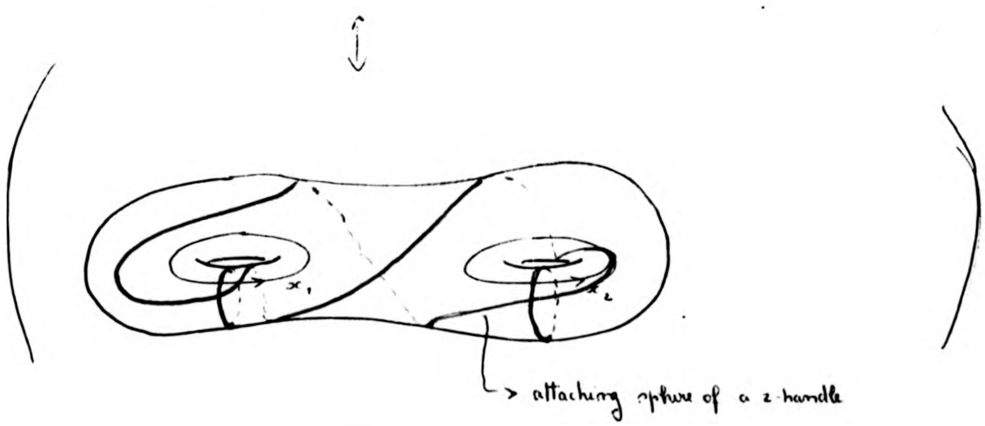
Given a link picture of a 4 manifold M^4 we can then read off from the link a presentation for $\pi_1(M^4)$ as follows:

Orient the curves of the link. Each 1-handle gives a generator which can be represented by an oriented unknotted circle linking the dotted circle once

Attaching spheres of the 2-handles determine the relations. They can be read off from the picture as suggested below:



$\pi_1(M^4) = \{x_1, x_2 : x_1 x_2^{-1} = 1\}$



Tietze moves and handle moves.

Given two finite presentations of a group it is known that by a sequence of moves - the Tietze moves - one can pass from one presentation to the other.

Tietze moves are the following:

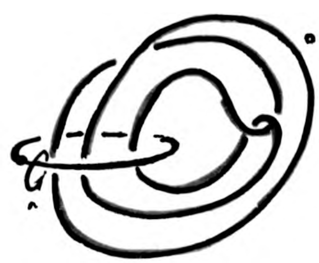
(I) add a generator and a relation which expresses that generator as a word in other generators

(I)' inverse

(II) add a relation which is a consequence of other relations

(II)' inverse.

Some of these moves can be done by handle moves. However, cancellation is not always possible as the following counter-example shows:



I

II

Picture I represents the Mazur Manifold M. M is contractible and $\{a : a^2a^{-1} = 1\}$ is a presentation for π_1 (Here we have the link pictures representing manifolds with boundary). Picture II is a link picture for the 4-ball B^4. A presentation for $\pi_1(B^4)$ is given by $\{a : a = 1\}$. If it were possible to pass from one representation to the other by handle moves this would lead to $M^4 \cong B^4$ which is false ($\pi_1(\partial M^4) \neq 0$).

Remarks : Cancellation can be done when homotopy implies isotopy.

Move II can be done by introducing a complementary (2,3) pair and then sliding the 2-handle over other 2-handles till we get the relation - this is possible since the new relation is a consequence of the other relations of the presentation.

Move I replaces a presentation (x;r) by $(x,y\ ;\ r,yw(x)^{-1})$ where y is a generator not in x (x denotes the generators, r the relations) expressible as a word w(x) in the generators x. It can be done by handle moves in the link pictures as follows:

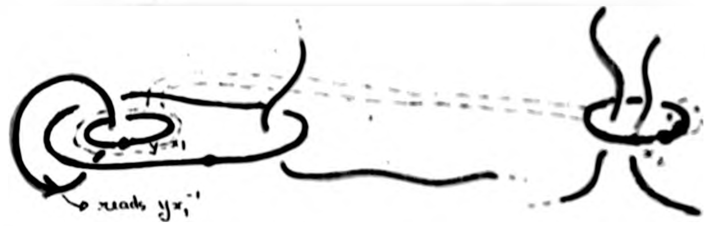
(a) Introduce a complementary (1,2) pair. This changes (x;r) to (x,y : r,y).

( can be assumed to be in a 1-connected part, hence can be numbered - otherwise take  parallel curve).

(b) Slide the new 1-handle over the other 1-handles according to the word w. In the picture dotted circles slide in the opposite directions and with a change of sign

e.g. if $y = x_1 x_2$

1. Slide over x_1



2. Slide over x_2

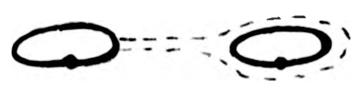
3. Relation between the links pictures given by two handle decompositions.

We have so far, associated a link picture to M^4 for a given handle decomposition. The next natural question is to ask if there is any relation between the links pictures associated with two different handle decompositions.

Let, then $\mathcal{H}_2$ be another nice handle decomposition which as before has only one 0,4-handles and L_2 the associated link picture.

We now define an equivalence relation on the link pictures associated to a manifold M generated by the following Γ - moves.

(a) Trivial slides of the dotted curves over the dotted curves,
i.e. slides of this type



(b) Slides of undotted curves over dotted curves

(c) Slides of undotted curves over undotted curves

(d) Introducing or deleting

(e) Introducing or deleting

(same comment whenever this appears)

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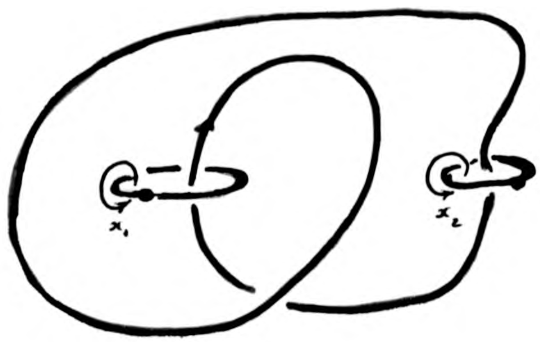

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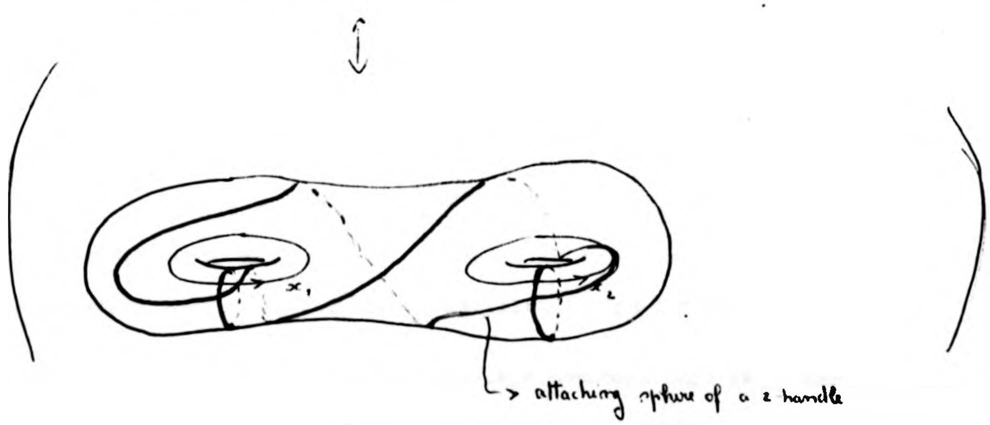

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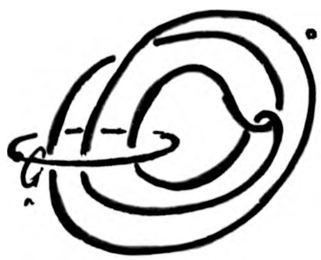

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e.g. if y = x_1x_2

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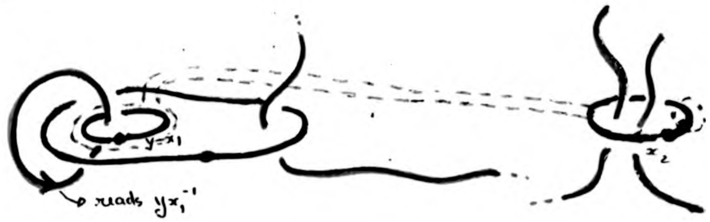

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We have so far, associated a link picture to M^4 for a given handle decomposition. The next natural question is to ask if there is any relation between the links pictures associated with two different handle decompositions.

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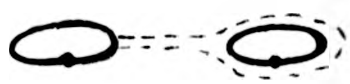
(a) Trivial slides of the dotted curves over the dotted curves, i.e. slides of this type 

(b) Slides of undotted curves over dotted curves

(c) Slides of undotted curves over undotted curves

(d) Introducing or deleting 

(e) Introducing or deleting  / or  → parallel curve

(same comment whenever this appears)

(f) Isotopies of the link picture in S^3.

Moves (a-e) correspond, respectively, to 1-handle slides (in the opposite direction), isotopy of the attaching sphere of the 2-handle (dotted curves bound discs), 2-handle slides, introducing or cancelling (1,2) complementary pairs and introducing or cancelling complementary (2,3) pairs. Move (f) corresponds to an isotopy of the attaching curves of the handles. (Note move (c) is a particular case of (f)). Thus none of them changes the (orientation preserving) homeomorphism class of M.

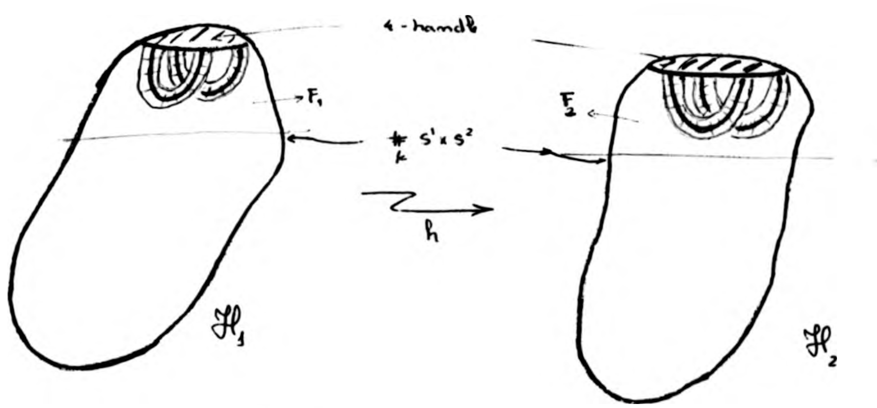
If two link pictures are related by Γ-moves we say they are Γ-equivalent. We now show that L_1, L_2 are Γ-equivalent:

(a) First we will see that at the expenses of some handle moves any homeomorphism $h: M \longrightarrow M$, up to isotopy, preserves the 3 and 4 handles.

By the disc theorem, we can assume that the 4-handle in $\mathcal{H}_1$ goes to the 4 handle in $\mathcal{H}_2$. Considering then the dual decompositions, 3,2 handles give a presentation for π_1. Let $\{x_1 \ldots x_n ; r\}$ $\{y_1 \ldots y_q ; s\}$ be the corresponding presentations for $\mathcal{H}_1, \mathcal{H}_2$ respectively. $y_i = w_i(x)$ $i = 1, \ldots, q$ so by a sequence of Tietze moves I we change $\mathcal{H}_1$ (introducing complementary (2,3) pairs and sliding handles) so that the new presentation has generators $x_1 \ldots x_n, y_1 \ldots y_q$. Similarly as $x_i = w_i'(y)$ we change $\mathcal{H}_2$ so as to get a new presentation with generators $y_1 \ldots y_q, x_1 \ldots x_n$ (Relations can be different !).

We denote still by L_1, L_2 the new link pictures, $\mathcal{H}_1, \mathcal{H}_2$, the new handlebody decompositions and h the new homeomorphism. $\mathcal{H}_1, \mathcal{H}_2$ have now the same number of 3-handles.

Let F_i, i = 1,2, be the union of the 4,3 handles. Both F_1, F_2 are homeomorphic to the same connected sum (along the boundary) of $S^1 \times B^3$'s, but the way this connected sum is embedded in M might be different. However, there is no problem, in our case, as we have already made sure that the 3 handles give (by reading off in the dual decomposition) the same elements of π_1, and as in dimension 4, homotopy implies (ambient) isotopy, there is an ambient isotopy of M which carries the cocore of the 3 handles in $\mathcal{H}_1$ to the cocore of the 3 handles in $\mathcal{H}_2$. Then by the regular neighbourhood theorem there is also an ambient isotopy of M which carries F_1 onto F_2 as required. Thus we can assume h preserves the 4,3 handles.



Then $\overline{M - F_1} \underset{h|}{\cong} \overline{M - F_2}$

(b) Let $W_i = \overline{M - F_i}$ i = 1,2 and let $W \cong W_i$, i = 1,2.

W_1, W_2 give two handle decompositions of W with associated link pictures L_1, L_2. Using the basic starting theorem of Cerf theory [3] [5] (transfer to the smooth category and use transversality), we can assume that the two handle decompositions are related by a sequence of the following moves:

(1) Births and deaths of complementary handle pairs.

(2) Handle slides.

We note the following :

(i) As the two handle decompositions have only 0,1,2 handles we will have to introduce and cancel the same number of (2,3) and (3,4) complementary pairs.

(ii) We can assume all the births take place first all the deaths last – move (f).

(iii) We can eliminate <u>0-handles</u> (and dually <u>4-handles</u>) at the expenses of some 1 (resp.3) handle slides (move (a)). So we are reduced to:

(1) Introducing and cancelling complementary (1,2) and (2,3) pairs.

In the link picture ; introduce or cancel 

(2) 2-handle slides - these correspond to slides of undotted curves - move (c)

(3) 1 handle slides - move (a)

(4) 3 handle slides - we don't see them in the link picture and by remark(i) all 3 handles disappear in the end.

i.e. <u>L_1, L_2 are equivalent by Γ-moves.</u>

(Recall we are always working up to isotopy hence move(b) is allowed).

As Tietze move I can be done by Γ-moves (e) and slidings of 3-handles don't affect the link picture we have proved the following :

<u>THEOREM 2.1</u>

<u>Orientation preserving homeomorphism classes of oriented closed 4-manifolds correspond bijectively to equivalence classes of "special framed links" in S^3, where the equivalence class is generated by Γ-moves.</u>

4. Stable equivalence and link pictures.

We now "stabilise" our result by allowing connected summing with $S^2 \tilde{\times} S^2$ or $S^2 \times S^2$.

The stable equivalence relation on the special framed link pictures is then generated by Γ-moves and by introducing or deleting  ,  (corresponding, resp. to connected summing with $S^2 \tilde{\times} S^2$ or $S^2 \times S^2$). If two link pictures are in the same class we will say that they are Γ_s-equivalent. Hence we have :

THEOREM 2.2

Orientation preserving stable homeomorphism classes of oriented closed 4-manifolds correspond bijectively to Γ_s-equivalence classes of special framed links in S^3.

In particular, if π is the fundamental group of an oriented closed 4-manifold certain Γ_s-equivalence classes of special framed links in S^3 are in 1-1 correspondence with the elements of $\Omega_4(\pi)$.

The non-orientable case

Let M^4 be a non-orientable closed 4-manifold, $\mathcal{H}_1$ a nice handle decomposition of it with only one 0,4 handles. As in the orientable case as any homeomorphism of $\#_k S^1 \tilde{\times} S^2 \#_j S^1 \times S^2$ extends, the cobordism formed by the full 2-handles determines the manifold uniquely up to homeomorphism.

We would like to associate, as in the orientable case, a special framed link picture to $(M^4, \mathcal{H}_1)$ and then define an equivalence class on such pictures so that homeomorphism classes of non-orientable manifolds are in 1-1 correspondence with such equivalence classes.

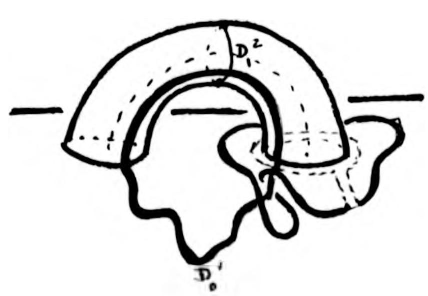
The main problem is that unlike the orientable case, we cannot trade a non-orientable 1-handle for a 2-handle (recall that in the orientable case this fact was used to represent a 1-handle by an unknotted dotted curve). However, we will show that a "certain similarity" between the two cases will enable us to choose an "unknotted curve" to "represent" a non-orientable 1-handle.

Once we have the link pictures for a certain handle decomposition we relate the pictures given by two different handle decompositions. As in the orientable case (proof is the same) we can assume that the 3 and 4-handles are embedded in the same way and thus we only have to interpret on the pictures the moves that relate the decompositions : slides and births and deaths of complementary pairs.

Finally we will consider the stable case.

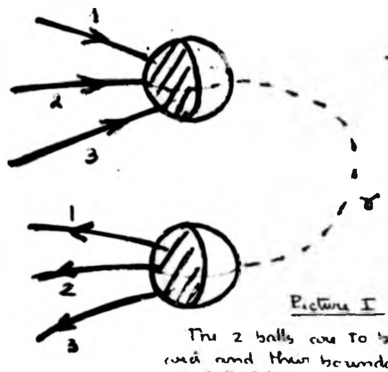
1. <u>Representing the non-orientable 1-handles</u>.

We first note that we cannot trade a non-orientable 1-handle into a 2-handle as $S^1 = D^1 \times (pt) \underset{\partial}{\cup} D^1_0$ (cf notation of orientable case) is a non-orientable curve and so it cannot be the attaching sphere of a 2-handle.



But we still can assume that D^1_0 doesn't have little knots in it and that two handles do not link our S^1 by sliding around one of the ends of the handle.

Think of attaching a non-orientable 1-handle to a manifold as removing two 3-balls from it and identifying their boundaries along an orientation preserving map (e.g. the identity).

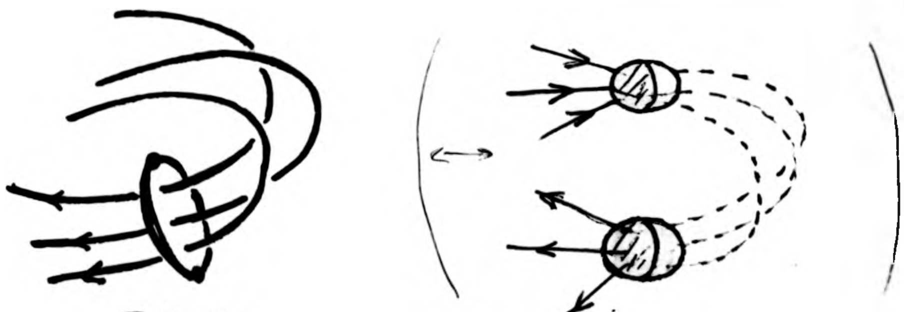


Picture I

The 2 balls are to be removed and their boundaries identified by an orientation preserving homeomorphism

Consider a meridian and let D_1^2, D_2^2 be the 2-discs into which it divides S^2. Clearly we can assume that all the curves attaching spheres of the 2-handles that pass through the handle pierce only one of the discs, D_1, say.

We can therefore think of replacing picture I by the following:



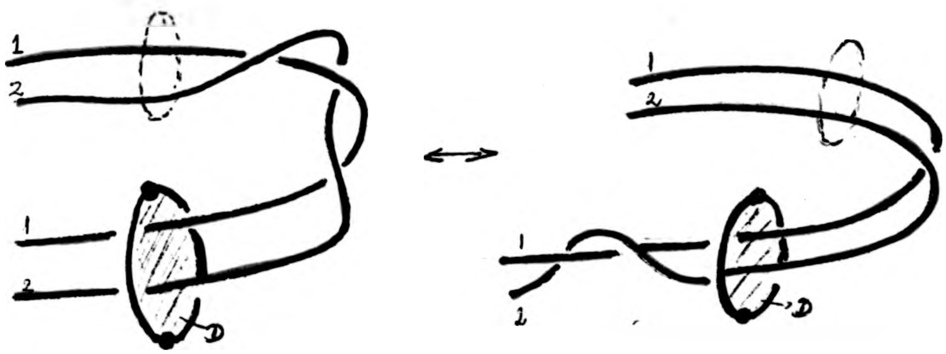
Picture II

where the curves piercing the 1-handle are joined up along a simple path γ (dotted in picture I) with a half twist and ringed by a <u>double dotted circle</u> (corresponding to the meridian that separates D_1 from D_2) with <u>the following conventions</u>:

(a_1) As any curve attaching sphere of a 2-handle passing between the two ends of the 1-handle can be unlinked from other curves, as already mentioned, we allow trivial slides over the double dotted curve to unlink and unknot things (trivial types of slides with no effect on the framings). Move (a_1) is therefore

-any 2 handle can slide over  without alteration of framing. (as it corresponds to an isotopy of attaching curve).

(a_2) As we pass through the 1 handle the space twists : a left hand twist in one side becomes a right hand twist in the other and vice versa.



i.e. these two pictures are equivalent.

The best way to visualise this is to consider the disc D bounded by the dotted curve ; then as we pass through the disc from one side to another the space twists.

2. A special t-framed link picture for a non-orientable closed 4-manifold and a relation between any two such link pictures.

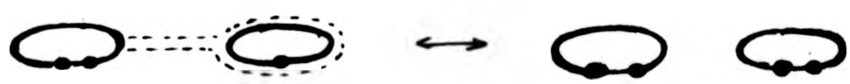
Given M^4 non-orientable closed 4-manifold with a nice handle decomposition we represent orientable 1-handles and 2-handles as in the orientable case (no difference in the arguments). Non-orientable 1-handles are represented as just described with conventions $(a_1),(a_2)$. We then have what we call a "special t-framed link" (t is for twisted). Framings on the undotted curves are given by parallel curves. As in the orientable case we only need to represent 1,2 handles in the link picture , and also we can assume that the link pictures of two different handle decompositions are related by slides of 1,2,3 handles and introducing and cancelling complementary (1,2) or (2,3) pairs.

(i) 1-handle slides.

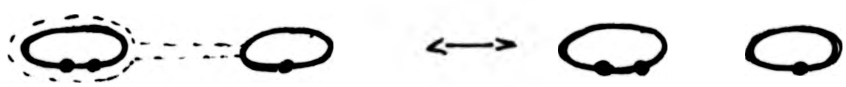
Whenever a 1-handle slides over a non-orientable handle it becomes either orientable or non-orientable if it was respectively non-orientable or orientable before the slide. We claim that again, 1-handle slides correspond to slides of the dotted and double dotted circles in the opposite direction. To see this look at (c_3) (cf orientable case) where it was shown that the dotted circles slide in opposite direction without using the complementary handles (which we cannot use for the non-orientable handles since they do not exist). Same proof works for non-orientable handles.

Thus 1-handle slides can be pictured as follows:

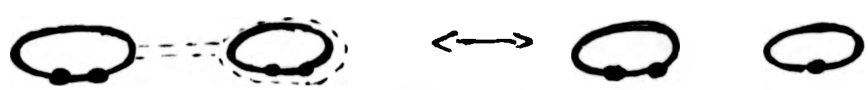
(a_3)



(a_4)



(a_5)



(a_6)

(ii) 2-handle slides and isotopies of attaching curves.

2-handle slides are the same as in the orientable case and as already said any 2 handle can slide over  or  with no alteration on framings (effects of slides on framings are determined by parallel curves.)

(iii) 3-handle slides - again we don't see them in the pictures.

(iv) Introducing or deleting complementary (1,2) pairs - in the picture : introducing or deleting 

(v) Introducing or deleting complementary (2,3) pairs - in the picture : introducing or deleting 

Call Γ_t-moves the Γ-moves together with a_1 — a_6 but with move (f) replaced by isotopies of the link picture subject to (a_2). Γ_t-moves generate an equivalence relation in special framed t-link pictures and as none of them changes the homeomorphism class of the manifold, from the above we get ;

THEOREM 2.3.

Homeomorphism classes of non-orientable 4-manifolds are in 1-1 correspondence with equivalence classes of special framed t-link pictures in S^3 where the equivalence class is generated by Γ_t-moves.

3. The stable case

Again, as in the orientable case, if we allow introducing or deleting  or  and define Γ_t^S moves to be Γ_t moves plus these we get :

THEOREM 2.4

Stable homeomorphism classes of non-orientable closed 4-manifolds are in 1-1 correspondence with Γ_t^S -equivalence classes of special framed t-link pictures in S^3.

REFERENCES

[1] ALEXANDER, "On the deformation of an n-cell," Proc. Nat. Acad. Sci. (USA) 9, (1923), 406-407

[2] S.E. CAPPELL and J.L. SHANESON, "On 4-dimensional surgery and applications," Comm. Math. Helv. 46 (1971), 500-528

[3] J. CERF, "La stratification naturelle des espaces de fonctions differentiables reelles et le théorème de la pseudo-isotopie", Publ. Math. I.H.E.S. 39 (1970)

[4] E. CÉSAR de SÁ', "3 manifolds as numbered links", MsC dissertation, Warwick University (1975)

[5] R. FENN and C.P. ROURKE, "On Kirby's Calculus of Links" (to appear)

[6] D.I. FUCHS and RABINOVITCH, "On the Automorphism Group of Free Products I", Math. Sbornik 50 (NS8) (1940), 265-276

[7] D.I. FUCHS and RABINOVITCH, "On the Automorphism group of Free Products II", Math. Sbornik 9 (1941), 183-220

[8] H. GLUCK, "The Embeddings of 2-spheres in the 4-sphere", Trans. Am. Math. Soc. 104 (1962), 308-333

[9] A. HAEFLIGER and V. POENARU, "La classification des immersions combinatoires", Publ. I.H.E.S. 23 (1964), 75-91.

[10] A. HATCHER, "Homeomorphisms of sufficiently large P^2-irreducible 3 manifolds", Topology 15 n°4 , (1976), 343-348.

[11] J. HEMPEL, "3-Manifolds", Annals of Math. Studies 86, Princeton (1976)

[12] H. HENDRIKS, "Applications de la théorie d'obstruction en dimension 3, C.R. Acad. Sci. Paris, t.276, série A. (1973), 1101-1104.

[13] J. HUDSON, "Extending Piecewise Linear Isotopies", Proc. London Math. Soc. (3) 16 (1966) 651-663.

[14] J. HUDSON and E.C. ZEEMAN, "On Combinatorial Isotopy" Pub. I.H.E.S. 19 (1964), 69-94

[15] R. KIRBY, "A Calculus for Framed Links" (to appear)

[16] H. KNESER, "Geschlossen Flachen in dreidimensionalen Mannifaltigkeiten", Jahresbericht der Deutschen Mathematiker Vereimigung, 38 (1929), 248-260

[17] F. LAUDENBACH, "Topologie de la dimension trois : homotopie et isotopie" Asterisque 12, Soc. Math de France

[18] J. MILNOR, "A unique decomposition theorem for 3-manifolds", Amer. J. Math. 84 (1962), 1-7.

[19] J. MILNOR, "Microbundles - Part I", Topology 3 suppl. (1964) 53-80

[20] J. NIELSEN, "Die Isomorphismengruppe der frein Gruppen", Math Ann. 91 (1924)

[21] C.P. ROURKE and B.J. SANDERSON, "Introduction to Piecewise Linear Topology", Springer-Verlag 69 (1972)

[22] G.P. SCOTT, "Some problems in Topology", Ph.D Thesis, Warwick University (1969)

[23] C.T.C. WALL", On simply connected 4-manifolds, Journal London Math. Soc. 39 (1964), 141-149

[24] C.T.C. WALL, "Diffeomorphisms of 4-manifolds" Journal London Math. Soc. 39 (1964), 131-140