# Embracing monotonicity

Luis Felipe Bartolo Alegre

UN Mayor de San Marcos, Lima, Perú
luis.bartolo@unmsm.edu.pe

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**Abstract**

A *non-embracing* consequence relation is one such that no set of wffs closed under it is equal to the set of all wffs. I prove that these relations have no deductive power if they are also extensive and monotonic.

Let $\alpha, \beta$, ... stand for sentences of a set $\mathbf{L}$ of wffs. $\mathbf{L}$ can be inductively defined in the usual way, although we are not going to presuppose any particular definition. Also let $\mathbf{A}, \mathbf{B}$, ... stand for subsets of $\mathbf{L}$, which may be finite or not. A *consequence relation* $\vdash: \wp\mathbf{L} \times \mathbf{L}$ has in its extension those pairs $\langle \mathbf{A}, \alpha \rangle$, such that $\alpha$ is a logical consequence of $\mathbf{A}$. The expression $\langle \mathbf{A}, \alpha \rangle \in \vdash$ is usually abbreviated by $\mathbf{A} \vdash \alpha$. Finally, the *consequence set* of $\mathbf{A}$, or $\mathbf{A}^{\vdash}$, is just the set of all the logical consequences of $\mathbf{A}$.

**Definition 1.** $\mathbf{A}^{\vdash} = \{\alpha \mid \mathbf{A} \vdash \alpha\}$

For now, let us only consider the following properties of $\vdash$.

**Axiom 2** (Extension)**.** $\mathbf{A} \subseteq \mathbf{A}^{\vdash}$

**Axiom 3** (Monotonicity)**.** $\mathbf{A}^{\vdash} \subseteq (\mathbf{A} \cup \mathbf{B})^{\vdash}$

**Corollary 4.** $\mathbf{A} \subseteq \mathbf{B} \Rightarrow \mathbf{A}^{\vdash} \subseteq \mathbf{B}^{\vdash}$ $\{3\}$

Languages with negation usually have other properties concerning consistent and trivial sets, which I will define as follows:

**Definition 5** (Consistency)**.** $\mathbf{A}$ is *consistent* $\Leftrightarrow \alpha, \neg\alpha \in \mathbf{A}$ holds for no $\alpha$, and *inconsistent* otherwise.

**Definition 6** (Triviality)**.** $\mathbf{A}$ is *trivial* $\Leftrightarrow \mathbf{A} = \mathbf{L}$.

To assert a trivial set of statements is the same as asserting everything and nothing, which justifies us in regarding any such assertion as absurd [1]. The *principle of explosion* transfers this absurdity to inconsistent sets since, according to it, *ex contradictione sequitur quodlibet* (ECQ), i.e. anything follows from a contradiction. Consequently, any formula whatsoever follows from an inconsistent set.

**Postulate ECQ** (Explosion)**.** $\mathbf{A}^{\vdash} = \mathbf{L}$, for any inconsistent $\mathbf{A}$.

Postulate ECQ has been questioned both on philosophical and logical grounds. These criticisms reached their highest point with the formulation of formal systems of logic that restricted its validity: the so-called *paraconsistent logics*. Such logics may be characterised by the following postulate, which is just the negation of ECQ.

**Postulate P** (Paraconsistency)**.** $\mathbf{A}^{\vdash} \neq \mathbf{L}$, for some inconsistent $\mathbf{A}$.

This postulate does not forbid that some inconsistent sets satisfy ECQ, for it only requires that at least one does not. And yet, we can define a class of logics where it is impossible to trivialise a set on grounds of it being inconsistent. Following Perzanowski's [3] terminology, these may be called *strongly paraconsistent logics*.

**Postulate SP** (Strong paraconsistency)**.** $\{\alpha, \neg\alpha\}^{\vdash} \neq \mathbf{L}$

Postulate SP does not completely rule out trivialisation. In principle, for every set $\mathbf{A}$ there could be some sophisticated wff $\alpha$ capable of trivialising it, that is, such that $(\mathbf{A} \cup \alpha)^{\vdash} = \mathbf{L}$. For example, in the paraconsistent calculus $\mathsf{C}_1$ of da Costa [2], a set $\mathbf{A}$ such that $\mathbf{A} \vdash *\neg\alpha$ could be trivialised by $\alpha$. Notwithstanding that, this postulate suggests an even stronger principle that would make it impossible to trivialise any set. Following Popper [4], I will call this the *principle of non embracingness.*

**Postulate $\bar{\mathrm{E}}$** (Non embracingness)**.** $\mathbf{A}^{\vdash} \neq \mathbf{L}$, for all $\mathbf{A} \subset \mathbf{L}$.

It clearly follows from this that no $\mathbf{A}$ obtained by subtracting a singleton from $\mathbf{L}$ can entail formula that is not already a member of $\mathbf{A}$.

**Lemma 7.** $\mathbf{L} - \mathbf{A}$ is a singleton $\Rightarrow \mathbf{A}^{\vdash} = \mathbf{A}$. $\{2, 3, \bar{\mathrm{E}}\}$

*Proof.* Assuming that $\mathbf{L} - \mathbf{A}$ results in that $\mathbf{A} \subseteq \mathbf{L}$ and that $\mathbf{L} - \mathbf{A}$ is not empty. From these, it follows that $\mathbf{A} \subset \mathbf{L}$, which by postulate $\bar{\mathrm{E}}$ entails that $\mathbf{A}^{\vdash} \neq \mathbf{L}$. This, by extension, entails that $\mathbf{A}^{\vdash} = \mathbf{A}$.

This lemma extends to every subset of $\mathbf{L}$.

**Theorem 8.** $\mathbf{A}^{\vdash} = \mathbf{A}$ $\{2, 3, \bar{\mathrm{E}}\}$

*Proof.* In order for $\mathbf{A}^{\vdash} = \mathbf{A}$ not to hold in general, there needs to exist at least one $\mathbf{A}$ for which there is at least one $\alpha \notin \mathbf{A}$ such that $\mathbf{A} \vdash \alpha$. If $\mathbf{A}$ and $\alpha$ are such, then we have that $\mathbf{A} \subseteq \mathbf{L} - \{\alpha\}$. From this, corollary 4 guarantees that $\mathbf{A}^{\vdash} \subseteq (\mathbf{L} - \{\alpha\})^{\vdash}$, which implies that $\mathbf{L} - \{\alpha\} \vdash \alpha$, contradicting lemma 7. Hence, no $\mathbf{A}$ entails an $\alpha$ that is not already a member of it.

This leads to the general result that a non-embracing relation of consequence satisfying monotonicity (and extension) has no deductive power, for it can only deduce as a theorem what already is a premise. In such case, what we may call the *principle of embracingness* would be essential in the realm of monotonic logics, where everything must follow from something.

We can, of course, restrict non-embracingness only to finite subsets of $\mathbf{L}$. This would make sense on intuitionistic grounds, since it does not seem plausible that a consequence relation, intended to formalise finitistic reasoning, can be applied to any arbitrary finite set. Whether intuitionism can offer a new opportunity for monotonicity and embracingness, or whether this theorem holds outside of the realm of monotonicity requires further research.

# References


[1] Newton Carneiro Affonso da Costa. *Ensaio sobre os fundamentos da lógica.* 2nd ed. São Paulo: Hucitec, 1994.

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