

POSITIVE TRANSVERSALITY VIA TRANSFER OPERATORS AND HOLOMORPHIC MOTIONS WITH APPLICATIONS TO MONOTONICITY FOR INTERVAL MAPS

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ABSTRACT. In this paper we will develop a general approach which shows that generalized “critical relations” of families of locally defined holomorphic maps on the complex plane unfold transversally. The main idea is to define a transfer operator, which is a local analogue of the Thurston pullback operator, using holomorphic motions. Assuming a so-called lifting property is satisfied, we obtain information about the spectrum of this transfer operator and thus about transversality. An important new feature of our method is that it is not global: the maps we consider are only required to be defined and holomorphic on a neighbourhood of some finite set.

We will illustrate this method by obtaining transversality for a wide class of one-parameter families of interval and circle maps, for example for maps with flat critical points, but also for maps with complex analytic extensions such as certain polynomial-like maps. As in Tsujii’s approach [48, 49], for real maps we obtain *positive* transversality (where > 0 holds instead of just $\neq 0$), and thus monotonicity of entropy for these families, and also (as an easy application) for the real quadratic family.

This method additionally gives results for unimodal families of the form $x \mapsto |x|^\ell + c$ for $\ell > 1$ not necessarily an even integer and c real.

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1. INTRODUCTION

This paper is about bifurcations in families of (real and complex) one dimensional dynamical systems.¹ For example, for real one-dimensional dynamical systems, we have a precise combinatorial description on the dynamics in terms of the so-called kneading sequences. One simple but very important question is how the kneading sequence varies in families of such systems. For the real quadratic family $f_a(x) = x^2 + a$, it is known that the kneading sequence depends monotonically on the parameter a (with respect to the natural order defined for kneading sequences). Interestingly the proofs of this result, by Milnor-Thurston, Douady-Hubbard and Sullivan, make

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use of Teichmüller theory, uniqueness in Thurston's realisation theorem, or quasiconformal rigidity theory and that the map f_a is quadratic, see [37, 47] and also [11].

To answer the above monotonicity question it is enough to show that when $f_{a'}^n(0) = 0$ for some $a' \in \mathbb{R}$, then there exists no other parameter $a'' \in \mathbb{R}$ for which $f_{a''}^n(0) = 0$ and for which $f_{a'}$ and $f_{a''}$ also have the same (periodic) kneading sequence.

If $\frac{d}{da}f_a^n(0)|_{a=a'} \neq 0$ (which is called *transversality*) then one has local uniqueness in the following sense: there exists $\epsilon > 0$ so that the kneading invariant of f_a for $a \in (a' - \epsilon, a')$, for $a \in (a', a' + \epsilon)$ and for $a = a'$ are all different. It turns out that *global* uniqueness and monotonicity follows from

$$Q := \frac{\frac{d}{da}f_a^n(0)|_{a=a'}}{Df_{a'}^n(f_{a'}(0))} > 0$$

(which we call *positive transversality*). Tsujii gave an alternative proof of the above monotonicity for the quadratic family by showing that this inequality holds [48, 49].

For general (complex) holomorphic families of maps with several critical points which all are eventually periodic there exists a similar expression Q . Again $Q \neq 0$ implies that the bifurcations are non-degenerate and hence the corresponding critical relations unfold transversally. In this paper we show that the inequality $Q \neq 0$ holds provided the spectrum of some operator \mathcal{A} does not contain 1, and that $Q > 0$ holds if additionally the spectrum of \mathcal{A} is contained in the closed unit disc and the family of maps is real.

We define this operator \mathcal{A} by considering how the speed of a (holomorphic) motion of the orbits of critical points is lifted under the dynamics. The novelty of our method, described in Proposition 5.1 and Theorem 6.1, is to show if these holomorphic motions have the lifting property, i.e. can be lifted infinitely many times over the same domain, then the operator \mathcal{A} has the above spectral properties.

It turns out that the lifting property makes minimal use of the global dynamics of the holomorphic extension of the dynamics. Thus we can obtain transversality properties of families f_t of maps defined on open subsets of the complex plane so that f_0 has a finite invariant marked set, e.g., f_0 is 'critically finite'.

The methods developed in this paper give a new and simple proof of well-known results for families of polynomial maps, rational or entire maps, but also applies to many other families for which no techniques were available. For example we obtain monotonicity for the family of maps

$$f_c(x) = be^{-1/|x|^\ell} + c$$

having a flat critical point at 0. We also obtain partial monotonicity for the family

$$f_c(x) = |x|^\ell + c$$

when ℓ is large.

As mentioned, the aim of this paper is to deal with families of maps which are only locally holomorphic. This means that the approach pioneered by Thurston, and developed by Douady and Hubbard in [13], cannot be applied. In Thurston's approach, when f is a globally defined holomorphic map, P is a finite f -forward invariant set containing the postcritical set and the Thurston map $\sigma_f: \text{Teich}(\widehat{\mathbb{C}} \setminus P) \rightarrow \text{Teich}(\widehat{\mathbb{C}} \setminus P)$ is defined by pulling back an almost holomorphic structure. It turns out that σ_f is contracting, see [13, Corollary 3.4]. In Thurston's result on the topological realisation of rational maps, Douady & Hubbard [13] use that the dual of the derivative of the Thurston map σ_f is equal to the Thurston pushforward operator f_* .

The pushforward operator acts on space of quadratic differentials, and the injectivity of $f_* - id$ has been used to obtain transversality results for rational maps or even more general spaces of maps, see Tsujii [48, 49], Epstein [16], Levin [28, 29, 30], Makienko [32], Buff & Epstein [6] and Astorg [1]. See also earlier [31], [14, Proposition 22²], [15]. For a short and elementary proof that critical relations unfold transversally in the setting of rational maps, see [22]. For the spectrum of f_* see also [25], [26], [27], [18] and [7]. In [45], transversality is shown for rational maps for

²This reference was provided by the author.

which each critical point is mapped into a hyperbolic set, using the uniqueness part of Thurston's realisation result.

However if for example $f: U \rightarrow V$ is a polynomial-like map then each point in the boundary of V is a singular value, $\text{Teich}(V \setminus P)$ is infinite dimensional, Thurston's algorithm is only locally well-defined and it is not clear whether it is locally contracting.

The purpose of our paper is to bypass this issue, by going back to the original Milnor-Thurston approach. Milnor and Thurston [37] associated to the space of quadratic maps and the combinatorial information of a periodic orbit, a map which assigns to a q tuple of points a new q tuple of points,

$$F : (z_1, \dots, z_q) \mapsto (\hat{z}_1, \dots, \hat{z}_q)$$

where $\hat{z}_q = 0$ and

$$f_{z_1}(\hat{z}_i) = z_{i+1 \bmod q}$$

where $f_c(z) \equiv z^2 + c$. Since F is many-valued, Milnor & Thurston considered a lift \tilde{F} of this map to the universal cover and apply Teichmüller theory to show that \tilde{F} is strictly contracting in the Teichmüller metric of the universal cover.

We bypass this issue by rephrasing their approach locally (via holomorphic motions). This is done in the set-up of so-called marked maps (and their local deformations) which include particularly critically finite maps with any number of critical (singular) points. In the first part of the paper we prove general results, notably the Main Theorem, which show that under the assumption that some lifting property holds for the deformation, either some critical relation persists along some non-trivial manifold in parameter space or one has transversality, i.e. the critical relation unfolds transversally. Here the lifting property is an assumption that sequences of successive lifts of holomorphic motions are compact. In the second part of the paper, we then show that this lifting property holds not only in previously considered global cases but also for interesting classes of maps where the 'pushforward' approach breaks down.

More precisely, we define a transfer operator \mathcal{A} by its action on infinitesimal holomorphic motions on P . It turns out that if the lifting property holds, then the spectrum of the operator \mathcal{A} lies inside the unit disc. Moreover, if the operator \mathcal{A} has no eigenvalue 1 then transversality holds; in the real case one has even positive transversality (the sign of some determinant is positive). One of the main steps in the proof of the Main Theorem is then to show that if the operator has an eigenvalue 1 then the critical relation persists along a non-trivial manifold in parameter space. It turns out that for globally defined critically finite maps f the transfer operator \mathcal{A} can be identified with (dual of) f_* .

By verifying the lifting property we recover previous results such as transversality for rational maps, but also obtain transversality for many interesting families of polynomial-like mappings and families of maps with essential singularities. For real local maps our approach gives the 'positive transversality' condition which first appeared in [48, 49] and therefore monotonicity of entropy for certain families of real maps.

2. STATEMENT OF RESULTS

2.1. The notion of a marked map. A *marked map* is a map g from the union of a finite set P_0 and an open set $U \subset \mathbb{C}$ mapping into \mathbb{C} such that

- there exists a finite set $P \supset P_0$ such that $g(P) \subset P$ and $P \setminus P_0 \subset U$;
- $g|_U$ is holomorphic and $g'(x) \neq 0$ for $x \in P \setminus P_0$.

Let $\{c_{0,1}, \dots, c_{0,\nu}\}$ denote the (distinct) points in P_0 and write $\mathbf{c}_0 = \mathbf{c}_0(g) = (c_{0,1}, \dots, c_{0,\nu})$ and

$$\mathbf{c}_1 = \mathbf{c}_1(g) = (g(c_{0,1}), \dots, g(c_{0,\nu})) := (c_{1,1}, c_{1,2}, \dots, c_{1,\nu}).$$

Remark 2.1. *So P is a forward invariant set for g , and g is only required to be holomorphic (and defined) on a neighbourhood of $P \setminus P_0$. A marked map g does not need to be defined in a neighbourhood of P_0 . In applications, points in P_0 will be where some extension of g has a critical point, an (essential) singularity or even where g has a discontinuity. In this sense marked maps correspond to a generalisation of the notion of critically finite maps.*

2.2. Holomorphic deformations. A *local holomorphic deformation* of g is a triple $(g, G, \mathbf{p})_W$ with the following properties:

- (1) W is an open connected subset of \mathbb{C}^ν containing $\mathbf{c}_1(g)$;
- (2) $\mathbf{p} = (p_1, p_2, \dots, p_\nu) : W \rightarrow \mathbb{C}^\nu$ is a holomorphic map, so that $\mathbf{p}(\mathbf{c}_1) = \mathbf{c}_0(g)$ (and so all coordinates of $\mathbf{p}(\mathbf{c}_1)$ are distinct).
- (3) $G : (w, z) \in W \times U \mapsto G_w(z) \in \mathbb{C}$ is a holomorphic map such that $G_{\mathbf{c}_1} = g|_U$.

Example 2.2. (i) *The simplest local holomorphic deformation of g is of course the trivial one: $G_w(z) = g(z)$, $p(w) = c_0$, $\forall w$.*

(ii) *If g is a marked map with $\nu = 1$, i.e., $P_0 = \{c_0\}$, then $G_w(z) = g(z) + (w - g(c_0))$, $p(w) \equiv c_0$, defines a local holomorphic deformation of g .*

(iii) *If g is rational map with ν critical points c_1, \dots, c_ν with multiplicity μ_1, \dots, μ_ν then there exists a local holomorphic deformation $(g, G, \mathbf{p})_W$ of g so that $w_i = G_w(\mathbf{p}_i(w))$ is a critical value of G_w for each $w = (w_1, \dots, w_\nu) \in W$ and each $i = 1, \dots, \nu$, see Appendix C.*

2.3. Transversal unfolding of critical relations. Let us fix $(g, G, \mathbf{p})_W$ as above. Since $g(P) \subset P$ and P is a finite set, for each $j = 1, 2, \dots, \nu$, exactly one of the following *critical relations* holds:

- (a) There exists an integer $q_j > 0$ and $\mu(j) \in \{1, 2, \dots, \nu\}$ such that $g^{q_j}(c_{0,j}) = c_{0,\mu(j)}$ and $g^k(c_{0,j}) \notin P_0$ for each $1 \leq k < q_j$;
- (b) There exist integers $1 \leq l_j < q_j$ such that $g^{q_j}(c_{0,j}) = g^{l_j}(c_{0,j})$ and $g^k(c_{0,j}) \notin P_0$ for all $1 \leq k \leq q_j$.

Relabelling these points $c_{0,j}$, we assume that there is r such that the first alternative happens for all $1 \leq j \leq r$ and the second alternative happens for $r < j \leq \nu$.

Define the map

$$\mathcal{R} = (R_1, R_2, \dots, R_\nu)$$

from a neighbourhood of $\mathbf{c}_1 \in \mathbb{C}^\nu$ into \mathbb{C}^ν as follows: for $1 \leq j \leq r$,

$$(2.1) \quad R_j(w) = G_w^{q_j-1}(w_j) - p_{\mu(j)}(w)$$

and for $r < j \leq \nu$,

$$(2.2) \quad R_j(w) = G_w^{q_j-1}(w_j) - G_w^{l_j-1}(w_j),$$

where $w = (w_j)_{j=1}^\nu$.

Definition 2.3. *We say that the holomorphic deformation $(g, G, \mathbf{p})_W$ of g satisfies the transversality property, if the Jacobian matrix $DR(\mathbf{c}_1)$ is invertible.*

Example 2.4. (i) *Assume that $(g, G, \mathbf{p})_W$ is a local holomorphic deformation so that for each $w = (w_1, \dots, w_\nu) \in W$, the critical values of G_w are w_1, \dots, w_ν and $\mathbf{p}_1(w), \dots, \mathbf{p}_\nu(w)$ are the critical points of G_w . Then $G_w(\mathbf{p}_i(w)) = w_i$. Hence $R_j \equiv 0$ in (2.1) or (2.2) correspond to*

$$G_w^{q_j}(\mathbf{p}_j(w)) - p_{\mu(j)}(w) \equiv 0 \text{ resp. } G_w^{q_j}(\mathbf{p}_j(w)) - G_w^{l_j}(\mathbf{p}_j(w)) \equiv 0.$$

These equations define the set of parameters w for which the corresponding ‘critical relation’ is satisfied within the family G_w .

(ii) *In Example 2.6(ii) we will consider a holomorphic deformation $(g, G, \mathbf{p})_W$ of a map g so that G_w is not defined (as an analytic map) in $\mathbf{p}_j(w)$, but nevertheless the above interpretation is valid.*

(iii) *If we take the trivial deformation $G_w(z) = g(z)$, $\mathbf{p}(w) = \mathbf{c}_0$, $\forall w \in W$, then definitions (2.1) or (2.2) take the form $R_j(w) = g^{q_j-1}(w_j) - c_{0,j'}$ respectively $R_j(w) = g^{q_j-1}(w_j) - g^{l_j-1}(w_j)$. Then $DR(\mathbf{c}_1)$ is a diagonal matrix with entries $Dg^{q_j-1}(w_j)$ for $1 \leq j \leq r$ and $Dg^{l_j-1}(w_j)[Dg^{q_j-l_j}(g^{l_j-1}(w_j)) - 1]$ for $r < j \leq \nu$. So the matrix $DR(\mathbf{c}_1)$ is non-degenerate iff $Dg^{q_j-l_j}(g^{l_j-1}(w_j)) \neq 1$ for $r < j \leq \nu$. It follows immediately that the holomorphic deformation $(g, G, \mathbf{p})_W$ satisfies the transversality property if and only if $Dg^{q_j-l_j}(g^{l_j}(c_{0,j})) \neq 1$ for $r < j \leq \nu$. We should emphasise that in this setting the condition $\mathcal{R}(w) = 0$ has nothing to do with the presence of critical relations.*

2.4. Real marked maps and positive transversality. A marked map g is called *real* if $P \subset \mathbb{R}$ and for any $z \in U$ we have $\bar{z} \in U$ and $\overline{g(z)} = g(\bar{z})$. Similarly, a local holomorphic deformation $(g, G, \mathbf{p})_W$ of a real marked map g is called *real* if for any $w = (w_1, w_2, \dots, w_\nu) \in W$, $z \in U$ and $j = 1, 2, \dots, \nu$, we have $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_\nu) \in W$, $G_{\bar{w}}(\bar{z}) = \overline{G_w(z)}$, and $p_j(\bar{w}) = \overline{p_j(w)}$.

Definition 2.5. Let $(g, G, \mathbf{p})_W$ be a real local holomorphic deformation of a real marked map g . We say that the unfolding $(g, G, \mathbf{p})_W$ satisfies the ‘positively oriented’ transversality property if

$$(2.3) \quad Q := \frac{\det(D\mathcal{R}(\mathbf{c}_1))}{\prod_{j=1}^{\nu} Dg^{q_j-1}(\mathbf{c}_{1,j})} > 0.$$

The sign in the previous inequality means that the intersection of the analytic sets $R_j = 0$ $j = 1, \dots, \nu$, is not only in general position (i.e. ‘transversal’), but that the intersection pattern is *everywhere* ‘positively oriented’.

2.5. The lifting property. Let $X \subset \mathbb{C}$ and Λ be a domain in \mathbb{C} which contains 0. As usual, we say that h_λ is a *holomorphic motion of X over $(\Lambda, 0)$* , if $h_\lambda: X \rightarrow \mathbb{C}$ satisfies:

- (i) $h_0(x) = x$, for all $x \in X$,
- (ii) $h_\lambda(x) \neq h_\lambda(y)$ whenever $x \neq y$ and $\lambda \in \Lambda$, $x, y \in X$ and
- (iii) $\Lambda \ni \lambda \mapsto h_\lambda(z)$ is holomorphic.

Given a holomorphic motion h_λ of $g(P)$ over $(\Lambda, 0)$, where Λ is a domain in \mathbb{C} which contains 0, we say that \hat{h}_λ is a *lift of h_λ over $\Lambda_0 \subset \Lambda$* (with $0 \in \Lambda_0$) with respect to $(g, G, \mathbf{p})_W$ if for all $\lambda \in \Lambda_0$,

- $\hat{h}_\lambda(c_{0,j}) = p_j(\mathbf{c}_1(\lambda))$ for each $j = 1, 2, \dots, \nu$, with $c_{0,j} \in g(P)$, where

$$\mathbf{c}_1(\lambda) = (h_\lambda(c_{1,1}), h_\lambda(c_{1,2}), \dots, h_\lambda(c_{1,\nu}));$$

- $G_{\mathbf{c}_1(\lambda)}(\hat{h}_\lambda(x)) = h_\lambda(g(x))$ for each $x \in g(P) \setminus P_0$.

Clearly such a lift exists, provided Λ_0 is contained in a sufficiently small neighbourhood of 0.

We say that the triple $(g, G, \mathbf{p})_W$ has the *lifting property* if for each holomorphic motion $h_\lambda^{(0)}$ of $g(P)$ over $(\mathbb{D}, 0)$ there exist $\varepsilon > 0$ and a sequence of holomorphic motions $h_\lambda^{(k)}$, $k = 1, 2, \dots$ of $g(P)$ over $(\mathbb{D}_\varepsilon, 0)$ such that for each $k \geq 0$,

- $h_\lambda^{(k+1)}$ is a lift of $h_\lambda^{(k)}$ over $(\mathbb{D}_\varepsilon, 0)$;
- there exists $M > 0$ such that $|h_\lambda^{(k)}(x)| \leq M$ for all $x \in g(P)$ and all $\lambda \in \mathbb{D}_\varepsilon$.

In the case $(g, G, \mathbf{p})_W$ is real, we say it has the *real lifting property* if the corresponding property holds for any real-symmetric holomorphic motions $h_\lambda^{(0)}$.

2.6. Statement of the Main Theorem.

Main Theorem. Assume that g does not have a parabolic periodic point in $P \setminus P_0$ and that $(g, G, \mathbf{p})_W$ satisfies the lifting property. Then exactly one of the following holds:

- (1) the holomorphic deformation $(g, G, \mathbf{p})_W$ of g satisfies the transversality property;
- (2) there exists a neighborhood W' of \mathbf{c}_1 such that $\{w \in W' : \mathcal{R}(w) = 0\}$ is a smooth complex manifold of positive dimension.

Moreover, if $(g, G, \mathbf{p})_W$ is real and satisfies the real lifting property then in (1) ‘the transversality property’ can be replaced by ‘the ‘positively oriented’ transversality property’.

The statement of this theorem is a combination of the more detailed statements in Theorems 4.1 and 6.1.

2.7. Classical settings where the lifting property holds. In many cases it is easy to check that the lifting property holds, and therefore the previous theorem applies. Indeed, it is easy to see that this holds in the setting of polynomial or rational maps, see Section C.

2.8. Transversality for new families of maps corresponding to classes \mathcal{F} , \mathcal{E} , \mathcal{E}_o . In this subsection we will discuss two new settings where the current approach can be applied to obtain transversality.

Let us first consider families of maps $f_c(z) = f(z) + c$. Here f is contained in the space \mathcal{F} of holomorphic maps $f: U \rightarrow V$ where

- (a) U is a bounded open set in \mathbb{C} such that $0 \in \bar{U}$;
- (b) V is a bounded open set in \mathbb{C} ;
- (c) $f: U \setminus \{0\} \rightarrow V \setminus \{0\}$ is an unbranched covering;
- (d) $V \supset B(0; \text{diam}(U)) \supset U$.

Examples of such families are

Example 2.6. (i) $f_c(z) = z^d + c$, where U, V are suitably large balls and $c \in U$.

(ii) $f_c(x) = be^{-1/|x|^\ell} + c$ where $\ell \geq 1$, $b > 2(e\ell)^{1/\ell}$, $c \in U$. Here $U = U^- \cup U^+$, $U^+ = -U^-$ are topological disks symmetric w.r.t. the real axis and V is a punctured disc. That $f_0 \in \mathcal{F}$ is proved in Corollary 7.3.

Theorem 2.1. Let $f \in \mathcal{F}$, let $g = f_{c_1}$ and for each $w \in W := \mathbb{C}$ define $G_w(z) = g(z) + (w - c_1)$ and $\mathbf{p}(w) = 0$ where we assume that $c_1 \in U$. Moreover, assume that there exists q so that $c_n = g^{n-1}(c_1) \in U$ for all $n < q$ and either $c_q = 0$ or $c_q \in \{c_1, \dots, c_{q-1}\}$. Moreover, assume $c_n \notin \{0, c_1, \dots, c_{n-1}\}$ for $0 < n < q$. Then

- $(g, G, \mathbf{p})_W$ satisfies the lifting property and transversality holds.
- if $(g, G, \mathbf{p})_W$ is real, then positive transversality holds.

Our methods also apply to families of the form $f_w(z) = wf(z)$ where f is contained in the spaces \mathcal{E} and \mathcal{E}_o defined as follows. Consider holomorphic maps $f: D \rightarrow V$ such that:

- (a) D, V are open sets which are symmetric w.r.t. the real line so that $f(D) = V$
- (b) Let $I = D \cap \mathbb{R}$ then there exists $c > 0$ so that $I \cup \{c\}$ is a (finite or infinite) open interval and $0 \in \bar{I}$, $c \in \text{int}(\bar{I})$. Moreover, f extends continuously to \bar{I} , $f(I) \subset \mathbb{R}$ and $\lim_{z \in D, z \rightarrow 0} f(z) = 0$.
- (c) Let D_+ be the component of D which contains $I \cap (c, \infty)$, where D_+ might be equal to D . Then $u \in D \setminus \{0\}$ and $v \in D_+ \setminus \{0\}$, $v \neq u$, implies $u/v \in V$.

Let \mathcal{E} be the class of maps which satisfy (a),(b),(c) and assumption (d):

- (d) $f: D \rightarrow V$ has no singular values in $V \setminus \{0, 1\}$ and $c > 0$ is minimal such that f has a positive local maximum at c and $f(c) = 1$.

Similarly let \mathcal{E}_o be the class of maps which satisfy (a),(b),(c) and assumption (e):

- (e) f is odd, $f: D \rightarrow V$ has no singular values in $V \setminus \{0, \pm 1\}$ and $c > 0$ is minimal such that f has a positive local maximum at c and $f(c) = 1$.

Here, as usual, we say that $v \in \mathbb{C}$ is a *singular value* of a holomorphic map $f: D \rightarrow \mathbb{C}$ if it is a critical value, or an asymptotic value where the latter means the existence of a path $\gamma: [0, 1) \rightarrow D$ so that $\gamma(t) \rightarrow \partial D$ and $f(\gamma(t)) \rightarrow v$ as $t \uparrow 1$. Note that we do not require here that $V \supset D$.

Classes \mathcal{E} and \mathcal{E}_o are rich even in the case $D = \mathbb{C}$. See [19] for a general method of constructing entire (or meromorphic) functions with prescribed asymptotic and critical values. These classes are also non-empty when $V = \mathbb{C}$ and the domain D is a topological disk or even if D not simply-connected [17]. V can also be a bounded subset of \mathbb{C} , see example (v) below.

Concrete examples of functions f of the class \mathcal{E} are, where in (i)-(iv) we have $D = V = \mathbb{C}$,

- (i) $f(z) = 4z(1 - z)$,
- (ii) $f(z) = 4 \exp(z)(1 - \exp(z))$,
- (iii) $f(z) = [\sin(z)]^2$,
- (iv) $f(z) = m^{-m}(ez)^m \exp(-z)$ when m is a positive even integer.
- (v) the unimodal map $f: [0, 1] \rightarrow [0, 1]$ defined by

$$f(x) = \exp(2^\ell) \left(-\exp(-1/|x - 1/2|^\ell) + \exp(-2^\ell) \right)$$

satisfies $f(0) = f(1) = 0$, $f(1/2) = 1$ and has a flat critical point at $c = 1/2$; this map has an extension $f: D \rightarrow V$ which is in \mathcal{E} and for which V a punctured bounded disc

provided ℓ is big enough and $D = D_- \cup D_+$ are disjoint open topological discs so that $D_- \cap \mathbb{R} = (0, 1/2)$ and $D_+ \cap \mathbb{R} = (1/2, 1)$, see Lemma 7.5.

Examples of maps in the class \mathcal{E}_o are

- (vi) $f(z) = \sin(z)$ and
- (vii) $f(z) = (m/2)^{-m/2} e^{m/2} z^m \exp(-z^2)$ when m is a positive odd integer.

Theorem 2.2. *Let $f \in \mathcal{E} \cup \mathcal{E}_o$ and for each $w \in W := D_+$ define $G_w(z) = w \cdot f(z)$ and $\mathbf{p}(w) = c$. Take $c_1 \in D$, $g = G_{c_1}$ and assume that there exists q so that $c_n = g^{n-1}(c_1) \in D$ for all $n \leq q$ and either $c_q = c$ or $c_q \in \{c_1, \dots, c_{q-1}\}$. Moreover, assume $c_n \notin \{c_0, c_1, \dots, c_{n-1}\}$ for $0 < n < q$. Then*

- $(g, G, \mathbf{p})_W$ satisfies the lifting property and transversality holds.
- if $(g, G, \mathbf{p})_W$ is real, then positive transversality holds.

2.9. Applications to monotonicity of topological entropy of interval maps.

Corollary 2.7. *Take $f \in \mathcal{F}$ and consider the family $f_c = f + c$, $c \in J = U \cap \mathbb{R}$. Then the kneading sequence $\mathcal{K}(f_c)$ is monotone increasing in $c \in J$. Moreover, whenever $c_* \in J$ is so that $f_{c_*}^q(0) = 0$ and $f_{c_*}^k(0) \neq 0$ for all $1 \leq k < q$ the following positive transversality condition*

$$(2.4) \quad Q = \frac{\frac{d}{dc} f_c^q(0) \big|_{c=c_*}}{Df_{c_*}^{q-1}(c_*)} > 0.$$

holds and the topological entropy of f_c is decreasing in $c \in J$.

The same statement holds for $f_c = c \cdot f$ for $f \in \mathcal{E} \cup \mathcal{E}_0$, except in this case we consider the topological entropy of the unimodal map $f|_{(0, b)}$ where $b = \sup\{b' \in I : b' > 0, f(y) > 0 \forall y \in (0, b')\}$.

Monotonicity of entropy was proved in the case $f_c(x) = x^2 + c$ in the 1980s as a major result in unimodal dynamics. By now there are several proofs, see [37, 47, 11, 48, 49]. All these proofs use complex analytic methods and rely on the fact that f_c extends to a holomorphic map on the complex plane. These methods work well for $f_c(x) = |x|^\ell + c$ when ℓ is a positive even integer but break down for general $\ell > 1$ and also for other families of non-analytic unimodal maps. No approach using purely real-analytic method has so far been successful in proving monotonicity for any $\ell > 1$. The approach to prove monotonicity via the inequality (2.4) was also previously used by Tsujii [48, 49] for real maps of the form $z \mapsto z^2 + c$, $c \in \mathbb{R}$.

Remark 2.8. *Let \mathcal{U} denote the collection of unimodal maps $f : \mathbb{R} \rightarrow \mathbb{R}$ which are strictly decreasing in $(-\infty, 0]$ and strictly increasing in $[0, \infty)$. The Milnor-Thurston kneading sequence of $f \in \mathcal{U}$ is defined as a word $\mathcal{K}(f) = i_1 i_2 \dots \in \{1, 0, -1\}^{\mathbb{Z}^+}$, where*

$$i_k = \begin{cases} 1 & \text{if } f^k(0) > 0 \\ 0 & \text{if } f^k(0) = 0 \\ -1 & \text{if } f^k(0) < 0. \end{cases}$$

For $g \in \mathcal{U}$ with $\mathcal{K}(g) = j_1 j_2 \dots$, we say that $\mathcal{K}(f) \prec \mathcal{K}(g)$ if there is some $n \geq 1$ such that $i_k = j_k$ for all $1 \leq k < n$ and $\prod_{k=1}^n i_k < \prod_{k=1}^n j_k$.

Remark 2.9 (Positive transversality and topological entropy). *Because f has a minimum at 0, $x \mapsto f_{c_*}^q(x)$ has a local maximum (minimum) at 0 if $Df_{c_*}^{q-1}(f_{c_*}(0)) < 0$ (resp. > 0). Hence Equation (2.4) implies that if 0 has (precisely) period q at some parameter c_* , then*

$$\begin{aligned} \frac{d}{dc} f_c^q(0) \big|_{c=c_*} < 0 & \quad \text{if } f_{c_*}^q \text{ has a local maximum at 0,} \\ \frac{d}{dc} f_c^q(0) \big|_{c=c_*} > 0 & \quad \text{if } f_{c_*}^q \text{ has a local minimum at 0.} \end{aligned}$$

Hence the number of laps of f_c^n (and therefore the topological entropy) is non-increasing when c increases. These inequalities also show that the multiplier $\lambda(c)$ of the (local) analytic continuation $p(c)$ of this periodic point of period q is strictly increasing. Note that there is a result of Douady-Hubbard which asserts that in each hyperbolic component of the family of quadratic maps, the multiplier of the periodic attractor is a univalent function of the parameter. Proving

(2.4) complements this by also showing that on the real line the multiplier of the periodic point is increasing.

When $f_c(x) = |x|^\ell + c$, and ℓ is not an integer, we have not been able to prove the lifting property. The next theorem, which will be proved in Appendix A, gives monotonicity when ℓ is a large real number (not necessarily an integer), but only if not too many points in the critical orbit are in the orientation reversing branch.

Theorem A.1. *Let $\ell_-, \ell_+ \geq 1$ and consider the family of unimodal maps $f_c = f_{c, \ell_-, \ell_+}$ where*

$$f_c(x) = \begin{cases} |x|^{\ell_-} + c & \text{if } x \leq 0 \\ |x|^{\ell_+} + c & \text{if } x \geq 0. \end{cases}$$

For any integer $L \geq 1$ there exists $\ell_0 > 1$ so that for any $q \geq 1$ and any periodic sequence $\mathbf{i} = i_1 i_2 \cdots \in \{-1, 0, 1\}^{\mathbb{Z}^+}$ of period q so that

$$(2.5) \quad \#\{1 \leq j < q; i_j = -1\} \leq L,$$

and any pair $\ell_-, \ell_+ \geq \ell_0$ there is at most one $c \in \mathbb{R}$ for which the kneading sequence of f_c is equal to \mathbf{i} . Moreover, if \mathbf{i} is realisable (i.e. if $c = c_$ exists) and \mathbf{i} has minimal period q then positive transversality holds*

$$(2.6) \quad Q = \frac{d}{dc} f_c^q(0) \Big|_{c=c_*} > 0.$$

The proof of this theorem uses delicate geometric arguments, see Appendix A. Note that there is an elegant algebraic proof of transversality for critically finite quadratic polynomials in [12, Chapter 19]. This proof also works for $x \mapsto |x|^{2n+1} + c$ provided n is a positive integer, but it does not give the sign, so no monotonicity for this family can be deduced.

2.10. Monotonicity along curves with one free critical point. The above results require that all critical points are eventually periodic. Nevertheless, they also give information about the bifurcations that occur for example along a curve L_* in parameter space corresponding to $(\nu - 1)$ -critical relations. The results in Section 8 informally state:

Informal Statement of Theorem 8.1. *Critical relations unfold everywhere in the same direction along L_* .*

This makes it possible to obtain information about monotonicity of entropy along the *bone curves* considered in [38, Figure 11] and [40, Figure 8]. Indeed we obtain an alternative proof for one of the main technical steps in [38] in Theorem 8.2. Could such a simplification be made in the case with at least three critical points?

Indeed, it would be interesting to know whether the sign in (2.3) makes it possible to simplify the existing proofs of Milnor's conjecture. This conjecture is about the space of real polynomials with only real critical points, all of which non-degenerate, and asks whether the level sets of constant topological entropy are connected. The proof of this conjecture in [38] in the cubic case and in [5] for the general case relies on quasi-symmetric rigidity, but does having a positive sign in (2.3) everywhere allow for a simplification of the proof of this conjecture?

2.11. Other applications. Our approach can also be applied to many other settings, such as families of Arnol'd maps, families of piecewise linear maps and to families of intervals maps with discontinuities (i.e. Lorenz maps), see [23, 24].

Even though we deal with the polynomial and rational case in Appendix C, since it is so important, in a separate paper [22] we have given a very elementary proof of transversality and related results in that setting, but without the sign in (2.6) and (2.3). In that paper the postcritical set is allowed to be *infinite*. See [16] for an alternative discussion on transversality for maps of finite type, and [6] when the postcritical set is finite.

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3. ORGANISATION OF THIS PAPER AND OUTLINE OF THE PROOF

In this paper we consider holomorphic maps $g: U \rightarrow \mathbb{C}$ where U is an open subset of the complex plane, together with a finite forward invariant marked set P , for example the postcritical set. These maps do not necessarily have to be rational or transcendental. The aim is to show that critical relations of such a marked map unfold transversally under a holomorphic deformation G of g . We do this as follows. First, in Section 4, we associate a linear operator $\mathcal{A}: \mathbb{C}^{\#g(P)} \rightarrow \mathbb{C}^{\#g(P)}$ by the action of G induced by lifting holomorphic motions on $g(P)$ and show

$$1 \notin \text{spec}(\mathcal{A}) \Leftrightarrow \text{transversality},$$

$$\text{spec}(\mathcal{A}) \subset \overline{\mathbb{D}} \setminus \{1\} \text{ and } G \text{ real} \implies \text{positive transversality}.$$

More precisely, it is shown in Theorem 4.1 that the dimension of kernel of $D\mathcal{R}(c_1)$ is equal to the geometric multiplicity of the eigenvalue 1 of \mathcal{A} . In Section 5 we then show

$$\text{lifting property} \implies \text{spec}(\mathcal{A}) \subset \overline{\mathbb{D}}$$

Then in Section 6 we show that provided the lifting property holds, $\{w; \mathcal{R}(w) = 0\}$ is locally a smooth submanifold whose dimension is equal to the geometric multiplicity of the eigenvalue 1 of \mathcal{A} . In applications, it is usually quite easy to show that the parameter set $\{w; \mathcal{R}(w) = 0\}$ cannot be a manifold of dimension > 0 , and therefore that $1 \notin \text{spec}(\mathcal{A})$ and so transversality holds.

It follows that transversality essentially follows from the lifting property. In Section C we show that the lifting property holds in some classical settings. In Sections 7 we will show the lifting property holds for polynomial-like mappings from a separation property, and for maps from the classes \mathcal{E} , \mathcal{E}_o . In this way, we derive transversality for many families of interval maps, for example for a wide class of one-parameter families of the form $f_\lambda(x) = f(x) + \lambda$ and $f_\lambda(x) = \lambda f(x)$. As an easy application, we will recover known transversality results for the family of quadratic maps, and address some conjectures from the 1980's about families of interval maps of this type.

In Appendix A we will study the family $x \mapsto |x|^\ell + c$. When ℓ is not an even integer, we have not been able to prove the lifting property in general. Nevertheless we will obtain the lifting property under additional assumptions.

In Appendix B we give some examples for both transversality and the lifting property fails to hold.

In a companion paper we show that the methods developed in this paper also apply to other families, including some for which separation property does not hold, such as the Arnol'd family. We also obtain positively oriented transversality for piecewise linear interval maps and interval maps with discontinuities (i.e. Lorenz maps), see also [23]

4. THE SPECTRUM OF A TRANSFER OPERATOR \mathcal{A} AND TRANSVERSALITY

In this section we define a transfer operator \mathcal{A} associated to the analytic deformation of a marked map, and show that if 1 is not an eigenvalue of \mathcal{A} then transversality holds. If the spectrum of \mathcal{A} is inside the closed unit circle, we will obtain additional information about transversality, see Section 4.3.

4.1. A transfer operator associated to a deformation of a marked map. In §2.5, we defined lift of holomorphic motions of $g(P)$ associated to $(g, G, \mathbf{p})_W$. Obviously there is a linear map $\mathcal{A}: \mathbb{C}^{\#g(P)} \rightarrow \mathbb{C}^{\#g(P)}$ such that whenever \hat{h}_λ is a lift of h_λ , we have

$$\mathcal{A} \left(\left\{ \frac{d}{d\lambda} h_\lambda(x) \Big|_{\lambda=0} \right\}_{x \in g(P)} \right) = \left\{ \frac{d}{d\lambda} \hat{h}_\lambda(x) \Big|_{\lambda=0} \right\}_{x \in g(P)}.$$

We will call \mathcal{A} the *transfer operator* associated to the holomorphic deformation $(g, G, \mathbf{p})_W$ of g .

If both g and $(g, G, \mathbf{p})_W$ are real, then $\mathcal{A}(\mathbb{R}^\nu) \subset \mathbb{R}^\nu$. In this case, we shall often consider real holomorphic motions, i.e. $\Lambda \ni 0$ is symmetric with respect to \mathbb{R} and $h_\lambda(x) \in \mathbb{R}$ for each $x \in g(P)$ and $\lambda \in \Lambda \cap \mathbb{R}$. Clearly, a lift of a real holomorphic motion is again real.

Example 4.1. Let g be a marked map with $P \supset P_0 = \{c_0 = 0\}$, so that $P = \{c_0, \dots, c_{q-1}\}$, $c_i = g^i(c_0)$, $0 \leq i < q$ are distinct and $g^q(c_0) = c_0$. Consider a deformation $(g, G, \mathbf{p})_W$ where W is a neighbourhood of c_1 and let $\mathbf{p}: W \rightarrow \mathbb{C}$ be so that $\mathbf{p} \equiv 0$. Consider the holomorphic motion of $g(P) = P$ defined by $h_\lambda(c_i) = c_i + v_i \lambda$. Then $\hat{h}_\lambda(c_i) = c_i + \hat{v}_i \lambda + O(\lambda^2)$ is defined by $\hat{h}_\lambda(c_0) = 0$, $G_{h_\lambda(c_1)}(\hat{h}_\lambda(c_i)) = h_\lambda(c_{i+1}) = c_{i+1} + v_{i+1} \lambda$, $i = 1, \dots, q-1$ where we take $c_q = c_0$, $v_q = v_0$. Writing $L_i = \frac{\partial G_w(c_i)}{\partial w}$ and $D_i = Dg(c_i)$ we obtain

$$\hat{v}_0 = 0, \quad L_i v_1 + D_i \hat{v}_i = v_{i+1}.$$

So

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -L_1/D_1 & 1/D_1 & 0 & & 0 \\ 0 & -L_2/D_2 & 0 & 1/D_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -L_{q-2}/D_{q-2} & 0 & 0 & \dots & 1/D_{q-2} \\ 1/D_{q-1} & -L_{q-1}/D_{q-1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence

$$I - \rho \mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 + \rho L_1/D_1 & -\rho/D_1 & 0 & & 0 \\ 0 & \rho L_2/D_2 & 1 & -\rho/D_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \rho L_{q-2}/D_{q-2} & 0 & 0 & \dots & -\rho/D_{q-2} \\ -\rho/D_{q-1} & \rho L_{q-1}/D_{q-1} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

So

$$(4.1) \quad \det(I - \rho \mathcal{A}) = 1 + \frac{L_1}{D_1} \rho + \frac{L_2}{D_1 D_2} \rho^2 + \dots + \frac{L_{q-1}}{D_1 D_2 \dots D_{q-1}} \rho^{q-1}.$$

So if the spectrum of \mathcal{A} is contained in the open unit disc and L_i, D_i are real, then (4.1) is strictly positive for all $\rho \in [-1, 1]$. Note that when $G_w(z) = g(z) + (w - c_1)$, the expression (4.1) agrees with (2.4) for $\rho = 1$.

4.2. Relating the transfer operator with transversality. It turns out that transversality is closely related to the eigenvalues of \mathcal{A} :

Theorem 4.1. Assume the following holds: for any $r < j \leq \nu$, $Dg^{qj-l_j}(c_{1,j}) \neq 1$. Then the following statements are equivalent:

- (1) 1 is an eigenvalue of \mathcal{A} ;
- (2) $D\mathcal{R}(c_1)$ is degenerate.

More precisely, the dimension of kernel of $D\mathcal{R}(c_1)$ is equal to the dimension of the eigenspace of \mathcal{A} associated with eigenvalue 1.

Proof. We first show that (1) implies (2), even without the assumption. So suppose that 1 is an eigenvalue of \mathcal{A} and let $\mathbf{v} = (v(x))_{x \in g(P)}$ be an eigenvector associated with 1. For $t \in \mathbb{D}$, define $h_t(x) = x + tv(x)$ for each $x \in g(P)$ and $w(t) = (c_{1,j} + tv(c_{1,j}))'_{j=1}$. Then for each $x \in g(P) \setminus P_0$,

$$(4.2) \quad G_{w(t)}(h_t(x)) - h_t(g(x)) = O(t^2),$$

and for each $x = c_{0,j} \in g(P) \cap P_0$, we have

$$(4.3) \quad h_t(x) - p_j(w(t)) = O(t^2).$$

For each $1 \leq j \leq \nu$, and each $1 \leq k < q_j$, applying (4.2) repeatedly, we obtain

$$(4.4) \quad G_{w(t)}^k(h_t(c_{1,j})) = h_t(g^k(c_{1,j})) + O(t^2).$$

Together with (4.3), this implies that

$$R_j(w(t)) = O(t^2),$$

holds for all $1 \leq j \leq \nu$. It remains to show $w'(0) \neq \mathbf{0}$. Indeed, otherwise, by (4.4), it would follow that $v(g^k(c_{1,j})) = (g^k)'(c_{1,j})v(c_{1,j}) = 0$ for each $1 \leq j \leq \nu$ and $1 \leq k < q_j$, and hence $v(x) = 0$ for all $x \in g(P)$, which is absurd. We completed the proof that (1) implies (2).

Now let us prove that (2) implies (1) under the assumption of the lemma. Suppose that $D\mathcal{R}(c_1)$ is degenerate. Then there exists a non-zero vector $(w_1^0, w_2^0, \dots, w_\nu^0)$ in \mathbb{C}^ν such that $R_j(w(t)) = O(t^2)$ as $t \rightarrow 0$ for all $j = 1, \dots, \nu$, where $w(t) = (w_j(t))_{j=1}^\nu = (c_{1,j} + tw_j^0)_{j=1}^\nu$. We claim that $w_j^0 = w_{j'}^0$ holds whenever $c_{1,j} = c_{1,j'}$, $1 \leq j, j' \leq \nu$. Indeed,

Case 1. If $1 \leq j \leq r$ then $1 \leq j' \leq r$ and $\mu(j) = \mu(j')$, $q_j = q_{j'}$. Then

$$G_{w(t)}^{q_j-1}(w_j(t)) - G_{w(t)}^{q_j-1}(w_{j'}(t)) = R_j(w(t)) - R_{j'}(w(t)) = O(t^2)$$

which implies that $w_j(t) - w_{j'}(t) = O(t^2)$, i.e. $w_j^0 = w_{j'}^0$.

Case 2. If $r < j \leq \nu$ then $r < j' \leq \nu$ and $\hat{l}_j = \hat{l}_{j'}$, $\hat{q}_j = \hat{q}_{j'}$ where we define for any $r < j \leq \nu$ the integers $\hat{l}_j < \hat{q}_j$ minimal so that $g^{\hat{q}_j}(c_{0,j}) = g^{\hat{l}_j}(c_{0,j})$. By the chain rule it follows that

$$G_{w(t)}^{q_j-1}(w_j(t)) - G_{w(t)}^{\hat{l}_j-1}(w_j(t)) = O(t^2)$$

implies

$$G_{w(t)}^{\hat{q}_j-1}(w_j(t)) - G_{w(t)}^{\hat{l}_j-1}(w_j(t)) = O(t^2)$$

under the assumption $Dg^{\hat{q}_j-l_j}(c_{l_j,j}) \neq 1$.

Thus we obtain

$$G_{w(t)}^{\hat{q}_j-1}(w_j(t)) - G_{w(t)}^{\hat{q}_j-1}(w_{j'}(t)) = G_{w(t)}^{\hat{l}_j-1}(w_j(t)) - G_{w(t)}^{\hat{l}_j-1}(w_{j'}(t)) + O(t^2),$$

which implies that

$$(Dg^{\hat{q}_j-1}(c_{l_j,j}) - Dg^{\hat{l}_j-1}(c_{l_j,j}))(w_j(t) - w_{j'}(t)) = O(t^2).$$

If such j and j' exist then $c_{l_j,j}$ is a hyperbolic periodic point, hence $Dg^{\hat{q}_j-1}(c_{l_j,j}) \neq Dg^{\hat{l}_j-1}(c_{l_j,j})$. It follows that $w_j^0 = w_{j'}^0$.

Thus the Claim is proved. To obtain an eigenvector for \mathcal{A} with eigenvalue 1, define $v(c_{1,j}) = w_j^0$, $v(c_{0,j}) = \frac{d}{dt}p_j(w(t))|_{t=0}$. For points $x \in g(P) \setminus P_0$, there is j and $1 \leq s < q_j$ such that $x = g^s(c_{0,j})$, define $v(x) = \frac{d}{dt}G_{w(t)}^{s-1}(w_j(t))|_{t=0}$. Note that $v(x)$ does not depend on the choice of j and s . (This can be proved similarly as the claim.)

The above argument builds an isomorphism between $\{v \in \mathbb{C}^\nu : D\mathcal{R}(c_1, v) = 0\}$ and the eigenspace of \mathcal{A} associated with eigenvalue 1. So these two spaces have the same dimension. \square

4.3. The spectrum of \mathcal{A} and the determinant of some matrix $D(\rho)$. Define $D(\rho) = (D_{j,k}(\rho))_{1 \leq j, k \leq \nu}$ as follows: Put

$$L_k(z) = \frac{\partial G_w(z)}{\partial w_k} \Big|_{w=c_1}; \quad p_{j,k} = \frac{\partial p_j}{\partial w_k}(c_1);$$

$$\mathcal{L}_{j,k}^0 = 0 \text{ and } \mathcal{L}_{j,k}^m = \sum_{n=1}^m \frac{\rho^n L_k(c_{n,j})}{Dg^n(c_{1,j})} \text{ for } m > 0;$$

$$D_{jk}(\rho) = \delta_{jk} + \mathcal{L}_{j,k}^{q_j-1} - \rho^{q_j} \frac{p_{\mu(j),k}}{Dg^{q_j-1}(c_{1,j})}$$

when $1 \leq j \leq r$ and

$$D_{jk}(\rho) = \delta_{jk} + \mathcal{L}_{j,k}^{q_j-1} - \frac{\rho^{q_j-l_j}}{Dg^{q_j-l_j}(c_{l_j,j})} \left(\mathcal{L}_{j,k}^{l_j-1} + \delta_{j,k} \right)$$

when $r < j \leq \nu$. Note that

$$(4.5) \quad \det(D\mathcal{R}(\mathbf{c}_1)) = \prod_{j=1}^{\nu} Dg^{q_j-1}(c_{1,j}) \det(D(1)).$$

We say that $\rho \in \mathbb{C}$ is an *exceptional value* if there exists $r < j \leq \nu$ such that $Dg^{q_j-l_j}(c_{1,j}) = \rho^{q_j-l_j}$.

Proposition 4.2. *For each non-exceptional $\rho \in \mathbb{C}$, we have*

$$(4.6) \quad \det(I - \rho\mathcal{A}) = 0 \Leftrightarrow \det(D(\rho)) = 0.$$

Proof. For $\rho = 0$, $\det(I) = \det(D(0)) = 1$. Assume $\rho \neq 0$. Define a new triple $(g^\rho, G^\rho, \mathbf{p}^\rho)$ as follows.

- For each $x \in P \setminus P_0$, $G_w^\rho(z) = G_w(x) + \frac{Dg(x)}{\rho}(z-x)$ in a neighbourhood of x ;
- $g^\rho(x) = g(x)$ for each $x \in P_0$ and $g^\rho(z) = G_{\mathbf{c}_1}^\rho(z)$ in a neighbourhood of $P \setminus P_0$;
- $\mathbf{p}^\rho(w) = \mathbf{c}_0 + \rho \frac{\partial \mathbf{p}}{\partial w}(\mathbf{c}_1) \cdot (w - \mathbf{c}_1)$.

Let \mathcal{A}^ρ be the transfer operator associated with the triple $(g^\rho, G^\rho, \mathbf{p}^\rho)$. Then it is straightforward to check that

$$\mathcal{A}^\rho = \rho\mathcal{A}.$$

We can define a map $\mathcal{R}^\rho = (R_1^\rho, R_2^\rho, \dots, R_\nu^\rho)$ for each $\rho \neq 0$ in the obvious way:

$$R_j^\rho(w) = (G_w^\rho)^{q_j-1}(w_j) - p_{\mu(j)}^\rho(w)$$

for $1 \leq j \leq r$ and

$$R_j^\rho(w) = (G_w^\rho)^{q_j-1}(w_j) - (G_w^\rho)^{l_j-1}(w_j)$$

for $r < j \leq \nu$. As long as ρ is non-exceptional for the triple (g, G, \mathbf{p}) , the new triple $(g^\rho, G^\rho, \mathbf{p}^\rho)$ satisfies the assumption of Theorem 4.1, thus

$$\det(I - \mathcal{A}^\rho) = 0 \Leftrightarrow D\mathcal{R}^\rho(\mathbf{c}_1) \text{ is degenerate.}$$

Direct computation shows that the (j, k) -th entry of $D\mathcal{R}^\rho(\mathbf{c}_1)$ is equal to $D_{j,k}(\rho) Dg^{q_j-1}(c_{1,j}) / \rho^{q_j-1}$. Indeed, for each $1 \leq j \leq r$,

$$\begin{aligned} D_{j,k}^\rho(\rho) &= \frac{\partial (G_w^\rho)^{q_j-1}(c_{1,j})}{\partial w_k} \Big|_{w=\mathbf{c}_1} + \frac{Dg^{q_j-1}(c_{1,j})}{\rho^{q_j-1}} \delta_{jk} - \rho \frac{\partial p_{\mu(j)}}{\partial w_k} \\ &= \frac{Dg^{q_j-1}(c_{1,j})}{\rho^{q_j-1}} \left(\delta_{jk} + \sum_{n=1}^{q_j-1} \frac{\rho^n L_k(c_{n,j})}{Dg^n(c_{1,j})} - \rho^{q_j} \frac{p_{\mu(j),k}}{Dg^{q_j-1}(c_{1,j})} \right), \end{aligned}$$

and for $r < j \leq \nu$,

$$\begin{aligned} &D_{j,k}^\rho(\rho) \\ &= \frac{\partial ((G_w^\rho)^{q_j-1}(c_{1,j}) - (G_w^\rho)^{l_j-1}(c_{1,j}))}{\partial w_k} \Big|_{w=\mathbf{c}_1} + \delta_{jk} \left(\frac{Dg^{q_j-1}(c_{1,j})}{\rho^{q_j-1}} - \frac{Dg^{l_j-1}(c_{1,j})}{\rho^{l_j-1}} \right) \\ &= \frac{Dg^{q_j-1}(c_{1,j})}{\rho^{q_j-1}} \mathcal{L}_{j,k}^{q_j-1} - \frac{Dg^{l_j-1}(c_{1,j})}{\rho^{l_j-1}} \mathcal{L}_{j,k}^{l_j-1} + \delta_{jk} \left(\frac{Dg^{q_j-1}(c_{1,j})}{\rho^{q_j-1}} - \frac{Dg^{l_j-1}(c_{1,j})}{\rho^{l_j-1}} \right) \end{aligned}$$

Therefore $\det(I - \rho\mathcal{A}) = 0$ if and only if $\det(D(\rho)) = 0$. □

4.4. Positive transversality in the real case. To illustrate the power of the previous proposition we state:

Corollary 4.3 (Positive transversality). *Let $(g, G, \mathbf{p})_W$ be a real local holomorphic deformation of a real marked map g . Assume that one has $|Dg^{q_j-l_j}(c_{1,j})| > 1$ for all $r < j \leq \nu$. Assume furthermore that all the eigenvalues of \mathcal{A} lie in the set $\{|\rho| \leq 1, \rho \neq 1\}$. Then the ‘positively oriented’ transversality condition holds.*

Proof. Write the polynomial $\det(D(\rho))$ in the form $\prod_{i=1}^N (1 - \rho \rho_i)$, where $\rho_i \in \mathbb{C} \setminus \{0\}$. Because of (4.5) it suffices to show that $\det(D(1)) > 0$. Since $\det(D(\rho))$ is a real polynomial in ρ , this follows from $\rho_i \notin [1, \infty)$ for each i . Arguing by contradiction, assume that $\rho_i \geq 1$ for some i . Then $1/\rho_i$ is a zero of $\det(D(\rho))$. As $|1/\rho_i| \leq 1$, $1/\rho_i$ is not an exceptional value. Thus $\det(I - \mathcal{A}/\rho_i) = 0$, which implies that $\rho_i \geq 1$ is an eigenvalue of \mathcal{A} , a contradiction! \square

4.5. A remark on an alternative transfer operator. Proposition 4.2 shows that for non-exceptional ρ , one has (4.6). One can also associate to (g, G, \mathbf{p}) another linear operator \mathcal{A}_J for which

$$(4.7) \quad \det D(\rho) = \det(I - \rho \mathcal{A}_J)$$

holds for all $\rho \in \mathbb{C}$. Here J denotes a collection of all pairs (i, j) such that $1 \leq j \leq \nu$, $0 \leq i \leq q_j - 1$ and if $i = 0$ then $j = \mu(j')$ for some $1 \leq j' \leq \nu$. Given a collection of functions $\{c_{i,j}(\lambda)\}_{(i,j) \in J}$ which are holomorphic in a small neighbourhood of $\lambda = 0$, there is another collection of holomorphic near 0 functions $\{\hat{c}_{i,j}(\lambda)\}_{(i,j) \in J}$ such that $\hat{c}_{0,j}(\lambda) = p_j(c_1(\lambda))$ where $c_1(\lambda) = (c_{1,1}(\lambda), \dots, c_{1,\nu}(\lambda))$ and, for $i \neq 0$, $G(c_1(\lambda), \hat{c}_{i,j}) = c_{i+1,j}(\lambda)$. Here we set $c_{q_j,j}(\lambda) = c_{0,\mu(j)}(\lambda)$ for $1 \leq j \leq r$ and $c_{q_j,j}(\lambda) = c_{l_j,j}(\lambda)$ for $r < j \leq \nu$. Define the linear map $\mathcal{A}_J : \mathbb{C}^{\#J} \rightarrow \mathbb{C}^{\#J}$ by taking the derivative at $\lambda = 0$: $\mathcal{A}_J(\{c'_{i,j}(0)\}_{(i,j) \in J}) = \{\hat{c}'_{i,j}(0)\}_{(i,j) \in J}$. Explicitly, we get:

$$\hat{c}'_{i,j}(0) = \begin{cases} \sum_{k=1}^{\nu} p_{j,k} & \text{if } i = 0 \text{ and } j = \mu(j') \text{ for some } j' \\ \frac{\sum_{k=1}^{\nu} p_{j,k}}{Dg(c_{i,j})} (v_{i+1,j} - \sum_{k=1}^{\nu} L_k(c_{i,j})v_{1,k}) & \text{if } 1 \leq i < q_j - 1, 1 \leq j \leq \nu \\ \frac{\sum_{k=1}^{\nu} p_{j,k}}{Dg(c_{q_j-1,j})} (v_{0,\mu(j)} - \sum_{k=1}^{\nu} L_k(c_{q_j-1,j})v_{1,k}) & \text{if } i = q_j - 1, 1 \leq j \leq r \\ \frac{\sum_{k=1}^{\nu} p_{j,k}}{Dg(c_{q_j-1,j})} (v_{l_j,j} - \sum_{k=1}^{\nu} L_k(c_{q_j-1,j})v_{1,k}) & \text{if } i = q_j - 1, r < j \leq \nu \end{cases}$$

Elementary properties of determinants being applied to the matrix $I - \rho \mathcal{A}_J$ lead to (4.7). Observe that $\mathcal{A}_J = \mathcal{A}$ if (and only if) all points $c_{i,j}$, $(i, j) \in J$ are pairwise different. Therefore, we have:

$$\det(I - \rho \mathcal{A}) = \det D(\rho)$$

for every $\rho \in \mathbb{C}$ provided $\sum_{j=1}^{\nu} (q_j - 1) + r = \#P$.

5. THE LIFTING PROPERTY AND THE SPECTRUM OF \mathcal{A}

The next proposition shows that the lifting property implies that the spectrum of \mathcal{A} is in the closed unit disc.

Proposition 5.1. *If $(g, G, \mathbf{p})_W$ has the lifting property, then the spectral radius of the associated transfer operator \mathcal{A} is at most 1 and every eigenvalue of \mathcal{A} of modulus one is semisimple (i.e. its algebraic multiplicity coincides with its geometric multiplicity). Moreover, for $(g, G, \mathbf{p})_W$ real, we only need to assume that the lifting property with respect to real holomorphic motions.*

Proof. For any $\mathbf{v} = (v(x))_{x \in g(P)}$, construct a holomorphic motion $h_\lambda^{(0)}$ over $(\Lambda, 0)$ for some domain $\Lambda \ni 0$, such that $\frac{d}{d\lambda} h_\lambda^{(0)}(x) |_{\lambda=0} = v(x)$ for all $x \in g(P)$. Then

$$\mathcal{A}^k(\mathbf{v}) = \left(\frac{d}{d\lambda} h_\lambda^{(k)}(x) |_{\lambda=0} \right)_{x \in g(P)}$$

for every $k > 0$. By Cauchy's integral formula, there exists $C = C(M, \varepsilon)$ such that $|\frac{d}{d\lambda} h_\lambda^{(k)}(x) |_{\lambda=0}| \leq C$ holds for all $x \in g(P)$ and all k . It follows that for any $\mathbf{v} \in \mathbb{C}^{\#g(P)}$, $\mathcal{A}^k(\mathbf{v})$ is a bounded sequence. Thus the spectral radius of \mathcal{A} is at most one and every eigenvalue of \mathcal{A} of modulus one is semisimple.

Suppose $(g, G, \mathbf{p})_W$ is real. Then for any $\mathbf{v} \in \mathbb{R}^{\#g(P)}$, the holomorphic motion $h_\lambda^{(0)}$ can be chosen to be real. Thus if $(g, G, \mathbf{p})_W$ has the real lifting property, then $\{\mathcal{A}^k(\mathbf{v})\}_{k=0}^\infty$ is bounded for each $\mathbf{v} \in \mathbb{R}^{\#g(P)}$. The conclusion follows. \square

To obtain that the radius is strictly smaller than one, we shall apply the argument to a suitable perturbation of the map g . For example, we have the following:

Proposition 5.2 (Robust spectral property). *Let $(g, G, \mathbf{p})_W$ be as above. Let Q be a polynomial such that $Q(c_{0,j}) = 0$ for $1 \leq j \leq \nu$ and $Q(x) = 0$, $Q'(x) = 1$ for every $x \in g(P)$. Let $\varphi_\xi(z) = z - \xi Q(z)$ and for $\xi \in (0, 1)$ let $\psi_\xi(w) = (\varphi_\xi^{-1}(w_1), \dots, \varphi_\xi^{-1}(w_\nu))$ be a map from a neighbourhood of \mathbf{c}_1 into a neighbourhood of \mathbf{c}_1 . Suppose that there exists $\xi \in (0, 1)$ such that the triple $(\varphi_\xi \circ g, \varphi_\xi \circ G, \mathbf{p} \circ \psi_\xi)$ has the lifting property. Then the spectral radius of \mathcal{A} is at most $1 - \xi$.*

Proof. Note that $\tilde{g} := \varphi_\xi \circ g$ is a marked map with the same sets $P_0 \subset P$. Furthermore, $\tilde{g}^i(c_{0,j}) = g^i(c_{0,j}) = c_{i,j}$, $D\tilde{g}(c_{i,j}) = (1 - \xi)Dg(c_{i,j})$, $\frac{\partial \varphi_\xi \circ G}{\partial w_k}(\mathbf{c}_1, z) = (1 - \xi)\frac{\partial G}{\partial w_k}(\mathbf{c}_1, z)$ for each $z \in g(P) \setminus P_0$, and $\frac{\mathbf{p} \circ \psi_\xi}{\partial w_k}(\mathbf{c}_1) = (1 - \xi)^{-1}\frac{\partial \mathbf{p}}{\partial w_k}(\mathbf{c}_1)$. Therefore, the operator which is associated to the triple $(\varphi_\xi \circ g, \varphi_\xi \circ G, \mathbf{p} \circ \psi_\xi)$ is equal to $(1 - \xi)^{-1}\mathcal{A}$. Since the latter triple has the lifting property, by Proposition 5.1, the spectral radius of $(1 - \xi)^{-1}\mathcal{A}$ is at most 1. \square

For completeness we include:

Lemma 5.3. *Assume that the spectrum radius of \mathcal{A} is strictly less than 1. Then the lifting property holds.*

Proof. Let $\Phi(Z) = (\varphi_x(Z))_{x \in g(P)}$ be the holomorphic map defined from a neighbourhood V of the point $\mathbf{z} := g(P) \in \mathbb{C}^{\#g(P)}$ by

$$\begin{aligned} G_{Z_1}(\varphi_x(Z)) &= z_{g(x)}, \quad x \in g(P) \setminus P_0, \\ \varphi_{c_{0,j}}(Z) &= p_j(Z_1), \quad 1 \leq j \leq \nu, \end{aligned}$$

where $\mathbf{z}_1 = (z_{c_{1,j}})_{j=1}^\nu$, $Z = (z_x)_{x \in g(P)}$. Then $\Phi(\mathbf{z}) = \mathbf{z}$. Moreover, if h_λ is a holomorphic motion of $g(P)$ over $(\Lambda, 0)$ with $\mathbf{h}_\lambda = (h_\lambda(x))_{x \in g(P)} \in V$ for all λ , then $\hat{h}_\lambda(x) := (\Phi(\mathbf{h}_\lambda))_x$ is the lift of h_λ over $(\Lambda, 0)$.

So the derivative of Φ at \mathbf{z} is equal to \mathcal{A} , and hence \mathbf{z} is a hyperbolic attracting fixed point of Φ . Therefore, there exist $N > 0$ and a neighborhood \mathcal{U} of \mathbf{z} such that Φ^N is well-defined on \mathcal{U} and such that $\Phi^N(\mathcal{U})$ is compactly contained in \mathcal{U}_0 . It follows Φ^n converges uniformly to the constant \mathbf{z} in \mathcal{U} .

Let us prove that $(g, G, \mathbf{p})_W$ has the lifting property. Indeed, if h_λ is a holomorphic motion of $g(P)$ over $(\mathbb{D}, 0)$, then there exists $\varepsilon > 0$ such that $\mathbf{h}_\lambda := (h_\lambda(x))_{x \in g(P)} \in \mathcal{U}$, so that $\mathbf{h}_\lambda^{(k)} := \Phi^k(\mathbf{h}_\lambda)$ is well-defined. Let $h_\lambda^{(k)}(x)$, $x \in g(P)$, be such that $\mathbf{h}_\lambda^{(k)} = (h_\lambda^{(k)}(x))_{x \in g(P)}$. Then for each $k \geq 0$, $h_\lambda^{(k)}$ is a holomorphic motion of $g(P)$ over $(\mathbb{D}_\varepsilon, 0)$ and $h_\lambda^{(k+1)}$ is the lift of $h_\lambda^{(k)}$. \square

6. THE LIFTING PROPERTY AND PERSISTENCE OF CRITICAL RELATIONS

The main technical result in this paper is the following theorem:

Theorem 6.1. *Assume that either the triple $(g, G, \mathbf{p})_W$ has the lifting property or $(g, G, \mathbf{p})_W$ is real and has the real lifting property. Assume also that for all $r < j \leq \nu$, $Dg^{q_j - l_j}(c_{l_j, j}) \neq 1$. Then*

- (1) *All eigenvalues of \mathcal{A} are contained in $\overline{\mathbb{D}}$.*
- (2) *There is a neighborhood W' of \mathbf{c}_1 in W such that*

$$(6.1) \quad \{\mathbf{w} \in W' \mid \mathcal{R}(\mathbf{w}) = 0\}$$

is a smooth submanifold of W' , and its dimension is equal to the geometric multiplicity of the eigenvalue 1 of \mathcal{A} .

The second statement is useful to conclude that $D\mathcal{R}$ is non-degenerate at \mathbf{c}_1 , or equivalently, that 1 is not an eigenvalue of \mathcal{A} . Indeed, if $\nu = 1$ and if 1 is an eigenvalue of \mathcal{A} , the manifold (6.1) must contain a neighbourhood of \mathbf{c}_1 and hence $\mathcal{R}(\mathbf{w}) = 0$ holds for every $\mathbf{w} \in \mathbb{C}$ near $\mathbf{c}_1 \in \mathbb{C}$, which only happens for trivial family $(g, G, \mathbf{p})_W$. It is also possible to apply this statement in a more subtle way, see [24].

Let Λ be a domain in \mathbb{C} which contains 0. A holomorphic motion $h_\lambda(x)$ of $g(P)$ over $(\Lambda, 0)$ is called *asymptotically invariant of order m* (with respect to $(g, G, \mathbf{p})_W$) if there is a subdomain

$\Lambda_0 \subset \Lambda$ which contains 0 and a holomorphic motion $\widehat{h}_\lambda(x)$ which is the lift of h_λ over $(\Lambda_0, 0)$, such that

$$(6.2) \quad \widehat{h}_\lambda(x) - h_\lambda(x) = O(\lambda^{m+1}) \text{ as } \lambda \rightarrow 0.$$

Obviously,

Lemma 6.1. *1 is an eigenvalue of \mathcal{A} if and only if there is a non-degenerate holomorphic motion which is invariant of order 1.*

Here, a holomorphic motion $h_\lambda(x)$ is called *non-degenerate* if $\frac{d}{d\lambda}h_\lambda(x)|_{\lambda=0} \neq 0$ holds for some $x \in g(P)$.

A crucial step in proving this theorem is the following Lemma 6.4 whose proof requires the following easy fact and its corollary.

Fact 6.2. *Let $F : \mathcal{U} \rightarrow \mathbb{C}$ be a holomorphic function defined in an open set \mathcal{U} of \mathbb{C}^N , $N \geq 1$. Let $\gamma, \tilde{\gamma} : \mathbb{D}_\varepsilon \rightarrow \mathcal{U}$ be two holomorphic curves such that*

$$\gamma(\lambda) - \tilde{\gamma}(\lambda) = O(\lambda^{m+1}) \text{ as } \lambda \rightarrow 0.$$

Then

$$F(\gamma(\lambda)) - F(\tilde{\gamma}(\lambda)) = \sum_{i=1}^N \frac{\partial F}{\partial z_i}(\gamma(0))(\gamma_i(\lambda) - \tilde{\gamma}_i(\lambda)) + O(\lambda^{m+2}) \text{ as } \lambda \rightarrow 0.$$

Proof. For fixed λ small, define $\delta(t) = (1-t)\tilde{\gamma}(\lambda) + t\gamma(\lambda)$ and $f(t) = F(\delta(t))$. Then

$$f'(t) = \sum_{i=1}^N \frac{\partial F}{\partial z_i}(\delta(t))(\gamma_i(\lambda) - \tilde{\gamma}_i(\lambda)).$$

Since $\delta(t) - \gamma(0) = O(\lambda)$, and $F(\gamma(\lambda)) - F(\tilde{\gamma}(\lambda)) = \int_0^1 f'(t)dt$, the equality follows. \square

Corollary 6.3. *A holomorphic motion h_λ of $g(P)$ is asymptotically invariant of order m if and only if*

(1) For each $x \in g(P) \setminus P_0$,

$$G_{h_\lambda(c_{1,1}), \dots, h_\lambda(c_{1,\nu})}(h_\lambda(x)) = h_\lambda(g(x)) + O(\lambda^{m+1}) \text{ as } \lambda \rightarrow 0.$$

(2) For $x = c_{0,j} \in g(P)$,

$$p_j(h_\lambda(c_{1,1}), \dots, h_\lambda(c_{1,\nu})) = h_\lambda(c_{0,j}) + O(\lambda^{m+1}) \text{ as } \lambda \rightarrow 0.$$

Proof. If h_λ is asymptotically invariant of order m , we get (1) applying Fact 6.2 to the function $F(z_1, z_2, \dots, z_\nu, z_{\nu+1}) = G_{(z_1, z_2, \dots, z_\nu)}(z_{\nu+1})$, and we get (2) applying it to the function $F(z_1, z_2, \dots, z_\nu) = p_j(z_1, z_2, \dots, z_\nu)$. Vice versa, assume that (1)-(2) hold. Given $x \in g(P) \setminus P_0$, let $F(z_1, z_2, \dots, z_\nu, z_{\nu+1})$ be a local branch of $G_{(z_1, z_2, \dots, z_\nu)}^{-1}(z_{\nu+1})$ which is a well defined holomorphic function in a neighborhood of $(\mathbf{c}_1, g(x))$. Let $V(\lambda) = G_{h_\lambda(c_{1,1}), \dots, h_\lambda(c_{1,\nu})}(h_\lambda(x))$ and $\widehat{V}(\lambda) = h_\lambda(g(x))$. By (1), $V(\lambda) - \widehat{V}(\lambda) = O(\lambda^{m+1})$. Hence, by Fact 6.2,

$$h_\lambda(x) - \widehat{h}_\lambda(x) = F(h_\lambda(c_{1,1}), \dots, h_\lambda(c_{1,\nu}), V(\lambda)) - F(h_\lambda(c_{1,1}), \dots, h_\lambda(c_{1,\nu}), \widehat{V}(\lambda)) = O(\lambda^{m+1}).$$

For $x = c_{0,j}$, the claim is straightforward. \square

Lemma 6.4. *One has the following:*

(1) *Assume $(g, G, \mathbf{p})_W$ has the lifting property. Suppose that there is a holomorphic motion h_λ of $g(P)$ over $(\Lambda, 0)$ which is asymptotically invariant of order m for some $m \geq 1$. Then there is a non-degenerate holomorphic motion H_λ of $g(P)$ over some $(\tilde{\Lambda}, 0)$ which is asymptotically invariant of order $m+1$. Besides, $H_\lambda(x) - h_\lambda(x) = O(\lambda^{m+1})$ as $\lambda \rightarrow 0$ for all $x \in g(P)$.*

(2) *Assume $(g, G, \mathbf{p})_W$ is real and has the real lifting property. Suppose that there is a real holomorphic motion h_λ of $g(P)$ over $(\Lambda, 0)$ which is asymptotically invariant of order m for some $m \geq 1$. Then there is a non-degenerate real holomorphic motion H_λ of $g(P)$ over some $(\tilde{\Lambda}, 0)$ which is asymptotically invariant of order $m+1$. Besides, $H_\lambda(x) - h_\lambda(x) = O(\lambda^{m+1})$ as $\lambda \rightarrow 0$ for all $x \in g(P)$.*

Proof. We shall only prove the first statement as the proof of the second is the same with obvious change of terminology. Let h_λ be a non-degenerate holomorphic motion of $g(P)$ over $(\Lambda, 0)$ which is asymptotically invariant of order m . By assumption that $(g, G, \mathbf{p})_W$ has the lifting property, there exists a smaller domain $\Lambda_0 \subset \Lambda$ and holomorphic motions $h_\lambda^{(k)}$ over Λ_0 , $k = 0, 1, \dots$ such that $h_\lambda^{(0)} = h_\lambda$ and such that $h_\lambda^{(k+1)}$ is the lift of $h_\lambda^{(k)}$ over $(\Lambda_0, 0)$ for each $k \geq 0$. Moreover, the functions $h_\lambda^{(k)}$ are uniformly bounded. For each $k \geq 1$, define

$$\psi_\lambda^{(k)}(x) = \frac{1}{k} \sum_{i=0}^{k-1} h_\lambda^{(i)}(x),$$

and

$$\varphi_\lambda^{(k)}(x) = \frac{1}{k} \sum_{i=1}^k h_\lambda^{(i)}(x).$$

By shrinking Λ_0 , we may assume that there exists $k_n \rightarrow \infty$, such that $\psi_\lambda^{(k_n)}(x)$ converges uniformly in $\lambda \in \Lambda_0$ as $k_n \rightarrow \infty$ to a holomorphic function $H_\lambda(x)$. Shrinking Λ_0 furthermore if necessary, H_λ defines a holomorphic motion of $g(P)$ over $(\Lambda_0, 0)$. Clearly, $\varphi_\lambda^{(k_n)}(x)$ converges uniformly to $H_\lambda(x)$ as well.

Let us show that H_λ is asymptotically invariant of order $m+1$ by applying the fact above. Due to Corollary 6.3 (and taking $k = k_n \rightarrow \infty$ in the next equations) this amounts to showing:

(i) For each $x \in g(P) \setminus P_0$, and any $k \geq 1$,

$$G_{\psi_\lambda^{(k)}(c_{1,1}), \dots, \psi_\lambda^{(k)}(c_{1,\nu})}(\varphi_\lambda^{(k)}(x)) = \psi_\lambda^{(k)}(g(x)) + O(\lambda^{m+2}) \text{ as } \lambda \rightarrow 0.$$

(ii) For $x = c_{0,j} \in g(P)$,

$$p_j(\psi_\lambda^{(k)}(c_{1,1}), \dots, \psi_\lambda^{(k)}(c_{1,\nu})) = \varphi_\lambda^{(k)}(c_{0,j}) + O(\lambda^{m+2}) \text{ as } \lambda \rightarrow 0.$$

Let us prove (i). Fix $x \in g(P) \setminus P_0$ and $k \geq 1$. Let $F(z_1, z_2, \dots, z_\nu, z_{\nu+1}) = G_{(z_1, z_2, \dots, z_\nu)}(z_{\nu+1})$. By the construction of $h_\lambda^{(k)}$, we have

$$F(h_\lambda^{(i)}(c_{1,1}), \dots, h_\lambda^{(i)}(c_{1,\nu}), h_\lambda^{(i+1)}(x)) = h_\lambda^{(i)}(g(x))$$

for every $i \geq 0$. Thus

$$(6.3) \quad \psi_\lambda^{(k)}(g(x)) = \frac{1}{k} \sum_{i=0}^{k-1} F(h_\lambda^{(i)}(c_{1,1}), \dots, h_\lambda^{(i)}(c_{1,\nu}), h_\lambda^{(i+1)}(x)).$$

Since all the functions $h_\lambda^{(i)}(x), \psi_\lambda^{(k)}(x), \varphi_\lambda^{(k)}(x)$ have the same derivatives up to order m at $\lambda = 0$, applying Fact 6.2, we obtain

$$\begin{aligned} & F(h_\lambda^{(i)}(c_{1,1}), \dots, h_\lambda^{(i)}(c_{1,\nu}), h_\lambda^{(i+1)}(x)) - F(\psi_\lambda^{(k)}(c_{1,1}), \dots, \psi_\lambda^{(k)}(c_{1,\nu}), \varphi_\lambda^{(k)}(x)) \\ &= \sum_{j=1}^{\nu} \frac{\partial F}{\partial z_j}(\mathbf{c}_1, x)(h_\lambda^{(i)}(c_{1,j}) - \psi_\lambda^{(k)}(c_{1,j})) + \frac{\partial F}{\partial z_{\nu+1}}(\mathbf{c}_1, x)(h_\lambda^{(i+1)}(x) - \varphi_\lambda^{(k)}(x)) + O(\lambda^{m+2}), \end{aligned}$$

as $\lambda \rightarrow 0$. Summing over $i = 0, 1, \dots, k-1$ and using the definition of $\psi_\lambda^{(k)}(x)$ and $\varphi_\lambda^{(k)}(x)$ we obtain

$$\begin{aligned} & \frac{1}{k} \sum_{i=0}^{k-1} F(h_\lambda^{(i)}(c_{1,1}), \dots, h_\lambda^{(i)}(c_{1,\nu}), h_\lambda^{(i+1)}(x)) \\ &= F(\psi_\lambda^{(k)}(c_{1,1}), \dots, \psi_\lambda^{(k)}(c_{1,\nu}), \varphi_\lambda^{(k)}(x)) + O(\lambda^{m+2}). \end{aligned}$$

Together with (6.3), this implies the equality in (i).

For (ii), we use $F(z_1, \dots, z_\nu) = p_j(z_1, \dots, z_\nu)$ and argue in a similar way. \square

Proof of the Main Theorem. By Proposition 5.1, all eigenvalues of \mathcal{A} are contained in $\overline{\mathbb{D}}$. It remains to prove (2). Let $L = \{v \in \mathbb{C}^\nu : D\mathcal{R}(\mathbf{c}_1, v) = 0\}$ and let d be the dimension of L . By Theorem 4.1, L has the same dimension as the eigenspace of \mathcal{A} associated with eigenvalue 1. Moreover, by Lemma 6.4, for each $m \geq 1$ and each $v \in L$, there is a holomorphic motion $h_\lambda(x)$ of $g(P)$ over \mathbb{D}_ε for some $\varepsilon > 0$ which is asymptotic invariant of order m and satisfies $h_\lambda(c_{1,j}) = c_{1,j} + v(c_{1,j})\lambda + O(\lambda^2)$ as $\lambda \rightarrow 0$. Putting $w_j(\lambda) = h_\lambda(c_{1,j})$, we obtain a holomorphic curve $\lambda \mapsto w(\lambda) = (w_j(\lambda))_{j=1}^\nu$, $\lambda \in \mathbb{D}_\varepsilon$, such that $w'_j(0) = v(c_{1,j})$ and

$$R_j(w(\lambda)) = O(\lambda^{m+1}) \text{ for all } j = 1, \dots, \nu.$$

If $d = 0$, i.e., $L = \{0\}$, then $D\mathcal{R}(\mathbf{c}_1)$ is invertible, so \mathcal{R} is a local diffeomorphism, and for a small neighborhood W' of \mathbf{c}_1 , the set in (6.1) consists of a single point \mathbf{c}_1 .

Now assume $d = \nu$, i.e., $L = \mathbb{C}^\nu$. We claim that $\mathcal{R}(w) \equiv 0$. Otherwise, there exists $m \geq 1$ such that $R(w) = \sum_{k=m}^\infty P_k(w - \mathbf{c}_1)$ in a neighborhood of \mathbf{c}_1 , where $P_k(u)$ is a homogeneous polynomial in u of degree k and $P_m(u) \not\equiv 0$. Therefore, there exists $\mathbf{v} \in \mathbb{C}^d$ such that $P_m(\lambda\mathbf{v}) = A\lambda^m$ for some $A \neq 0$. By the argument above, there is holomorphic curve $\lambda \mapsto w(\lambda)$ passing through \mathbf{c}_1 and tangent to \mathbf{v} at $\lambda = 0$, such that $|\mathcal{R}(w(\lambda))| = O(\lambda^{m+1})$. However,

$$\mathcal{R}(w(\lambda)) = P_m(w(\lambda) - \mathbf{c}_1) + \sum_{k>m} P_k(w - \mathbf{c}_1(\lambda)) = A\lambda^m + O(\lambda^{m+1}),$$

a contradiction.

The case $0 < d < \nu$ can be done similarly. To be definite, let us assume that

$$(6.4) \quad \left. \frac{\partial(R_1, R_2, \dots, R_{d'})}{\partial(w_1, w_2, \dots, w_{d'})} \right|_{w=\mathbf{c}_1} \neq 0,$$

where $d' = \nu - d$. By the Implicit Function Theorem, there is holomorphic map $\Phi : B \rightarrow \mathbb{C}^{d'}$, where B is a neighborhood of $\mathbf{u}_0 = (c_{1,d'+1}, c_{1,d'+2}, \dots, c_{1,\nu})$ in \mathbb{C}^d such that

$$(6.5) \quad R_j(\Phi(u), u) = 0 \text{ for all } 1 \leq j \leq d', u \in B,$$

and $t = \Phi(u)$ is the only solution of $R_j(t, u) = 0, 1 \leq j \leq d'$, in a fixed neighborhood of $(c_{1,1}, c_{1,2}, \dots, c_{1,d})$. It suffices to prove that

$$R_j(\Phi(u), u) = 0 \text{ for } u \text{ close to } \mathbf{u}_0, d' < j \leq \nu.$$

To this end, we only need to show that for any $m \geq 1$, and any $\mathbf{e} \in \mathbb{C}^{d'}$, there is a holomorphic curve $u(\lambda)$ in B which passes through \mathbf{u}_0 at $\lambda = 0$, such that $u'(0) = \mathbf{e}$ and

$$(6.6) \quad R_j(\Phi(u(\lambda)), u(\lambda)) = O(\lambda^{m+1}), \text{ as } \lambda \rightarrow 0, d' < j \leq \nu.$$

Indeed, the curve $\tilde{w}(\lambda) = (\Phi(\mathbf{u}_0 + \lambda\mathbf{e}), \mathbf{u}_0 + \lambda\mathbf{e})$ is tangent to L at \mathbf{c}_1 . Thus by the argument in the first paragraph of the proof, there is a curve $w(\lambda) = (t(\lambda), u(\lambda)) \in \mathbb{C}^{d'} \times \mathbb{C}^d$, tangent to $\tilde{w}(\lambda)$ at $\lambda = 0$ such that

$$(6.7) \quad R_j(w(\lambda)) = O(\lambda^{m+1}) \text{ as } \lambda \rightarrow 0, 1 \leq j \leq \nu.$$

Together with (6.4) and (6.5), this implies that

$$|\Phi(u(\lambda)) - t(\lambda)| = O(\lambda^{m+1}).$$

Finally, by (6.7), we obtain (6.6), completing the proof of the Main Theorem. \square

7. FAMILIES OF THE FORM $f_\lambda(x) = f(x) + \lambda$ AND $f_\lambda(x) = \lambda f(x)$

In this section we will apply these techniques to show that one has monotonicity and the transversality properties (2.4) and (2.3) within certain families of real maps of the form $f_\lambda(x) = f(x) + \lambda$ and $f_\lambda(x) = \lambda \cdot f(x)$ where $x \mapsto f(x)$ has one critical value (and is unimodal - possibly on a subset \mathbb{R}) or satisfy symmetries. There are quite a few papers giving examples for which one has non-monotonicity for such families, see for example [4, 21, 39, 50]. In this section we will prove several theorems which show monotonicity for a fairly wide class of such families.

In Subsection 7.1 we show that the methods we developed in the previous section apply if one has something like a polynomial-like map $f : U \rightarrow V$ with sufficiently 'big complex bounds'.

This gives yet another proof for monotonicity for real families of the form $z^\ell + c$, $c \in \mathbb{R}$ in the setting when ℓ is an even integer. We also apply this method to a family of maps with a flat critical point in Subsection 7.2. In Subsection 7.3 we show how to obtain the lifting property in the setting of one parameter families of the form $f_a(x) = af(x)$ with f in some rather general class of maps.

7.1. Families of the form $f_\lambda(x) = f(x) + \lambda$ with a single critical point. Let $f: U \rightarrow V$ be a map from the class \mathcal{F} defined in Subsection 2.8. Consider a marked map g with $g = f + g(0)$ for some $f \in \mathcal{F}$ from a finite set P into itself with $P \supset P_0 = \{0\}$, $P \setminus P_0 \subset U$. In other words, g extends to a holomorphic map $g: U_g \rightarrow V_g$ where

- U_g is a bounded open set in \mathbb{C} such that $U_g \supset P \setminus \{0\}$ and $0 \in \overline{U}_g$;
- V_g is a bounded open set in \mathbb{C} such that $c_1 := g(0) \in V_g$;
- $g: U_g \setminus \{0\} \rightarrow V_g \setminus \{c_1\}$ is an unbranched covering;
- $V_g \supset B(c_1; \text{diam}(U_g)) \supset U_g$.

Next define a local holomorphic deformation $(g, G, \mathbf{p})_W$ of g as follows: $G_w(z) = g(z) + (w - g(0))$ and $\mathbf{p}(w) = 0$ for all $w \in W := \mathbb{C}$.

Theorem 7.1. *Let $(g, G, \mathbf{p})_W$ be as above. Then*

- (1) $(g, G, \mathbf{p})_W$ satisfies the lifting property;
- (2) the spectrum of the operator \mathcal{A} is contained in $\overline{\mathbb{D}} \setminus \{1\}$.

If, in addition, the robust separation property $V_g \supset \overline{B(c_1; \text{diam}(U_g))} \supset U_g$ holds, then the spectral radius of \mathcal{A} is strictly smaller than 1 and

$$\det(I - \rho \mathcal{A}) = \sum_{i=0}^{q-1} \frac{\rho^i}{Dg^i(c_1)} \neq 0$$

holds for all $|\rho| \leq 1$. In particular, if g, G are real then $\sum_{i=0}^{q-1} \frac{1}{Dg^i(c_1)} > 0$.

Proof. Let us show that $(g, G, \mathbf{p})_{\mathbb{C}}$ satisfies the lifting property. For each domain $\Delta \ni 0$ in \mathbb{C} , let \mathcal{M}_Δ denote the collection of all holomorphic motions h_λ of $g(P)$ over $(\Delta, 0)$ such that for all $\lambda \in \Delta$,

$$(7.1) \quad h_\lambda(x) \in U_g \text{ for all } x \in g(P) \setminus \{0\} \text{ and } h_\lambda(0) = 0.$$

Claim. Let $\Delta \ni 0$ be a simply connected domain in \mathbb{C} . Then any holomorphic motion h_λ in \mathcal{M}_Δ has a lift \widehat{h}_λ which is again in the class \mathcal{M}_Δ .

Indeed, $g(P \setminus \{0\}) \subset g(U_g \setminus \{0\}) = V_g \setminus \{g(0)\}$. So for each $x \in P \setminus \{0\}$, $g(x) \neq g(0)$, thus for any $\lambda \in \Delta$,

$$0 < |h_\lambda(g(x)) - h_\lambda(g(0))| < \text{diam}(U_g),$$

hence by $V_g \supset B(c_1; \text{diam}(U_g)) \supset U_g$,

$$h_\lambda(g(x)) - h_\lambda(g(0)) + g(0) \in V_g \setminus \{g(0)\}.$$

Since $g: U_g \setminus \{0\} \rightarrow V_g \setminus \{g(0)\}$ is an unbranched covering and Δ is simply connected, there is a holomorphic function $\lambda \mapsto \widehat{h}_\lambda(x)$, from Δ to $U_g \setminus \{0\}$, such that $\widehat{h}_0(x) = x$ and

$$g(\widehat{h}_\lambda(x)) = h_\lambda(g(x)) - h_\lambda(g(0)) + g(0), \text{ i.e. } G_{h_\lambda(g(0))}(\widehat{h}_\lambda(x)) = h_\lambda(g(x)).$$

Define $\widehat{h}_\lambda(0) = 0$ if $0 \in g(P)$. Then \widehat{h}_λ is a lift of h_λ over Δ .

For any holomorphic motion h_λ of $g(P)$ over $(\Delta, 0)$ with $h_\lambda(0) = 0$, there is a simply connected sub-domain $\Delta \ni 0$ such that the restriction of h_λ to Δ belongs to the class \mathcal{M}_Δ . It follows that $(g, G, \mathbf{p})_W$ has the lifting property.

Therefore the assumptions of the Main Theorem are satisfied. The operator \mathcal{A} cannot have an eigenvalue 1 because otherwise for all parameters $w \in W$ the G_w would have the same dynamics. Hence, (2) in the conclusion of the theorem follows.

If the robust separation property $V_g \supset \overline{B(c_1; \text{diam}(U_g))} \supset U_g$ holds, then Proposition 5.2 applies and therefore the spectral radius of \mathcal{A} is strictly smaller than 1. As in Example 4.1 the conclusion follows. \square

Corollary 7.1. For any even integer d , transversality condition (2.3) holds and the topological entropy of $g_c(z) = z^d + c$ depends monotonically on $c \in \mathbb{R}$.

7.2. A unimodal family map $f \in \mathcal{F}$ with a flat critical point. Fix $\ell \geq 1$, $b > 2(\ell e)^{1/\ell}$ and consider

$$f_c(x) = \begin{cases} be^{-1/|x|^\ell} + c & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ c & \text{for } x = 0. \end{cases}$$

Note that $\mathbb{R}^+ \ni x \mapsto 2xe^{1/x^\ell}$ has a unique critical point at $x = \ell^{1/\ell}$ corresponding to a minimum value $2(\ell e)^{1/\ell}$. Therefore the assumption on b implies that $b = 2xe^{1/x^\ell}$ has a unique solution $x = \beta \in (0, \ell^{1/\ell})$. This implies in particular that the map $f_{-\beta}$ has the Chebeshev combinatorics: $f_{-\beta}(0) = -\beta$ and $f_{-\beta}(\beta) = \beta$. Note that

$$Df_{-\beta}(\beta) = be^{-1/\beta^\ell} \frac{\ell}{\beta^{\ell+1}} = \frac{2\ell}{\beta^\ell} > 2.$$

Therefore, there exists $x_1 > x_0 > \beta$ such that $f_{-\beta}(x_0) = x_1$ and $x_1 - \beta > 2(x_0 - \beta)$. Choosing x_0 close enough to β , we have

$$R := f_0(x_0) = x_1 + \beta < b.$$

For a bounded open interval $J \subset \mathbb{R}$, let $D_*(J)$ denote the Euclidean disk with J as a diameter. This set corresponds to the set of points for which the distance to J w.r.t. the Poincaré metric on $\mathbb{C}_J = \mathbb{C} \setminus (\mathbb{R} \setminus J)$ is at most some $k_0 > 0$. Also, let $B^*(x, R) = B(x, R) \setminus \{x\}$ where $B(x, R)$ is the open disc with radius R and centre at x .

Lemma 7.2. The map $f_0 : (-x_0, 0) \cup (0, x_0) \rightarrow (0, R)$ extends to an unbranched holomorphic covering map $F_0 : U \rightarrow B^*(0, R)$, where $U \subset D_*((-x_0, 0)) \cup D_*((0, x_0))$. In particular, $\text{diam}(U) = 2x_0 < R$.

Proof. Let $\Phi(re^{i\theta}) = r^\ell e^{i\ell\theta}$ denote the conformal map from the sector $\{re^{i\theta} : |\theta| < \pi/(2\ell)\}$ onto the right half plane, let $U^+ = \Phi^{-1}(D_*((0, x_0^\ell)))$. Since $\Phi^{-1} : \mathbb{C}_{(0, x_0^\ell)} \rightarrow \mathbb{C}_{(0, x_0)}$ is holomorphic, by the Schwarz Lemma Φ^{-1} contracts the Poincaré metrics on these sets, and therefore $U^+ \subset D_*((0, x_0))$. Define $U^- = \{-z : z \in U^+\}$, $U = U^+ \cup U^-$ and

$$F_0(z) = \begin{cases} be^{-1/\Phi(z)} & \text{if } z \in U^+ \\ be^{-1/\Phi(-z)} & \text{if } z \in U^- \end{cases}$$

It is straightforward to check that F_0 maps U^+ (resp. U^-) onto $B^*(0, R)$ as an un-branched covering. \square

Corollary 7.3. $F_0 \in \mathcal{F}$. Moreover, if $c \in U$ then $F_c = F_0 + c$ satisfies the robust separation property in Theorem 7.1. In particular, the kneading sequence of F_c depends monotonically on $c \in [-\beta, \infty)$.

Proof. Take U as in the previous lemma and take $V = B(c, R)$. Then $F_c : U \rightarrow V \setminus \{c\}$ is an unbranched covering, and since $\text{diam}(U) < R$ the robust separation property in Theorem 7.1 is satisfied. \square

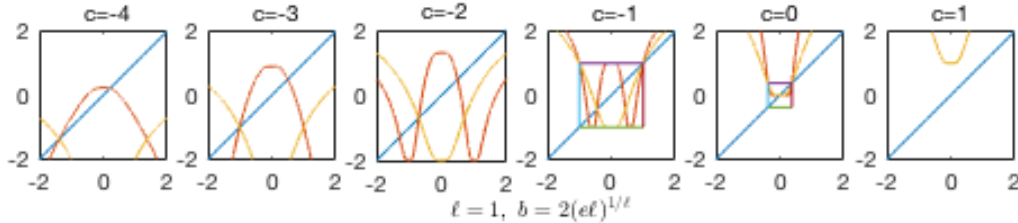


FIGURE 1. The graphs of $f_c : [-2, 2] \rightarrow [-2, 2]$ (in orange) and $f_c^2 : [-2, 2] \rightarrow [-2, 2]$ (in red) for various choices of c when $b = 2(\ell e)^{1/\ell}$. For $b < 2(\ell e)^{1/\ell}$ there exists no Chebeshev parameter c , and $b > 2(\ell e)^{1/\ell}$ there exists two such parameters.

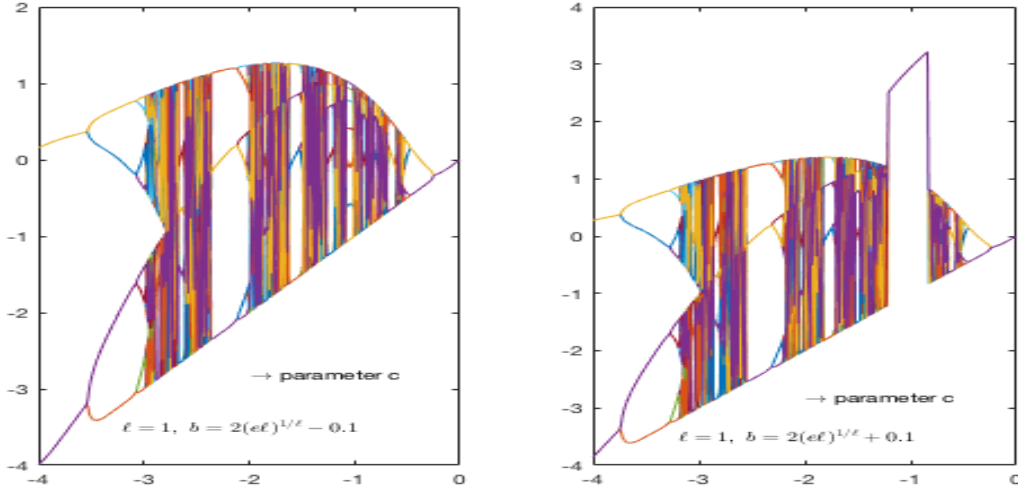


FIGURE 2. A bifurcation diagram. Here for $c \in [-4, 0]$ (drawn in the horizontal axis), the last 100 iterates from the set $\{f^k(0)\}_{k=0}^{1000}$ are drawn (in the vertical direction). Notice that for $b = 2(e\ell)^{1/\ell} + 0.1$ the interval $[-\beta_c, \beta_c]$, where $\beta_c > 0$ is the repelling fixed point of f_c (when it exists), is not invariant for parameters $c \approx -1$ and for these parameter almost every point is the basin of an attracting fixed point in \mathbb{R}_+ . The entropy decreases for $c \in [-\beta, 0]$ where for $b = 2(e\ell)^{1/\ell} + 0.1$, we have $\beta \approx -0.831$.

Remark 7.4. When $b < 2(e\ell)^{1/\ell}$, there no longer exists c so that f_c is a Chebychev map, i.e. so that $f_c^2(0) = \beta_c$ where $\beta_c > 0$ is a fixed point. Indeed, otherwise $f_c^2(0) = \beta_c$ and therefore since β_c is a fixed point and f_c is symmetric, $c = f_c(0) = -\beta_c = -f_c(\beta_c) = -f_{-\beta_c}(\beta_c)$ which implies that β_c is a solution of $b = 2xe^{1/x^\ell}$. As we have shown above, this equation has no solution when $b < 2(e\ell)^{1/\ell}$. Also note that when $b = 2(e\ell)^{1/\ell}$ then $c = -\beta = -\ell^{1/\ell}$ and $\frac{d\beta_c}{dc} = 1/(1 - Df_c(\beta_c)) = 1/(1 - 2) = -1$, and therefore the positive oriented transversality of f_c fails at $c = -\beta$.

7.3. Families of the form $f_a(x) = af(x)$. There are quite a few papers which ask the question:

For which interval maps f , has one monotonicity of the entropy for the family $x \mapsto f_a(x)$, $a \in \mathbb{R}$?

This question is subtle, as the counter examples to various conjectures show, see [39, 21, 4, 50]. In this section we will obtain monotonicity and transversality for such families provided f is contained in the large classes $\mathcal{E}, \mathcal{E}_o$ defined in Subsection 2.8. For convenience, let us recapitulate the definitions of the spaces \mathcal{E} and \mathcal{E}_o . Consider holomorphic maps $f : D \rightarrow V$ such that:

- (a) D, V are open sets which are symmetric w.r.t. the real line so that $f(D) = V$
- (b) Let $I = D \cap \mathbb{R}$ then there exists $c > 0$ so that $I \cup \{c\}$ is a (finite or infinite) open interval and $0 \in \bar{I}$, $c \in \text{int}(\bar{I})$. Moreover, f extends continuously to \bar{I} , $f(I) \subset \mathbb{R}$ and $\lim_{z \in D, z \rightarrow 0} f(z) = 0$.
- (c) Let D_+ be the component of D which contains $I \cap (c, \infty)$, where D_+ might be equal to D . Then $u \in D \setminus \{0\}$ and $v \in D_+ \setminus \{0\}$, $v \neq u$, implies $u/v \in V$.

Let \mathcal{E} be the class of maps which satisfy (a),(b),(c) and assumption (d):

- (d) $f : D \rightarrow V$ has no singular values in $V \setminus \{0, 1\}$ and $c > 0$ is minimal such that f has a positive local maximum at c and $f(c) = 1$.

Similarly let \mathcal{E}_o be the class of maps which satisfy (a),(b),(c) and assumption (e):

- (e) f is odd, $f : D \rightarrow V$ has no singular values in $V \setminus \{0, \pm 1\}$ and $c > 0$ is minimal such that f has a positive local maximum at c and $f(c) = 1$.

Here, as usual, we say that $v \in \mathbb{C}$ is a *singular value* of a holomorphic map $f : D \rightarrow \mathbb{C}$ if it is a critical value, or an asymptotic value where the latter means the existence of a path $\gamma : [0, 1) \rightarrow D$ so that $\gamma(t) \rightarrow \partial D$ and $f(\gamma(t)) \rightarrow v$ as $t \uparrow 1$. Note that we do not require here that $V \supset D$.

Using qs-rigidity, it was already shown in [41] that the topological entropy of $\mathbb{R} \ni x \mapsto af(x)$ is monotone a , where $f(x) = \sin(x)$ or more generally f is real, unimodal and entire on the complex plane and satisfies a certain sector condition. Here we strengthen and generalise this result as follows:

Theorem 7.2. *Let f be either in \mathcal{E} or in \mathcal{E}_o . Assume that the local maximum $c > 0$ is periodic for $f_a(x) = af(x)$ where $0 < a < b$. Then the following ‘positive-oriented’ transversality property holds:*

$$(7.2) \quad \frac{\frac{d}{d\lambda} f_\lambda^q(c) |_{\lambda=a}}{Df_a^{q-1}(f_a(c))} > 0.$$

(A similar statement holds when c is pre-periodic for f_a .) In particular, the kneading sequence of the family $f_a(x) : J \rightarrow \mathbb{R}$ is monotone increasing.

Proof. Let $f \in \mathcal{E} \cup \mathcal{E}_o$. Denote $g(x) = af(x)$ and let $V_a = a \cdot V$. Let $P_0 = \{c\}$, $P = \{c_i = g^i(c) : i \geq 0\}$. Since $0 < a < b$, f_a maps $(0, b)$ into itself, and so $P \subset (0, b)$. We may also assume that $g(c) > c$ because otherwise $q = 1$ and the result is again trivial. By the assumptions, g is a holomorphic map $g : D \rightarrow V_a$, $g(P) \subset P$ and $Dg(x) \neq 0$ for any $x \in P \setminus P_0$. In particular, g is a real marked map. For each $w \in W := \mathbb{C}_* = \mathbb{C} \setminus \{0\}$, $G_w(z) := wf(z)$ is a branched covering from $U := D \setminus \{0\}$ into \mathbb{C} . Define $p(w) \equiv c$. Then $(g, G, p)_W$ is a local holomorphic deformation of g . It suffices to prove that $(g, G, p)_W$ has the lifting property so that the Main Theorem applies. Indeed, if 1 is an eigenvalue of \mathcal{A} then by the Main Theorem $\{\mathcal{R}(w) = 0\}$ is an open set and therefore this critical relation holds for all parameters, which clearly is not possible.

Let us first consider the case $f \in \mathcal{E}$. In this case, w is the only critical or singular value of G_w . Given a simply connected domain $\Delta \ni 0$ in \mathbb{C} , let \mathcal{M}_Δ denote the collection of all holomorphic motions h_λ of $g(P)$ over $(\Delta, 0)$ with the following property that for all $\lambda \in \Delta$ we have $h_\lambda(x) \in U$ for all $x \in g(P) \setminus \{c\}$ and $h_\lambda(c) = c$. Given such a holomorphic motion, for each $x \in g(P)$ there is a holomorphic map $\lambda \mapsto \widehat{h}_\lambda(x)$, $\lambda \in \Delta$, with $\widehat{h}_0(x) = x$ and such that $f(\widehat{h}_\lambda(x)) = h_\lambda(g(x))/h_\lambda(g(c))$. Indeed, for $x = c$, take $\widehat{h}_\lambda(x) \equiv c$ and for $x \in g(P) \setminus \{c\}$, we have by property (c) that $h_\lambda(g(x))/h_\lambda(g(c)) \in V \setminus \{0, 1\}$. Note that we use here that $g(c) \in D_+$ since $c < g(c) < b$. So the existence of \widehat{h}_λ follows from the fact that $f : D \setminus f^{-1}\{0, 1\} \rightarrow V \setminus \{0, 1\}$ is an unbranched covering. Clearly, \widehat{h}_λ is a holomorphic motion in \mathcal{M}_Δ and it is a lift of h_λ over Δ . It follows that $(g, G, p)_W$ has the lifting property. Indeed, if h_λ is a holomorphic motion of $g(P)$ over $(\Lambda, 0)$ for some domain $\Lambda \ni 0$ in \mathbb{C} , then we can take a small disk $\Delta \ni 0$ such that the restriction of h_λ on $(\Delta, 0)$ is in the class \mathcal{M}_Δ . Therefore, there exists a sequence of holomorphic motions $h_\lambda^{(k)}$ of $g(P)$ over $(\Delta, 0)$ such that $h_\lambda^{(0)} = h_\lambda$ and $h_\lambda^{(k+1)}$ is a lift of $h_\lambda^{(k)}$ over Δ for each $k \geq 0$. If $x = c$ then $h_\lambda^{(k)}(x) \equiv c$ for each $k \geq 1$ while if $x \in g(P) \setminus \{c\}$, $h_\lambda^{(k)}(x)$ avoids values 0 and c . Restricting to a small disk, we conclude by Montel’s theorem that $\lambda \mapsto h_\lambda^{(k)}(x)$ is bounded.

The case $f \in \mathcal{E}_o$ is similar. In this case, G_w has two critical or singular values w and $-w$, but it has additional symmetry being an odd function. Given a simply connected domain $\Delta \ni 0$ in \mathbb{C} , let \mathcal{M}_Δ^o denote the collection of all holomorphic motions h_λ of $g(P)$ over $(\Delta, 0)$ with the following properties: for each $\lambda \in \Delta$,

- $h_\lambda(x) \in U$ for all $x \in g(P) \setminus \{c\}$ and $h_\lambda(c) = c$;
- $h_\lambda(x) \neq -h_\lambda(y)$ for $x, y \in g(P)$ and $x \neq y$.

Then similar as above, we show that each h_λ in \mathcal{M}_Δ^o has a lift which is again in the class \mathcal{M}_Δ^o . It follows that $(g, G, p)_W$ has the lifting property. \square

Let us now show that this theorem applies to a unimodal family with a flat critical point:

Lemma 7.5. *The unimodal map $f : [0, 1] \rightarrow [0, 1]$ defined by*

$$(7.3) \quad f(x) = \exp(2^\ell) \left(-\exp(-1/|x - 1/2|^\ell) + \exp(-2^\ell) \right)$$

has a holomorphic unbranched extension $f : D_- \cup D_+ \rightarrow B^*(1, 1)$. Here D_-, D_+ are domains with $D_- \subset B_*(0, 1/2)$, $D_+ \subset B_*(1/2, 1)$ and $B^*(x, r)$ is the ball with radius r centered and

punctured at x . There exists $\ell_0 > 0$ so that for all $\ell > \ell_0$, one can slightly enlarge V , D_- , D_+ and obtain a map in \mathcal{E} .

Proof. Let us consider f as a composition of a number of maps. First $z \mapsto (1/2 - z)^\ell$ and $z \mapsto (z - 1/2)^\ell$ map some domains U_-, U_+ which are contained in $D_*(0, 1/2)$ and $D_*(1/2, 1)$ onto $D_*(0, 1/2^\ell)$ when $\ell \geq 1$. Next $z \mapsto -1/z$ maps $D_*(0, 1/2^\ell)$ onto a half-plane $\operatorname{Re}(z) \leq -2^\ell$. Then $z \mapsto \exp(z)$ maps this half-plane onto the punctured disc $B^*(0, \exp(-2^\ell))$ centered at 0 and with radius $\exp(-2^\ell)$ (and with a puncture at 0). Applying the translation $z \mapsto \exp(-2^\ell)$ to this punctured disc we obtain the punctured disc centered at $B^*(\exp(-2^\ell), \exp(-2^\ell))$. Then multiplying this disc by $\exp(2^\ell)$ shows that f maps U_-, U_+ onto $B^*(1, 1)$. (Note that this final punctured disc touches the imaginary axis.) Since 0 is a repelling fixed point of f with multiplier > 2 , and U_-, U_+ are close to the intervals $(0, 1/2)$ and $(1/2, 1)$ when ℓ is large, we can enlarge the domain and range, and obtain a map as in (a)-(d). \square

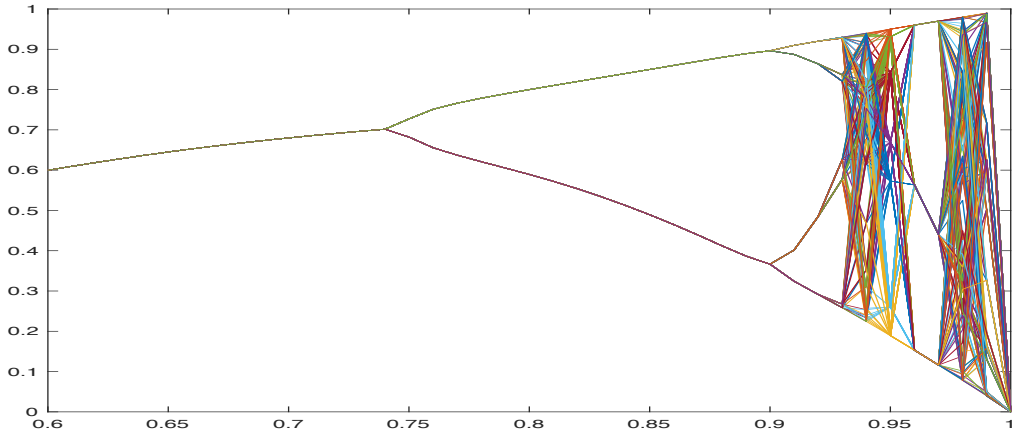


FIGURE 3. A bifurcation diagram for the family $f_\lambda(x) = \lambda f(x)$, $\lambda \in [0, 1]$ where f is as formula (7.3). Note that this unimodal family with a flat critical point is monotone.

8. APPLICATION TO FAMILIES WITH ONE FREE CRITICAL POINT

Let us apply the method along a curve in parameter space corresponding to where some ν -parameter family of maps G_w has $\nu - 1$ critical relationships.

Let us say that $(G, \mathbf{p})_W$ is a *partially marked family of maps* if

- (1) W is an open connected subset of \mathbb{C}^ν and U is an open subset of \mathbb{C} ;
- (2) $\mathbf{p} = (p_1, p_2, \dots, p_\nu) : W \rightarrow \mathbb{C}^\nu$ is a holomorphic map and so that all coordinates of $\mathbf{p}(w)$ are distinct. Let $P_{0,w} = \{p_1(w), \dots, p_\nu(w)\}$.
- (3) $G : (w, z) \in W \times U \mapsto G_w(z) \in \mathbb{C}$ is a holomorphic map and $DG_w(z) \neq 0$ for each $w \in W$ and $z \in U$.
- (4) associated to each $j = 1, \dots, \nu - 1$ there exists a positive integer q_j so that $G_w^k(w_j) \in U$ for $k = 1, \dots, q_j - 1$ and $w = (w_1, \dots, w_{\nu-1}, w_\nu) \in W$.

Assume that the family map G_w is real. That is, assume that the properties defined below Definition 2.3 hold: for any $w = (w_1, w_2, \dots, w_\nu) \in W$, $z \in U$ and $j = 1, 2, \dots, \nu$, we have $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_\nu) \in W$, $\bar{z} \in U$, and

$$G_{\bar{w}}(\bar{z}) = \overline{G_w(z)}, \text{ and } p_j(\bar{w}) = \overline{p_j(w)}.$$

Choose for each $j = 1, \dots, \nu - 1$ either $\mu(j) \in \{1, 2, \dots, \nu\}$ or $1 \leq l_j < q_j$. Given this choice, let L be the set of $w = (w_1, \dots, w_\nu) \in W$ for which the following hold:

- (5) if $\mu(j)$ is defined then $G_w^{\mu(j)-1}(w_j) = p_{\mu(j)}(w)$ and $G_w^k(w_j) \notin P_{0,w}$ for each $1 \leq k < \mu(j)$;
- (6) if l_j is defined then $G_w^{q_j-1}(w_j) = G_w^{l_j-1}(w_j)$ and $G_w^k(w_j) \notin P_{0,w}$ for all $0 \leq k \leq q_j - 1$.

Relabelling these points $w_1, \dots, w_{\nu-1}$, we assume that there is r such that the first alternative happens for all $1 \leq j \leq r$ and the second alternative happens for $r < j \leq \nu - 1$.

Remark 8.1. So for each $w \in L$, G_w has $\nu-1$ critical relations which start with $p_1(w), \dots, p_{\nu-1}(w)$. Hence the terminology of partially marked family of maps.

Define for $w \in W$, for $1 \leq j \leq r$,

$$(8.1) \quad R_j(w) = G_w^{q_j-1}(w_j) - p_{\mu(j)}(w)$$

and for $r < j \leq \nu-1$,

$$(8.2) \quad R_j(w) = G_w^{q_j-1}(w_j) - G_w^{l_j-1}(w_j),$$

where $w = (w_j)_{j=1}^\nu$. Then L is precisely the set

$$L = \{w \in W; R_1(w) = \dots = R_{\nu-1}(w) = 0\}.$$

Let L_* be a maximal connected subset of $L \cap \mathbb{R}^\nu$ such that for each $w \in L_*$, the $\nu \times (\nu-1)$ matrix

$$(8.3) \quad V_w := \left[\frac{1}{DG_w^{q_1-1}(w_1)} \nabla R_1(w), \dots, \frac{1}{DG_w^{q_{\nu-1}-1}(w_{\nu-1})} \nabla R_{\nu-1}(w) \right]$$

has rank $\nu-1$. Here $\nabla R_i(w)$ is the gradient of R_i . By the implicit function theorem, L_* is a real analytic curve.

Now, let us assume that for some $\mathbf{c}_1 = (c_{1,1}, \dots, c_{1,\nu}) \in L_*$, $G_{\mathbf{c}_1}$ has an additional critical relation starting with $p_\nu(w)$, i.e., $g = G_{\mathbf{c}_1}$ is a *marked map* and G extends to a local holomorphic deformation of g : there is a neighborhood $W_{\mathbf{c}_1} \subset W$ of \mathbf{c}_1 , an open set $U_{\mathbf{c}_1} \supset U$ such that $G : W_{\mathbf{c}_1} \times U \rightarrow W_{\mathbf{c}_1} \times \mathbb{C}$ extends to a holomorphic map $G : (w, z) \in W_{\mathbf{c}_1} \times U_{\mathbf{c}_1} \mapsto (w, G_w(z)) \in W_{\mathbf{c}_1} \times \mathbb{C}$ and $DG_w|U_{\mathbf{c}_1} \neq 0$ for each $w \in W_{\mathbf{c}_1}$ and there exists either $\mu(\nu) \in \{1, 2, \dots, \nu\}$ or $l_\nu < q_\nu$ so that either

- if $\mu(\nu)$ is defined then $g^{q_\nu}(p_\nu(\mathbf{c}_1)) = p_{\mu(\nu)}(\mathbf{c}_1)$, $g^k(c_{1,\nu}) \notin P_{0,\mathbf{c}_1}$ for each $1 \leq k < q_\nu$ then define

$$R_\nu(w) = G_w^{q_\nu-1}(w_\nu) - p_{\mu(\nu)}(w)$$

- if l_ν exists then $g^{q_\nu}(p_\nu(\mathbf{c}_1)) = g^{l_\nu}(p_\nu(\mathbf{c}_1))$ and $g^k(w_\nu) \notin P_{0,\mathbf{c}_1}$ for all $1 \leq k \leq q_\nu$, then we define

$$R_\nu(w) = G_w^{q_\nu-1}(w_\nu) - G_w^{l_\nu-1}(w_\nu).$$

Notice that R_ν is only defined in a small neighbourhood $W_{\mathbf{c}_1}$ of \mathbf{c}_1 .

The following theorem gives a condition implying that along the curve L_* all bifurcations are in the same direction.

Theorem 8.1. For each $w \in L_*$, define $E_w \in T_w \mathbb{C}^\nu$ to be the unique unit vector in \mathbb{R}^ν orthogonal to the range of the matrix V_w and so that

$$\det \left[\frac{1}{DG_w^{q_1-1}(w_1)} \nabla R_1(w), \dots, \frac{1}{DG_w^{q_{\nu-1}-1}(w_{\nu-1})} \nabla R_{\nu-1}(w), E_w \right] > 0.$$

Then

- E_w is a tangent vector to L_* at w and $L_* \ni w \mapsto E_w$ is real analytic. In particular, E_w defines an orientation on the entire curve L_* which we will call ‘positive’.
- If for some $\mathbf{c}_1 \in L_*$ the corresponding map $g = G_{\mathbf{c}_1}$ is a marked map as above and the positively oriented transversality property (2.3) holds for the local holomorphic deformation $(g, G, p)_{W_{\mathbf{c}_1}}$, then

$$\frac{1}{Dg^{q_\nu-1}(c_{1,\nu})} \nabla_E R_\nu(\mathbf{c}_1) > 0$$

where $Dg^{q_\nu-1}(c_{1,\nu})$ is the spatial derivative, and $\nabla_E R_\nu(\mathbf{c}_1)$ is the derivative in the direction of the tangent vector $E = E_{\mathbf{c}_1}$ of L_* at \mathbf{c}_1 .

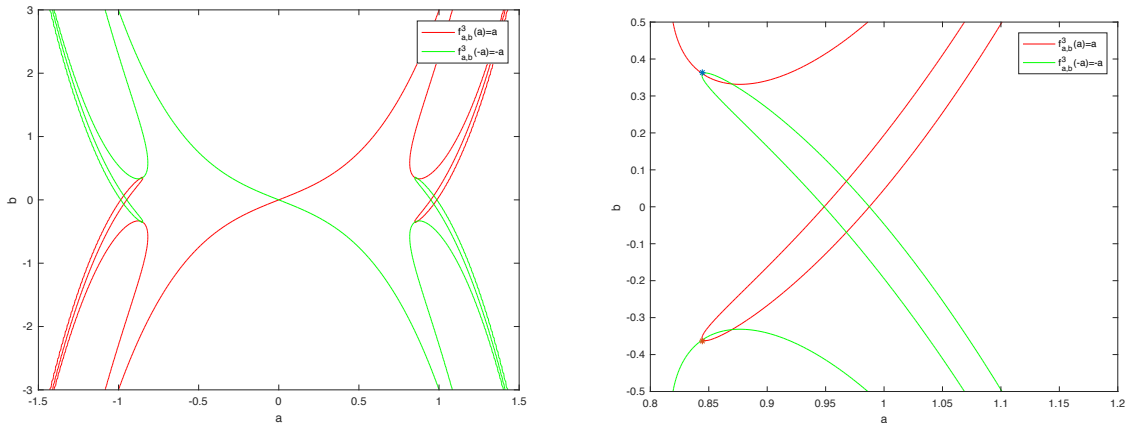


FIGURE 4. The sets $\Gamma_3(\pm a) = \{(a, b); f_{a,b}^3(\pm a) = \pm a\}$ for the cubic family $f_{a,b} = x^3 - 3a^2x + b$ (with critical points $\pm a$). The two curves in the left panel crossing at $(0, 0)$ correspond to the sets where one of the critical points of $f_{a,b}$ is a fixed point, i.e. where $\{f_{a,b}(\pm a) = \pm a\}$. The two points marked with $*$ in the magnification on the right are where the two critical points $\pm a$ lie on the same orbit. By Theorem 8.2, the topological entropy is monotone on the components of $\Gamma_3(\pm a)$ minus these points. This information was already obtained by Milnor and Tresser in [37] but here we derive it from the local methods derived in this paper.

Proof. Let $A = [A_1, \dots, A_\nu]$ be the $\nu \times \nu$ matrix with the first $(\nu - 1)$ -columns equal to the columns of $V_{\mathbf{c}_1}$ and the last column equal to $\frac{1}{DG_{\mathbf{c}_1}^{q\nu-1}(\mathbf{c}_{1,\nu})} \nabla R_\nu(\mathbf{c}_1)$. Note that the determinant of this matrix is positive by the positive oriented transversality condition (2.3). There exists λ so that $A_\nu = \lambda E_{\mathbf{c}_1} + v$ where v is in the range of the matrix $V_{\mathbf{c}_1}$. So $\det(A_1, \dots, A_{\nu-1}, A_\nu) = \det(A_1, \dots, A_{\nu-1}, \lambda E_{\mathbf{c}_1} + v) = \lambda \det(A_1, \dots, A_{\nu-1}, E_{\mathbf{c}_1})$. Since the first and the last determinants are positive, we have $\lambda > 0$ and so $A_\nu \cdot E_{\mathbf{c}_1} > 0$. This is precisely the expression claimed to be positive in the theorem. \square

Remark 8.2. *Applying this theorem to the setting of a family of globally defined real analytic maps, we obtain monotonicity of entropy along such curves L_* . This holds because the topological entropy is equal to the growth rate of the number of laps for continuous piecewise monotone interval or circle maps, [36].*

8.1. Application to ‘bone’ curves in the space of real cubic maps. For every $(a, b) \in \Sigma := \mathbb{R}^2 \setminus \{a = 0\}$, let $f_{a,b}(x) = x^3 - 3a^2x + b$. Then $f_{a,b}$ has critical points $\pm a$. It is clear that for any $(a, b) \in \mathbb{C}^2 \setminus \{a = 0\}$, $f_{a,b}$ is locally parametrized by its (different) critical values $w_{1,2} = \pm 2a^3 + b$. For $q > 0$ consider a connected component L_q of the set

$$\{(a, b) \in \Sigma : f_{a,b}^q(a) \in \{\pm a\}, f_{a,b}^k(a) \notin \{\pm a\}, k = 1, \dots, q-1\}.$$

By [22], the corresponding 2×1 matrix (8.3) has rank one and hence $L_q = L_{q*}$ is a simple smooth curve. In fact, the positively oriented transversality property holds for any critically-finite $f_{a,b}$; this follows similar to the proof of Theorem 7.1 of the next Section (see [24] for a general result though). By Theorem 8.1 we have a positive orientation on $L_q = L_q^*$ and the entropy increases or decreases along this curve as mentioned in Remark 8.2.

For $q > 0$ consider a connected component Γ_q of the set

$$\{(a, b) \in \Sigma : f_{a,b}^q(a) = a\}$$

which was called a *bone* in [37]. The next theorem proves a crucial property of this set, which was derived in [37] using global considerations (including Thurston rigidity for postcritically finite maps). Here we will derive this property from positive transversality.

Theorem 8.2 (Properties of bones). *Assume that for some $(\tilde{a}, \tilde{b}) \in \Gamma_q$ the integer $q > 0$ is minimal so that $f_{\tilde{a}, \tilde{b}}^q(\tilde{a}) = \tilde{a}$. Then for all $(a, b) \in \Gamma_q$ one has that $f_{a, b}^i(a) \neq a$ for all $0 < i < q$. Moreover,*

- (1) *there exists at most one $(a_*, b_*) \in \Gamma_q$ so that $f_{a_*, b_*}^i(a_*) = -a_*$ for some $0 \leq i \leq q$.*
- (2) *the kneading sequence of $f_{a, b}$ is monotone on each of the components of $\Gamma_q \setminus \{(a_*, b_*)\}$; more precisely, it is non-decreasing on one component and non-increasing on the other component.*

Proof. That $f_{a, b}^i(a) \neq a$ for all $0 < i < q$, $(a, b) \in \Gamma_q$ follows from the implicit function theorem because the multiplier of this q -periodic orbit is not equal to 1. It is well known that Γ_q is a smooth curve (this also follows for example from [22]). The curve Γ_q is a component of the zero set of

$$\tilde{R}(a, b) = f_{a, b}^q(a) - a.$$

As remarked, the critical values $w = (w_1, w_2)$ are local parameters along the curve Γ_q and we define a direction on the curve Γ_q by the tangent vector $V_{(a, b)} = \left(-\frac{\partial \tilde{R}}{\partial w_2}, \frac{\partial \tilde{R}}{\partial w_1}\right)$.

Assume that for some $(a_*, b_*) \in \Gamma_q$ the orbit of a_* contains the other critical point, i.e. assume that $f_{a_*, b_*}^i(a_*) = -a_*$ for some $0 < i < q$. The idea of the proof below is as follows. We will show that as the point $(a, b) \in \Gamma_q$ passes through (a_*, b_*) , the point $-a$ crosses $f_{a, b}^i(a)$ in a direction which depends only on the sign of $\Delta_i(a_*, b_*)$ where

$$(8.4) \quad \Delta_i(a, b) := \prod_{1 \leq k < q, k \neq i} Df_{a, b}(f_{a, b}^k(a))$$

so

$$\Delta_i(a_*, b_*) = Df_{a_*, b_*}^{i-1}(f_{a_*, b_*}(a_*)) Df_{a_*, b_*}^{q-i-1}(f_{a_*, b_*}(-a_*)).$$

Now as (a, b) moves further along Γ_q , on the one hand (for the same reason) $-a$ cannot cross $f_{a, b}^i(a)$ in the opposite direction, and on the other hand $-a$ cannot cross a neighbour $f_{a, b}^j(a)$ of $f_{a, b}^i(a)$ at some other $(a_\bullet, b_\bullet) \in \Gamma_q$ because $\Delta_i(a_*, b_*)$ and $\Delta_j(a_\bullet, b_\bullet)$ have opposite signs. Let us explain this in more detail.

Since the map f_{a_*, b_*} is critically finite, one has positive transversality at this parameter. More precisely, define

$$R_1(a, b) = f_{a, b}^i(a) - (-a), R_2(a, b) = f_{a, b}^{q-i}(-a) - a.$$

Now observe that

$$\left(\frac{\partial R_2}{\partial w_1}, \frac{\partial R_2}{\partial w_2}\right)_{(a_*, b_*)} = \left(\frac{\partial \tilde{R}}{\partial w_1}, \frac{\partial \tilde{R}}{\partial w_2}\right)_{(a_*, b_*)}.$$

Hence the positive transversality condition can be written as

$$\frac{1}{\Delta_i(a_*, b_*)} \det \begin{pmatrix} \frac{\partial R_1}{\partial w_1} & \frac{\partial R_1}{\partial w_2} \\ \frac{\partial \tilde{R}}{\partial w_2} & \frac{\partial \tilde{R}}{\partial w_1} \end{pmatrix}_{(a_*, b_*)} > 0$$

As in the proof of Theorem 8.1 we obtain

$$(8.5) \quad \frac{1}{\Delta_i(a_*, b_*)} D_{V_{(a_*, b_*)}} R_1(a_*, b_*) > 0$$

where $D_{V_{(a_*, b_*)}}$ stands for the directional derivative of R_1 in the direction $V_{(a_*, b_*)}$. To be definite, let us consider the case that

$$(8.6) \quad \Delta_i(a_*, b_*) < 0.$$

This implies that

$$(8.7) \quad D_{V_{(a_*, b_*)}} R_1(a_*, b_*) < 0$$

and so the derivative of

$$(8.8) \quad \Gamma_q \ni (a, b) \mapsto f_{a,b}^i(a) - (-a) \text{ is negative at } (a_*, b_*).$$

By contradiction, assume that there exists another parameter $(a_\bullet, b_\bullet) \in \Gamma_q$ which is the nearest to the right of (a_*, b_*) for which there exists $0 < j < q$ so that $f_{a_\bullet, b_\bullet}^j(a_\bullet) = -a_\bullet$. In what follows we use that the ordering of the points $a, \dots, f_{a,b}^{q-1}(a)$ in \mathbb{R} does not change along the curve Γ_q . Let Γ_q^\bullet be the open arc between (a_*, b_*) and (a_\bullet, b_\bullet) . Notice that because of (8.6)

$$(8.9) \quad \Delta_i(a, b) < 0$$

for all $(a, b) \in \Gamma_q^\bullet$. Observe that $j \neq i$ because when $j = i$ then (8.9) holds on the closure of Γ_q^\bullet and so $D_{V_{(a_\bullet, b_\bullet)}} R_1(a_\bullet, b_\bullet) < 0$. Therefore (8.8) also holds at (a_\bullet, b_\bullet) which is clearly a contradiction. Therefore, $j \neq i$ and

$$(8.10) \quad f_{a,b}^i(a) < -a < f_{a,b}^j(a)$$

along the open arc Γ_q^\bullet and there are no points of the orbit of a between $f_{a,b}^i(a), f_{a,b}^j(a)$. The sign of

$$(8.11) \quad \Delta_j(a, b) := Df_{a,b}^{j-1}(f_{a,b}(a)) Df_{a,b}^{q-j-1}(f_{a,b}(-a))$$

is constant for (a, b) near (a_\bullet, b_\bullet) . Therefore for $(a, b) \in \Gamma_q^\bullet$,

$$\text{sign } \Delta_j(a_\bullet, b_\bullet) = \text{sign } \Delta_j(a, b) = \text{sign} \left[\Delta_i(a, b) \frac{Df_{a,b}(f_{a,b}^i(a))}{Df_{a,b}(f_{a,b}^j(a))} \right].$$

Because of (8.9) we therefore have

$$\text{sign } \Delta_j(a_\bullet, b_\bullet) = -\text{sign} \left[\frac{Df_{a,b}(f_{a,b}^i(a))}{Df_{a,b}(f_{a,b}^j(a))} \right].$$

The key point is that the sign of the ratio in the r.h.s. of this expression is negative because $-a$ is a folding critical point and because of (8.10). It follows that $\Delta_j(a_\bullet, b_\bullet) > 0$ and so arguing as before the derivative of

$$(8.12) \quad \Gamma_q \ni (a, b) \mapsto f_{a,b}^j(a) - (-a) \text{ is positive at } (a_\bullet, b_\bullet).$$

But by (8.10) we have that $f_{a,b}^j(a) - (-a) > 0$ on Γ_q^\bullet . This and (8.12) imply that $f_{a_\bullet, b_\bullet}^j(a_\bullet) - (-a_\bullet) > 0$ which is a contradiction.

The 2nd assertion follows immediately from Theorem 8.1 and Remark 8.2. \square

Remark 8.3. *The proof of the previous theorem can also be applied to the setting of polynomials of higher degrees.*

APPENDIX A. THE FAMILY $f_c(x) = |x|^{\ell_\pm} + c$ WITH $\ell_\pm > 1$ LARGE

In this section we obtain monotonicity for unimodal (not necessary symmetric!) maps in the presence of critical points of large non-integer order, but only if not too many points in the critical orbit are in the orientation reversing branch.

A.1. Unimodal family with high degrees.

Theorem A.1. *Fix real numbers $\ell_-, \ell_+ \geq 1$ and consider the family of unimodal maps $f_c = f_{c, \ell_-, \ell_+}$ where*

$$f_c(x) = \begin{cases} |x|^{\ell_-} + c & \text{if } x \leq 0 \\ |x|^{\ell_+} + c & \text{if } x \geq 0. \end{cases}$$

For any integer $L \geq 1$ there exists $\ell_0 > 1$ so that for any $q \geq 1$ and any periodic kneading sequence $\mathbf{i} = i_1 i_2 \dots \in \{-1, 0, 1\}^{\mathbb{Z}^+}$ of period q so that

$$\#\{0 \leq j < q; i_j = -1\} \leq L,$$

and any pair $\ell_-, \ell_+ \geq \ell_0$ there is at most one $c \in \mathbb{R}$ for which the kneading sequence of f_c is equal to \mathbf{i} . Moreover,

$$(A.1) \quad \sum_{n=0}^{q-1} \frac{1}{Df_c^n(c)} > 0.$$

Notations. As usual, for any three distinct point $o, a, b \in \mathbb{C}$, let $\angle aob$ denote the angle in $[0, \pi]$ which is formed by the rays oa and ob . We shall often use the following obvious relation: for any distinct four points o, a, b, c ,

$$\angle aob + \angle boc \geq \angle aoc.$$

For $\theta \in (0, \pi)$, let

$$D_\theta = \{z \in \mathbb{C} \setminus \{0, 1\} : \angle 0z1 > \pi - \theta\}$$

and let

$$S_\theta = \{re^{it} : t \in (-\theta, \theta)\}.$$

For $0 < t < 1$, we shall only consider z^t in the case $z \notin (-\infty, 0)$ and z^t is understood as the holomorphic branch with $1^t = 1$.

Let us fix a map $f = f_{c, \ell_-, \ell_+}$ with a periodic critical point of period q and let $P = \{f^n(0) : n \geq 0\}$. So P is a forward invariant finite set. Denote

$$\ell = \min\{\ell_-, \ell_+\}.$$

Definition A.1. A holomorphic motion h_λ of P over $(\Omega, 0)$, is called θ -regular if

(A1). For $a \in P$,

$$h_\lambda(a) \in S_{4\theta/\ell}, \text{ if } a > 0$$

and

$$h_\lambda(a) \in -S_{4\theta/\ell}, \text{ if } a < 0;$$

(A2). For $a, b \in P$, $|a| > |b| > 0$ and $ab > 0$,

$$\frac{h_\lambda(b)}{h_\lambda(a)} \in D_\theta.$$

Given a θ -regular holomorphic motion h_λ of P over Ω , with $\theta \in (0, \pi)$, one can define another holomorphic motion \tilde{h}_λ of P over the same domain Ω as follows: $\tilde{h}_\lambda(0) = 0$; for $a \in P$ with $a > 0$,

$$\tilde{h}_\lambda(a) = (h_\lambda(f(a)) - h_\lambda(f(0)))^{1/\ell_+};$$

for $a \in P$ with $a < 0$, define

$$\tilde{h}_\lambda(a) = -(h_\lambda(f(a)) - h_\lambda(f(0)))^{1/\ell_-}.$$

The new holomorphic motion is called the *lift* of h_λ which clearly satisfies the condition (A1), but not necessarily (A2) in general.

Main Lemma. There is ℓ_0 depending only on the number L such that for any $\ell \geq \ell_0$ and each θ small enough, the following holds: If $\#\{0 \leq j < q; i_j = -1\} \leq L$ and if a θ -regular motion can be successively lifted $q - 1$ times and all these successive lifts are θ -regular, then the q -th lift of the holomorphic motion is $\theta/2$ -regular.

Proof of Theorem A.1. Given L , choose ℓ_0 as in the Main Lemma. It is enough to prove (A.1) provided $\ell \geq \ell_0$. Consider a local holomorphic deformation $(f_c, f_w, \mathbf{p})_W$ where $W \subset \mathbb{C}$ is a small neighbourhood of c , $f_w = f_c + (w - c)$ and $\mathbf{p} = 0$. Let h_λ be a holomorphic motion of P over $(\Delta, 0)$. Let us fix $\theta > 0$ small enough. Restricting h_λ to a smaller domain Δ_ε , we may assume that h_λ is θ -regular and that h_λ can be lifted successively for q times. Therefore by the Main Lemma, we obtain a sequence of holomorphic motions $h_\lambda^{(n)}$ of P over $(\Delta_\varepsilon, 0)$, such that $h_\lambda^{(0)} = h_\lambda$ and $h_\lambda^{(n+1)}$ is the lift of $h_\lambda^{(n)}$ and such that $h_\lambda^{(n)}(x) \in \pm S_{\theta_n}$ for all n and all $x \in P$ where $\theta_n \rightarrow 0$ as $n \rightarrow \infty$. Thus $(f_c, f_w, \mathbf{p})_W$ has the lifting property and by the Main Theorem, the transversality condition (A.1) holds.

Alternatively, the uniqueness of c follows directly from the Main Lemma. Indeed, let $\tilde{f} = f_{\tilde{c}}$ be a map with the same kneading sequence as f_c . Then one can define a real holomorphic motion h_λ over some domain $\Omega \ni 0, 1$ such that $h_\lambda(f^n(0)) = \tilde{f}^n(0)$ for $\lambda = 1$. As above, for $i > 0$ let $h_\lambda^{(i)}$ be the lift of $h_\lambda^{(i-1)}$. As we have just shown, $h_\lambda^{(i)}(c)$ is contained in the sector $-S_{\theta_n}$ with $\theta_n \rightarrow 0$, this sequence of functions $\lambda \rightarrow h_\lambda^{(i)}(c)$ has to converge to a constant function. Since by construction of the lifts $\tilde{c} = h_1^{(n)}(c)$ for each $n \geq 1$ we conclude that $\tilde{c} = c$. \square

A.2. Proof of the Main Lemma.

Lemma A.2. *For any $\theta \in (0, \pi)$ and $0 < t < 1$, if $z \in D_\theta$ then $z^t \in D_\theta$.*

Proof. This is a well-known consequence of the Schwarz lemma, due to Sullivan. \square

When $\angle 01z$ is much smaller than $\angle 10z$, we have the following improved estimate.

Lemma A.3. *For any $\varepsilon > 0$, there is $\delta > 0$ such that the following holds. For $z \in D_\theta$ with $\theta \in (0, \pi/2]$ and $\angle 01z < \delta\theta$ and for any $0 < t < 1$, we have $\angle 01z^t < \varepsilon\theta$.*

Proof. Write $z = re^{i\alpha}$ where $r > 0$ and $\alpha \in (0, \theta)$ and write $\alpha' = t\alpha$ and $\beta' = \angle 01z^t$. By assumption, $\alpha + \beta \leq \theta$. By the sine theorem,

$$r = \frac{\sin \beta}{\sin(\alpha + \beta)}$$

and

$$r^t = \frac{\sin \beta'}{\sin(t\alpha + \beta')}.$$

If $\alpha + \beta < \varepsilon\theta$ then by Lemma A.2, $\alpha' + \beta' \leq \alpha + \beta < \varepsilon\theta$. Assume now $\alpha + \beta \geq \varepsilon\theta$. Let $K > 0$ be a large constant such that

$$\frac{t}{K^t - 1} < \varepsilon \text{ for any } 0 < t < 1.$$

Assume $\beta < \delta\theta$ for δ small. Then $r < 1/K$. Thus

$$\tan \beta' = \frac{r^t \sin t\alpha}{1 - r^t \cos t\alpha} \leq \frac{tr^t}{1 - r^t} \alpha \leq \frac{t}{K^t - 1} \alpha < \varepsilon\theta.$$

\square

Lemma A.4. *Let φ_λ be a θ -regular motion with $\theta \in (0, \pi/10]$ and let ψ_λ be its lift. For $x, y \in P$ so that $xy \geq 0$ let $x_\lambda = \psi_\lambda(x)$, $y_\lambda = \psi_\lambda(y)$, $u_\lambda = \varphi_\lambda(f(x))$, $v_\lambda = \varphi_\lambda(f(y))$ and $c_\lambda = \varphi_\lambda(f(0))$.*

For any $\varepsilon > 0$ there is ℓ_0 and $\delta > 0$ such that if $\ell > \ell_0$ then the following hold.

- (1) *If $f(x) \leq 0 \leq f(y)$ then $\angle 0x_\lambda y_\lambda \geq \pi - \varepsilon\theta$ for all λ .*
- (2) *Let $0 < f(x) < f(y)$. Then (i) $\angle 0x_\lambda y_\lambda \geq \angle 0u_\lambda v_\lambda - \frac{8\theta}{\ell}$. If, moreover, $c_\lambda \in -S_{\theta_1}$ and $u_\lambda, v_\lambda \in S_{\theta_1}$ for some $\theta_1 \in (0, 4\theta/\ell]$ then (ii) $x_\lambda, y_\lambda \in \pm S_{\theta_1/\ell}$ and $\angle 0x_\lambda y_\lambda \geq \angle 0u_\lambda v_\lambda - 2\theta_1$.*
- (3) *Suppose $f(x) < f(y) < 0$ and*

$$\alpha = \pi - \min(\angle c_\lambda v_\lambda 0, \angle u_\lambda v_\lambda 0) < \delta\theta.$$

Then

$$\angle 0x_\lambda y_\lambda \geq \pi - \varepsilon\theta.$$

Proof. Note that $\triangle 0xy$ is the image of $\triangle c_\lambda u_\lambda v_\lambda$ under an appropriate branch of $z \mapsto (z - c_\lambda)^t$. Since $\angle xoy < 8\theta/\ell$, an upper bound on $\angle oxy$ implies a lower bound on $\angle oxy$.

- (1) In this case, we have $u_\lambda \in -S_{4\theta/\ell}$ and $v_\lambda \in S_{4\theta/\ell}$, so

$$\angle 0u_\lambda v_\lambda \leq 4\theta/\ell,$$

and

$$\angle 0v_\lambda u_\lambda \leq 4\theta/\ell.$$

In particular,

$$\angle c_\lambda u_\lambda v_\lambda \geq \angle c_\lambda u_\lambda 0 - \angle 0u_\lambda v_\lambda \geq \pi - \theta - 4\theta/\ell \geq \pi - 5\theta.$$

By Lemma A.3, the statement follows.

(2) In this case,

$$\angle c_\lambda u_\lambda v_\lambda \geq \angle 0u_\lambda v_\lambda - \angle 0u_\lambda c_\lambda \geq \angle 0u_\lambda v_\lambda - 8\theta/\ell.$$

Thus by Lemma A.2, the conclusion (i) follows; (ii) is similar.

(3) In this case,

$$\angle c_\lambda u_\lambda v_\lambda \geq \angle c_\lambda u_\lambda 0 - \angle v_\lambda u_\lambda 0 \geq \pi - \theta - \alpha \geq \pi - 2\theta$$

and

$$\angle c_\lambda v_\lambda u_\lambda \leq 2\pi - (\angle c_\lambda v_\lambda 0 + \angle 0v_\lambda u_\lambda) \leq 2\alpha.$$

So the conclusion follows from Lemma A.3. \square

Now suppose that we have a sequence of θ -regular holomorphic motions $h_\lambda^{(i)}$ of P , $i = 0, 1, \dots, q-1$ over the same marked domain $(\Omega, 0)$, such that $h_\lambda^{(i)}$ is a lift of $h_\lambda^{(i-1)}$ for all $1 \leq i < q$. Then $h_\lambda^{(q)}$, lift of $h_\lambda^{(q-1)}$ is well-defined and satisfies the condition (A1) with the same constant θ . For each $0 \leq i < q$, $\lambda \in \Omega$ and $x, y \in P$ so that $0 < |x| < |y|$ and $xy > 0$, let

$$\begin{aligned} \theta_\lambda^i(x, y) &= \pi - \\ &\inf\{\angle 0h_\lambda^{(i)}(z_1)h_\lambda^{(i)}(z_2) : z_1, z_2 \in P, 0 < |z_1| \leq |x| < |y| \leq |z_2|, xz_1 > 0, xz_2 > 0\} \\ &\geq \pi - \angle 0h_\lambda^{(i)}(x)h_\lambda^{(i)}(y). \end{aligned}$$

Furthermore, given any $x, y \in P$, $xy > 0$ (but not necessarily $|x| < |y|$), denote

$$\hat{\theta}^i(x, y) = \theta^i(x \wedge y, x \vee y)$$

where $x \wedge y = x/|x| \min(|x|, |y|)$ and $x \vee y = x/|x| \max(|x|, |y|)$.

Lemma A.5. Consider $0 \leq i < q$, $x, y \in P$ where $xy > 0$ and $\lambda \in \Omega$. For any $\varepsilon > 0$ there is $\delta > 0$ and $\ell_0 > 0$ such that if $\ell \geq \ell_0$, then the following hold.

(1) If $f(x) \leq 0 \leq f(y)$ then

$$\hat{\theta}_\lambda^{i+1}(x, y) \leq \varepsilon\theta.$$

(2) Let $r \geq 1$ be such that $i+r \leq q$. If $0 < f^j(x) < f^j(y)$ for all $1 \leq j \leq r$, then

$$\hat{\theta}_\lambda^{i+r}(x, y) \leq \hat{\theta}_\lambda^i(f^r(x), f^r(y)) + \varepsilon\theta.$$

(3) If $f(x) < f(y) < 0$ and $\hat{\theta}_\lambda^i(f(x), f(y)) < \delta\theta$, then

$$\hat{\theta}_\lambda^{i+1}(x, y) \leq 4 \max(\varepsilon\theta, \hat{\theta}_\lambda^i(f(x), f(y))).$$

Proof. Note that $f(x) < f(y)$ implies $|x| < |y|$.

(1) For each $0 < |z_1| \leq |x| < |y| \leq |z_2|$ as in the definition of $\theta_\lambda^i(x, y)$ we have $f(z_1) \leq 0$ and $f(z_2) \geq 0$. So by Lemma A.4 (1), (applying to $\varphi = h^{(i)}$ and $\psi = h^{(i+1)}$), $\angle 0h_\lambda^{(i+1)}(z_1)h_\lambda^{(i+1)}(z_2) \geq \pi - \varepsilon\theta$. Thus the statement holds.

(2) Consider $0 < |z_1| \leq |x| < |y| \leq |z_2|$ so that $z_1 z_2 > 0$. Then $f(z_2) > 0$. If $f(z_1) \leq 0$, then by Lemma A.4 (1), $\angle 0h_\lambda^{(i+r)}(z_1)h_\lambda^{(i+r)}(z_2) \geq \pi - \varepsilon\theta$. Assume $f(z_1) > 0$ and let $r_1 \in \{1, \dots, r\}$ be maximal such that $0 < f^j(z_1) \leq f^j(x) < f^j(y) \leq f^j(z_2)$ for all $1 \leq j \leq r_1$. Notice that then

$$0 < f^{r_1}(z_2) \leq f^{r_1-1}(f^2(0)) < f^{r_1-2}(f^2(0)) < \dots < f^2(0).$$

Let us show that for all $k \in \{0, \dots, r_1 - 1\}$,

$$(A.2) \quad h_\lambda^{(k+i+r-r_1)}(f(0)) \in -S_{4\theta/\ell^{k+1}},$$

and

$$(A.3) \quad h_\lambda^{(k+i+r-r_1)}(f^{r_1-k}(z_1)), h_\lambda^{(k+i+r-r_1)}(f^{r_1-k}(z_2)) \in S_{4\theta/\ell^{k+1}}.$$

Indeed, this holds for $k = 0$ as $h_\lambda^{(i+r-r_1)}$ is θ -regular. Now, for $1 \leq k \leq r_1 - 1$, (A.2)-(A.3) follows by a successive application of the second part of Lemma A.4 (2). This proves (A.2)-(A.3). In turn, using (A.2)-(A.3) and again applying successively Lemma A.4 (2),

$$\begin{aligned} \angle 0h_\lambda^{(i+r)}(z_1)h_\lambda^{(i+r)}(z_2) &> \angle 0h_\lambda^{(i+r-r_1)}(f^{r_1}(z_1))h_\lambda^{(i+r-r_1)}(f^{r_1}(z_2)) - 2 \sum_{k=0}^{\infty} \frac{4\theta}{\ell^{k+1}} = \\ &\angle 0h_\lambda^{(i+r-r_1)}(f^{r_1}(z_1))h_\lambda^{(i+r-r_1)}(f^{r_1}(z_2)) - \frac{8\theta}{\ell-1}. \end{aligned}$$

Consider two cases. If $r_1 < r$, then $f^{r_1+1}(z_1) \leq 0$ and $f^{r_1+1}(z_2) > 0$ and by Lemma A.4 (1),

$$\angle 0h_\lambda^{(i+r-r_1)}(f^{r_1}(z_1))h_\lambda^{(i+r-r_1)}(f^{r_1}(z_2)) \geq \pi - \varepsilon\theta$$

for any ℓ large enough. If $r_1 = r$,

$$\angle 0h_\lambda^{(i)}(f^r(z_1))h_\lambda^{(i)}(f^r(z_2)) \geq \pi - \theta_\lambda^i(f^r(x), f^r(y)).$$

In any case,

$$\angle 0h_\lambda^{(i+r)}(z_1)h_\lambda^{(i+r)}(z_2) > \pi - \theta_\lambda^i(f^r(x), f^r(y)) - \varepsilon\theta$$

provided ℓ is large enough. Thus the statement holds.

(3) Notice that in this case $\hat{\theta}_\lambda^i(f(x), f(y)) = \theta_\lambda^i(f(y), f(x))$. Consider $0 < |z_1| \leq |x| < |y| \leq |z_2|$ so that $z_1 z_2 > 0$. If $f(z_2) > 0$ then by Lemma A.4 (1), $\angle 0h_\lambda^{(i+1)}(z_1)h_\lambda^{(i+1)}(z_2) \geq \pi - \varepsilon\theta$. Assume $f(z_2) < 0$. Then $0 > f(z_2) \geq f(y) > f(x) > f(z_1) > c$. So

$$\angle h_\lambda^{(i)}(c)h_\lambda^{(i)}(f(z_2))0 \geq \pi - \theta_\lambda^i(f(y), f(x))$$

and

$$\angle h_\lambda^{(i)}(f(z_1))h_\lambda^{(i)}(f(z_2))0 \geq \pi - \theta_\lambda^i(f(y), f(x)).$$

By Lemma A.4 (3),

$$\angle 0h_\lambda^{(i+1)}(z_1)h_\lambda^{(i+1)}(z_2) \geq \pi - 4 \max(\theta_\lambda^i(f(y), f(x)), \varepsilon\theta),$$

provided that $\theta_\lambda^i(f(y), f(x))/\theta$ is small enough and ℓ is large enough. \square

Completion of proof of the Main Lemma. It is easy to check that h_λ^q satisfies the condition (A1) with $S_{4\theta/\ell}$ replaced by $S_{2\theta/\ell}$. It remains to check that for $x, y \in P$, $0 < |x| < |y|$ and $xy > 0$ implies $\angle 0h_\lambda^q(x)h_\lambda^q(y) > \pi - \theta/2$. Since the critical point is periodic, there is a minimal integer p , less than the period q of the critical point, such that

$$f^p([x, y]) \ni 0.$$

Let us define $p-1 = m_0 > m_1 > \dots > m_{j_0-1} > m_{j_0} = 0$ inductively as follows. Given m_i , let $m_{j+1} \in \{0, 1, \dots, m_j - 1\}$ be the maximal so that $f^{m_{j+1}}([x, y]) \subset \mathbb{R}^-$ if it exists and $m_{j+1} = 0$ otherwise. Note that $j_0 \leq L+1$. Let

$$\kappa_{m_j} = \hat{\theta}_\lambda^{q-m_j}(f^{m_j}(x), f^{m_j}(y))/\theta, j = 0, 1, \dots, j_0.$$

Fix $\varepsilon > 0$ small. Assume that ℓ is large. Then by Lemma A.5 (1),

$$\kappa_{m_0} = \kappa_{p-1} \leq \varepsilon.$$

For each $0 < j \leq j_0$, by Lemma A.5 (2) and (3),

$$\kappa_{m_{j+1}} \leq 4\kappa_{m_j} + 4\varepsilon$$

provided that κ_{m_j} is small enough and ℓ is large enough. Therefore, provided that ℓ is large enough, we have $\kappa_0 < 1/2$. It follows that

$$\angle 0h_\lambda^{(q)}(x)h_\lambda^{(q)}(y) \geq \pi - \kappa_0\theta \leq \pi - \theta/2.$$

\square

APPENDIX B. FAMILIES WITHOUT THE LIFTING PROPERTY

In this appendix we will give a few examples of families for which the lifting property does not hold.

B.1. Remark on the lifting property for the flat family. Using the notations of Section 7.2 let $b = 2(e\ell)^{1/\ell}$, $c = -\beta = -\ell^{1/\ell}$ and $f = F_c$ so that $0 \mapsto c \mapsto \beta \mapsto \beta$ by f . By Remark 7.4 the transversality fails for (f, F, \mathbf{p}) . (It can be also checked directly that the function $R(w) = F_w^2(w) - F_w(w)$ vanished at $w = c$, not identically zero but $R'(c) = 0$.) Therefore, by the Main Theorem, this triple does not have the lifting property.

The purpose of this remark is to give a more direct argument that the lifting property does not hold for $(f, F, \mathbf{p})_W$. So let us assume by contradiction that (f, F, \mathbf{p}) has the lifting property. Fix $r > 0$ small so that the function $E(z) := \exp\{-z^{-\ell}\}$ is univalent in a disk $B(\beta, r) \subset U^+$. By assumption, given an arbitrary holomorphic motion h_λ^0 of $\{c, \beta\}$ there exist $\epsilon > 0$ and a sequence of holomorphic motions $\{h_\lambda^{(k)}\}_{k=0}^\infty$ of $\{c, \beta\}$ over \mathbb{D}_ϵ such that for all $\lambda \in \mathbb{D}_\epsilon$ and all $k \geq 0$ we have that $-h_\lambda^{(k)}(c), h_\lambda^{(k)}(\beta) \in B(\beta, r)$. Therefore,

$$bE(-h_\lambda^{(k+1)}(c)) + h_\lambda^{(k)}(c) = h_\lambda^{(k)}(\beta),$$

$$bE(h_\lambda^{(k+1)}(\beta)) + h_\lambda^{(k)}(c) = h_\lambda^{(k)}(\beta)$$

and $h_\lambda^{(k+1)}(c) = -h_\lambda^{(k+1)}(\beta)$. Let us now choose $h_\lambda^0(c) = c - \lambda$ and $h_\lambda^0(\beta) = -h_\lambda^0(c) = \beta + \lambda$. Then $h_\lambda^{(k)}(c) = -h_\lambda^{(k)}(\beta)$ holds for all $k \geq 0$. Hence, for $a_k(\lambda) := h_\lambda^{(k)}(\beta)$ and all $\lambda \in \mathbb{D}_\epsilon$, $k \geq 0$,

$$b \exp\left\{-\frac{1}{a_{k+1}(\lambda)^\ell}\right\} = 2a_k(\lambda).$$

On the other hand, it is easy to check that the for function $f : (0, +\infty) \rightarrow (0, +\infty)$,

$$f(x) = \frac{b}{2} \exp\left\{-\frac{1}{x^\ell}\right\},$$

we have: $f(\beta) = \beta$, $Df(\beta) = 1$, $D^2f(\beta) < 0$, $f : [\beta, +\infty) \rightarrow [\beta, b/2)$ is an increasing homeomorphism so that $f^k(x) \rightarrow \beta$ as $k \rightarrow +\infty$ for all $x \in [\beta, +\infty)$. Let $U : [\beta, b/2) \rightarrow [\beta, +\infty)$ be a branch of f^{-1} such that $U(\beta) = \beta$. Since $a_k(0) = \beta$ and functions a_k are continuous in \mathbb{D}_ϵ , it follows that

$$a_k(\lambda) = U^k(a_0(\lambda))$$

for all $k > 0$ and all λ provided $a_0(\lambda) = \beta + \lambda \geq \beta$. Fix $\lambda_0 \in (0, \epsilon)$ so that $a_0(\lambda_0) = \beta + \lambda_0 > \beta$. It follows that $h_{\lambda_0}^{(k)}(\beta) = a_k(\lambda_0) = U^k(a_0(\lambda_0)) \rightarrow +\infty$ as $k \rightarrow \infty$, a contradiction with the definition of the lifting property.

B.2. Spectrum of the transfer operator and linear coordinate changes of the quadratic family. Consider the standard holomorphic deformation of a critically finite quadratic map (g, G, \mathbf{p}) , that is, $\mathbf{p}(w) = 0$, $G_w(z) = z^2 + w$ and $g = G_{c_1}$ is so that 0 is periodic for g of period $q \geq 2$.

Given a function ν which is locally holomorphic in a neighborhood of a point $v_1 \in \mathbb{C} \setminus \{0\}$ so that $\nu(v_1) = c_1$, let $w = \nu(v)$ and $\varphi(w) = \varphi(\nu(v)) = \nu(v)/v$. Define

$$G_v^\nu(z) = \frac{1}{\varphi(w)} G_w(\varphi(w)z) = \frac{\nu(v)}{v} z^2 + v$$

and $g_\nu = G_{v_1}^\nu$.

Then $(g_\nu, G^\nu, \mathbf{p})$ is a local holomorphic deformation of g_ν . Denote by \mathcal{A} respectively \mathcal{A}_ν the transfer operator of (g, G, \mathbf{p}) respectively of $(g_\nu, G^\nu, \mathbf{p})$. Note that $1/2$ is always in the spectrum of \mathcal{A} (see [26]).

Proposition B.1.

$$\det(1 - \rho \mathcal{A}_\nu) = \frac{1 - \frac{1}{2} \frac{v_1 \nu'(v_1)}{c_1}}{1 - \frac{\rho}{2}} \det(1 - \rho \mathcal{A}).$$

Proof. Let $L_\nu(z) = \partial G_\nu^\nu(z)/\partial v|_{v=v_1}$, $v_n = g_\nu^n(0)$,

$$D_\nu(\rho) = 1 + \sum_{n=1}^{q-1} \rho^n \frac{L_\nu(v_n)}{(g_\nu^n)'(v_1)},$$

$$D(\rho) = 1 + \sum_{n=1}^{q-1} \rho^n \frac{1}{(g^n)'(c_1)}.$$

We have to verify the following identity:

$$D_\nu(\rho) = \frac{1 - \rho(1 - \frac{1}{2} \frac{v_1 \nu'(v_1)}{\nu(v_1)})}{1 - \frac{\rho}{2}} D(\rho).$$

Let us indicate its proof. We have: $(g_\nu^n)'(v_1) = (g^n)'(c_1)$, $v_n = c_n/\varphi(c_1)$ (in particular, $\varphi(c_1) = c_1/v_1$, and

$$L_\nu(v_n) = (\frac{\nu(v)}{v})'|_{v=v_1} v_n^2 + 1.$$

Then the above identity turns out to be equivalent to the following one:

$$2(1 - \frac{\rho}{2}) \frac{1}{c_1} \sum_{n=1}^{q-1} \rho^{n-1} \frac{c_n^2}{(g^n)'(c_1)} = D(\rho)$$

which is checked directly using $c_{n+1} - c_n^2 = c_1$ for $1 \leq n \leq q-2$ and $-c_{q-1}^2 = c_1$. \square

Corollary B.2. *The transversality of $(g_\nu, G^\nu, \mathbf{p})$ fails if and only if $\nu'(v_1) = 0$.*

Corollary B.3. *Let c_1 and ν be real. Then $(g_\nu, G^\nu, \mathbf{p})$ has the lifting property if and only if the positive transversality property holds.*

Proof. By the Main Theorem, the lifting property yields the positive transversality. Conversely, by Proposition B.1, $D_\nu(1) > 0$ implies that the spectrum of \mathcal{A}_ν belongs to the open unit disk which in turn implies the lifting property. \square

APPENDIX C. THE LIFTING PROPERTY IN THE SETTING OF RATIONAL MAPPINGS

The goal of this section is to show how to apply the method developed in this paper to deal with transversality problems in the classical case of polynomials and rational maps. It is natural to assume that the maps involved are suitably normalized, so we shall only consider the following situations:

- (a) f is a map in \mathbf{P}_d , the family of monic centered polynomials of degree $d \geq 2$;
- (b) Z is a set with $\#Z = 3$ and \mathbf{Rat}_d^Z is the family of all rational maps f of degree d such that $f(Z) = Z$ and such that Z is disjoint from the critical orbit of f .

Note that for each rational map f of degree $d \geq 2$, it is possible to find Z , consisting of either a cycle of period 3, or a fixed point and a cycle of period 2, such that $f \in \mathbf{Rat}_d^Z$. Let $\mathcal{U} = \mathbf{P}_d$ in case (a) and $\mathcal{U} = \mathbf{Rat}_d^Z$ in case (b).

In case (b), we assume without loss of generality that critical points and their orbits avoids the point at ∞ . Let c_1, c_2, \dots, c_ν be the distinct (finite) critical points of f with multiplicities $\mu_1, \mu_2, \dots, \mu_\nu$ and let $v_j = f(c_j)$. In the following, we fix f and let \mathcal{U}^f denote the subcollection of maps in \mathcal{U} which have exactly ν critical points with multiplicity $\mu_1, \mu_2, \dots, \mu_\nu$. It is well-known that \mathcal{U}^f is a complex manifold of dimension ν and the critical values are holomorphic coordinates. See for example [22].

Proposition C.1. *There is a neighbourhood W of (v_1, v_2, \dots, v_ν) in \mathbb{C}^ν such that $W \ni w \mapsto f_w$ in \mathbf{P}_d (resp. \mathbf{Rat}_d^Z) is biholomorphism from W to a neighborhood of f in \mathcal{U}^f , and a holomorphic function $\mathbf{p} : W \rightarrow \mathbb{C}^\nu$, such that $p_j(w)$ is a critical point of f_w of multiplicity μ_j and*

$$w = (f_w(p_1(w)), f_w(p_2(w)), \dots, f_w(p_\nu(w))).$$

Now we assume that f is critically finite. Let $P = P(f)$, $P_0 = \{c_1, c_2, \dots, c_\nu\}$ and let U be a small neighborhood of $P \setminus P_0$. Then f , restricting to $P_0 \cup U$, is a marked map in the sense of §2.1. We shall use the same notation f for the marked map. Moreover, defining $F(w, z) = f_w(z)$, $(f, F, \mathbf{p})_W$ is a holomorphic deformation of f in the sense of §2.2.

Theorem C.1. *The holomorphic deformation $(f, F, \mathbf{p})_W$ satisfies the lifting property.*

Proof. Let $h_\lambda^{(0)}$ be an arbitrary holomorphic motion of $f(P)$ over $(\mathbb{D}, 0)$. By Bers-Royden [3], there exists $\varepsilon > 0$ such that $h_\lambda^{(0)}$, $\lambda \in \mathbb{D}_\varepsilon$, extends to a holomorphic motion of \mathbb{C} over $(\mathbb{D}_\varepsilon, 0)$ which satisfies the following: in case (a), for $|z|$ large enough, $h_\lambda^{(0)}(z)$ is holomorphic in z and $h_\lambda^{(0)}(z) = z + o(1)$ near infinity, moreover, fix a big enough disk V such that $f^{-1}(V) \subset V$ and such that the complex dilatation $\mu_\lambda^{(0)}$ of $h_\lambda^{(0)}$ is supported in V for all $\lambda \in \mathbb{D}_\varepsilon$, and in case (b), $h_\lambda(z) = z$ for all $z \in Z$. Moreover, the complex dilatation $\mu_\lambda^{(0)}$ of $h_\lambda^{(0)}$ depends holomorphically in λ . Define $\mu_\lambda^{(k)} = (f^k)^*(\mu_\lambda^{(0)})$ for each $k \geq 1$ (here f is considered as a globally defined map) and let $h_\lambda^{(k)}$ denote the unique qc map with complex dilatation $\mu_\lambda^{(k)}$ and such that the following holds: in case (a), $h_\lambda^{(k)}(z) = z + o(1)$ near infinity; and in case (b), $h_\lambda^{(k)}(z) = z$ for all $z \in Z$. Then by the Measurable Riemann Mapping Theorem, $h_\lambda^{(k)}$ is a holomorphic motion of \mathbb{C} over $(\mathbb{D}_\varepsilon, 0)$. Let us show that for each $k \geq 0$, $h_\lambda^{(k+1)}$, restricting to $f(P)$, is a lift of $h_\lambda^{(k)}$, restricting to $f(P)$, with respect to $(f, F, \mathbf{p})_W$. This amounts to prove the following:

Claim. For $|\lambda|$ small enough, we have

$$h_\lambda^{(k)} \circ f \circ (h_\lambda^{(k+1)})^{-1} = f_{(h_\lambda^{(k)}(v_1), h_\lambda^{(k)}(v_2), \dots, h_\lambda^{(k)}(v_\nu))}.$$

Proof of Claim: for each $\lambda \in \mathbb{D}_\varepsilon$, the complex dilatation of $h_\lambda^{(k+1)}$ is the f -pull back of that of $h_\lambda^{(k)}$, and therefore the function $g_\lambda := h_\lambda^{(k)} \circ f \circ (h_\lambda^{(k+1)})^{-1}$ is holomorphic in \mathbb{C} . It is a branched covering of degree d , so it is either a polynomial or a rational function of degree d . By the normalization of both $h_\lambda^{(j)}$, $j = k, k+1$, in case (a), $g_\lambda \in \mathcal{U}$, and hence $g \in \mathcal{U}^f$. Clearly the critical values of g_λ are $h_\lambda^{(k)}(v_i)$. The claim follows.

To complete the proof, notice that in the case (a), $\mu_\lambda^{(k)}$ are supported in $f^{-k}(V) \subset V$ and by compactness of K-qc maps the conclusion follows. \square

Corollary C.2. *Under the circumstances above, either $(f, F, \mathbf{p})_W$ satisfies the transversality property or f is a flexible Lattés map.*

Proof. Suppose that $(f, F, \mathbf{p})_W$ does not satisfies the transversality property. Then by the Main Theorem, there is a non-trivial local holomorphic family f_t passing through f of critically finite rational maps. By McMullen-Sullivan [34], f carries an invariant line field in its Julia set. Since the postcritical set of f is finite, f must be a flexible Lattés map, see e.g. Corollary 3.18 of [33]. \square

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