

GEOMETRICALLY FINITE TRANSCENDENTAL ENTIRE FUNCTIONS

MASHAEL ALHAMED, LASSE REMPE-GILLEN, AND DAVE SIXSMITH

ABSTRACT. For polynomials, local connectivity of Julia sets is a much-studied and important property. Indeed, when the Julia set of a polynomial of degree $d \geq 2$ is locally connected, the topological dynamics can be completely described as a quotient of a much simpler system: angle d -tupling on the circle.

For a transcendental entire function, local connectivity is less significant, but we may still ask for a description of the topological dynamics as the quotient of a simpler system. To this end, we introduce the notion of *docile* functions (Definition 1.1): a transcendental entire function with bounded postsingular set is docile if it is the quotient of a suitable *disjoint-type* function. Moreover, we prove docility for the large class of *geometrically finite* transcendental entire functions with *bounded criticality* on the Julia set. This can be seen as an analogue of the local connectivity of Julia sets for geometrically finite polynomials, first proved by Douady and Hubbard, and extends previous work of the second author and of Mihaljević-Brandt for more restrictive classes of entire functions.

We deduce a number of further results for geometrically finite functions with bounded criticality, concerning bounded Fatou components and the local connectivity of Julia sets. In particular, we show that the Julia set of the sine function is locally connected, answering a question raised by Osborne.

1. INTRODUCTION

Suppose that f is an entire function, and denote by f^n the n th iterate of f . The *Fatou set* $F(f)$ consists of those points where the dynamics of f is “regular”, or, more precisely, where the iterates form a normal family. Its complement is the *Julia set* $J(f)$; it is the locus of “chaotic” behaviour.

Let f be a polynomial of degree $d \geq 2$ with connected Julia set. By the Riemann mapping theorem there is a conformal isomorphism ϑ mapping the complement of the unit disc in the Riemann sphere to the attracting basin of infinity for f , in such a way that $\vartheta(\infty) = \infty$. This isomorphism is in fact a conjugacy between f and the action of $z \mapsto z^d$ [Mil06, Theorem 9.5].

If ϑ extends continuously to the unit circle, then the extension is a semiconjugacy between the action of z^d on the circle (i.e., angle d -tupling) and the action of f on its Julia set. Therefore we obtain a complete description of the interesting topological dynamics of f . By a classical theorem of Carathéodory and Torhorst [Car13, Tor21], such a continuous extension exists if and only if $J(f)$ is locally connected; see [Pom92, Theorem 2.1]. It is for this reason that local connectivity of Julia sets is one of the

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central topics of study in polynomial dynamics. Many, but not all, polynomial Julia sets are locally connected; see e.g. [Mil00] for a discussion.

The study of the dynamics of *transcendental entire functions* has also received considerable attention. Here infinity is an essential singularity, rather than an attracting fixed point, and the *escaping set* $I(f)$, consisting of points whose orbits under f converge to this singularity, is no longer open. Furthermore, the set of *singular values* $S(f)$, which generalises the set of critical values of a polynomial, may be very complicated.

The *Eremenko-Lyubich class* \mathcal{B} consists of those transcendental entire functions for which $S(f)$ is bounded; see [EL92, Six18]. It is characterised by strong expansion properties near infinity [EL92, Lemma 1], which in turn lead to a certain rigidity of the structure of the set $I(f)$ [Rem09]. As discussed above, many methods in the polynomial setting rely on studying the dynamics via structures in the escaping set. Therefore the class \mathcal{B} is perhaps the best setting for adapting these methods to transcendental entire functions – despite the major differences that remain.

A function in \mathcal{B} is said to be of *disjoint type* if its Fatou set is connected and compactly contains the postsingular set. (Recall that the *postsingular set* $P(f)$ is the closure of the forward orbit of the set of singular values.) The dynamical properties of these maps were studied in detail in [Rem16], which gave a comprehensive topological description of their Julia sets.

It is natural to consider functions of disjoint type to be the simplest maps in their given parameter space. Here by the *parameter space* of a map f , we mean the class of all entire functions that are *quasiconformally equivalent* to f . In other words, they are obtained from f by pre- and post-composition with quasiconformal homeomorphisms; see, for example, [Rem09, p. 245]. It is straightforward to show that if $f \in \mathcal{B}$, then $g(z) := f(\lambda z)$ is of disjoint type for all sufficiently small values of $\lambda \neq 0$ [BK07, p.392]; in particular, every such f contains a disjoint-type function in its parameter space. Moreover, any two quasiconformally equivalent functions of disjoint type are quasiconformally conjugate on their Julia sets [Rem09, Theorem 3.1].

So we may consider a function of disjoint type to play the same role for its parameter space as $z \mapsto z^d$ does for polynomials of degree d . It appears natural to ask for which $f \in \mathcal{B}$ the dynamical properties of the disjoint type function g can be “transferred” to the original map using a semiconjugacy from the Julia set of g to that of f .

The first general result in this direction was given in [Rem09, Theorem 1.4], which states that such a semiconjugacy can be constructed for all *hyperbolic* functions. (A transcendental entire function f is hyperbolic if $F(f) \cap P(f)$ is compact, and $J(f) \cap P(f)$ is empty; see [RS17].) This result was extended by Mihaljević-Brandt [Mih12], who showed that such a construction is possible for a larger class of functions she calls *strongly subhyperbolic*. Here a transcendental entire function f is called *subhyperbolic* if the set $F(f) \cap P(f)$ is compact, and $J(f) \cap P(f)$ is finite. It is called *strongly subhyperbolic* if, in addition, f has *bounded criticality* on the Julia set; that is, $J(f)$ contains no finite asymptotic values of f , and the local degree of f at the points in $J(f)$ is uniformly bounded. It is easy to see that the result becomes false in general when there are asymptotic values in the Julia set. In fact, even for the postsingularly finite function $z \mapsto 2\pi i e^z$, many questions about the topological dynamics remain unanswered; see [Wor18].

We can formalise the desired properties of an entire function f with bounded postsingular set $P(f)$ as follows. Let λ be sufficiently small to ensure that $g(z) := f(\lambda z)$ is of disjoint type. It follows from the results of [Rem09] that there is a natural (but not necessarily continuous) bijection $\vartheta: I(g) \rightarrow I(f)$ between the *escaping sets* of g and f ; see [BR17, Section 7] and Section 2 below. (Recall that a point belongs to the escaping set if its orbit tends to infinity.)

We may think of ϑ as an analogue of the Riemann map in the case of polynomials. Hence it makes sense to define the following, as an analogue of local connectivity of the Julia set.

1.1. Definition (Docile functions). The postsingularly bounded function f is called *docile* if the bijection $\vartheta: I(g) \rightarrow I(f)$ extends to a continuous function

$$\vartheta: J(g) \cup \{\infty\} \rightarrow J(f) \cup \{\infty\}.$$

With this new terminology, we may state the main result of Mihaljević-Brandt in [Mih12] as follows.

1.2. Theorem. *A strongly subhyperbolic entire function is docile.*

A particularly important class of polynomials consists of those for which each critical point in the Julia set has a finite orbit; see, for example, [CG93, Theorem 4.3] and [McM00]. These maps are called *geometrically finite*. The natural extension of this definition to transcendental entire functions was given by Mihaljević-Brandt [Mih10]. We say that a transcendental entire function f is *geometrically finite* if $F(f) \cap S(f)$ is compact and $J(f) \cap P(f)$ is finite. We also say that f is *strongly geometrically finite* if it is geometrically finite with bounded criticality on the Julia set. As noted in [Mih10], all subhyperbolic maps are geometrically finite, while a geometrically finite map is subhyperbolic if and only if it has no parabolic orbits. Clearly all geometrically finite maps are in the class \mathcal{B} . Our main result is the following extension of Theorem 1.2.

1.3. Theorem. *A strongly geometrically finite entire function is docile.*

An immediate consequence can be stated more directly. (See Proposition 2.3.)

1.4. Corollary (Semiconjugacies for geometrically finite functions). *Suppose that f is strongly geometrically finite, and that $\lambda \neq 0$ is such that the map $g(z) := f(\lambda z)$ is of disjoint type. Then there exists a continuous surjection $\vartheta: J(g) \rightarrow J(f)$, such that:*

- (a) $f \circ \vartheta = \vartheta \circ g$;
- (b) $\lim_{z \rightarrow \infty} \vartheta(z) = \infty$;
- (c) $\vartheta: I(g) \rightarrow I(f)$ is a homeomorphism and $\vartheta^{-1}(I(f)) = I(g)$;
- (d) For each component C of $J(g)$ the map $\vartheta: C \rightarrow \vartheta(C)$ is a homeomorphism.

Remark. Note that this statement makes explicit some properties of the semiconjugacy ϑ that were only implicit in [Rem09, Mih12].

The key idea of the proof of Theorem 1.2 given in [Mih12] is to show that f is uniformly expanding near the Julia set, with respect to a certain *orbifold metric*. The desired semiconjugacy is then obtained by a standard construction: for a point $z \in J(g)$, we iterate forward n steps under g , then pull back the resulting point n times under

f , using appropriate branches. The uniform expansion property then ensures that this process converges, and that $\vartheta(z)$ depends continuously on z . See also [Par19], in which the orbifold metric is used in a similar way to study a class of maps for which the postsingular set is unbounded.

When f has parabolic orbits, the function cannot be uniformly expanding near the parabolic point for any metric, and new techniques are required. The crux of the argument in this paper is to modify a suitable orbifold metric in a neighbourhood of each parabolic cycle in such a way that there is still sufficient expansion for the construction to converge. This idea is also used in the proof of local connectivity of the Julia set of a geometrically finite polynomial; see, for example, [CG93, Theorem 4.3], and [TY96] which studies the case of a geometrically finite rational map. However, there are additional challenges to overcome in the transcendental case. In particular, the proofs in the polynomial and rational case obtain uniform expansion arguments away from the parabolic points by relying, in an essential way, upon the fact that the degree of f is finite. Therefore new arguments are needed for transcendental entire functions.

Docile functions have many strong dynamical properties, which will be explored in a subsequent paper. Here, we give three simple examples of such properties in our setting. Recall that a *Cantor bouquet* is a certain type of uncountable union of arcs to infinity, called *hairs*, and a *pinched Cantor bouquet* is the quotient of a Cantor bouquet by a closed equivalence relation defined on its endpoints, embedded in a plane in a way that preserves the cyclic ordering of its hairs at infinity. Compare [Mih12, BJR12]. The following result then follows from Corollary 1.4 and [BJR12].

1.5. Corollary (Pinched Cantor Bouquets). *Suppose that f is strongly geometrically finite, and that additionally f is either of finite order, or a finite composition of class \mathcal{B} functions of finite order. Then $J(f)$ is a pinched Cantor bouquet.*

The second result also follows immediately from Corollary 1.4, together with well-known properties of disjoint-type functions.

1.6. Corollary (Components of the escaping set). *Suppose that f is strongly geometrically finite. Then $I(f)$ has uncountably many connected components.*

Remark. For $f(z) = 2\pi i e^z$, which is subhyperbolic but has an asymptotic value in the Julia set, the escaping set $I(f)$ is connected [Jar11]. It is conjectured that, in general, the escaping set of a transcendental entire function is either connected or has uncountably many components; see [RS19] for a partial result in this direction.

Our third result concerns the case that the Fatou set of a geometrically finite map is *connected*.

1.7. Theorem (Geometrically finite maps with connected Fatou set). *Suppose that f is geometrically finite, and that $F(f)$ is connected and non-empty. Then f is strongly geometrically finite, and the function ϑ in Corollary 1.4 is a homeomorphism between $J(g)$ and $J(f)$. In particular, if f is either of finite order, or a finite composition of class \mathcal{B} functions of finite order, then $J(f)$ is a Cantor bouquet.*

In [BFR15] the uniform expansion property of hyperbolic maps was used to study the Fatou and Julia sets of these functions. The results of that paper included various sufficient conditions for all Fatou components to be bounded, and other sufficient conditions

for the Julia set to be locally connected. Using the expansion properties for strongly geometrically finite functions obtained in the course of the proof of Theorem 1.3, we are able to deduce that many of the results of [BFR15] can be generalised to this larger class. We give two such results. The first concerns the boundedness of the components of the Fatou set, and is a generalisation of [BFR15, Theorem 1.2].

1.8. Theorem (Bounded Fatou components). *Suppose that f is strongly geometrically finite. Then the following are equivalent:*

- (a) *every component of $F(f)$ is a bounded Jordan domain;*
- (b) *the map f has no asymptotic values, and every component of $F(f)$ contains at most finitely many critical points.*

The second result concerns the local connectedness of the Julia set, and generalises [BFR15, Corollary 1.8].

1.9. Theorem (Bounded degree implies local connectivity). *Suppose that f is strongly geometrically finite with no asymptotic values. Suppose, furthermore, that there is a uniform bound on the number of critical points, counting multiplicity, that are contained in any single Fatou component of f . Then $J(f)$ is locally connected.*

As in [BFR15, Corollary 1.9(a)], when each Fatou component contains at most one critical value, Theorem 1.9 implies the following.

1.10. Corollary (Locally connected Julia sets). *Suppose that f is strongly geometrically finite with no asymptotic values, and that every component of $F(f)$ contains at most one critical value. Suppose additionally that the multiplicity of the critical points of f is uniformly bounded. Then $J(f)$ is locally connected.*

The function $f(z) := \sin z$ has a triple fixed point at the origin, and each of the two immediate parabolic basins contains one of the two critical values ± 1 . Hence Corollary 1.10 gives a positive answer to Osborne’s question [Osb13, remark following Example 6.2] whether $J(f)$ is locally connected. See Example 11.2 and Figure 3(c) below.

Remark. It is known that if f is a transcendental entire function and $J(f)$ is locally connected, then $J(f)$ has the additional topological property of being a *spider’s web*; see [Osb13, Theorem 1.4], and see [RS12] for the definition of a spider’s web. In particular, all Fatou components of f are bounded.

Finally, we discuss some further consequences of Theorem 1.3. The *exponential family* consists of the maps

$$(1.1) \quad E_a(z) := ae^z, \quad \text{for } a \in \mathbb{C} \setminus \{0\}.$$

The strongly geometrically finite exponential maps are exactly those that have either an attracting or a parabolic orbit. For this setting, Corollary 1.4 was stated in [Rem06], but the proof was given only for the case of attracting orbits.

The simplest geometrically finite map in this family that is not also subhyperbolic is $f(z) := E_{1/e}(z) = e^{z-1}$, which has a parabolic fixed point at 1 and connected Fatou set; see Example 11.1 below. For this function, it was observed by Devaney and Krych [DK84] that the Julia set is an uncountable collection of arcs to infinity. Furthermore, Aarts and Oversteegen [AO93] proved that the Julia set of f is a Cantor bouquet. It

follows from Theorem 1.7 that, indeed, $f|_{J(f)}$ is topologically conjugate to any disjoint type exponential map on its Julia set, for example to $g(z) = e^{z-2}$.

These results are related to the following more general phenomenon. Suppose that f is a polynomial, or even a rational function, that is geometrically finite. Then it is possible to perturb f such that all parabolic orbits become attracting, without changing the topological dynamics on the Julia set. More precisely, there is a path γ in the space of geometrically finite rational functions of the same degree, ending at f , such that each $g \in \gamma$ is conjugate to $f|_{J(f)}$ on its Julia set, and has no parabolic cycles. Furthermore, the conjugacy depends continuously on g and tends to the identity as g tends to f along γ . See [CT18], [Ha98, Ha00, Ha02] and [Kaw03, Kaw06] for more details.

It seems likely that, using our techniques, these results can be extended to the case of strongly geometrically finite functions. Here, we restrict ourselves to noting a direct consequence of Theorem 1.3. Suppose that f_1 and f_2 are two strongly geometrically finite functions in the same quasiconformal equivalence class, and suppose that g_1 and g_2 are disjoint-type functions with $g_1(z) = f_1(\lambda_1 z)$ and $g_2(z) = f_2(\lambda_2 z)$, for some $\lambda_1, \lambda_2 \neq 0$. Let ϑ_1 be the semiconjugacy between g_1 and f_1 from Corollary 1.4. Note that g_1 and g_2 are quasiconformally equivalent and hence, by [Rem09, Theorem 3.1], quasiconformally conjugate. So there is also a semiconjugacy ϑ_2 between g_1 and f_2 . Suppose that we can additionally conclude, say by combinatorial means, that the identifications between non-escaping points of $J(g_1)$ induced by ϑ_1 are the same as those induced by ϑ_2 . Then f_1 and f_2 are conjugate on their Julia sets, since they are, topologically, the quotient of the same map, by the same equivalence relation.

In particular, using prior results from [RS08], we can conclude the following for the case of exponential maps E_a as defined in (1.1). A *hyperbolic component* is a connected component of the set of parameters $a \in \mathbb{C}$ such that E_a is hyperbolic. The *hairs* or *external rays* of an exponential map E_a are the path-connected components of the escaping set $I(E_a)$; these hairs are ordered “vertically” according to how they tend to infinity; compare [RS08, Section 2].

1.11. Theorem (Conjugate exponential maps). *Suppose that a_0 is a parameter such that E_{a_0} has a parabolic cycle. Then there exists a unique hyperbolic component W such that, for $a \in W$, the functions E_a and E_{a_0} are topologically conjugate when restricted to their respective Julia sets, by a conjugacy that respects the vertical order of hairs. Moreover, $a_0 \in \partial W$.*

Structure. We now discuss the structure of this paper, including a brief outline of the proof of Theorem 1.3, which is long and quite complicated. We begin by discussing docility, and the natural bijection between escaping sets, in more detail in Section 2. In particular, Proposition 2.3 reduces the proof of Corollary 1.4 to the proof of Theorem 1.3. The discussion also sets the stage for the proof of Theorem 1.3.

Next, we present some preliminaries for the construction of an expanding metric. Section 3 discusses the dynamics of an entire function near a parabolic point; Section 4 gives an introduction to Riemann surface orbifolds. We collect basic results concerning geometrically finite functions in Section 5.

The proof of Theorem 1.3 proceeds, very roughly, as follows. Suppose that f is strongly geometrically finite. We work with an iterate f of f . Specifically, we choose the

smallest value $m \in \mathbb{N}$ such that all parabolic points of the map $\mathfrak{f} := f^m$ are fixed and of multiplier one. Note that f and \mathfrak{f} have the same Julia set, Fatou set and parabolic periodic points; we shall use these facts without comment. It is not difficult to verify that \mathfrak{f} is also strongly geometrically finite.

Recall that, in the polynomial case, the analogous property to docility (continuous extension of the Riemann map) can be expressed in terms of a topological property of the Julia set (local connectivity), and therefore it holds for f if and only if it holds for any iterate. For transcendental entire functions, it is less straightforward to see that docility of \mathfrak{f} implies docility of f . Hence we instead prove docility of the original function f directly, but using expansion estimates only known to hold for the iterate \mathfrak{f} .

More precisely, in Section 6 we construct certain hyperbolic orbifolds $\tilde{\mathcal{O}}$, \mathcal{O}' and \mathcal{O} such that the maps $\mathfrak{f}: \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ and $f: \mathcal{O}' \rightarrow \mathcal{O}$ are orbifold covering maps. This construction is closely related to that in [Mih12]. Next, in Section 7 we modify the orbifold metric on $\tilde{\mathcal{O}}$ close to the parabolic fixed points of \mathfrak{f} , and in Section 8 we show that \mathfrak{f} is suitably expanding in this metric. We are then able to use this expansion to prove Theorem 1.3. We also give the proofs of Theorem 1.7 and Theorem 1.11 in this section.

In Section 10 we use the expansion properties of \mathfrak{f} to prove the remaining results outlined above. Finally, Section 11 discusses a number of examples.

Notation and terminology. We denote the Riemann sphere by $\hat{\mathbb{C}}$, and the unit disc centred at the origin by \mathbb{D} . The (Euclidean) open ball of radius $r > 0$ around a point $a \in \mathbb{C}$ is denoted by

$$B(a, r) := \{z : |z - a| < r\}.$$

Unless stated otherwise, topological operations such as closure are taken in the plane. If $A \subset B \subset \mathbb{C}$, then we write $A \Subset B$ if the closure of A is compact and contained in B . We denote the Euclidean distance between two sets $A, B \subset \mathbb{C}$ by $\text{dist}(A, B)$. When A is the singleton $A = \{a\}$ we just write $\text{dist}(a, B)$.

We refer to [Ber93, Sch10] for background on transcendental dynamics. Also see [Six18] for background on the Eremenko-Lyubich class \mathcal{B} , as well as for elementary definitions and properties used in this paper, which we omit for reasons of brevity.

Suppose that f is a transcendental entire function. We denote by $\text{Attr}(f)$ and $\text{Par}(f)$ the set of attracting and parabolic periodic points, respectively. We let $F_{\text{Attr}}(f) \subset F(f)$ denote the union of attracting Fatou components; i.e., the set of points whose orbit converges to an attracting periodic orbit. The set $F_{\text{Par}}(f)$ is defined analogously. For $z \in \mathbb{C}$ we define $d_{\text{par}}(z) := \text{dist}(z, \text{Par}(f))$.

2. DOCILE FUNCTIONS

Let f be a transcendental entire function with bounded postsingular set $P(f)$, and let $\lambda > 0$ be sufficiently small to ensure that $g: z \mapsto f(\lambda z)$ is of disjoint type. As mentioned in the introduction, there is a natural bijection $\vartheta: I(g) \rightarrow I(f)$. This bijection arises from a certain conjugacy between g and f on the set of points whose orbits remain close to infinity, as constructed in [Rem09]. The existence of this bijection is implicit in [Rem07, Proof of Theorem 3.5], but is first made explicit in [BR17, Proof of Theorem 7.2]. Here, we give a slightly different description, for suitably small λ , which avoids “external

addresses", which are used in [BR17]. The construction will also allow us to explain the strategy of the proof of Theorem 1.3.

We begin by choosing a value of λ . Choose $L > K > 0$ sufficiently large that

$$(2.1) \quad P(f) \subset B(0, K) \quad \text{and} \quad B(0, L) \supset \overline{f(B(0, K))}.$$

Set $\lambda := K/L$, and consider g defined as above. Then $S(g) = S(f) \subset P(f) \subset B(0, K)$. Furthermore,

$$g(\overline{B(0, L)}) = \overline{f(B(0, K))} \subset B(0, L).$$

Hence $\overline{B(0, L)}$ is contained in the immediate basin of an attracting fixed point of g , and g is of disjoint type; compare also [Mih12, Proposition 2.8].

Next we define

$$\mathcal{T}^k := f^{-k}(\mathbb{C} \setminus \overline{B(0, L)}) \quad \text{and} \quad \mathcal{U}^k := g^{-k}(\mathbb{C} \setminus \overline{B(0, L)}), \quad \text{for } k \geq 0.$$

By construction $\mathcal{T}^k \cap P(f) \neq \emptyset$ for all $k \geq 0$, and $\overline{\mathcal{U}^{k+1}} \subset \mathcal{U}^k$, for $k \geq 0$. Since every point in the Fatou set $F(g)$ eventually iterates into $B(0, L)$, we deduce that

$$J(g) = \bigcap_{k \geq 0} \mathcal{U}^k.$$

We now construct a sequence $(\vartheta^k)_{k \geq 0}$ of conformal isomorphisms

$$(2.2) \quad \vartheta^k : \mathcal{U}^k \rightarrow \mathcal{T}^k, \quad \text{for } k \geq 0$$

such that

$$(2.3) \quad f \circ \vartheta^k = \vartheta^{k-1} \circ g, \quad \text{for } k \geq 1.$$

We first set $\vartheta^0(z) := z$ and $\vartheta^1(z) := \lambda z$, and note that (2.2) and (2.3) both hold with $k = 1$. Also observe that ϑ^1 is defined on \mathcal{U}^0 and takes values in $\mathbb{C} \setminus \overline{B(0, K)} \subset W := \mathbb{C} \setminus P(f)$ there. Consider the isotopy

$$\vartheta^t : \mathcal{U}^0 \rightarrow W; \quad \vartheta^t(z) := (1 - (1 - \lambda)t)z \quad (0 \leq t \leq 1)$$

between ϑ^0 and ϑ^1 on \mathcal{U}^0 . Since $f : f^{-1}(W) \rightarrow W$ is a covering map, $(\vartheta^t \circ g)_{t \in [0, 1]}$ lifts to an isotopy $(\vartheta^t)_{t \in [1, 2]}$ on \mathcal{U}^1 between ϑ^1 and a map $\vartheta^2 : \mathcal{U}^1 \rightarrow W$. The restriction of ϑ^2 to \mathcal{U}^2 takes values in \mathcal{T}^2 and satisfies (2.3) by construction. We continue this procedure inductively to define ϑ^k for all k . Observe that each ϑ^k extends to a homeomorphism

$$\vartheta^k : \overline{\mathcal{U}^k} \cup \{\infty\} \rightarrow \overline{\mathcal{T}^k} \cup \{\infty\}$$

with $\vartheta^k(\infty) = \infty$.

Let U^k be a connected component of \mathcal{U}^k . Then, for $k' < k$, there is a unique connected component $U^{k'}$ of $\mathcal{U}^{k'}$ such that $U^k \subset U^{k'}$. Consider the connected components $T^j := \vartheta^j(U^j)$ of \mathcal{T}^j , for $j = k, k'$. It follows from the construction that all sufficiently large points of T^k also lie in $T^{k'}$. We say that T^k *tends to ∞ in $T^{k'}$* . If $k' \geq 2$, then $f|_{T^{k'}}$ is injective, and hence T^k is the unique connected component of $f^{-1}(f(T^k))$ that tends to infinity within $T^{k'}$.

The following is a consequence of [Rem09, Section 3] and [BR17, Proposition 4.4]. Let us denote by $J_{\geq R}(f)$ the set of all points of $J(f)$ whose orbits do not enter the disc $B(0, R)$.

2.1. Theorem (Conjugacy near infinity). *There is $R \geq L$ such that the restrictions $\vartheta^k|_{J_{\geq R}(g) \cup \{\infty\}}$ converge uniformly in the spherical metric to a function*

$$\vartheta: J_{\geq R}(g) \cup \{\infty\} \rightarrow J(f) \cup \{\infty\},$$

which is a homeomorphism onto its image, and satisfies $\vartheta \circ g = f \circ \vartheta$ on $J_{\geq R}(g)$.

For sufficiently large Q , $J_{\geq Q}(f) \subset \vartheta(J_{\geq R}(g))$. Moreover, let $z \in J_{\geq R}(g)$ and denote the connected component of \mathcal{T}^k containing $\vartheta(z)$ by T^k . Then, for $k > 0$, T^k tends to infinity within T^{k-1} , and the unbounded connected component of $T^k \cap T^{k-1}$ contains $\vartheta(z)$, as well as $\vartheta^j(z)$ for $j \geq k$.

As mentioned in the introduction, this result leads to the existence of a natural bijection between escaping sets.

2.2. Theorem (Natural bijection). *For every $z \in I(g)$, the values $\vartheta^k(z)$ converge to a limit $\vartheta(z) \in I(f)$. The function $\vartheta: I(g) \rightarrow I(f)$ is a bijection, and satisfies $\vartheta \circ g = f \circ \vartheta$.*

Proof. Let R be as in Theorem 2.1. Let $z \in I(g)$, and write $z_n := g^n(z)$. For $n \geq 0$ and $k \geq 1$, let T_n^k be the connected component of \mathcal{T}^k such that

$$w_n^k := \vartheta^k(z_n) \in T_n^k.$$

That is, if U_n^k is the connected component of \mathcal{U}^k containing z_n , then $T_n^k = \vartheta^k(U_n^k)$. In particular, T_n^{k+1} tends to infinity within T_n^k . Let G_n^{k+1} denote the unbounded connected component of $T_n^{k+1} \cap T_n^k$.

Since $z \in I(g)$, there is $n_0 \geq 0$ such that $z_{n_0} \in J_{\geq R}(g)$. Then, by Theorem 2.1, the sequence $w_{n_0}^k$ converges to a point w_{n_0} . Moreover, by the final statement of the theorem, $w_{n_0} = \vartheta(z_{n_0})$ and $w_{n_0}^k = \vartheta^k(z_{n_0})$ belong to the same connected component of \mathcal{T}^k for $k \geq 1$. In other words, $w_{n_0} \in T_{n_0}^k$, and we see that $w_{n_0}, w_{n_0}^j \in G_{n_0}^k$ for $j \geq k \geq 2$.

The function f maps $T_{n_0}^{k+1}$ conformally to a component of \mathcal{T}^k . By the functional relation (2.3),

$$f(w_{n_0}^{k+1}) = f(\vartheta^{k+1}(z_{n_0})) = \vartheta^k(z_{n_0+1}) = w_{n_0+1}^k.$$

So $f(T_{n_0}^{k+1}) = T_{n_0+1}^k$, and $f: T_{n_0}^{k+1} \rightarrow T_{n_0+1}^k$ is a conformal isomorphism. It follows that $f(G_{n_0}^{k+1}) = G_{n_0+1}^k$ for $k \geq 2$.

Inductively we see that $f^{n_0}: T_0^k \rightarrow T_{n_0}^{k-n_0}$ is a conformal isomorphism for $k \geq n_0 + 1$, with $f^{n_0}(G_0^{k+1}) = G_{n_0}^{k+1-n_0}$. Let

$$\varphi^k: T_{n_0}^{k-n_0} \rightarrow T_0^k$$

be the corresponding branch of f^{-n_0} . Since G_0^{k+1} is also a subset of T_0^{k+1} , it follows that $\varphi^k = \varphi^{k+1}$ on $G_{n_0}^{k+1-n_0}$.

Hence we see that

$$w_0^k = \varphi^k(w_{n_0}^{k-n_0}) = \varphi^{k-1}(w_{n_0}^{k-n_0}) = \dots = \varphi^{n_0+1}(w_{n_0}^{k-n_0})$$

for all $k \geq n_0 + 1$. In particular, w_0^k converges to $w := \varphi^{n_0+1}(w_{n_0}) \in I(f)$. Setting $\vartheta(z) := w$, we have proved the existence of the limit function $\vartheta: I(g) \rightarrow I(f)$, which satisfies the desired functional relation by (2.3).

It remains to show that the map ϑ is a bijection. Firstly suppose that $z, \tilde{z} \in I(g)$ with $\vartheta(z) = \vartheta(\tilde{z}) =: w$. Let n_0 be chosen sufficiently large that $f^{n_0}(z), f^{n_0}(\tilde{z}) \in J_{\geq R}(g)$. Then

$$\vartheta(g^{n_0}(z)) = \vartheta(g^{n_0}(\tilde{z})) = f^{n_0}(w).$$

Since the restriction of ϑ to $I(g) \cap J_{\geq R}(g)$ is injective, we see that $g^{n_0}(z) = g^{n_0}(\tilde{z})$.

Now use the same notation as above. Then $T_0 := T_0^{n_0+1}$ is the component of \mathcal{T}^{n_0+1} containing w_n , and $T_{n_0} := T_{n_0}^1$ is the component of \mathcal{T}^1 containing w_{n_0} , with $f^{n_0}(T_0) = T_{n_0}$. Let U_0 and \tilde{U}_0 be the components of \mathcal{U}^{n_0+1} containing z and \tilde{z} , respectively. By construction, we have

$$\vartheta^{n_0+1}(U_0) = T_0 = \vartheta^{n_0+1}(\tilde{U}_0).$$

So $U_0 = \tilde{U}_0$. As $g^{n_0}|_{U_0}$ is injective, we conclude that $z = \tilde{z}$, as desired.

Similarly, let $w \in I(f)$, and write $w_n := f^n(w)$. By Theorem 2.1, there is n_0 and $z_{n_0} \in J_{\geq R}(g)$ such that $\vartheta(z_{n_0}) = w_{n_0}$. Let T_0 and T_{n_0} be as above, and set

$$U_0 := (\vartheta^{n_0+1})^{-1}(T_0) \subset \mathcal{U}^{n_0+1} \quad \text{and} \quad U_{n_0} := (\vartheta^1)^{-1}(T_{n_0}) \subset \mathcal{U}^1.$$

Then $z_{n_0} \in U_{n_0}$, and $g^{n_0}: U_0 \rightarrow U_{n_0}$ is a conformal isomorphism by the functional relation (2.3). Let $z \in U_0$ be the unique point with $g^{n_0}(z) = z_{n_0}$. According to the construction of $\vartheta(z)$ in the first part of the proof, $\vartheta(z)$ is the unique point in $\vartheta^{n_0+1}(U_0) = T_0$ with $f^{n_0}(\vartheta(z)) = \vartheta(z_{n_0}) = w_{n_0}$. Thus $\vartheta(z) = w$ and ϑ is surjective, as claimed. \blacksquare

Recall from the introduction that f is said to be *docile* if the bijection ϑ is continuous and extends to a continuous function on $J(g) \cup \{\infty\}$. Observe that ϑ is not, strictly speaking, canonical, as we could have chosen a different isotopy between ϑ^0 and ϑ^1 , leading to a different choice of ϑ . Moreover, the initial choice of ϑ^0 and ϑ^1 provides a “quasiconformal equivalence” between g and f ; there may be other such equivalences which can be used for the same purpose. However, using the results of [Rem09], it can be shown that any two possible bijections that arise in this manner differ by a quasiconformal self-conjugacy of g on its Julia set, much in the same way that the Böttcher map for a polynomial of degree d is defined only up to multiplication with a $(d-1)$ -th root of unity. Similarly, a different choice of λ , and hence of g , will lead to the same function ϑ , up to a quasiconformal conjugacy between the corresponding disjoint-type functions on their Julia sets. In particular, the definition of docility is independent of these choices.

Properties of docile functions will be studied in greater detail in a forthcoming separate article. Here, we restrict to showing that, for a docile function f , the conjugacy ϑ necessarily has the properties stated in Corollary 1.4.

2.3. Proposition (Semiconjugacies for docile functions). *Let f, g and ϑ be as above. Suppose that f is docile. Then the continuous extension $\vartheta: J(g) \cup \{\infty\} \rightarrow J(f) \cup \{\infty\}$ has the following properties.*

- (a) $\vartheta(\infty) = \infty$.
- (b) ϑ is surjective onto $J(f) \cup \{\infty\}$.
- (c) $\vartheta(J(g)) = J(f)$ and $f \circ \vartheta = \vartheta \circ g$ on $J(g)$.
- (d) $\vartheta: I(g) \rightarrow I(f)$ is a homeomorphism and $\vartheta^{-1}(I(f)) = I(g)$.
- (e) For each component C of $J(g)$ the map $\vartheta: C \cup \{\infty\} \rightarrow \vartheta(C) \cup \{\infty\}$ is a homeomorphism.

Proof. Part (a) follows immediately from Theorem 2.1.

In the following, let us write $\hat{J}(g) := J(g) \cup \{\infty\}$, and similarly for $\hat{J}(f)$. By Theorem 2.2, ϑ is a bijection between $I(g)$ and $I(f)$. Since $I(f)$ is dense in $\hat{J}(f)$, and $\hat{J}(g)$ is compact, it follows that ϑ is indeed surjective onto $\hat{J}(f)$, establishing (b).

To prove (c), let $z \in J(g)$, and let $z^j \in I(g)$ be a sequence converging to z . Set $w := \vartheta(z)$ and $w^j := \vartheta(z^j)$; so $w^j \rightarrow w$. If $w \in \mathbb{C}$, then we have

$$(2.4) \quad \vartheta(g(z)) = \lim_{j \rightarrow \infty} \vartheta(g(z^j)) = \lim_{j \rightarrow \infty} f(\vartheta(z^j)) = \lim_{j \rightarrow \infty} f(w^j) = f(w) \in \mathbb{C}.$$

Conversely, if $\vartheta(g(z)) = \infty$, then $\vartheta(z) = \infty$. Hence the set

$$X := \vartheta^{-1}(\infty) \cap \mathbb{C} = \{z \in J(g) : \vartheta(z) = \infty\}$$

is backwards-invariant. The Julia set of the disjoint-type function g contains no Fatou exceptional points, and hence the backwards orbit of any $z \in J(g)$ is dense in $J(g)$. As X is closed in $J(g)$, we have either $X = \emptyset$ or $X = J(g)$. The latter is impossible since ϑ maps $I(g)$ to $I(f)$.

So $\vartheta(J(g)) \subset J(f)$, and by (a) and (b), we have equality. The second part of (c) now also follows from (2.4).

As for (d), we already know that $\vartheta: I(g) \rightarrow I(f)$ is a continuous bijection. Moreover, by compactness of $\hat{J}(g)$, together with (a) and (c), we have

$$z_n \rightarrow \infty \iff \vartheta(z_n) \rightarrow \infty$$

for any sequence z_n in $\hat{J}(g)$. In particular, $I(g) = \vartheta^{-1}(I(f))$. It remains to show that the inverse of the restriction $\vartheta|_{I(g)}$ is continuous. This is a consequence of the following fact: If $h: X \rightarrow Y$ is a continuous surjection between compact metric spaces, and $h: h^{-1}(B) \rightarrow B$ is injective for some $B \subset Y$, then this restriction is a homeomorphism. See e.g. [Kah98, Lemma 2.2.13].

Finally, we prove (e). Let C be a connected component of $J(g)$. Since $\hat{C} = C \cup \{\infty\}$ is compact, it remains to show that ϑ is injective on the set of non-escaping points in C . So let $z, w \in C \setminus I(g)$ with $z \neq w$, and set $z_n := g^n(z)$, $w_n := g^n(w)$. By assumption, there is a subsequence (z_{n_k}) such that $\sup |z_{n_k}| < \infty$. On the other hand, we have

$$||z_n| - |w_n|| \rightarrow \infty$$

as $n \rightarrow \infty$; see e.g. [RRRS11, Lemmas 3.1 and 3.2]. In particular, $|w_{n_k}| \rightarrow \infty$. So $\vartheta(w_{n_k}) \rightarrow \infty$, while $\vartheta(z_{n_k})$ does not accumulate on ∞ by (c). So $\vartheta(w_{n_k}) \neq \vartheta(z_{n_k})$ for sufficiently large k ; by the semiconjugacy relation, this implies $\vartheta(w) \neq \vartheta(z)$, as claimed. \blacksquare

We conclude the section by observing that, in order to establish docility, it is enough to see that the maps ϑ^n converge uniformly.

2.4. Observation. *If the convergence of $\vartheta^n|_{I(g)} \rightarrow \vartheta$ is uniform with respect to the spherical metric, then f is docile.*

Proof. Recall that ϑ^n is defined and continuous on $\mathcal{U}^n \cup \{\infty\} \supset \hat{J}(g) := J(g) \cup \{\infty\}$. Since ϑ^n converges uniformly on $I(g)$, and $I(g)$ is dense in $\hat{J}(g)$, the sequence (ϑ^n) is Cauchy on $\hat{J}(g)$. Therefore it converges uniformly to a continuous function ϑ as desired. \blacksquare

In [Rem09, Section 5] and [Mih12], this is in fact how the semiconjugacy is constructed: expansion in a suitable hyperbolic metric is used to establish uniform convergence of the ϑ^n . As mentioned in the introduction, we follow the same strategy, but need to take particular care about the definition of the metric near parabolic points.

3. PARABOLIC PERIODIC POINTS

In this section, we collect some local results concerning parabolic points. These are well-known, but we know of no reference that contains them precisely in the form that we require.

Recall that a periodic point $\zeta \in \mathbb{C}$ of period k is *parabolic* if

$$(f^k)'(\zeta) = e^{2\pi is}, \quad \text{with } s \in \mathbb{Q}.$$

The orbit $\zeta, f(\zeta), \dots, f^{k-1}(\zeta)$ is called a *parabolic cycle*. If we replace f with a sufficiently large iterate, then we obtain

$$f(\zeta) = \zeta \quad \text{and} \quad f'(\zeta) = 1,$$

and now we say that ζ is a *multiple fixed point*.

We begin with some definitions and results relating to a function g , analytic in a neighbourhood N of the origin, such that the origin is a multiple fixed point. All definitions and results we give here can be transferred to the original function f by a conjugacy; we usually do so without comment. We can assume that N is sufficiently small that g has a well-defined and analytic inverse on N . For more background on the dynamics of an analytic function in a neighbourhood of a parabolic fixed point, we refer to, for example, [Bea91, Mil06].

Note first that there exist $a \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{N}$ such that

$$(3.1) \quad g(z) = z + az^{p+1} + O(z^{p+2}).$$

Here $p+1$ is termed the *multiplicity* of the parabolic fixed point.

The dynamics of g are determined by attracting and repelling vectors, which are defined as follows. A *repelling vector* \mathbf{v} is a complex number such that $pav^p = 1$, and an *attracting vector* \mathbf{v} is a complex number such that $pav^p = -1$.

Following [Mil06, Section 10], if ζ is a parabolic fixed point of multiplier one and multiplicity $p+1$, and \mathbf{v} is an attracting vector at ζ , then we say that the orbit of a point z *converges to ζ in the direction \mathbf{v}* if $(f^n(z) - \zeta) \sim \mathbf{v}/n^{1/p}$ as $n \rightarrow \infty$. We shall work with “petals” around the attracting (and repelling) directions that are sufficiently “thick”, in the following sense.

3.1. Proposition (Attracting petals). *Suppose that N is a neighbourhood of the origin, that g is an analytic function on N , and that the origin is a multiple fixed point, with attracting vector \mathbf{v} and of multiplicity $p+1$. Suppose finally that $\alpha \in (0, 2\pi/p)$. Then there exist a Jordan domain $P \subset N$ and $r_0 > 0$ such that*

- (a) $\overline{g(P)} \subset P \cup \{0\}$;
- (b) if $z \in N$, then the orbit of z converges to 0 in the direction \mathbf{v} if and only if $g^n(z) \in P$ for all sufficiently large values of n ;
- (c) $A_{\mathbf{v}}(\alpha, r_0) := \{z \in \mathbb{C} : |\arg z - \arg \mathbf{v}| < \alpha/2 \text{ and } |z| < r_0\} \subset P$.

Proof. We can suppose that g has the form (3.1). To conjugate g to a function which is close to $z \mapsto z + 1$ near infinity, we define functions

$$(3.2) \quad \kappa(z) := \frac{-1}{paz^p},$$

defined in a neighbourhood of 0. It is elementary to see that there is a function G , defined on a neighbourhood of ∞ , such that

$$(3.3) \quad G \circ \kappa = \kappa \circ g.$$

Shrinking N and restricting g if necessary, we may assume that G is defined on $\kappa(N)$. A calculation shows that

$$(3.4) \quad G(w) = w + 1 + o(1), \quad \text{as } w \rightarrow \infty.$$

It can be readily deduced from (3.4) that for each $\varphi \in (0, \pi/4)$, if $L = L(\varphi) > 0$ is sufficiently large then

$$P := \kappa^{-1}(\{w \in \mathbb{C} : |\pi - \arg(w - L)| > \varphi\})$$

is a subset of N that satisfies (a) and (b). (Here the values of \arg are taken in $[0, 2\pi)$.) If $\varphi < \alpha p/2$, then clearly $A_{\mathbf{v}}(\alpha, r_0) \subset P$ for sufficiently small r_0 . \blacksquare

We call P an *attracting petal* for g (of opening angle at least α) at \mathbf{v} . We also call the set $A = A_{\mathbf{v}}(\alpha, r_0)$ an *attracting sector of angle α and radius r_0* . The corresponding sets for the inverse g^{-1} are called *repelling petals* and *repelling sectors*, respectively.

We will need to work with repelling sectors that are “thin”, in the sense that their opening angle α is sufficiently small. To be definite, let \mathbf{v} be a repelling vector, and let us call the sector $A_{\mathbf{v}}(r_0) := A_{\mathbf{v}}(\pi/4p, r_0)$ a *thin repelling sector*. (The angle $\pi/4p$ could be replaced by any number strictly less than $\pi/2p$ in the following.)

The reason for this terminology is that we have good control of g and its derivative in a thin repelling sector.

3.2. Proposition (Thin repelling sectors). *Suppose that N is a neighbourhood of the origin, that g is an analytic function on N , that the origin is a multiple fixed point of g , and that \mathbf{v} is a repelling vector of f at 0. Then there exists $r_0 > 0$ such that*

$$(3.5) \quad |g(z)| > |z| \quad \text{and} \quad |g'(z)| > \frac{|g(z)|}{|z|} > 1, \quad \text{for } z \in A_{\mathbf{v}}(r_0).$$

Moreover, if $w \in A_{\mathbf{v}}(r_0)$ is such that $g(w)$ lies in a thin repelling sector $A_{\mathbf{v}'}(r)$, then $\mathbf{v} = \mathbf{v}'$.

Proof. Without loss of generality, we can assume that the argument of the repelling vector is zero. Let the multiplicity of the parabolic fixed point be $p + 1$. It follows that there exists $a > 0$ such that, as $z \rightarrow 0$,

$$g(z) = z \cdot (1 + az^p + o(z^p)) \quad \text{and} \quad g'(z) = 1 + a(p + 1)z^p + o(z^p).$$

Since a is real and positive and, by assumption, $|\arg(z^p)| \leq \pi/8$, the first part of (3.5) is immediate, as is the final claim of the proposition.

Furthermore,

$$\begin{aligned} \frac{z \cdot g'(z)}{g(z)} &= \frac{1 + a(p+1)z^p + o(z^p)}{1 + az^p + o(z^p)} \\ &= (1 + a(p+1)z^p + o(z^p)) \cdot (1 - az^p + o(z^p)) = 1 + apz^p + o(z^p). \end{aligned}$$

This implies the second part of (3.5). \blacksquare

We also need to know that, when some number n of iterates stay within the same thin repelling sector, the derivative of g^n grows sufficiently quickly.

3.3. Proposition (Cascades within repelling sectors). *Suppose that N is a neighbourhood of the origin, that g is an analytic function on N , and that the origin is a multiple fixed point of g of multiplicity $p+1$. Let \mathbf{v} be a repelling vector at the origin for g . Then there exists $r_0 > 0$ with the following property. If $n \in \mathbb{N}$, and $z \in N$ is such that $z_j := g^j(z) \in A_{\mathbf{v}}(r_0)$, for $0 \leq j \leq n$, then*

$$(3.6) \quad |z| < C_1 \cdot n^{-1/p} \quad \text{and}$$

$$(3.7) \quad |(g^n)'(z)| > C_2 \cdot |z|^{-(1+p)} |z_n|^{1+p},$$

where $C_1, C_2 > 0$ are constants independent of z .

Proof. Recall that g has the form (3.1), and that p is the number of repelling vectors of g at the origin. By Proposition 3.1, let P be a repelling petal at the origin in direction \mathbf{v} , which contains a repelling sector $A_{\mathbf{v}}(\pi/p, r')$.

We begin by fixing $r_0 > 0$. By [Mil06, Theorem 10.9], there is a conformal map $\varphi: P \rightarrow \mathbb{C}$, known as the *Fatou coordinate*, such that

$$(3.8) \quad \varphi(g(z)) = \varphi(z) + 1, \quad \text{for } z \in P \cap g^{-1}(P).$$

Let κ and G be as defined in (3.2) and (3.3). Let ψ be the branch of κ^{-1} that maps a left half-plane $H = \{a + ib : a < -\rho\}$, with $\rho > 0$, into P . Set

$$(3.9) \quad \Phi = \varphi \circ \psi.$$

The conformal map Φ is known as the *Fatou coordinate at infinity*. Note that it follows from (3.8) that

$$(3.10) \quad \Phi(G(w)) = \Phi(w) + 1, \quad \text{for } w \in H \cap G^{-1}(H),$$

and so, if $n \in \mathbb{N}$,

$$\Phi(G^n(w)) = \Phi(w) + n, \quad \text{for } w \in \bigcap_{j=0}^n G^{-j}(H),$$

from which we deduce that

$$(3.11) \quad \Phi'(G^n(w)) \cdot (G^n)'(w) = \Phi'(w), \quad \text{for } w \in \bigcap_{j=0}^n G^{-j}(H).$$

By (3.4) we can choose $r'_0 \in (0, r')$ sufficiently small that

$$(3.12) \quad |G(w) - w - 1| < 1/4, \quad \text{for } w \in W := \kappa(B(0, r'_0) \cap P).$$

Set $W' := \{w \in W : \text{dist}(w, \partial W) > 2\}$. We can then fix $r_0 \in (0, r'_0)$ sufficiently small that, for the thin repelling sector $A := A_{\mathbf{v}}(r_0) \subset P$, we have

$$\kappa(A) \subset W' \cap H.$$

This completes our choice of r_0 .

Our next goal is to find an estimate on the derivative of the Fatou coordinate at infinity; see (3.15) below. Suppose that $w \in W'$. Define a map $\Psi_w : B(0, 2) \rightarrow \mathbb{C}$ by

$$(3.13) \quad \Psi_w(a) := \frac{\Phi(w+a) - \Phi(w)}{\Phi'(w)}.$$

Then Ψ_w is a conformal map on $B(0, 2)$ such that $\Psi_w(0) = 0$ and $\Psi'_w(0) = 1$. An application of the Koebe distortion theorem gives that

$$(3.14) \quad \frac{4|a|}{(2+|a|)^2} \leq |\Psi_w(a)| \leq \frac{4|a|}{(2-|a|)^2}, \quad \text{for } a \in B(0, 2).$$

By (3.12), we have that $3/4 < |G(w) - w| < 5/4$. Substituting $a = G(w) - w$ in (3.13) and (3.14) then gives that

$$\frac{49}{169} < \frac{|\Phi(G(w)) - \Phi(w)|}{|\Phi'(w)|} < \frac{80}{9}, \quad \text{for } w \in W'.$$

Hence, by (3.10), we deduce that

$$(3.15) \quad \frac{9}{80} < |\Phi'(w)| < \frac{169}{48}, \quad \text{for } w \in W' \cap \kappa(P \cap g^{-1}(P)).$$

This is the required estimate on the derivative of Φ .

We now use the estimate (3.15) to prove our result. Suppose that $n \in \mathbb{N}$, and also that $z \in N$ is such that $z_j := g^j(z) \in A$, for $0 \leq j \leq n$. Let $w = \kappa(z)$. By the choice of r_0 , and by assumption, we have $w \in W' \cap \bigcap_{j=0}^n G^{-j}(H)$. It follows by (3.11) and (3.15) that

$$|(G^n)'(w)| = \frac{|\Phi'(w)|}{|\Phi'(G^n(w))|} > \frac{9}{80} \cdot \frac{48}{169} = \frac{27}{845}.$$

By the definitions (3.2) and (3.3), we conclude that

$$|(g^n)'(z)| > \frac{27}{845} |z|^{-(1+p)} |z_n|^{1+p}.$$

This establishes (3.6). Furthermore, it follows from (3.12) that

$$|G^n(w) - w - n| < n/4.$$

Since $g^n(z) \in A$, we have $\text{Re } G^n(w) < 0$, and deduce that

$$\text{Re}(w) = \text{Re} \left(-\frac{1}{paz^p} \right) < -3n/4,$$

and hence

$$\frac{1}{|z|} > \left(\frac{3}{4} p |a| \right)^{1/p} n^{1/p},$$

as desired. ■

We can combine the above results as follows, to obtain a statement for a global entire function having (possibly several) multiple fixed points. For reasons that will become apparent later, it will be convenient to measure expansion near the parabolic points with respect to a metric whose density is given by

$$(3.16) \quad \omega(z) = \omega_s(z) := d_{\text{Par}}(z)^{-s},$$

where s is a parameter, $0 < s < 1$.

3.4. Proposition (Behaviour near multiple fixed points). *Suppose that \mathfrak{f} is a transcendental entire function with finitely many parabolic points, all of which are multiple fixed points. Let $s \in (0, 1)$. Then there exist $r_{\min} > 0$, $K > 0$, $\ell > 1$ and $\tau > 1$ with the following properties. Consider a thin repelling sector $A = A_{\mathbf{v}}(r_{\min})$, for a repelling vector \mathbf{v} at a multiple fixed point ζ . Then*

(a) *We have that*

$$|\mathfrak{f}(z) - \zeta| > |z - \zeta| \quad \text{and} \quad \frac{\omega(\mathfrak{f}(z))}{\omega(z)} \cdot |\mathfrak{f}'(z)| > 1, \quad \text{for } z \in A.$$

(b) *Suppose that $z \in A$ is such that $\mathfrak{f}(z)$ lies in a thin repelling sector of radius r_{\min} . Then $\mathfrak{f}(z) \in A$.*

(c) *Suppose that $n \in \mathbb{N}$, and $z \in A$ are such that $z_j := \mathfrak{f}^j(z) \in A$, for $0 \leq j \leq n$. Then*

$$\frac{\omega(z_n)}{\omega(z)} \cdot |(\mathfrak{f}^n)'(z)| > K |z_n - \zeta|^\ell n^\tau.$$

Remark. We remark that only the values K, ℓ and τ depend on the choice of s .

Proof of Proposition 3.4. The proposition follows easily by applying the preceding results separately at each of the finitely many multiple fixed points of \mathfrak{f} , and its finitely many repelling directions. Note that (a) follows from Proposition 3.2, since

$$\frac{\omega(\mathfrak{f}(z))}{\omega(z)} = \frac{|z - \zeta|^s}{|\mathfrak{f}(z) - \zeta|^s} > \frac{|z - \zeta|}{|\mathfrak{f}(z) - \zeta|}$$

for the parabolic point ζ closest to z , provided r_{\min} was chosen sufficiently small. Similarly, (c) follows from Proposition 3.3. Indeed, let us set $w_n := z_n - \zeta$ and $w := z - \zeta$, with ζ as above. Then

$$\begin{aligned} \frac{\omega(z_n)}{\omega(z)} \cdot |(\mathfrak{f}^n)'(z)| &> C_2 \cdot \frac{|w|^s}{|w_n|^s} \cdot |w|^{-(1+p)} \cdot |w_n|^{1+p} \\ &= C_2 \cdot |w|^{-(p+1-s)} \cdot |w_n|^{p+1-s} > C_1 \cdot C_2 \cdot n^{-(1+\frac{1-s}{p})} \cdot |w_n|^{p+1-s}, \end{aligned}$$

where $p + 1$ is the multiplicity of the fixed point ζ . So the claim follows with $\tau = 1 + (1 - s)/P > 1$ and $\ell = P + 1 - s > 1$, where $P + 1$ is the maximal multiplicity of a fixed point of \mathfrak{f} . \blacksquare

4. RIEMANN ORBIFOLDS

In general, an *orbifold* is a space locally modeled on the quotient of an open subset of \mathbb{R}^n by the linear action of a finite group; see [Thu79, §13]. In this paper we will use the following definition, which is rather more specialised.

4.1. Definition. An *orbifold* is a pair (S, v) , where $S \subset \hat{\mathbb{C}}$ is a Riemann surface and $v : S \rightarrow \mathbb{N}$ a map such that $\{z \in S : v(z) > 1\}$ is a discrete set. The map v is called the *ramification map*. A point $z \in S$ such that $v(z) > 1$ is called a *ramified point*.

Note that S may be disconnected, in which case properties such as the type of the surface are understood component by component.

Suppose that \tilde{S}, S are Riemann surfaces, and that $f : \tilde{S} \rightarrow S$ is holomorphic. The map f is called a *branched covering* if each point of S has a neighbourhood U with the property that f maps each component of $f^{-1}(U)$ onto U as a proper map. We wish to define holomorphic and covering maps of orbifolds. This requires the following definitions.

4.2. Definition. Suppose that $f : \tilde{S} \rightarrow S$ is a holomorphic map of Riemann surfaces. If $w \in \tilde{S}$, then the *local degree* of f at w , which we denote by $\deg(f, w)$, is the value $n \in \mathbb{N}$ such that, in suitable local coordinates,

$$f(z) = f(w) + a(z - w)^n + O(|z - w|^{n+1}),$$

where $a \in \mathbb{C} \setminus \{0\}$. In particular, w is a critical point of f if and only if $\deg(f, w) > 1$.

4.3. Definition. Suppose that $f : \tilde{S} \rightarrow S$ is a holomorphic map of Riemann surfaces, and that $\tilde{\mathcal{O}} = (\tilde{S}, \tilde{v})$ and $\mathcal{O} = (S, v)$ are orbifolds.

- The map $f : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ is *holomorphic* if

$$v(f(z)) \text{ divides } \deg(f, z) \cdot \tilde{v}(z), \quad \text{for } z \in \tilde{S}.$$

- The map $f : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ is an *orbifold covering* if $f : \tilde{S} \rightarrow S$ is a branched covering, and

$$v(f(z)) = \deg(f, z) \cdot \tilde{v}(z), \quad \text{for } z \in \tilde{S}.$$

- If f is an orbifold covering and \tilde{S} is simply connected, then we call $\tilde{\mathcal{O}}$ a *universal covering orbifold* of \mathcal{O} .

Every Riemann surface has a universal cover that is conformally equivalent to either $\hat{\mathbb{C}}, \mathbb{C}$ or \mathbb{D} . The same is true for almost all orbifolds; see [McM94, Theorem A2]. With two exceptions (which do not occur in this paper) each orbifold has a universal cover whose underlying surface is either $\hat{\mathbb{C}}, \mathbb{C}$ or \mathbb{D} . In these cases we say the orbifold is *elliptic*, *parabolic* or *hyperbolic* respectively. All orbifolds considered in this paper are hyperbolic, so we restrict to this case.

Denote by $\rho_{\mathbb{D}}(z)|dz|$ the unique complete conformal metric of constant curvature -1 . Since this metric is invariant under conformal automorphisms, it descends to a well-defined metric on \mathcal{O} . We call this metric the *orbifold metric*, and denote it by $\rho_{\mathcal{O}}(w)|dw|$. For simplicity we will omit the $|dw|$, but still refer to $\rho_{\mathcal{O}}$ as the metric. If S is hyperbolic, v is identically one on S , and $\mathcal{O} = (S, v)$, then the usual universal cover of S as a Riemann surface is also a holomorphic covering map of orbifolds, and hence $\rho_{\mathcal{O}}$ is identical to the hyperbolic metric in S . On the other hand, $\rho_{\mathcal{O}}(z)$ becomes infinite at any ramified point.

The well-known Pick Theorem for hyperbolic surfaces generalizes to hyperbolic orbifolds [Thu84, Proposition 17.4].

4.4. Proposition. *Suppose that $f : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ is a holomorphic map between hyperbolic orbifolds. Then distances, as measured in the hyperbolic orbifold metric, are strictly decreased, unless f is an orbifold covering map in which case f is a local isometry.*

The next result follows by applying Proposition 4.4 to the inclusion map.

4.5. Corollary. *Suppose that $\tilde{\mathcal{O}} = (\tilde{S}, \tilde{v}), \mathcal{O} = (S, v)$ are hyperbolic orbifolds with metrics $\rho_{\tilde{\mathcal{O}}}$ and $\rho_{\mathcal{O}}$ respectively. Suppose also that $\tilde{S} \subset S$, and that the inclusion $\tilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic. Then*

$$\rho_{\tilde{\mathcal{O}}}(z) \geq \rho_{\mathcal{O}}(z), \quad \text{for } z \in \tilde{S}.$$

We also use the following observation, which gives us a form of expansion for certain orbifold coverings; this is [Mih12, Proposition 3.1].

4.6. Proposition. *Suppose that $\tilde{\mathcal{O}}, \mathcal{O}$ are hyperbolic orbifolds such that $\tilde{\mathcal{O}} \subset \mathcal{O}$, and with metrics $\rho_{\tilde{\mathcal{O}}}$ and $\rho_{\mathcal{O}}$ respectively. Suppose that $f : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ is an orbifold covering map, and that the inclusion $\tilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic but not an orbifold covering. Then*

$$\|Df(z)\|_{\mathcal{O}} := |f'(z)| \frac{\rho_{\mathcal{O}}(f(z))}{\rho_{\tilde{\mathcal{O}}}(z)} > 1, \quad \text{for } z \in \tilde{\mathcal{O}}.$$

In order to prove an important expansion result, see Proposition 6.2 below, we require two preliminary results. The first is [BR17, Lemma 3.2].

4.7. Lemma (Preimages in annuli). *Let f be an entire transcendental function which is bounded on an unbounded connected set. Let $z_1, z_2 \in \mathbb{C}$. Then, for all $C > 1$, and all sufficiently large R , $f^{-1}(\{z_1, z_2\})$ contains a point of modulus between R/C and $C \cdot R$.*

Remark. If z belongs to the unbounded connected component of $\mathbb{C} \setminus S(f)$, then the conclusion holds even for the preimage $f^{-1}(z)$ of the single point z ; compare the proof of [Rem09, Lemma 5.1].

The second result is a generalisation of statements at the start of the proof of [Mih12, Theorem 4.1].

4.8. Proposition. *Let $\tilde{\mathcal{O}} = (\tilde{S}, \tilde{v})$ and $\mathcal{O} = (S, v)$ be hyperbolic orbifolds such that the union of $S \setminus \tilde{S}$ with the set of ramified points in \mathcal{O} is bounded. Suppose furthermore that there is $C > 1$ such that, for every sufficiently large R for which $\{R/C < |z| < CR\} \subset \tilde{S}$, this annulus contains a ramified point z with $\tilde{v}(z)$ even. Then*

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \rightarrow \infty \text{ as } z \rightarrow \infty \text{ with } z \in \tilde{S}.$$

Remark. It is likely that the condition that $\tilde{v}(z)$ be even can be omitted, but it is easy to satisfy in our context, and leads to a simple proof.

Proof. The assumption implies that there is a sequence (z_i) with $|z_i| < |z_{i+1}| < C^2|z_i|$, such that each z_i is either in $\mathbb{C} \setminus \tilde{S}$, or $\tilde{v}(z)$ is even. Let \mathcal{O}' be the orbifold (\mathbb{C}, v') , where

$$v'(z) = \begin{cases} 2, & \text{for } z \in \{z_i\}_{i \in \mathbb{N}}, \\ 1, & \text{otherwise.} \end{cases}$$

Then the inclusion map from $\tilde{\mathcal{O}}$ to \mathcal{O}' is holomorphic. It follows by [Mih12, Theorem 4.3], together with Corollary 4.5, that

$$\frac{1}{\rho_{\tilde{\mathcal{O}}}(z)} \leq \frac{1}{\rho_{\mathcal{O}'}(z)} = O(|z|) \text{ as } z \rightarrow \infty.$$

We next estimate the metric in \mathcal{O} . Let D be a large disc such that $v(z) = 1$, for $z \notin D$. Once again by Corollary 4.5, we can estimate $\rho_{\mathcal{O}}(z)$ above by the hyperbolic density on $\mathbb{C} \setminus \bar{D}$, which can be computed explicitly (see e.g. [Hay89, Example 9.10]). It follows that

$$\rho_{\mathcal{O}}(z) \leq \rho_{\mathbb{C} \setminus \bar{D}}(z) = O\left(\frac{1}{|z| \log |z|}\right) \text{ as } z \rightarrow \infty.$$

Combining these two estimates gives

$$\frac{\rho_{\mathcal{O}}(z)}{\rho_{\tilde{\mathcal{O}}}(z)} = O(1/\log |z|) \text{ as } z \rightarrow \infty. \quad \blacksquare$$

5. DYNAMICS OF GEOMETRICALLY FINITE MAPS

In this section we give three general results about the dynamical properties of geometrically finite maps. The first is [Mih09, Proposition 3.1]. Here, and throughout this paper, by *Jordan domain* we always mean a simply-connected complementary component of a Jordan curve on the sphere; so a Jordan domain may be bounded or unbounded. Also, if $C \subset \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$, then we define the *forward orbit* of C by $O^+(C) := \bigcup_{n \geq 0} f^n(C)$.

5.1. Proposition (Absorbing domains in attracting Fatou components). *Suppose that f is a transcendental entire function, and also that $C \subset F_{\text{Attr}}(f)$ is compact. Then there exist pairwise disjoint bounded Jordan domains D_1, \dots, D_n with pairwise disjoint closures such that if $D := \bigcup_{i=1}^n D_i$, then*

$$f(D) \Subset D \Subset F_{\text{Attr}}(f) \quad \text{and} \quad O^+(C) \Subset D.$$

We also need the following, which is an analogous result for parabolic cycles.

5.2. Proposition (Absorbing domains in parabolic Fatou components). *Suppose that f is a transcendental entire function with finitely many parabolic points. Suppose also that $C \subset F_{\text{Par}}(f)$ is compact. Then there exist bounded Jordan domains D'_1, \dots, D'_n such that if $D' := \bigcup_{i=1}^n D'_i$, then the following all hold.*

- (a) $f(D') \subsetneq D'$.
- (b) $\partial D' \cap \partial f(D') = \text{Par}(f)$.
- (c) $\overline{O^+(C)} \subsetneq D' \cup \text{Par}(f)$.
- (d) $\overline{D'} \setminus \text{Par}(f) \subset F_{\text{Par}}(f)$.
- (e) If $z \in D'_j \cap \text{Par}(f)$, then D'_j is an attracting petal for z .
- (f) The sets $D'_j \setminus \text{Par}(f)$ are pairwise disjoint.
- (g) There exists $r_1 > 0$ with the following property. If $z \in \mathbb{C} \setminus D'$ is such that $0 < d_{\text{Par}}(z) < r_1$, then z belongs to a thin repelling sector of f .

Proof. The proof of this result is almost exactly as the proof of [Mih09, Proposition 3.2]. Indeed, parts (a), (c) and (d) are already explicitly stated in that result, and parts (b), (e) and (f) are implicit in the construction in [Mih09]. (It should be noted that in [Mih09] the domains are only stated to be simply-connected, but it is easy to see that we can shrink them and obtain Jordan domains with the same property; even Jordan domains whose boundary is analytic except possibly at the parabolic points.)

That leaves only part (g). This can be obtained by a very small modification to the proof of [Mih09, Proposition 3.2]. That proof uses [Mil06, Theorem 10.7] to obtain attracting petals which are contained in the domains D'_i contained in the immediate attracting basins of the parabolic fixed points. Instead we use Proposition 3.1 (applied to a suitable iterate of f) with $\alpha > 7\pi/4p$. Then every point sufficiently close to a parabolic fixed point of f either belongs to one of these petals, or to a thin repelling sector. ■

Finally, we use the following properties of the Fatou set of a geometrically finite map; see [Mih10, Proposition 2.5].

5.3. Proposition. *Suppose that f is a geometrically finite entire transcendental function. Then the Fatou set of f is either empty, or consists of finitely many attracting and parabolic basins. Furthermore, every periodic cycle in the Julia set is repelling or parabolic. In particular, $P(f)$ is bounded.*

6. CONSTRUCTING THE ORBIFOLDS

In the next four sections we prove Theorem 1.3. Throughout these sections f is strongly geometrically finite, and $\mathfrak{f} = f^m$ is the smallest iterate such that all the parabolic points of \mathfrak{f} are fixed and of multiplier one. We now define an orbifold $\mathcal{O}_f = (S, v)$ associated to f . The construction generalises that in [Mih12, Section 3], where there are no parabolic cycles.

By Proposition 5.3, the Fatou set of f consists of finitely many attracting and periodic cycles. Let D be the set from Proposition 5.1, applied to $C = S(f) \cap F_{\text{Attr}}(f)$, and let D' be the set from Proposition 5.2, applied to $C = S(f) \cap F_{\text{Par}}(f)$. The underlying surface of our orbifold is

$$S := \mathbb{C} \setminus \overline{D \cup D'}.$$

The set of ramified points is precisely $(P(f) \cap J(f)) \setminus \text{Par}(f)$, and the ramification of $z \in P(f) \cap J(f)$ is given by

$$v(z) := 2 \cdot \text{lcm}\{\deg(f^m, w), \text{ where } f^m(w) = z\}.$$

Observe that $v(z)$ is finite for all z . Indeed, by assumption on f there are only finitely many critical values in $J(f)$, and the local degree of f at their preimages is uniformly bounded.

6.1. Observation (Properties of \mathcal{O}_f). *We have*

$$(6.1) \quad J(f) \setminus \text{Par}(f) \subset S, \quad \text{Par}(f) \cap S = \emptyset, \quad \text{and} \quad F(f) \cap P(f) \cap S = \emptyset.$$

Furthermore, \mathcal{O}_f is hyperbolic.

Proof. The first three claims are immediate from the definition. If $S \neq \mathbb{C}$, then S is hyperbolic, and therefore \mathcal{O}_f is also. Otherwise, the set of orbifold points is $P(f) \subset J(f)$. Observe that $\#P(f) \geq 2$ for any transcendental entire function (see e.g. [BR17, Proposition 3.1]). Furthermore, every point $z \in P(f)$ is the iterated image of a critical point, and therefore $v(z) \geq 4$ by definition of v . The plane with two marked points of valence 4 is hyperbolic (see [Mil06, Remark E.6]), and therefore \mathcal{O}_f is hyperbolic. \blacksquare

6.2. Proposition (Preimage orbifold). *There is a unique orbifold $\mathcal{O}' = (S', v')$, with $S' = f^{-1}(S)$, such that $f: \mathcal{O}' \rightarrow \mathcal{O}_f$ is an orbifold covering map.*

The inclusion $\mathcal{O}' \rightarrow \mathcal{O}_f$ is holomorphic, but not an orbifold covering. For every $r > 0$, there exists $\mu = \mu(r) > 1$ such that

$$\|Df(z)\|_{\mathcal{O}_f} \geq \mu, \quad \text{for } z \in \tilde{S} \text{ such that } d_{\text{Par}}(z) > r.$$

Proof. The orbifold \mathcal{O}' is defined by the ramification index

$$(6.2) \quad v'(z) := \frac{v(f(z))}{\deg(f, z)}, \quad \text{for } z \in S'.$$

Note that, if $f(z)$ is a ramified point of \mathcal{O}_f , then $v'(z)$ is an even integer by definition of $v(f(z))$. Since $f: S' \rightarrow S$ is a branched covering, $f: \mathcal{O}' \rightarrow \mathcal{O}_f$ is an orbifold covering map.

Suppose that $z \in S'$. The definition of v , together with the fact that

$$\deg(f^m, w) = \deg(f, w) \cdot \deg(f, f(w)) \cdot \dots \cdot \deg(f, f^{m-1}(w))$$

implies that the product $v(z) \cdot \deg(f, z)$ divides $v(f(z))$. It then follows by (6.2) that $v(z)$ divides $v'(z)$, which proves that the inclusion is a holomorphic map.

By Lemma 4.7, the orbifolds \mathcal{O}' and \mathcal{O}_f satisfy the hypotheses of Proposition 4.8. So

$$(6.3) \quad \|Df(z)\|_{\mathcal{O}_f} = \frac{\rho_{\mathcal{O}'}(z)}{\rho_{\mathcal{O}_f}(z)} \rightarrow \infty$$

as $z \rightarrow \infty$ with $z \in S'$. In particular, the inclusion of \mathcal{O}' into \mathcal{O}_f is not an orbifold covering, and we have

$$\|Df(z)\|_{\mathcal{O}_f} > 1$$

for all $z \in S'$. This derivative tends to ∞ as $z \rightarrow \partial\tilde{S} \setminus \partial S$, and $\partial\tilde{S} \cap \partial S = \text{Par}(f)$. The claim thus follows from (6.3). \blacksquare

The orbifolds $\mathcal{O} := \mathcal{O}_f = (S, v)$ and $\mathcal{O}' = (S', v')$ will remain in place throughout this paper, along with the sets D, D' used in the construction of S . Observe that (using the same choice of D and D'), we also get the same orbifold $\mathcal{O}_{\mathfrak{f}} = \mathcal{O}$ for the iterate \mathfrak{f} . We can therefore also fix the corresponding preimage orbifold $\tilde{\mathcal{O}} = (\tilde{S}, \tilde{v})$ of \mathcal{O} under \mathfrak{f} .

7. CONSTRUCTING THE METRIC

When $\text{Par}(f) = \emptyset$ – i.e. in the setting of [Mih12] – Proposition 6.2 implies that f is uniformly expanding in the metric $\rho_{\mathcal{O}}$. This is sufficient to establish docility in this case.

In our setting, where $\text{Par}(f)$ may be non-empty, the expansion with respect to the orbifold metric will degenerate rapidly near the parabolic points, which are shared boundary

points of S and of S' . Accordingly we modify the metric $\rho_{\mathcal{O}}$ on S near these points to a metric σ .

Let us begin by fixing the number

$$n_{\sigma} := \text{lcm}\{\deg(\mathfrak{f}, w) \cdot v(w) : w \in S \cap \mathfrak{f}^{-1}(\text{Par}(\mathfrak{f}))\}.$$

Note that n_{σ} is finite since the set of ramified points of \mathcal{O} is finite, and since the local degree of f at the points in $J(f)$ is uniformly bounded. We also fix

$$(7.1) \quad s := s_{\sigma} := 1 - \frac{1}{2n_{\sigma}},$$

and recall the definition of

$$\omega : \mathbb{C} \setminus \text{Par}(f) \rightarrow \mathbb{R}; \quad \omega(z) := d_{\text{Par}}(z)^{-s}$$

from (3.16).

For $\epsilon_{\sigma} > 0$, we define a metric $\sigma = \sigma_{\epsilon_{\sigma}}$ on S by setting

$$(7.2) \quad \sigma(z) := \begin{cases} \rho_{\mathcal{O}}(z), & \text{for } d_{\text{Par}}(z) \geq \epsilon_{\sigma}, \\ \omega(z), & \text{otherwise.} \end{cases}$$

Observe that σ has singularities at the ramified points of \mathcal{O} , and that σ is not complete on S , since any parabolic point has finite distance from a point of S in the metric ω . However, σ does induce a complete metric on the Julia set $J(f)$. The main result of this section is the following. Recall that $\tilde{\mathcal{O}} = (\tilde{S}, \tilde{v})$ is the preimage orbifold of \mathcal{O} under the iterate \mathfrak{f} of f for which all parabolic points are multiple fixed points. If $z \in \tilde{S}$, then the derivative of \mathfrak{f} with respect to the metric σ is denoted

$$\|\text{D}\mathfrak{f}(z)\|_{\sigma} = |\mathfrak{f}'(z)| \cdot \frac{\sigma(\mathfrak{f}(z))}{\sigma(z)}.$$

Observe that $\|\text{D}\mathfrak{f}(z)\|_{\sigma}$ may become infinite at preimages of ramified points of \mathcal{O} . Let r_{\min} be the constant from Proposition 3.4, with the choice of s from (7.1). We may assume, and will do so from now on, that r_{\min} is chosen smaller than the constant r_1 from Proposition 5.2. Our main result is the following.

7.1. Proposition (The metric σ is expanding). *There exists $\epsilon_{\sigma} \in (0, r_{\min})$ with the following properties. Let $z \in \tilde{S}$. Then*

$$(7.3) \quad \|\text{D}\mathfrak{f}(z)\|_{\sigma} \geq \frac{\omega(\mathfrak{f}(z))}{\omega(z)} \cdot |\mathfrak{f}'(z)| > 1, \quad \text{for } 0 < d_{\text{Par}}(z) < \epsilon_{\sigma}.$$

Moreover, if $0 < \delta < \epsilon_{\sigma}$, then there exists $\chi = \chi(\delta) > 1$ with the property that if $z \in \tilde{S}$, then

$$(7.4) \quad \|\text{D}\mathfrak{f}(z)\|_{\sigma} \geq \chi, \quad \text{for } d_{\text{Par}}(z) \geq \delta.$$

To prove Proposition 7.1 we need the following. Recall that B denotes the set of ramified points of \mathcal{O} .

7.2. Proposition. *There exists $r_0 \in (0, r_{\min})$ with the following property. If $\zeta \in \text{Par}(\mathfrak{f})$ and $0 < r < r_0$, then the following all hold, where V is the component of $\mathfrak{f}^{-1}(B(\zeta, r))$ containing ζ .*

- (a) The punctured disc $B(\zeta, r) \setminus \{\zeta\}$ does not meet $S(\mathfrak{f})$.
- (b) If $Q \neq V$ is a component of $\mathfrak{f}^{-1}(B(\zeta, r))$, then $Q \subset S$, and Q contains no ramified points of \mathcal{O} , with the possible exception of the unique preimage of ζ in Q .
- (c) If $z \in V \cap \tilde{S}$, then $z \in B(\zeta, r)$.

Proof. Since \mathfrak{f} only has finitely many parabolic fixed points, we may prove the results separately for each parabolic point ζ .

Since \mathfrak{f} is geometrically finite, ζ is not an accumulation point of $S(\mathfrak{f})$. Therefore condition (a) holds whenever $r_0 > 0$ is chosen sufficiently small.

Recall that the set B of ramified points of \mathcal{O} is finite and, by construction, $\mathbb{C} \setminus S$ is compact. It follows by [BR17, Lemma 2.1] that only finitely many preimages of $B(\zeta, r_0)$ can meet $(\mathbb{C} \setminus S) \cup B$. Moreover, by the Definition of \mathcal{O} in Section 6, we have $\mathfrak{f}^{-1}(\zeta) \setminus \{\zeta\} \subset S$. Property (b) follows, reducing r_0 if necessary.

Reducing r_0 one last time, if necessary, we can assume that $V \subset B(\zeta, r_{\min})$. Recall that $r_0 < r_1$, where r_1 is the constant from Proposition 5.2. Suppose that $z \in V \cap \tilde{S}$. The fact that $z \in V$ implies that $|\mathfrak{f}(z) - \zeta| < r$. The fact that $z \in V \cap \tilde{S} \subset V \cap S$ implies, by Proposition 5.2(g), that z is in a thin repelling sector. Hence $|z - \zeta| < |\mathfrak{f}(z) - \zeta|$ by Proposition 3.4(a). Part (c) follows. \blacksquare

Proof of Proposition 7.1. Recall that the metric $\rho_{\mathcal{O}}$ is expanding at every point of \tilde{S} , while the metric ω is expanding in thin repelling sectors near parabolic point. Our main goal is, therefore, to show that, for sufficiently small choice of ϵ_{σ} , the function \mathfrak{f} is also expanding in the metric σ when one of $d_{\text{Par}}(z)$ and $d_{\text{Par}}(\mathfrak{f}(z))$ is less than ϵ_{σ} , and the other is not.

To this end, let $r_0 > 0$ be the constant from Proposition 7.2. For each parabolic fixed point, ζ , define a ramification map on the disc $B(\zeta, r_0)$ by

$$v^{\zeta}(z) = \begin{cases} n_{\sigma}, & \text{for } z = \zeta, \\ 1, & \text{otherwise.} \end{cases}$$

Let ρ_{ζ} denote the hyperbolic orbifold metric of $(B(\zeta, r_0), v^{\zeta})$.

Claim. There is $\epsilon_{\zeta} < r_0$ with the following property. If V is the connected component of $\mathfrak{f}^{-1}(B(\zeta, r_0))$ containing ζ , then

$$(7.5) \quad \frac{\omega(\mathfrak{f}(z))}{\rho_{\mathcal{O}}(z)} \cdot |\mathfrak{f}'(z)| \geq 2, \quad \text{for } z \in \mathfrak{f}^{-1}(B(\zeta, \epsilon_{\zeta})) \setminus V \subset S.$$

Proof. Note that, by Proposition 7.2(b), we indeed have $\mathfrak{f}^{-1}(B(\zeta, \epsilon_{\zeta})) \setminus V \subset S$. So the left-hand side of (7.5) is defined, except possibly in the case where $\mathfrak{f}(z) = \zeta$; as we shall see below, in the latter case the quantity becomes infinite.

Let $Q \neq V$ be a connected component of $\mathfrak{f}^{-1}(B(\zeta, r_0))$, and let $\xi \in Q$ be such that $\mathfrak{f}(\xi) = \zeta$. Recall by Proposition 7.2(b) that $Q \subset S$ and that no point of Q , except possibly ξ , is a ramified point of the orbifold \mathcal{O} . Define another ramification map on $B(\zeta, r_0)$ by

$$v^{\#}(z) = \begin{cases} \deg(\mathfrak{f}, \xi) \cdot \nu(\xi), & \text{for } z = \zeta, \\ 1, & \text{otherwise,} \end{cases}$$

and let $\rho_{\#}$ denote the metric on $(B(\zeta, r_0), v^{\#})$.

It follows by Proposition 7.2(a) that $\mathbf{f}: (Q, v) \rightarrow (B(\zeta, r_0), v^\#)$ is an orbifold covering map. Let ρ_Q denote the metric on (Q, v) . The definition of n_σ ensures that the inclusion $(B(\zeta, r_0), v^\zeta) \hookrightarrow (B(\zeta, r_0), v^\#)$ is holomorphic. It follows, by Proposition 4.4 and Corollary 4.5, that

$$(7.6) \quad 1 = \frac{\rho_\#(\mathbf{f}(z))}{\rho_Q(z)} \cdot |\mathbf{f}'(z)| < \frac{\rho_\#(\mathbf{f}(z))}{\rho_{\mathcal{O}}(z)} \cdot |\mathbf{f}'(z)| \leq \frac{\rho_\zeta(\mathbf{f}(z))}{\rho_{\mathcal{O}}(z)} \cdot |\mathbf{f}'(z)|, \quad \text{for } z \in Q.$$

The density $\rho_\zeta(w)$ is comparable to $|w - \zeta|^{-(1-1/n_\sigma)}$ as $w \rightarrow \zeta$. Therefore,

$$(7.7) \quad \frac{\omega(w)}{\rho_\zeta(w)} \equiv |w - \zeta|^{-\frac{1}{n_\sigma}} \rightarrow \infty$$

as $w \rightarrow \zeta$. Hence we can choose ε_ζ so small that the quotient in (7.7) is at least 2 for $|w - \zeta| \leq \varepsilon_\zeta$. The claim (7.5) then follows from (7.6). \triangle

Claim. There exists $\varepsilon_0 \in (0, r_0/2)$ such that

$$(7.8) \quad \rho_{\mathcal{O}}(\mathbf{f}(z)) \geq \omega(\mathbf{f}(z)), \quad \text{for } z \in \tilde{S} \text{ such that } d_{\text{Par}}(z) < \varepsilon_0.$$

Proof. Suppose that $\zeta \in \text{Par}(\mathbf{f})$. Let ρ denote the *hyperbolic* metric in S . Since S is the complement of a finite collection of bounded Jordan domains, it follows from [BP78, Theorem 1] that there is a constant $c > 0$ such that

$$\rho_{\mathcal{O}}(w)|w - \zeta| \geq \rho(w)|w - \zeta| \geq c, \quad \text{for } w \in S,$$

However, by definition of the function ω , as $w \rightarrow \zeta$ in S , we have that

$$\omega(w)|w - \zeta| = |w - \zeta|^{1-s} = |w - \zeta|^{2n_\sigma}.$$

Hence $\rho_{\mathcal{O}}(w) \geq \omega(w)$, whenever $d_{\text{Par}}(w)$ is sufficiently small. The claim follows, since ζ is a fixed point. \triangle

We now define

$$\varepsilon_\sigma := \min(\varepsilon_0, \min_{\zeta \in \text{Par}(\mathbf{f})} \varepsilon_\zeta).$$

Suppose that $z \in \tilde{S}$ and that $0 < \delta < \varepsilon_\sigma$. We must consider four cases, which depend on the sizes of $d_{\text{Par}}(z)$ and $d_{\text{Par}}(\mathbf{f}(z))$ compared to ε_σ .

- Suppose that $d_{\text{Par}}(z) < \varepsilon_\sigma$ and $d_{\text{Par}}(\mathbf{f}(z)) < \varepsilon_\sigma$. By Proposition 5.2(g) we have that z is in a thin repelling sector. Then, by Proposition 3.4(a),

$$\|\mathbf{Df}(z)\|_\sigma = \frac{\omega(\mathbf{f}(z))}{\omega(z)} \cdot |\mathbf{f}'(z)| > 1.$$

- Suppose that $d_{\text{Par}}(z) < \varepsilon_\sigma$ and $d_{\text{Par}}(\mathbf{f}(z)) \geq \varepsilon_\sigma$. Once again, z is in a thin repelling sector. Then, by (7.8) and Proposition 3.4(a),

$$\|\mathbf{Df}(z)\|_\sigma = \frac{\rho_{\mathcal{O}}(\mathbf{f}(z))}{\omega(z)} |\mathbf{f}'(z)| \geq \frac{\omega(\mathbf{f}(z))}{\omega(z)} \cdot |\mathbf{f}'(z)| > 1.$$

- Suppose that $d_{\text{Par}}(z) \geq \varepsilon_\sigma$ and $d_{\text{Par}}(\mathbf{f}(z)) \geq \varepsilon_\sigma$. Then, by Proposition 6.2, there exists $\mu = \mu(\varepsilon_\sigma) > 1$ such that

$$\|\mathbf{Df}(z)\|_\sigma = \frac{\rho_{\mathcal{O}}(\mathbf{f}(z))}{\rho_{\mathcal{O}}(z)} |\mathbf{f}'(z)| = \|\mathbf{Df}(z)\|_{\mathcal{O}} \geq \mu.$$

- Suppose that $d_{\text{Par}}(z) \geq \epsilon_\sigma$ and $d_{\text{Par}}(\mathbf{f}(z)) < \epsilon_\sigma$. By definition there exists a point $\zeta \in \text{Par}(\mathbf{f})$ such that $|\mathbf{f}(z) - \zeta| < \epsilon_\sigma$ and $|z - \zeta| \geq \epsilon_\sigma$. It follows by Proposition 7.2(c) that z does not lie in the component of $\mathbf{f}^{-1}(B(\zeta, \epsilon_\sigma))$ containing ζ . Hence, by (7.5) we obtain that

$$\|\mathbf{D}\mathbf{f}(z)\|_\sigma = \frac{\omega(\mathbf{f}(z))}{\rho_{\mathcal{O}}(z)} \cdot |\mathbf{f}'(z)| \geq 2.$$

Note that the first two cases above complete the proof of (7.3). To prove (7.4) we first choose $\alpha > 1$ sufficiently small that

$$\frac{\omega(\mathbf{f}(z))}{\omega(z)} \cdot |\mathbf{f}'(z)| > \alpha, \quad \text{for } \delta < d_{\text{Par}}(z) < r_0.$$

This is possible by Proposition 3.4(a), together with the fact that a continuous function attains a minimum on a compact set. We then set

$$\chi := \min\{2, \mu, \alpha\} > 1,$$

and the result follows from the four cases above. \blacksquare

We will fix the constant ϵ_σ from Proposition 7.1 and the corresponding metric σ throughout the rest of the paper.

8. EXPANSION

The following result gives a form of expansion for all sufficiently long orbits in \tilde{S} .

8.1. Proposition. *There exist constants $\ell, \tau > 1$ and $C > 0$ with the following property. Suppose that $z \in \tilde{S}$ and $k \geq 1$ are such that $\mathbf{f}^k(z) \in S$. Then*

$$(8.1) \quad \|\mathbf{D}\mathbf{f}^k(z)\|_\sigma \geq C \cdot \min\{d_{\text{Par}}(\mathbf{f}^k(z))^\ell, 1\} \cdot k^\tau.$$

Observe that the reciprocal of the estimate (8.1) is summable over k . This property of the estimate will allow us to construct our semiconjugacy in the next section.

Proof of Proposition 8.1. The idea of the proof is as follows. We know that the function \mathbf{f} is uniformly expanding away from the set of parabolic points. This suggests that the worst-case behaviour (in terms of least expansion) occurs for pull-backs that spend a long time near a parabolic point, and hence in a thin repelling petal. However, along such pull-backs, the derivative does indeed grow at least as described by (8.1), by Proposition 3.4.

To fill in the details, let $K > 0$ and $\ell, \tau > 1$ be the constants from Proposition 3.4; recall that we have fixed the constant ϵ_σ from Proposition 7.1.

First we introduce a number of constants. Begin by choosing a positive $\varepsilon < \epsilon_\sigma$ with the following property: if $d_{\text{Par}}(w) < \varepsilon$, then $d_{\text{Par}}(\mathbf{f}(w)) < \epsilon_\sigma$. This is possible as $\text{Par}(\mathbf{f})$ is finite and \mathbf{f} is continuous; we may suppose that $\varepsilon \leq 1$ and $K \cdot \varepsilon^\ell \leq 1$. Now define

$$\alpha_k := K \varepsilon^\ell k^\tau \leq k^\tau, \quad \text{for } k \in \mathbb{N},$$

and choose $k_0 \in \mathbb{N}$ sufficiently large that

$$(8.2) \quad \alpha_k \alpha_m > \alpha_{k+m}, \quad \text{for } k, m \geq k_0.$$

Then, similarly as in the choice of ε above, choose $\delta \in (0, \epsilon_\sigma)$ sufficiently small that

$$(8.3) \quad d_{\text{Par}}(\mathbf{f}^j(w)) < \epsilon_\sigma, \quad \text{for } d_{\text{Par}}(w) < \delta \text{ and } 0 \leq j \leq k_0.$$

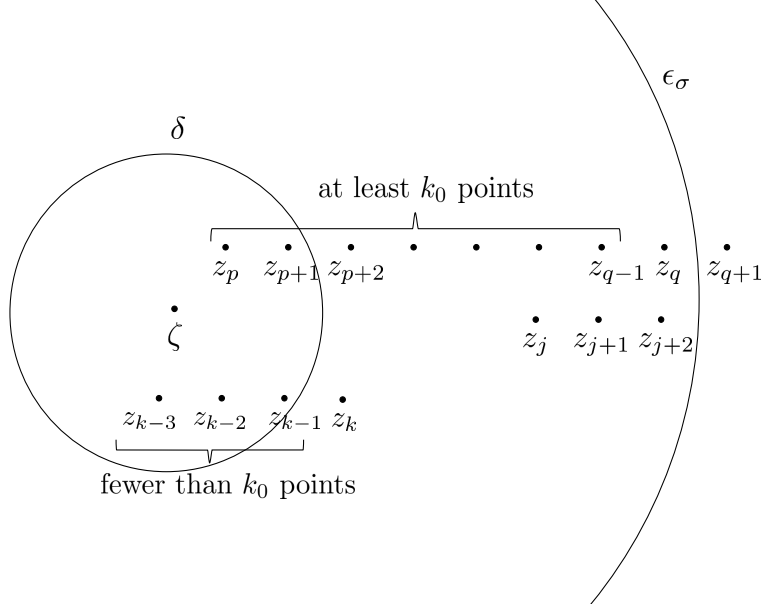


FIGURE 1. An approximate schematic of the three cases in the proof of Proposition 8.1; the top row represents case (A), the middle row case (B), and bottom row case (C).

Let $\chi = \chi(\delta) > 1$ be the constant from Proposition 7.1. We then choose $C' > 0$ sufficiently small that

$$(8.4) \quad \chi^k \geq C'(1 + k_0 + k)^\tau, \quad \text{for } k \in \mathbb{N}.$$

This completes the choice of constants.

Now, suppose that $z \in \tilde{S}$ and $k \geq 1$ satisfy the assumptions of the proposition. Define $z_j := \mathfrak{f}^j(z)$, for $j \geq 0$, so that

$$\|D\mathfrak{f}^k(z)\|_\sigma = \prod_{j=0}^{k-1} \|D\mathfrak{f}(z_j)\|_\sigma.$$

For each $0 \leq j < k$ exactly one of the following three cases holds; see Figure 1.

(A) There exist $p, q \geq 0$ such that the following all hold:

- $0 \leq p \leq j < q \leq k$;
- $k_0 \leq q - p$;
- $d_{\text{Par}}(z_p) < \delta$;
- $d_{\text{Par}}(z_{j'}) < \epsilon_\sigma$ for $p \leq j' \leq q$.

(B) Case (A) does not hold, and $d_{\text{Par}}(z_j) \geq \delta$.

(C) Cases (A) and (B) do not hold, and in particular $d_{\text{Par}}(z_j) < \delta$.

Let k_A , k_B and k_C respectively be the number of values of j for which each of the three cases above hold. Clearly $k_A + k_B + k_C = k$. Moreover, it follows from our choice of δ that case (C) can only occur if $j > k - k_0$; in particular, $k_C < k_0$.

We estimate $\|\mathrm{Df}(z_j)\|_\sigma$ in each of these three cases. In the case (C), we only use that $\|\mathrm{Df}(z_j)\|_\sigma > 1$ by (7.3). Estimating $\|\mathrm{Df}(z_j)\|_\sigma$ in the case (B) is also straightforward. Since $d_{\mathrm{Par}}(z_j) \geq \delta$ we have, by (7.4), that $\|\mathrm{Df}(z_j)\|_\sigma \geq \chi$.

In particular, if $k_A = 0$, then by (8.4) and since $k_0 > k_C$,

$$(8.5) \quad \|\mathrm{Df}^k(z)\|_\sigma = \prod_{j=0}^{k-1} \|\mathrm{Df}(z_j)\|_\sigma \geq \chi^{k_B} \geq C'(1 + k_0 + k_B)^\tau \geq C'k^\tau$$

and the proof is complete.

Hence it remains to suppose that $k_A > 0$, where the more complicated case (A) in fact occurs. Suppose that the conditions and terminology of that case all hold. Choose p minimal and q maximal with the properties stated in (A). Note that if $q < k$, then $d_{\mathrm{Par}}(z_{q+1}) \geq \epsilon_\sigma$, and hence $d_{\mathrm{Par}}(z_q) \geq \epsilon$.

Now

$$(8.6) \quad \prod_{j=p}^{q-1} \|\mathrm{Df}(z_j)\|_\sigma = \|\mathrm{Df}^{q-p}(z_p)\|_\sigma = \frac{\sigma(z_q)}{\sigma(z_p)} \cdot |(\mathfrak{f}^{q-p})'(z_p)| = \frac{\omega(z_q)}{\omega(z_p)} \cdot |(\mathfrak{f}^{q-p})'(z_p)|.$$

Let $\zeta \in \mathrm{Par}(\mathfrak{f})$ be the parabolic fixed point closest to z_p . Suppose that $p \leq j \leq q$. Since $z_j \in S$, it follows by Proposition 5.2(g) that z_j lies in a thin repelling sector at ζ . Let A be the thin repelling sector containing z_p . It then follows, by Proposition 3.4(b) that $z_j \in P$, for $p \leq j \leq q$. Hence we can apply Proposition 3.4(c), with $z = z_p$ and $n = q - p + 1$, and obtain

$$\prod_{j=p}^q \|\mathrm{Df}(z_j)\|_\sigma \geq K \cdot |z_q - \zeta|^\ell \cdot (q - p)^r \geq \alpha_{q-p} \cdot \left(\frac{d_{\mathrm{Par}}(z_q)}{\epsilon} \right)^\ell.$$

Note that $k_A > 0$ implies $k - k_B - k_C = k_A \geq k_0$. It follows by (8.2) and (8.4) that

$$\begin{aligned} \|\mathrm{Df}^k(z)\|_\sigma &= \prod_{j=0}^{k-1} \|\mathrm{Df}(z_j)\|_\sigma \geq \chi^{k_B} \cdot \alpha_{k_A} \cdot \min\left(1, \frac{d_{\mathrm{Par}}(z_k)^\ell}{\epsilon^\ell}\right) \\ &\geq C'(1 + k_0 + k_B)^\tau \cdot \alpha_{k_A} \cdot \min(1, d_{\mathrm{Par}}(z_k)^\ell) \\ &\geq C' \cdot \alpha_{k_0+k_B} \cdot \alpha_{k_A} \cdot \min(1, d_{\mathrm{Par}}(z_k)^\ell) \\ &\geq C' \cdot \alpha_k \cdot \min(1, d_{\mathrm{Par}}(z_k)^\ell) \\ &= (C' \cdot K \cdot \epsilon^\ell) \cdot \min(1, d_{\mathrm{Par}}(z_k)^\ell) \cdot k^\tau. \quad \blacksquare \end{aligned}$$

8.2. Remark. It follows from the proof, and Proposition 3.3, that we may let ℓ depend on z_n , taking $\ell = p + 1 - s_\sigma = p + 1/n_\sigma$, where $p + 1$ is the multiplicity of a parabolic fixed point ζ closest to $\mathfrak{f}^n(z)$. We use this observation in Section 10.

9. DOCILITY

We now use our earlier results to prove Theorem 1.3. Note that we now, in general, work directly with f , rather than the iterate $\mathfrak{f} = f^m$ which was used in the two preceding chapters. We continue to use the terminology and definitions from earlier in the paper, often without comment.

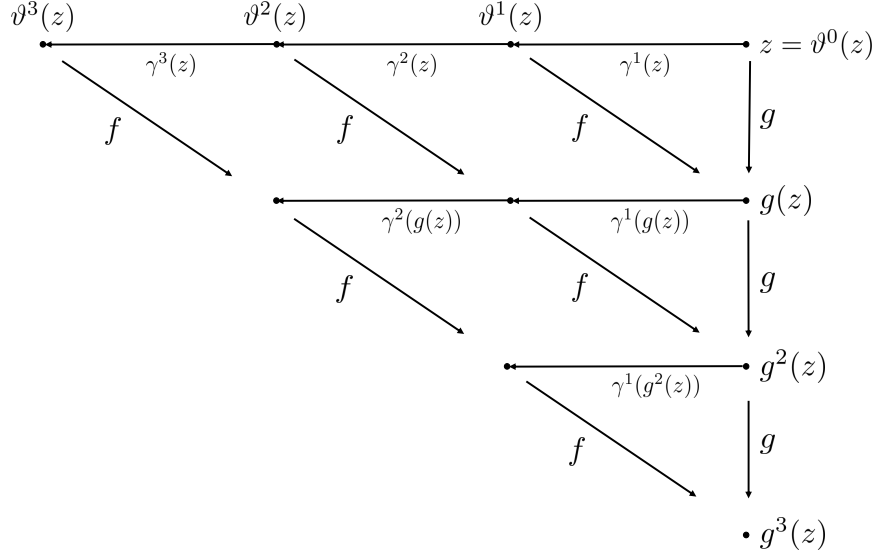


FIGURE 2. A schematic of the functions and curves used in the proof of Theorem 1.3.

Proof of Theorem 1.3. Let us use the notation from Section 2. Recall that we chose $K > 0$ sufficiently large that $P(f) \subset B(0, K)$, and then chose $L > K$ sufficiently large to ensure that, for $\lambda = K/L$, $g: z \mapsto f(\lambda z)$ is of disjoint type.

We may additionally suppose that $K > 2$, and that K was chosen sufficiently large that

$$(9.1) \quad P(f) \cup \bigcup_{n=0}^m f^n(\overline{D \cup D'} \cup \{z: d_{\text{Par}}(z) < \epsilon_\sigma\}) \subset B(0, K/2).$$

This is possible since all the sets in the union on the left hand side of (9.1) are bounded. Note that the definition of K ensures that if $0 \leq n \leq m$ and $|f^n(z)| \geq K/2$, then $z \in S$ and $d_{\text{Par}}(z) \geq \epsilon_\sigma$, and so $\sigma(z)$ is defined and equal to $\rho_{\mathcal{O}}(z)$. We will make frequent use of this observation.

We may also suppose that L is so large that

$$B(0, L) \supset \overline{f(B(0, K+1))}.$$

Recall that in Section 2 we defined a sequence $(\vartheta^k)_{k \geq 0}$ of conformal isomorphisms

$$\vartheta^k: \mathcal{U}^k := g^{-k}(\mathbb{C} \setminus \overline{B(0, L)}) \rightarrow f^{-k}(\mathbb{C} \setminus \overline{B(0, L)}) =: \mathcal{T}^k, \quad \text{for } j \geq 1,$$

such that

$$f \circ \vartheta^k = \vartheta^{k-1} \circ g, \quad \text{for } k \geq 1.$$

For $z \in \mathcal{U}^k$, let $\gamma^k(z) = \{\vartheta^t(z): t \in [k, k+1]\}$, where $(\vartheta^t)_{t \in [k, k+1]}$ is the isotopy between ϑ^k and ϑ^{k+1} used in the definition of ϑ^{k+1} . Then $\gamma^k(z) \subset \mathbb{C} \setminus P(f)$ is a curve connecting $\vartheta^k(z)$ and $\vartheta^{k+1}(z)$. Note that, for $k = 0$, $\gamma^0(z)$ is the straight line segment connecting $z = \vartheta^0(z)$ and $\lambda z = \vartheta^1(z)$. For $k > 1$, $\gamma^k(z)$ is the connected component of $f^{-k}(\gamma^0(g^k(z)))$ containing $\vartheta^k(z)$. (See Figure 2.)

Our goal is to show that, for each k , the σ -length $\ell_\sigma(\gamma^k(z))$ of $\gamma^k(z)$ is bounded independently of $z \in J(g)$, with the bound summable over k . This in turn means that the maps $\vartheta^j|_{J(g)}$ form a Cauchy sequence with respect to the metric σ . Indeed, let $d_\sigma(z_1, z_2)$ denote the σ -distance between points $z_1, z_2 \in S$, i.e. the infimum over the length of all curves connecting z_1 and z_2 . If $z \in \mathcal{U}^{k+1} \subset \mathcal{U}^k$, then by construction,

$$(9.2) \quad d_\sigma(\vartheta^{k+1}(z), \vartheta^k(z)) \leq \ell_\sigma(\gamma^{k+1}(z)).$$

We begin by estimating the length of $\gamma^k(z)$ for $k = 0$.

Claim. There is a constant $\alpha > 0$ such that $\ell_\sigma(\gamma^0(z)) \leq \alpha$ whenever $z \in \mathcal{U}^0$.

Proof. Suppose that $z \in \mathcal{U}^0$. By choice of K ,

$$\gamma^0(z) \subset \mathbb{C} \setminus \overline{B(0, K)} \subset \mathbb{C} \setminus \overline{B(0, K/2)} \subset \{z \in S : d_{\text{Par}}(z) \geq \epsilon_\sigma\}.$$

In particular, by the definition of σ , $\ell_\sigma(\gamma^0(z)) = \ell_{\mathcal{O}}(\gamma^0(z))$. Since all ramified points of \mathcal{O} lie in $B(0, K/2)$, it is a consequence of Proposition 4.4 that we can estimate $\ell_{\mathcal{O}}(\gamma^0(z))$ from above using the hyperbolic metric on $\mathbb{C} \setminus \overline{B(0, K/2)}$. This is given by

$$\frac{|d\zeta|}{|\zeta| \log \frac{2|\zeta|}{K}},$$

see e.g. [Hay89, Example 9.10]. For $\zeta \in \gamma^0(z) \subset \mathbb{C} \setminus \overline{D(0, K)}$, the denominator is bounded below by $\log 2 \cdot |\zeta|$. So

$$\ell_{\mathcal{O}}(\gamma^1(z)) \leq \frac{\log \lambda}{\log 2} =: \alpha.$$

This completes the proof of the claim. \triangle

Claim. We next claim that

$$(9.3) \quad \ell_\sigma(\gamma^j(z)) \leq \alpha, \quad \text{for } z \in J(g) \text{ and } 0 \leq j < m.$$

Proof. We prove the claim by induction on j ; note that we have just proved (9.3) when $j = 0$. So we can assume that $0 < j < m$ and that the claim holds for $j - 1$. Let $z \in J(g)$. Recall that f maps $\gamma^j(z)$ to $\gamma^{j-1}(g(z))$ in one-to-one fashion, and that $f^{j-1}(z) = \gamma^0(g^j(z)) \subset \mathbb{C} \setminus \overline{D(0, K)}$. By choice of K , and since $j < m$, this means that $\gamma^j(z)$ and $\gamma^{j-1}(g(z))$ lie in the region where σ agrees with the orbifold metric $\rho_{\mathcal{O}}$. Hence we can apply Proposition 4.6, and see that indeed

$$\ell_\sigma(\gamma^j(z)) = \ell_{\mathcal{O}}(\gamma^j(z)) \leq \ell_{\mathcal{O}}(\gamma^{j-1}(g(z))) = \ell_\sigma(\gamma^{j-1}(g(z))) \leq \alpha. \quad \triangle$$

Now, suppose that $z \in J(g)$ and that $k \geq m$. Write $k = \tilde{k}m + p$, where $\tilde{k} \geq 1$ and $0 \leq p < m$. Observe that $\gamma^k(z)$ is the pullback of $\gamma^p(g^{k-p}(z))$ by an inverse branch of $\tilde{f}^{\tilde{k}} = f^{\tilde{k}m}$. Moreover, $\gamma^p(g^{k-p}(z)) \subset \mathbb{C} \setminus \overline{D(0, K)} \subset S$. Hence the hypotheses of Proposition 8.1 are satisfied for points in $\gamma^k(z)$. It follows by Proposition 8.1, together with (9.3), that

$$\ell_\sigma(\gamma^k(z)) \leq \frac{\alpha}{C\epsilon_\sigma^\ell \tilde{k}^\tau},$$

where $C > 0$ and $\ell, \tau > 1$ are the constants from Proposition 8.1. Since these bounds are independent of z and summable over k , it follows from (9.2) that the maps $\vartheta^k|_{J(g)}$ form a Cauchy sequence with respect to the metric σ .

By definition of σ , the density of σ with respect to the spherical metric on $\hat{\mathbb{C}}$ is uniformly bounded from below. In particular, $\vartheta^k|_{J(g)}$ also forms a Cauchy sequence with respect to the spherical metric, and hence converges uniformly to a function ϑ in this metric. By Observation 2.4, f is docile. ■

Proof of Corollary 1.4. The corollary is an immediate consequence of Theorem 1.3 and Proposition 2.3. ■

Proof of Corollary 1.5. We use Corollary 1.4. By [BJR12], the Julia set $J(g)$ of the disjoint-type function g is a Cantor Bouquet. The Julia set $J(f)$ can be topologically described as the quotient of $J(g)$ by the closed equivalence relation

$$z \equiv w \quad \iff \quad \vartheta(z) = \vartheta(w).$$

Since $\vartheta: I(g) \rightarrow I(f)$ is a bijection, any nontrivial equivalence class is contained in the set of non-escaping points of g , which is a subset of the endpoints of the Cantor bouquet. (See [RRRS11, Theorem 5.10].)

Furthermore, ϑ preserves the cyclic order of hairs at infinity. (This is easy to see from the construction, but also follows from the fact that, for large R as in Theorem 2.1, the restriction of ϑ to $J_{\geq R}(g)$ extends to a quasiconformal homeomorphism of the complex plane by [Rem09].) Hence $J(g)$ is indeed a pinched Cantor Bouquet. ■

Proof of Corollary 1.6. The escaping set $I(g)$ has uncountably many connected components (see e.g. [Rem16, Proposition 3.10 and Corollary 3.11], and $I(f)$ is homeomorphic to $I(g)$ by Corollary 1.4. ■

To conclude the section we prove Theorem 1.7 and Theorem 1.11.

Proof of Theorem 1.7. Suppose that f is geometrically finite, and that $U := F(f) \neq \emptyset$ is connected. By assumption $U \cap S(f)$ is compact, and it is well-known that U is simply-connected. Hence, by [RS18, Corollary 8.5], $S(f) \subset U$. In particular f is strongly geometrically finite.

Let ϑ be the semiconjugacy from Corollary 1.4. Suppose, by way of contradiction, that ϑ is not injective. Then there are two distinct points $z_1, z_2 \in J(g)$ such that $\vartheta(z_1) = \vartheta(z_2)$. Let C_1 and C_2 be the connected components of $J(g)$ containing z_1 and z_2 respectively. Note that $C_1 \neq C_2$ by Corollary 1.4(d).

The set of non-escaping points in any component of $J(g)$ has zero Hausdorff dimension, see [Rem16, Theorem 2.3], and therefore is totally disconnected. Moreover, ϑ is an injection on the escaping set of g . Thus $\vartheta(C_1) \cap \vartheta(C_2)$ is totally disconnected. It is known that the union of two non-separating, closed, connected subsets of the sphere separates the sphere if and only if their intersection is disconnected; see [Mul22]. Hence $\vartheta(C_1) \cup \vartheta(C_2) \subset J(f)$ separates the plane. This is a contradiction, because the Julia set of f is nowhere dense, and so the fact that the Fatou set of f is connected implies that no closed subset of the Julia set of f can separate the plane.

The final claim of the theorem is immediate (again using the fact that the conjugacy preserves the order of hairs). ■

Proof of Theorem 1.11. We use the established notions of the symbolic dynamics of exponential maps; see, for example, [RS08] for definitions. As in [RS08], we shall use the

parameterisation of exponential maps as

$$f_\kappa: z \mapsto e^z + \kappa, \kappa \in \mathbb{C}.$$

Note that E_a as in (1.1) is conformally conjugate to f_κ if and only if $a = e^\kappa$. As first proved in [SZ03], the escaping set $I(f_\kappa)$ is an uncountable union of injective curves to infinity, called *hairs* or *dynamic rays*. These hairs can be described as the path-connected component of $I(f_\kappa)$ [FRS08, Corollary 4.3]. They are identified by infinite sequences of integers called *external addresses*, and the lexicographical ordering of addresses corresponds to the vertical ordering of hairs; see [RS08, Section 2]. If $\kappa_1, \kappa_2 \in \mathbb{C}$ are parameters for which the singular values do not belong to the escaping sets, then there is a natural bijection $\vartheta_{\kappa_1 \rightarrow \kappa_2}: I(\kappa_1) \rightarrow I(\kappa_2)$ that preserves the external addresses of hairs [Rem06, Theorem 1.1].

If κ_1 is of disjoint type, then the map $\vartheta_{\kappa_1 \rightarrow \kappa_2}$ is, by construction, the same bijection as in Theorem 2.2, up to a topological conjugacy between the two disjoint-type functions on their Julia sets. (We omit the details.) In particular, f_{κ_2} is docile if and only if $\vartheta_{\kappa_1 \rightarrow \kappa_2}$ extends continuously to $J(f_{\kappa_1}) \cup \{\infty\}$. Let us fix a disjoint-type parameter κ_1 for the remainder of the proof.

If f_κ has an attracting or parabolic orbit, then there is a unique associated *intermediate external address* $\text{addr}(\kappa)$; see [RS08, Section 2] or [Sch03, Lemma 3.3]. If κ has an attracting orbit, i.e. if κ is hyperbolic, then this address completely determines the dynamics on the Julia set as follows: Two hairs of E_κ land together if and only if their external addresses have the same *itinerary* with respect to $\text{addr}(\kappa)$ [Rem06, Proposition 9.2]. In particular, $\text{addr}(a)$ determines completely which endpoints of the Cantor bouquet $J(f_{\kappa_1})$ are identified by the extension of $\vartheta_{\kappa_1 \rightarrow \kappa}$.

The proof of [Rem06, Proposition 9.2] uses contraction properties of a suitable hyperbolic metric. Replacing the hyperbolic metric by our metric σ , and using Proposition 8.1, we see that the same result holds for any exponential map with a parabolic cycle. Now let $\kappa_0 \in \mathbb{C}$ be such a parabolic parameter. Then by [Sch03, Theorem 3.5], there is a unique hyperbolic component W consisting of parameters κ with $\text{addr}(\kappa) = \text{addr}(\kappa_0)$. (In [Sch03], the theorem is stated for the family E_a ; see [RS08, Proposition 4.2] for the corresponding statement in the κ -plane.) By Theorem 1.3 and the above, the bijection $\vartheta_{\kappa_0 \rightarrow \kappa} = \vartheta_{\kappa_1 \rightarrow \kappa} \circ \vartheta_{\kappa_1 \rightarrow \kappa_0}^{-1}$ of escaping sets extends to a conjugacy $J(\kappa_0) \cup \{\infty\} \rightarrow J(\kappa) \cup \{\infty\}$. By construction, this conjugacy preserves external addresses, and therefore the vertical order of hairs. This shows that W has the property claimed in the statement of Theorem 1.11. Moreover, by [RS08, Corollary 5.5], the component W is the unique “child component” of the parameter κ_0 , and in particular $\kappa_0 \in \partial W$.

To show uniqueness, let κ_0 be as above, and suppose that f_{κ_0} and f_κ are conjugate via a conjugacy that preserves the vertical order of hairs. Then by [Rem06, Proof of Corollary 8.4], there is $k \in \mathbb{Z}$ such that, for $\kappa' = \kappa + 2\pi ik$, the map $\vartheta_{\kappa_0 \rightarrow \kappa'}$ extends to a conjugacy between f_{κ_0} and $f_{\kappa'}$ on their Julia sets. In particular, the hairs at two given external addresses land together for f_{κ_0} if and only if they do so for $f_{\kappa'}$. It follows that f_{κ_0} and $f_{\kappa'}$ have the same *characteristic addresses* [RS08, Lemma 3.3 and Theorem 3.4], which in turn implies that $\text{addr}(\kappa_0) = \text{addr}(\kappa')$ [RS08, Lemma 3.10]. In other words, κ' belongs to the hyperbolic component W at address $\text{addr}(\kappa_0)$. As κ and κ' correspond

to the same parameter $a = \exp(\kappa)$ in our original parameterisation of the exponential family, the proof of the Theorem is complete. \blacksquare

10. FATOU COMPONENTS AND LOCAL CONNECTIVITY OF THE JULIA SET

We frequently use the following, which combines parts of Propositions 2.8 and 2.9 of [BFR15].

10.1. Proposition. *Suppose that f is an entire function, that $\Delta \subset \mathbb{C}$ is a simply connected domain, and that $\tilde{\Delta}$ is a component of $f^{-1}(\Delta)$. Then exactly one of the following holds.*

- (a) *The map $f : \tilde{\Delta} \rightarrow \Delta$ is proper, and hence has finite degree.*
- (b) *For $w \in \Delta$, with at most one exception, $f^{-1}(w) \cap \tilde{\Delta}$ is infinite. Also, either $\tilde{\Delta}$ contains an asymptotic curve corresponding to an asymptotic value in Δ , or $\tilde{\Delta}$ contains infinitely many critical points. Moreover, if $\Delta \cap S(f)$ is compact, then infinity is accessible in $\tilde{\Delta}$.*

If, in addition, $\Delta \cap S(f)$ is a singleton, then $\tilde{\Delta}$ contains at most one critical point of f .

We also use the following, which strengthens other parts of [BFR15, Proposition 2.9].

10.2. Proposition. *Suppose that f , Δ , and $\tilde{\Delta}$ are as in the statement of Proposition 10.1, and that case (a) of that proposition holds. Suppose also that Δ is a bounded Jordan domain. Then either $\partial\Delta$ contains an asymptotic value, or $\tilde{\Delta}$ is also a bounded Jordan domain.*

Proof. It is easy to see that, at any finite point, $\partial\tilde{\Delta}$ is locally an arc. In particular, if $\tilde{\Delta}$ is bounded, then it is a Jordan domain.

Now suppose that $\tilde{\Delta}$ is unbounded. Then the boundary of $\tilde{\Delta}$ in $\hat{\mathbb{C}}$ is locally connected. Indeed, this is true at every finite point by the above, and a continuum cannot fail to be locally connected at only a single point; see [Why42, (12.3) in Chapter I].

Let φ be a Riemann map from \mathbb{D} to $\tilde{\Delta}$ and let ψ be a Riemann map from \mathbb{D} to Δ . Then $\psi^{-1} \circ f \circ \varphi$ is a finite Blaschke product, say B , which extends continuously to $\partial\mathbb{D}$. By the Carathéodory-Torhorst theorem, φ and ψ , also extend continuously to $\partial\mathbb{D}$. Since $\tilde{\Delta}$ is unbounded, there is $\zeta \in \partial\mathbb{D}$ with $\varphi(\zeta) = \infty$. It follows that $\psi(B(\zeta)) \in \partial\Delta$ is an asymptotic value of f , as claimed. \blacksquare

A key result of this section is the following.

10.3. Theorem (Immediate basins of geometrically finite maps). *Suppose that f is strongly geometrically finite, and that U is a periodic Fatou component of f , of period $p \geq 1$. Then the following are equivalent:*

- (a) *U is a Jordan domain;*
- (b) *$\hat{\mathbb{C}} \setminus U$ is locally connected at some finite point of ∂U ;*
- (c) *U is bounded;*
- (d) *infinity is not accessible from U ;*
- (e) *the orbit of U contains no asymptotic curves and only finitely many critical points;*
- (f) *$f^p : U \rightarrow U$ is a proper map;*
- (g) *for at least two distinct choices of $z \in U$, the set $f^{-p}(z) \cap U$ is finite.*

10.4. Remark. Note that Theorem 10.3 is similar to [BFR15, Theorem 1.10]. That result has the stronger hypothesis that f is hyperbolic, and the stronger conclusion that U is, in fact, a quasidisc; in other words, the image of the unit disc under a quasiconformal map of the sphere. This stronger result does not hold in general when ∂U contains a parabolic point (even if U is an immediate attracting basin).

Proof of Theorem 10.3. The first parts of the proof are as in the proof of [BFR15, Theorem 1.10], and are included for completeness. That (a) \implies (b) is immediate. It follows from [BFR15, Theorem 2.6] that (b) \implies (c), and the fact that (c) \implies (d) is immediate. Note that, by assumption, $S(f) \cap U$ is compact. Suppose that infinity is not accessible in U . It follows from Proposition 10.1 that the orbit of U contains only finitely many critical points and no asymptotic curves. In other words, (d) \implies (e). The fact that (e) \implies (f) \iff (g) is also a consequence of Proposition 10.1.

It remains to show that (f) implies (a). As earlier in the paper, let \mathfrak{f} be the smallest iterate of f such that all periodic Fatou components of \mathfrak{f} are fixed, and all parabolic points are of multiplier one.

Suppose that U is a periodic Fatou component of f . If U is an immediate attracting basin and $\partial U \cap \text{Par}(f) = \emptyset$, then \mathfrak{f} is uniformly expanding in a neighbourhood of ∂U , with respect to the metric σ . In this case, we can follow the proof of [BFR15, Theorem 1.10], and obtain even that U is a quasidisc. If U is an immediate attracting basin whose boundary contains a parabolic point, then we may still follow the proof of [BFR15, Theorem 1.10], but replacing uniform expansion with Proposition 8.1. We again obtain that U is a Jordan domain (but no longer a quasidisc).

Suppose, then, that U is an immediate parabolic basin of \mathfrak{f} , with a parabolic fixed point $\zeta \in \partial U$. Recall that, by assumption, $\mathfrak{f} : U \rightarrow U$ is a proper map. Let $D'_j \subset U$ be the attracting petal from Proposition 5.2 that is contained in U . We may assume for simplicity that $\partial D'_j$ is piecewise analytic, and analytic except possibly at ζ . Define $U_n := \mathfrak{f}^{-(n+1)}(D'_j)$. Since D'_j contains $P(\mathfrak{f}) \cap U$, each ∂U_n is again a piecewise analytic Jordan curve, mapped as a covering map to ∂U_{n-1} .

Now define a homotopy between $\partial U_0 \setminus \{\zeta\}$ and $\partial U_1 \setminus \{\zeta\}$ in $U \setminus \bar{V}$. More precisely, we choose a continuous function

$$\vartheta : S^1 \times [0, 1] \rightarrow \bar{U}$$

such that:

- (a) $\vartheta(\cdot, 0) : S^1 \rightarrow \partial U_0$ is a homeomorphism;
- (b) $\vartheta(\cdot, 1) : S^1 \rightarrow \partial U_1$ is a homeomorphism;
- (c) $\vartheta(1, t) = \zeta$ for all $t \in [0, 1]$;
- (d) $\vartheta(x, t) \in U \setminus \bar{V}$ for $x \neq 1$ and $t < 1$.

Consider the curve $\gamma_0(x) := \vartheta(x, [0, 1])$, which connects $z_0(x) := \vartheta(x, 0)$ to $z_1(x) := \vartheta(x, 1)$. Clearly we may assume that ϑ is smooth, so that the Euclidean length of $\gamma_0(x)$ is uniformly bounded.

Moreover, for x sufficiently close to 1, we may suppose that $z_1(x)$ is the image of $z_0(x)$ under the branch of \mathfrak{f}^{-1} that fixes ζ . We have

$$|z_1(x) - z_0(x)| = O(|z_0(x) - \zeta|^{p+1}).$$

(Here $p+1$ is the multiplicity of ζ , as before.) It is easy to see that we can let the length of $\gamma_0(x)$ to be of the same order. In summary, the Euclidean length satisfies

$$\ell(\gamma_0(x)) = O(\min(1, d_{\text{Par}}(z_0(x))^{p+1})),$$

and the σ -length satisfies

$$(10.1) \quad \ell_\sigma(\gamma_0(x)) = O(\min(1, d_{\text{Par}}(z_0(x))^{p+1-s_\sigma})).$$

(Recall that $s = s_\sigma = 1 - \frac{1}{2n_\sigma}$ was defined in (7.1).)

We now extend ϑ to a map on $S^1 \times [0, \infty)$ as follows. For $x \in S^1$, let $x_1 \in S^1$ be the unique number such that $\mathfrak{f}(z_1(x)) = z_0(x_1)$. Since $f: U \setminus \overline{U_0} \rightarrow U \setminus \overline{D'_j}$ is a covering map, there is a unique continuous extension of ϑ to $S^1 \times [0, 2]$ such that

$$\mathfrak{f}(\vartheta(x, t+1)) = \vartheta(x_1, t)$$

for all $x \in S^1$.

Continuing inductively, we obtain the desired extension

$$\vartheta: S^1 \times [0, \infty) \rightarrow \overline{U}.$$

For $x \in S^1$ and $n \geq 0$, the curve $\gamma_n(x) := \vartheta(x, [n, n+1])$ connects ∂U_n and ∂U_{n+1} , and is a pullback of the curve $\gamma_0(x_n)$ for some $x_n \in S^1$.

By Proposition 8.1, Remark 8.2 and (10.1), it follows that

$$\ell_\sigma(\gamma_n(x)) = O\left(\frac{\ell_\sigma(\gamma_0(x_n))}{\min(d_{\text{Par}}(z_0(x))^{p+1-s_\sigma})} \cdot k^{-\tau}\right) = O(k^{-\tau}).$$

Since $\tau > 1$, these lengths are summable, and it follows that the functions $\vartheta_t = \vartheta(\cdot, t): S^1 \rightarrow \overline{U}$ converge uniformly in the metric σ to a continuous function $\varphi: S^1 \rightarrow \partial U$. Since ∂U_k converges to ∂U in the Hausdorff metric on the sphere, φ is surjective. In particular, U is bounded and ∂U is locally connected. By the maximum principle and Montel's theorem, \overline{U} is full with $\overline{\partial U} = \partial U$, and it follows that ∂U is indeed a Jordan curve. \blacksquare

We now use Theorem 10.3 to prove Theorem 1.8.

Proof of Theorem 1.8. Suppose that f is strongly geometrically finite. If f has an asymptotic value, then, by the definition of a strongly geometrically finite map, this lies in the Fatou set, and so f has an unbounded Fatou component. If a Fatou component contains infinitely many critical points, then these must accumulate at infinity and so the component must be unbounded. This completes the proof in one direction.

To prove the other direction, suppose that f has no asymptotic values and that each component of $F(f)$ contains at most finitely many critical points. It follows from Theorem 10.3 that every periodic Fatou component of f is a bounded Jordan domain.

Suppose that U is a Fatou component of f , and let V be the Fatou component of f containing $f(U)$. We claim that if V is a bounded Jordan domain, then so is U . Since, by Proposition 5.3, f has no wandering domains, it follows from this claim that all Fatou components of f are bounded Jordan domains, completing the proof.

To prove the claim, note that, by assumption, case (a) of Proposition 10.1 applies, with $\tilde{\Delta} = U$ and $\Delta = V$. Since f has no asymptotic values, it follows that U is a bounded Jordan domain by Proposition 10.2. \blacksquare

We can now deduce Theorem 1.9. The proof is a generalisation of [BFR15, Theorem 2.5] to strongly geometrically finite maps; see also [Mor99, Theorem 2].

Proof of Theorem 1.9. Suppose that f satisfies the hypotheses of the theorem; that is, f is strongly geometrically finite with no asymptotic values, and there is a uniform bound on the number of critical points, counting multiplicity, in each Fatou component. In particular, f satisfies the hypotheses of Theorem 1.8, and hence every Fatou component of f is a bounded Jordan domain.

It follows by [Why42, Thm. 4.4, Chapter VI], and see also [BFR15, comments after Lemma 2.3], that a compact subset of the sphere is locally connected if and only if the following conditions both hold:

- (a) the boundary of each complementary component is locally connected;
- (b) for each $r > 0$ there are only finitely many complementary components of spherical diameter greater than r .

By the above, the Julia set $J(f)$ satisfies (a). So it remains to show that (b) also holds. In other words, we have to show that for each $r > 0$ there are only finitely many Fatou components of spherical diameter greater than r . By passing to an iterate, we may assume that all periodic Fatou components are invariant; in particular, all parabolic points of f are fixed and of multiplier one.

Since f has only finitely many attracting or parabolic basins, it is enough to establish (b) for the connected components of each such basin separately. So suppose that V is an immediate attracting or parabolic basin. For each $z \in \partial V$, we choose a set U_z as follows.

- (a) If $z \notin \text{Par}(f)$, then we choose $U_z \subset S$ to be a round disc around z whose closure does not contain any postsingular points of f , except possibly z itself.
- (b) If $z \in \text{Par}(f)$, we choose a union $\tilde{U}_z \subset S$ of thin repelling petals at z , for each repelling direction at z , and set $U_z := \tilde{U}_z \cup \{z\}$. We may choose these sectors small enough that $\overline{U_z}$ contains no postsingular points other than z .

In each case, $U_z \cap J(f)$ is a relative neighbourhood of z in $J(f)$. Let $\delta > 0$ be such that every point in U_z can be connected to z by a curve of σ -length at most δ . Consider the collection $\mathcal{U}_n(z)$ of connected components of $f^{-n}(U_z)$. Each $U \in \mathcal{U}_n(z)$ contains exactly one element of $f^{-n}(z)$. Let $r(U)$ be the smallest number such that every point of U can be connected to this element by a curve of σ -length at most $r(U)$; we call $r(U)$ the *radius* of U . Since f does not expand the metric σ , we have $r(U) \leq \delta$, and moreover $r(U) \leq r(f(U))$ when $n \geq 1$.

Claim. The radius of $U \in \mathcal{U}_n(z)$ tends to zero uniformly as $n \rightarrow \infty$.

Proof. If $z \notin \text{Par}(f)$, then $d_{\text{par}}(w)$ is bounded from below for $w \in U_z$ by choice of U_z . It follows from Proposition 8.1 that

$$r(U) = O(\delta \cdot n^{-\tau})$$

for $U \in \mathcal{U}_n(z)$, showing that the radius tends to zero as desired.

On the other hand, suppose that $z \in \text{Par}(f)$, and let $U \in \mathcal{U}_1(z)$. If $z \notin U$, then there is $\varepsilon > 0$, independent of U , such that all points $w \in U$ have $d_{\text{par}}(w) \geq \varepsilon$. As above,

by Proposition 8.1, there is a sequence $\alpha_n > 0$ with $\alpha_n \rightarrow 0$ such that every connected component of $f^{-n}(U)$ has radius at most α_n , for all $n \geq 0$ and all $U \in \mathcal{U}_1(z)$ with $z \notin U$.

On the other hand, for each $n \geq 0$, there is exactly one $U_n \in \mathcal{U}_n(z)$ with $z \in U_n$, and we have $U_{n+1} \subset U_n$. Since U_z consists of repelling petals, together with z , we have $\bigcap U_n = \{z\}$. Therefore $\beta_n := r(U_n) \rightarrow 0$ as $n \rightarrow \infty$.

Now let $U \in \mathcal{U}_n(z)$, and let $k \leq n$ be minimal such that $f^k(U) \subset U_z$. Then, on the one hand,

$$r(U) \leq r(f^k(U)) = r(U_{n-k}) = \beta_{n-k}.$$

On the other hand, if $k \neq 0$, then $f^{k-1}(U)$ is contained in an element of $\mathcal{U}_1(z)$ that does not contain z , and therefore

$$r(U) \leq \alpha_{k-1}.$$

Therefore

$$r(U) \leq \gamma_n := \max_{k \leq n} \min(\alpha_{k-1}, \beta_{n-k}).$$

Clearly $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$, as desired. \triangle

Claim. Let \mathcal{V}_n denote the connected components of $f^{-n}(V)$, other than V itself. Then the σ -diameter of $V_n \in \mathcal{V}_n$ tends to zero uniformly as $n \rightarrow \infty$.

Proof. Since ∂V is compact and locally connected, we may choose finitely many points z_1, \dots, z_M such that the connected components of $U_{z_j} \cap \partial V$ containing z_j cover ∂V . Write $\mathcal{U}_n := \bigcap_{j=1}^M \mathcal{U}_n(z_j)$. By the claim, there is $\delta_n \rightarrow 0$ such that the σ -diameter of $U \in \mathcal{U}_n$ is at most δ_n .

If $V_n \in \mathcal{V}_n$, then we can cover ∂V_n by $M \cdot L$ sets from \mathcal{U}_n , where M is the degree of $f^n: V_n \rightarrow V$. By assumption, this degree is uniformly bounded. Indeed, since $S(f) \cap F(f)$ is compact, only a finite number, t say, of the Fatou components of f intersect $S(f)$. By the Riemann-Hurwitz formula, and by assumption, the degree of f on any Fatou component is bounded by $N + 1$. Thus $M \leq (N + 1)^t$, and therefore $\text{diam}_\sigma(V) \leq L \cdot (N + 1)^t \cdot \delta_n \rightarrow 0$. \triangle

In particular, the spherical diameter of $V \in \mathcal{V}_n$ tends to zero uniformly as $n \rightarrow \infty$. On the other hand, for fixed n the different elements of \mathcal{V}_n can accumulate only at ∞ ; see [BR17, Lemma 2.1]. In other words, for a given $\varepsilon > 0$, only finitely many \mathcal{V}_n contain elements of spherical diameter at least ε , and for each n the number of such elements is finite. The proof of the theorem is complete. \blacksquare

11. EXAMPLES

In this final section we give some examples. We begin with the simplest example of a transcendental entire function that is (strongly) geometrically finite but not subhyperbolic.

11.1. Example. Let $f_1(z) = e^{z-1}$. Then f_1 is strongly geometrically finite, but not subhyperbolic.

Proof. We have $AV(f_1) = S(f_1) = \{0\}$. Moreover, 1 is a parabolic fixed point of f_1 . Since its immediate basin must contain a singular value – or by elementary considerations – we have $f^n(0) \in F_{\text{par}}(f)$. Hence $J(f_1) \cap P(f_1) = \{1\}$, and f_1 is strongly geometrically finite, but $F(f_1) \cap P(f_1)$ is not compact and so f_1 is not subhyperbolic.

It is straightforward to show that $F(f)$ is connected. Hence, by Theorem 1.7, f is topologically conjugate on its Julia set to any map of the form $z \mapsto \lambda e^z$, for $\lambda \in (0, 1/e)$. In particular, as noted earlier, $J(f)$ is a Cantor bouquet. ■

Our second example, mentioned in the introduction, shows that the Julia set of the sine function is locally connected.

11.2. Example. Let $f_2(z) = \sin z$. Then f_2 is strongly geometrically finite, and $J(f_2)$ is locally connected.

Proof. Note that $S(f_2) = \{\pm 1\}$. The origin is a parabolic fixed point of f_2 , and it is easy to see that both points of $S(f_2)$ lie in the parabolic basin of this point. It follows that $J(f_2) \cap P(f_2) = \{0\}$, and f_2 is strongly geometrically finite. All points on the imaginary axis apart from the origin lie in $I(f_2)$, and so the whole imaginary axis lies in $J(f_2)$. Thus the two singular values of f_2 lie in different Fatou components. Hence the assumptions of Corollary 1.10 are satisfied, and so $J(f_2)$ is locally connected. ■

A transcendental entire function f is *parabolic* if $S(f) \subset F(f)$, and $J(f) \cap P(f)$ is finite and contained in $\text{Par}(f)$. Note that Examples 11.1 and 11.2 are both parabolic, and so covered by the results of [Alh18]. Our next example is a transcendental entire function that is geometrically finite but not parabolic.

11.3. Example. Let $f_3(z) = ze^z$. Then f_3 is geometrically finite, but not parabolic, not strongly geometrically finite, and not docile.

Proof. It can be seen that f_3 has one asymptotic value, at the origin, which is fixed and parabolic. It also has one critical value, at $-1/e$, and this is in the parabolic basin. Hence f_3 is geometrically finite, but not parabolic since $S(f_3) \cap J(f_3) \neq \emptyset$. Since $0 \in J(f_3)$ is an asymptotic value, f_3 is also not strongly geometrically finite.

To show that f_3 is not docile, let $g: z \mapsto f_3(\lambda z)$ be of disjoint type, and suppose by contradiction that $\vartheta: J(g) \cup \{\infty\} \rightarrow J(f_3)$ is a semiconjugacy as in Proposition 2.3. Let $z_0 \in J(g)$ with $\vartheta(z_0) = 0$. The compact set $\vartheta^{-1}(0) \subset \hat{\mathbb{C}}$ is a subset of \mathbb{C} by Proposition 2.3 (a). Since z_0 is not a Picard exceptional value of g , there is $z_1 \in g^{-1}(z_0) \setminus \vartheta^{-1}(0)$. But then $0 \neq w_1 := \vartheta(z_1) \in \mathbb{C}$ and

$$f_3(w_1) = f_3(\vartheta(z_1)) = \vartheta(g(z_1)) = \vartheta(z_0) = 0$$

by Proposition 2.3 (c). This is a contradiction since $f_3^{-1}(0) = \{0\}$. ■

Remark. A similar argument shows that, more generally, a function having a direct asymptotic value in the Julia set cannot be docile.

Our final example is a transcendental entire function which is not parabolic, but which is *strongly* geometrically finite.

11.4. Example. For each $a \in \mathbb{C}$ such that $\cos a \neq 0$, define the transcendental entire function

$$g_a(z) = \frac{\sin(z+a) - \sin a}{\cos a}.$$

For a suitable value of a , the function $f_4 := g_a$ is strongly geometrically finite, but neither subhyperbolic nor parabolic. The Julia set $J(f_4)$ is locally connected.

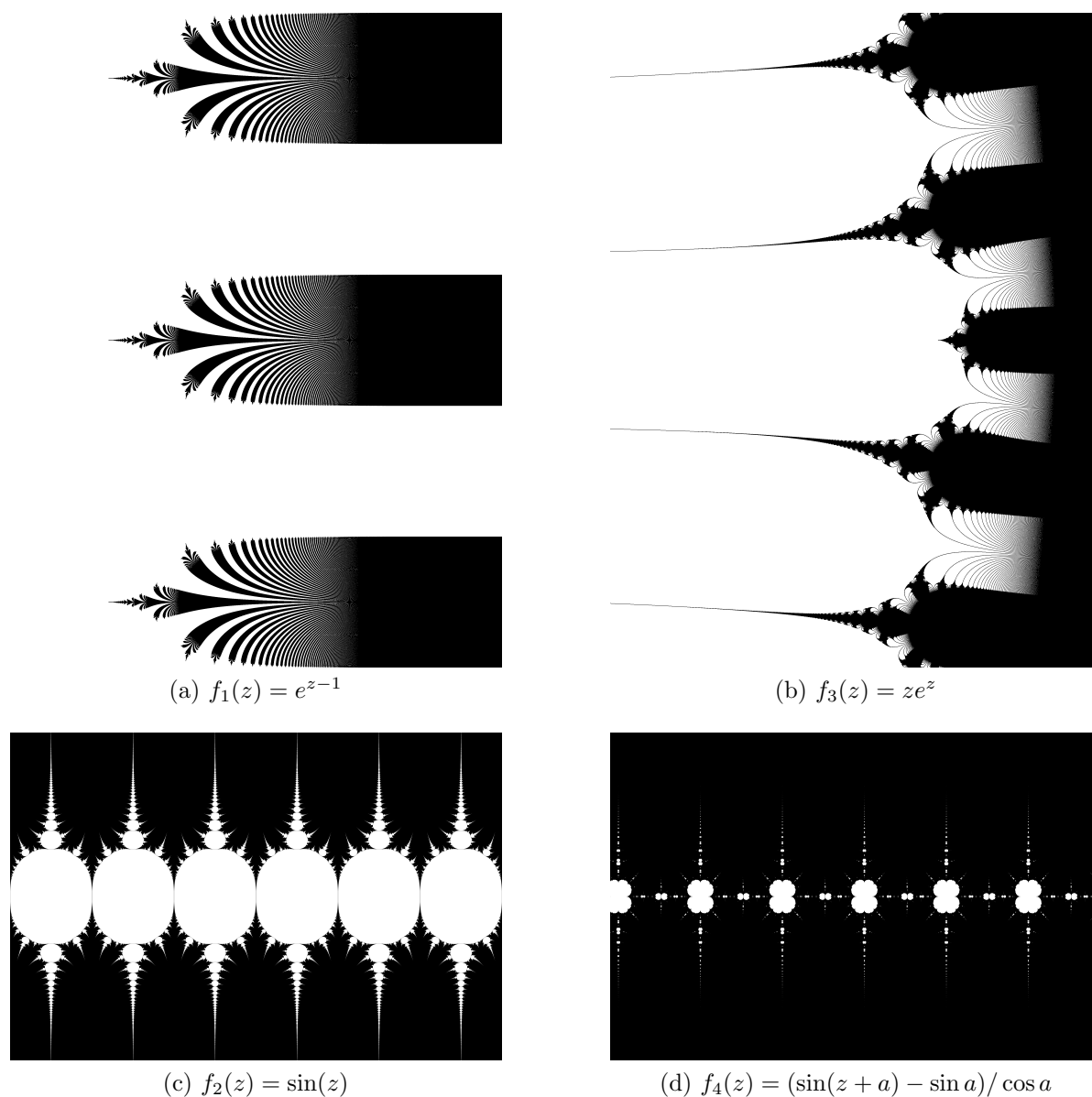


FIGURE 3. Julia sets (in black) of the four functions from Section 11.

Proof. Note that g_a has no asymptotic values, and

$$S(g_a) = CV(g_a) = \left\{ \frac{\pm 1 - \sin a}{\cos a} \right\}.$$

Also, the origin is a parabolic fixed point of g_a . We choose $a \in (0, \pi/2)$ such that

$$\frac{1 + \sin a}{\cos a} = 2\pi;$$

in fact, it can be seen that there is a unique such a , with $a \approx 1.255134$. It follows that one critical value of g_a is at -2π . Since $g_a(-2\pi) = 0$, this critical value lies in the Julia

set. It can be checked that the other critical value lies in the parabolic basin, and hence f_4 is geometrically finite. As f_4 is obtained from \sin up to pre- and post-composition with affine functions, it has no asymptotic values, and all critical points are simple. Therefore f_4 has bounded criticality, and f_4 is strongly geometrically finite. By Corollary 1.10, the Julia set $J(f_4)$ is locally connected. ■

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DEPARTMENT OF MATHEMATICS, PRINCESS NORA BINT ABDULRAHMAN UNIVERSITY, RIYADH 11538, SAUDI ARABIA


E-mail address: maalhamad@pnu.edu.sa

DEPT. OF MATHEMATICAL SCIENCES, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 7ZL, UK,

 <https://orcid.org/0000-0001-8032-8580>

E-mail address: l.rempe@liverpool.ac.uk

DEPT. OF MATHEMATICAL SCIENCES, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 7ZL, UK,

 <https://orcid.org/0000-0002-3543-6969>

E-mail address: david.sixsmith@open.ac.uk