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THE KOCHEN – SPECKER THEOREM IN QUANTUM MECHANICS: A PHILOSOPHICAL COMMENT (Part1)

Резюме: Обсъжда се фундаменталната теорема на Коушън и Шпекър (1967) в квантовата механика във философски контекст. Изходна точка за открояване на нейния смисъл е аналогичната предходна теорема на фон Нойман (1932) за отсъствие на скрити параметри в квантовата механика. Докато последната обвързва отсъствието на такива с наличието на некомутиращи оператори и със съответните им спрегнати физически величини, то двамата автори показват, че необходимо и достатъчно условие за това е самата дуалност на вълна и частица в квантовата механика. Обичайно е разглеждането на теоремата като обобщение на тази на Бел (1964). По такъв начин квантовите корелации, които следват от теоремата на Бел, могат да се изведат от дуалността на вълна и частица. Обсъжда се непосредственото следствие от теоремата в контекста на квантовата информация като невъзможността кюбит да се представи еквивалентно като крайно множество от битове и оттук – защо квантовият компютър е нетюрингова машина. Друга линия на възможна интерпретация на теоремата е като обобщение на Айнщайновия общ принцип на относителността(1916–1918): от дифеоморфизмите, за които той е валиден,към дискретни морфизми, тълкувани като квантови движения, при които скоростта не може да се дефинира като еднозначна крайна величина.По този начин квантовата механика може да се тълкува като обобщение на общата теория на относителността за дискретни морфизми, т.е. за квантови движения. От подобна позиция може да се предложи нестандартно тълкуване за съотношението на общата и специалната теория на относителността, при което първата е интерпретация на обобщения математически формализъм на втората за скорости, надвишаващи тази на светлината във вакуум.Това налага осмисляне на вероятността като физическа величина в контекста на скоростта. Проследява се начинът, по който Коушън и Шпекър интерпретират несъизмеримостта на физически величини в квантовата механика математически, като отсъствие на обща мярка, както и преходите между хилбертово и фазово пространство, залегнали във фундаменталните работи на Вайл (1927), Вигнер (1932) и Мойл (1947), връзката между обобщения на понятието за вероятност и отсъствието на скрити променливи. Специално внимание е обърнато на примера в §6 на статията от двамата автори за разликата между теоремата на фон Нойман и тяхната собствена, както и на следствието за „класическата тавтология, която е невярна, ако се замести с твърдения за съизмерими квантово-механични величини". Оттук се предлага понятие за логически скрити променливи. Засяга се въпросът за представимостта на причиността чрез случайността.

Key words: Kochen – Specker theorem, Bell's theorem, wave-particle duality, generalized relativity, Hilbert space, phase space, negative probability, complex probability, qubit, quantum computer, Ψ-function by qubits

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I. The Kochen-Specker Theorem and Its Context:

At first glance, the work of Kochen and Specker reiterates well-known results:

The main aim of this paper is to give proof of the nonexistence of hidden variables. This requires that we give at least a precise necessary condition for their existence (Kochen, Specker 1967: 59).

In fact, their work was revolutionary, fundamentally new in regard to the proof and the foundation of the claim given initially by von Neumann. Before it, the non-existence of the hidden parameters in quantum mechanics had been attributed to non-commuting operators and observables (e.g. in Dmitriev,2005:435, who summarizes the premises of von Neumann's theorem). Kochen and Specker demonstrated the impossibility of hidden parameters even with regard to commuting operators in quantum mechanics. Respectively, in the case of statements about commuting, and therefore commensurable, quantum-mechanical observables, classical logic is not always applicable, because in quantum mechanics its tautologies might prove refutable and even identically false.

Furthermore, after a more detailed look at their proof, we underline the fact that, according to their interpretation, the absence of hidden parameters is due to the necessity of common considering discrete and continual morphisms, i.e. to wave-particle duality in the last analysis.

Thereupon, they tacitly understand the hidden parameters as local ones, since the Lorentz invariance still remains in force, restricting the generalization of the continuous functions as Borel functions, and this enables the precise translation of the commensurability of quantum-mechanical observables into mathematical language as a common measure in the rigorous mathematical meaning of the concept 'measure'. Thus non-local hidden parameters – which are left outside the range of Kochen and Specker's article – are completely and implicitly ignored, on the grounds that their Lorentz non-invariance implies their mathematical and physical incommensurability with the quantities to whose functions they should serve as arguments.

On the other hand, Dirac delta functions or Schwartz distributions (generalized functions), which have long been involved in the apparatus of quantum mechanics, do not require such mathematical commensurability of the areas of the argument and the values of the generalized function. At times the local (Lorentz invariant) hidden parameters are unduly confused with hidden parameters in general (including the violation of Bell's inequalities opposite to Kochen and Specker's results), but this confusion does not evolve either explicitly or implicitly from their article.

Kochen and Specker's text – both rigorous and precise, heuristic, and containing radically new ideas and approach, not only gives rise to a great number of subsequent studies, but has still not exhausted its intrinsic potential. In the beginning of their article the authors present their conception concisely; it can be summed up as follows: if we look at the previous attempts to introduce hidden variables (e.g. the Bohm theory, 1952, or the description of the general model made by von Neumann - see Penchev 2009, ch.4), the paradigm of classical statistic mechanics appears:

The proposals in the literature for a classical reinterpretation usually introduce a phase space of hidden pure states in a manner reminiscent of statistical mechanics. The attempt is then shown to succeed in the sense that the quantum mechanical average of an observable is equal to the phase space average (Kochen, Specker 1967: 59).

Von Neumann used to underline quite explicitly that the half of the $2k$ variables of the configuration space of k micro-objects are „superfluous". redundant and simultaneously fully adequate to describe again the same micro-system if the other half of the same variables, k in number, used in the first description are now left aside as redundant. The two descriptions are incompatible, complementary, or dual in the intention of Bohr, but they both give the same probabilistic description of the micro-system, which, as Schrödinger (1935: 827) highlighted, is all that can be possibly known of it.

Hence the phase space must be modified, in order to be applicable in quantum mechanics: one modification was made by Wigner (1932) and Moyal (1949) on the base of the preceding fundamental work of Weyl (1927): e.g. the basic cell in the classical phase space is the product of quantities – position and momentum – which are non-commuting in quantum mechanics; therefore each cell is duplicated in order in which the quantities are multiplied. As this is independently valid for each of the cells in the phase space, the variants of the phase space that have to be referred to the same quantum system are found to be $2k$ as a number instead of the only single one in classical consideration.

Since the observables in the two sets are conjugated, each with the one to which it is relevant , and their operators do not commute (e.g. position and momentum for every particular micro-object according to the uncertainty relation), there may be propounded the hypothesis by analogy, unlawful as a strict logical inference, that the non-commutability of the operators (or the observables in quantum as contrasted to classical mechanics) is the premise, the precondition for the absence of hidden parameters. Hence it becomes obvious that, if hidden parameters exist, the physical quantities would commute with each other in the same way as in classical mechanics. As the non-commutability does not allow a physically relevant interpretation of the product and even the sum of two such non-commuting quantities (demonstrated in the Hermann (1935) – Bell (1966) argument), „the back door" of our ignorance, behind which the cherished „true" hidden variables could be found eventually, remains. Notice that we speak of another (second!) heuristic hypothesis by analogy.

Kochen and Specker showed categorically and unambiguously (i.e. by a counter-example) that the non-commutability of the observable variables is not the premise for the absence of hidden parameters: Commutability is not an indispensable condition for hidden variables, and thus they clear their way for formulating a logically strict indispensable condition, instead of the „heuristic", and in fact wrong, hypothesis based on a misleading analogy to classical statistic mechanics.

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Their interpretation of the commensurability of physical quantities in quantum mechanics by the mathematical concept of „commensurability" (and thereby of „measure") is a decisive step. The measure of a function does not require the latter to be continuous, but only almost continuous, i.e. the measure of the set of points where it is discontinuous must be zero. Two quantities of a common measure are commensurable and commutable.

The algebraic structure to be preserved is formalized ... in the concept of a partial algebra. The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute (Kochen, Specker 1967: 59-60).

Nevertheless, although commensurable and commutable, they do not allow hidden parameters, as Kochen and Specker show, since the indispensable condition for their presence is not fulfilled: the embeddability of „partial algebra" (according to the concept of the two authors, by which they formalise commensurable quantities) of quantum-mechanical quantities in commutative algebra. Respectively the statements on such quantities – so-called *partial* Boolean algebra – is not embeddable in Boolean algebra; in other words, to put it more contemporarily, one qubit is not embeddable in one bit, a quantum computer is not a Turing machine.

A necessary condition then for the existence of hidden variables is that this partial algebra be embeddable in a commutative algebra (such as the algebra of all real-valued functions on a phase space) (Kochen, Specker, 1967: 60).

Then

it is shown that there exists a finite partial algebra of quantum mechanical observables for which no such embedding exists. The physical description of this result may be understood in an intuitive fashion quite independently of the formal machinery introduced (Kochen, Specker 1967: 60).

So it comes natural to ask how one can explain the different behaviour of physical quantities in classical and in quantum mechanics – the determinism of the former and the indeterminism of the latter – if the demarcation "commutability – non-commutability" no longer has meaning. Obviously the only difference left is the continuality of the quantities in classical physics and their discrete character as a rule in quantum mechanics, or in other words – the validity of its field of the principle of quantum-mechanical duality. The real premise for the absence of hidden parameters could be formulated as invalidity of Einstein's principle of relativity (Einstein, 1918: 241) and, resulting from it, the suspension of Mach's principle (ibid.): the concepts of speed or resp., of diffeomorphism are not universal in regard to mechanical as well as physical movement.

Along with this, the requirement for the Lorentz invariance may remain in force, whereas the discontinuities appear to be in space-time and it corresponds to the velocity

confined to the same maximum, which is defined by the fundamental constant of the velocity of light in free space. This is precisely the implicit model with which Kochen and Specker comply, suggesting the ordinary consideration of hidden variables as local ones. That is the reason why their statement regarding the absence of hidden variables concerns only local ones and does not affect either the type of investigation made by Bell or the possibility of violating the inequalities introduced by him.

Here we should raise again the question of ineradicable insolubility, which faces any profound philosophical discussion of quantum mechanics. Because of the Skolemian relativity of the discrete and continual, the absence of hidden parameters also seems to be Skolemian relative, including the manner of their exposition in Kochen and Specker's article. After proving their famous theorem and its implications, they gave a counter-example introducing hidden parameters limiting their consideration to two-dimensional Hilbert space and a model of a single electron spin, emphasizing that it is completely artificial and even invalid in the case of two electrons in a potential field, according to their words. However, their intention was thus to show that von Neumann's theorem requires in that case the absence of hidden parameters, while their own consideration would demonstrate the possibility of introducing such parameters.

In turn, we may easily show that this counter-example is isomorphic to a qubit, since it represents a sphere in three-dimensional Euclidean space, and because of the qubit additivity, it can be transferred and consolidated for the whole Hilbert space. In other words, this is also a counter-example regarding their main theorem and is in direct contradiction to the immediate corollary. That is why the theorem should obtain the statute of yet another unsolvable claim in quantum mechanics – the one side of a complementary, dual relation whose other side is precisely its negation. Together, they demonstrate the same, suggesting that it is only one special case; on the basis of Skolemian-type relativity we can talk about a special kind of quantum duality: absence – presence of hidden parameters. But how then should we interpret the hidden parameter? According to the illustration that Kochen and Specker have given, this is a random position on a disc, i.e. on a large circle of the sphere. In the general case of ordered sum of qubits representing Hilbert space, the hidden parameter will be the angle formed from the „axes" of Hilbert space, which represent an infinite number of decreasing oscillators embedded into one another. That angle may be interpreted as an initial moment in time: for example, if we have chosen a zero point in time for all oscillators, then Hilbert space as an ordered set of qubits will be displayed in a simple and determined manner by the hidden parameters as an infinite strip. That is also a respectively ordered set of zeros and ones according to the following (not the only possible) rule: 0, if the hidden parameter determines a past moment in time corresponding to the chosen zero benchmark; and 1, if it determines present or future moment. In addition, „curved" Hilbert space can be compared in a simple manner with pseudo-Riemannian space and thus be so interpreted. Therefore the parameter can be construed as gravity. These two interpretations of the hidden parameter – temporal

and gravitational – again prove to be dual, which turns out to be a normal expectation in quantum mechanics.

At the end of their article, in the last § 7, the authors suggest that their consideration may be logically demonstrated as the impossibility of embedding (resp. weak embedding – homomorphism) the partial Boolean algebra of quantum-mechanical observables in Boolean algebra.

It is proved there that the embedding problem we considered earlier is equivalent to the question of whether the logic of quantum mechanics is essentially the same as classical logic (Kochen, Specker 1967: 60).

Thence they deduce that there is a classical tautology, φ, which is false even in meaningful substitution, i.e. the substitution with statements concerning commensurable quantum-mechanical variables:

Roughly speaking a propositional formula $\psi(x_1, \dots, x_n)$ is valid in quantum mechanics if for every „meaningful“ substitution of quantum mechanical propositions P_i for the variables x_i this formula is true, where a meaningful substitution is one such that the propositions P_i, are only conjoined by the logical connectives in $\psi(P_1, \dots, P_n)$ if they are simultaneously measurable. It then follows from our results that there is a formula $\psi(x_1, \dots, x_n)$ which is a classical tautology but is false for some meaningful substitution of quantum mechanical propositions. In this sense the logic of quantum mechanics differs from classical logic (Kochen, Specker, 1967: 60).

And they immediately give a simple example of such a tautology. According to our principle position we will pay attention once again to the alleged relativity of this statement, i.e. from what kind of dual, complementary, but also quite legitimate, position the opposite is true: the non-existence of such a classical tautology or no substitution of the quantum mechanical observables, which makes that classical tautology false.

For this purpose the concept of hidden parameter should be transferred to a properly logical consideration. Such would be the presence of a hidden unsolvable statement, in other words, a hidden axiom. Thus, whether it or its negation is accepted will determine whether the statement on quantum mechanical observable is true or false. Embeddability (weak embedment, homomorphism) is the necessary condition for the existence of such a logical hidden parameter. Respectively the absence of such embeddability ensures its non-existence. Then our propositional formula ψ, which is a classical tautology, will appear to also be true in substitution for commensurable quantum-mechanical observables. Therefore, the very formula ψ is of the desired type of unsolvable statement or a logically hidden parameter. In this case, any propositional formula that is true in a classical sense and false in a quantum-mechanical

substitution, as stated above, is such a logically hidden parameter, an unsolvable statement.

There emerges a common and fundamental hidden parameter of such logical type – an unsolvable statement: whether a randomly given formula will be considered classical or quantum-mechanical. That cannot surprise us at all, as it is built in the very foundation, in the structure and mathematical formalism of quantum mechanics as a theory about the system of a classical device and a quantum object. Accordingly, such insoluble, dually true propositions about the system can be solved when referred either only to the apparatus, or only to the micro-object. But the second reference itself contains an element of insolubility and its being a theory of the micro-object does not in itself seem possible.

With a similar reservation reducing the mere statement about the existence of hidden parameters to insolubility, however, the opposition or the duality between device and quantum object may be assumed and therefore interpreted in any special case as a universal hidden parameter in the logical sense.

Finally, the same situation can be demonstrated by the counter-example given by them, in which a sphere, as a qubit, will be compared to the propositional formula ψ of quantum-mechanical observables; and a usual bit, i.e. a binary unit, to the true value of the propositional formula ψ, classically interpreted.

Kochen and Specker's conclusion indicates the significance of their work to the overall development of thought in quantum mechanics, which we have already tried to sketch briefly:

> *This way of viewing the results of Sections 3 and 4, seems to us to display a new feature of quantum mechanics in its departure from classical mechanics. It is of course true that the Uncertainty Principle, say, already marks a departure from classical physics. However, the statement of the Uncertainty Principle involves two observables which are not commeasurable, and so may be refuted in the future with the addition of new states. This is the view of those who believe in hidden variables. Thus, the Uncertainly Principle as applied to the two-dimensional situation described in Section 6 becomes inapplicable once the system is imbedded in the classical one. The statement, we have constructed deals only in each of the steps of its construction with commeasurable observables, and so cannot be refuted at a later date (Kochen, Specker 1967: 86).*

Let us start our detailed discussion of the work of Kochen and Specker by proceeding from the possibility, the difficulties, and the ways to use the phase space of classical mechanics and thermodynamics, as it acts as a bridge between the statistical interpretations of the latter by the former, and thus sets a successful example for the introduction of „hidden parameters. Therefore, any confirmation of such impossibility must clarify precisely what exactly is the difference between classical and quantum me-

chanics that deters us from following this method. We also have the major works of Weyl (1927), Wigner (1932), Groenewold (1946) and Moyal (1949), which show with mathematical rigor the degree of correspondence between Hilbert and phase space. They demonstrate how and by what necessary generalizations of the classical phase space in the latter may be present and deployed by the standard formalism of quantum mechanics based on Hilbert space.

The work of Wigner is fundamental. As for the study of Weil, it is historically the earlier (1927) and is based on the theory of groups, some of the simplest and most fundamental objects of abstract algebra, equipped with a single binary operation, a reverse element to any, and a single neutral element coinciding with its reverse element. The theory of representations[1] is also interesting − Hermann Weil should be considered its founder[2] − and the study in question clarifies the meaning of such an abstract mathematical theory for quantum mechanics as well.

The main idea of the theory of representations − the identification under certain conditions, namely the availability of representation in general, of the groups and of (the transformations of) Hilbert space, will allow us to make a decisive step forward in studying the relativity of the continuous and discrete in a mathematical and in a physical, and in a philosophical sense as well. If the group is not only continuous but also smooth, i.e. differentiable, such as Lie groups are, we could equate it, at least mathematically, by its presentations, to Hilbert space of Ψ-functions, i.e. of quantum, therefore discrete, states. If the Lie group itself embodies Einstein's principle of general covariance (relativity), we should clarify how exactly (or namely) Ψ-function presents a quantum, discrete state. It will help us to move forward from a merely *qualitative* relativity of continuity and discreteness to a *quantitative (in a broad sense, by mathematical structures)* description of their unity and the transition between them.

Ψ-function presents the discrete by the random as follows. It is always a function of arguments consisting of exactly half the parameters in the configuration space as are in the classical case and those parameters may be considered as continuous ones. The other half – according to Heisenberg's uncertainty – prove to be completely vague, random, and discrete. Since there is a quantum leap (discreteness), that second half of the parameters appears to be a set of random variables, which may assume one value or other with different probability.

Then we will interpret Ψ-function, in the spirit of Bartlett's approach (1945), as the characteristic function of the discrete and therefore random coordinates in configuration space. The other half of the coordinates in configuration space simply do not need a description by Ψ-function, since, being continuous, they are not random.

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From this point of view „the problem of hidden parameters" appears to be a result of misunderstanding: Ψ-function does not summarize, but only complements the continual description of classical physics with its discrete „mate", where the discrete is represented by the accidental. The other „half". i.e. the continual description itself, is given by the presence – inevitable in quantum mechanics – of classical device. Hence the importance of the theory of representations for interpreting or creating the ontology of quantum mechanics: it provides the possibility, unity and quantitative equivalence of the discrete description of quantum phenomena in terms of micro-object, and their continual description in terms of device.

In such „translations" between both languages, we should pay special attention to the consubstantiality and the equivalent transformation of the speed from a smooth description (i.e. not only mathematically continual but also differentiable) in the probability from a discrete description. There comes the conclusion that Lorentz invariance (and respectively the postulate of no exceeding the speed of light in free space) is a direct result of a principle already involved in the previous sentence, which is valid for the imposed generalisation of Einstein's relativity principle for discrete motions: since gravity and inertia are treated equally in general relativity, velocity and probability should coincide as to the sketched more general view. However, this would be possible only if there is a fundamental constant of maximal velocity, in relation to which any velocity is converted to a dimensionless number that for all less or equal to the maximal velocity is respectively less than or equal to one and can therefore be interpreted as a standard probability.

If, however, we use Bartlett's approach and introduce negative probabilities (and hence those which are greater than one), then the speeds exceeding that of light should also be discussed, according to a principle of equivalence of velocity and probability. Conversely, the complex speed or other kinematic physical quantities, emerging from the mathematical formalism of special relativity, which get complex values, are immediately interpreted as the complex probabilities explained above or the physical quantities of entangled systems studied by quantum information. The tachyons theory developed in the second half of last century could be identified with quantum information or, more precisely, with its translation into the diffeomorphism language of classical physics. So the Wigner function (Wigner, 1932: 750) is in fact interpreted as the corresponding and earlier translation into the classical language of smooth transformation from the previously postulated discreteness of quantum mechanics.

On the one hand, our world, well-described by classical physics, allows an equivalent quantum description towards a sufficiently massive mega-object losing its *causality, **equivalently** replaced by randomness*. On the other hand, we could extrapolate the situation regarding micro-objects studied by quantum mechanics and information in hypothetically introducing an analogous classical physical description for them (by diffeomorfisms, causal, using as a hidden parameter the moment of time within the almost

eternity of their own present). A similar hidden parameter, of course, can nowise be defined in terms (quantities) of the massive object. That is why we can generalize in the spirit of Skolemian relativity that both following statements are valid: there is and there is not a 'hidden parameter' in quantum mechanics: the latter is from the viewpoint of the appliance, the former from that of the micro-object. In classical physics, in science or even in knowledge in general, the empirical and the objective never come to such direct impact between each other. The objective is also interpreted as the hidden, non-empirical, also as the random, non-causally impacting on the practical world, also as the ideal, non-material, and also as the numinous, sacral, non-profane. [ii]

NOTES

[1] And in particular, Lie groups in the automorphisms of Hilbert space.

[2]. A work (Peter, Weyl 1927) co-authored with his student, dating from the same year, should be mentioned in a properly mathematical aspect. Its main theorem essentially ensures that any group fulfilling certain broad conditions can be juxtaposed with one or even one-one Hilbert space determined by its orthonormal basis, if the group has a representation into it: In other words, representation is the condition (its boundaries of necessity or sufficiency could be investigated in different cases) for identifying a group with (a) Hilbert space.

LITERATURE

Пенчев, В. 2009. *Философия на квантовата информация*. Т. 1. *Айнщайн и Гьодел*. София: ИФИ–БАН [Penchev, V. Philosophy of Quantum Information. Vol. 1 Einstein and Gödel, in Bulgarian].

Bartlett, M. 1945. Negative probability. – *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 41, No 1 (June 1945), 71–73.

Bell, J. 1966. On the Problem of Hidden Variables in Quantum Mechanics. // *Reviews of Modern Physics*. Vol. 38, No 3 (July), 447–452. (Bell, J. *Speakable and unspeakable in quantum mechanics: collected papers in quantum mechanics*. Cambridge: University Press, 1987, 1–13)

http://www.ffn.ub.es/luisnavarro/nuevo_maletin/Bell%20(1966)_Hidden%20variables.pdf

Bohm, D. 1952. A Suggested Interpretation of the Quantum Theory in Terms of „Hidden" Variables. I. // *Physical Review*. Vol. 85, No 2, 166–179

http://dieumsnh.qfb.umich.mx/archivoshistoricosMQ/ModernaHist/david%20bohm%20I.pdf

Bohm, D. 1952. A Suggested Interpretation of the Quantum Theory in Terms of „Hidden" Variables. II. // *Physical Review*. Vol. 85, No 2, 180–193. –

http://dieumsnh.qfb.umich.mx/archivoshistoricosMQ/ModernaHist/david%20bohm%20II.pdf

Kochen, S., E. Specker. 1967. The Problem of Hidden Variables in Qunatum Mechanics. – *Journal of Mathematics and Mechanics*. Vol. 17, No 1, 59–87 –

http://viper.princeton.edu/~mcdonald/examples/QM/kochen_iumj_17_59_68.pdf .

Einstein, A. 1918. Prinziplelles zur allgemeinen Relativitätstheorie. – *Annnalen der Physik.* Bd. 55, № 4, 241–244. – http://www.physik.uni-augsburg.de/annalen/history/einstein-papers/1918_55_241–244.pdf

Groenewold, H. 1946. On the principles of elementary quantum mechanics. – *Physica.* Vol. 12, No 7 (October 1946), 405–460.

Hermann, G. 1935. The circularity in von Neumann's proof. (Translation by Michiel Seevinck of „Der Zirkel in NEUMANNs Beweis". section 7 from the essay by Grete Hermann, *Die Naturphilosophischen Grundlagen de Quantenmechanik.* Abhandlungen der fries'schen Schule,6,1935.) http://www.phys.uu.nl/igg/seevinck/trans.pdf).

Moyal, J. 1949. Quantum mechanics as a statistical theory. – *Proceedings of the Cambridge Philosophical Society.* Vol. 45, No 1, 99–124.
http://epress.anu.edu.au/maverick/mobile_devices/apc.html .

Neumann, J. von. 1932. *Mathematische Grundlagen der Quantenmechanik.* Berlin: Verlag von Julius Springer (http://www.calameo.com/read/000187019bb347837a6bf). In English: J. von Neumann. 1955. *Mathematical Foundations of Quantum Mechanics.* Princeton: University Press. Й. фон Нейман. 1964. *Математические основы квантовой механики.* Москва: „Наука" (in Russian).

Peter, F., H. Weyl. 1927. Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. – *Mathematische Annalen.* Vol. 97, No 1, 737–755.

Schrödinger, E. 1935. Die gegenwärtige situation in der Quantenmechanik. – *Die Naturwissenschaften*, Bd. 48, 807–812; Bd. 49, 823–828, Bd. 50, 844–849. (In English: http://www.tu-harburg.de/rzt/rzt/it/QM/cat.html; Russian translation: Шредингер, Э. 1971. Современное положение в квантовой механике. – В: Э. Шредингер. *Новые пути в физике.* Москва: „Наука 1971, 66–106.)

Weyl, H. 1927. Quantenmechanik und Gruppentheorie. – *Zeitschrift für Physik.* Vol. 46, No 1–2, 1–46. (H. Weyl. *Gesammelte Abhandlungen.* B. III. Berlin – Heidelberg – New York: Springer.)

Wigner, E. 1932. On the quantum correction for thermodynamic equilibrium. – *Physical Review.* Vol. 40, No 5 (June 1932), 749–759.

REFERENCES

Penchev, V. 2009. Filosofija na kvantovata informacija [Philosophy of quantum information]. IFI-BAN [IPhR-BAS], Sofia.