Iteration in Tracts

Thesis

ITERATION IN TRACTS

submitted by

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DECLARATION

This thesis is the result of my own work, except where explicit reference is made to the work of others, and has not been submitted for another qualification to this or any other university.

Milton Keynes, United Kingdom, October 29, 2019

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ABSTRACT

This thesis is focused on the iteration of transcendental entire and meromorphic functions within a particular domain of the complex plane called a direct tract. Many new results are given on the rates of escape and the dimension of points of bounded orbit in a direct tract, as well as the geometry of direct tracts.

First, we study iterates of points within one of these tracts. The points that escape to infinity are of particular interest in the iteration of entire functions due both to their simple definition and, in contrast to rational functions, the interesting phenomena and structures they exhibit. We expand on work of Rippon and Stallard to show that in many cases there exist points that escape to infinity within a direct tract as slowly as desired. In order to accomplish this, we develop several tools based both on the expansion of the hyperbolic metric and estimates on the function value in these direct tracts.

Next, we relate the geometry of a direct tract to how well behaved the entire function is inside this direct tract. We consider a particular type of direct tract, called a logarithmic tract. Many results are known for functions with this type of direct tract, so the ability to identify them from their geometric properties is important. In particular, we give new descriptions of when a direct tract is a logarithmic tract or contains logarithmic tracts.

Finally, we show that, for functions with a specific restriction on the geometry of the direct tract, the Hausdorff dimension of the set of points with bounded orbit in the Julia set is strictly greater than one. To do this, we prove new results related to Wiman–Valiron theory.

PUBLICATIONS

Much of the contents of this thesis has previously been published in the following papers:

1. J. Waterman. Slow escape in tracts. *Proc. Amer. Math. Soc.*, 147(7):3087–3101, 2019.

2. J. Waterman. Identifying logarithmic tracts. To appear in *Ann. Acad. Sci. Fenn.* Preprint arXiv:1902.04330.

3. J. Waterman. Wiman–Valiron discs and the dimension of Julia sets. To appear in *International Mathematics Research Notices*. Preprint arXiv:1910.08474.

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INTRODUCTION

The subject of this thesis lies in the area of complex dynamics, which more broadly sits between complex analysis and dynamical systems. The field of complex dynamics was established in the early 20th century by Pierre Fatou and Gaston Julia and has been widely studied since. An introduction to the history of complex dynamics, including the life of Fatou and Julia, can be found in [2]. In this thesis, we more specifically study the iteration of transcendental entire and meromorphic functions. We will be concerned with the behavior inside a particular domain, called a tract.

1.1 COMPLEX DYNAMICS

A function $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function if it is holomorphic and not a polynomial. Complex dynamics is concerned with the behavior of the iterates f^n of f, that is, f composed with itself n times, for $n = 0, 1, 2 \ldots$.

Under iteration the complex plane $\mathbb{C}$ splits into two dynamically interesting and important sets named after the originators of the field, the *Fatou set* $F(f)$ and its complement, which is known as the *Julia set* $J(f)$. Loosely speaking, the Fatou set is the domain where points have stable behavior under iteration and the Julia set is where chaotic behavior occurs. Formally, $F(f)$ is the set of points $z \in \mathbb{C}$ such that $\{f^n\}$ forms a normal family on a neighborhood of z, and $J(f)$ is its complement. A family of transcendental entire functions $\mathcal{F}$ on a domain $D \in \mathbb{C}$ is called *normal* if every sequence (f_n) in $\mathcal{F}$ has a subsequence which converges locally uniformly in D to either an entire function or infinity. The Fatou set is open, while the Julia set is closed. Both the Fatou set and the Julia set are *completely invariant*, that is $z \in F(f)$ if and only if $f(z) \in F(f)$ and similarly for $J(f)$. This gives the property that $F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in N$. For a survey of the iteration of entire and meromorphic functions, see [10]. For a survey of the iteration of rational functions, see [8] and [46].

1.1.1 *Fixed and periodic points*

If $z_0 \in \mathbb{C}$, then we call z_0 a *fixed point* if $f(z_0) = z_0$ and z_0 a *periodic point* if $f^p(z_0) = z_0$ for some $p \in \mathbb{N}$. The smallest such p is then called the *period* of the point z_0. Much of the local behavior about these points is determined by what is called the *multiplier* of a periodic point, which is defined to be $(f^p)'(z_0)$. A periodic point is then called *attracting*, *indifferent*, or *repelling* depending on whether the modulus of the multiplier is less than, equal to, or greater than 1 respectively. We further differentiate the cases where the modulus of the multiplier is 0 and the case where it is equal to 1. In the first case the point is called *superattracting*. In the latter case, the multiplier is of the form $e^{2\pi i \alpha}$, where $0 \leqslant \alpha < 1$. We call z_0 a *rationally indifferent* periodic point if α is rational and *irrationally indifferent* otherwise. Rationally indifferent fixed points are also called *parabolic*. Finally, we call a point *pre-periodic* if $f^n(z_0)$ is periodic for some $n \in \mathbb{N}$. For a more detailed discussion of fixed points and their properties, see [8].

Many of these points only occur in either the Fatou set or the Julia set respectively. Attracting periodic points only occur in the Fatou set, while repelling and rationally indifferent periodic points only exist in the Julia set. Irrationally indifferent points can occur in both the Julia set and Fatou set, with very different dynamical behavior occurring depending on which set they fall in.

1.1.2 *Singularities of the inverse function*

There is a very strong connection between the dynamics of a function and the singularities of the inverse function. First, we distinguish the following two kinds of singularities:

- If there exists z such that $f'(z) = 0$, then $f(z)$ is called a *critical value* and z is known as a *critical point*.
- If there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ with $|\Gamma(t)| \to \infty$ as $t \to \infty$ and $f(\Gamma(t)) \to a$ as $t \to \infty$, then a is called an *asymptotic value*. Further, we call Γ an *asymptotic path*.

The critical and asymptotic values are collectively known as the *singular values* of an entire or meromorphic function and sometimes denoted by $\text{sing}(f^{-1})$.

Due to the importance of the singular values in the dynamics of an entire function, two specific classes of functions are routinely studied with restrictions on their set of critical and asymptotic values. The *Speiser class* $\mathcal{S}$ is the class of transcendental entire functions with a finite number of singular values. The *Eremenko-Lyubich class* $\mathcal{B}$ is the class of transcendental entire functions with a bounded set of singular values. Many important results have been proven for each of these classes, as they are in many ways the simplest entire functions to study.

1.1.3 *The Fatou set*

Connected components of the Fatou set, *Fatou components*, can have varied and interesting behavior and have been the subject of major study. Let U be a Fatou component and denote by U_p the component of $F(f)$ in which $f^p(U)$ is contained. A Fatou component U is called *periodic* if there exists a p such that $U_p = U$. However, if U is not periodic, but U_n is periodic for some $n \in \mathbb{N}$, then we say that U is *pre-periodic*. A Fatou component which is not periodic or pre-periodic is called a *wandering domain*. Importantly, wandering domains do not exist for rational functions by a major result of Sullivan [70]. This argument was generalized to functions in the class $\mathcal{S}$ by Eremenko and Lyubich [27], and Goldberg and Keen [33], independently.

If f is an entire function and U is periodic, there is a well known classification of the behavior of the iterates f^n inside the periodic Fatou component U, essentially due to Cremer [24] and Fatou [29, 30, 31]. We have one of the following four possibilities:

- U contains an attracting periodic point z_0 of period p, then $f^{np}(z) \to z_0$ for all $z \in U$ and we call U an *immediate attracting basin* of z_0.
- The boundary of U contains a parabolic periodic point z_0 of period p, where $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$. We call U a *Leau domain* or a *parabolic basin*.
- U contains an irrationally indifferent periodic point and f can be conjugated by an analytic homeomorphism to an irrational rotation in $\mathbb{D}$. In this case, we call U a *Siegel disc*.
- There exists a point $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for all $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case, we call U a *Baker domain*.

Baker domains do not exist for rational functions. Further, if U is a periodic Fatou component of a rational or meromorphic function f, then there is a further possibility that does not occur for entire functions, and so plays no role in this thesis:

- U contains an irrationally indifferent periodic point z_0 of period p and f can be conjugated by an analytic homeomorphism to an irrational rotation in an annulus. In this case, we call U a *Herman ring*.

1.1.4 *The Julia set*

The Julia set has many interesting properties in its own right. Near points in the Julia set, we have what is known as the blowing up property. Any open neighborhood about a point in the Julia set will "blow up" to cover the entire plane except at

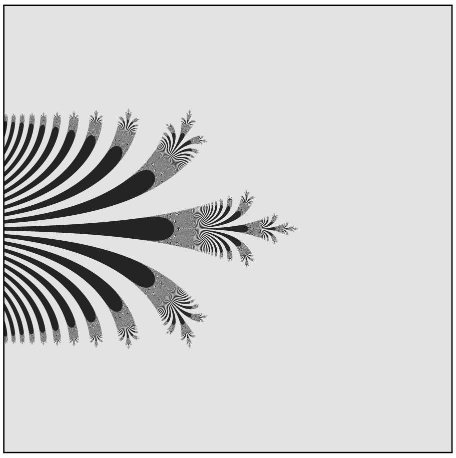

Figure 1.1: The Julia set of Fatou's function, $z + 1 + e^{-z}$, in black. The Fatou set, in gray, is a Baker domain in which points tend to infinity.

most one point. More formally, if U is an open set that intersects the Julia set then there is an exceptional set $E(f)$ so that for any compact set $K \subset \mathbb{C} \setminus E(f)$ there exists $N \in \mathbb{N}$ such that

$$f^n(U) \supset K,$$

for all $n \geqslant N$; see, for instance, [8, Theorem 4.2.5]. Note that for a transcendental entire function, $E(f)$ contains at most one point. As a result, backward iterates of a point in the Julia set are dense in the Julia set. Further, the Julia set is either the entire complex plane, or $J(f)$ has empty interior. For example, the Julia set of the exponential function is the entire complex plane [47]. In general, the Julia set is a perfect set, that is a closed, nonempty set which does not contain any isolated points [10, Theorem 3]. A major result of Baker [3] is that the Julia set contains non-degenerate continua (non-empty compact connected sets). These continua were first studied by Devaney and Tangerman [25] who showed that they take the form of *Cantor bouquets* for functions such as $f(z) = \lambda e^z$ for a suitable value of λ. In fact, the Julia set is a Cantor bouquet for many functions, including the exponential function and Fatou's function $z + 1 + e^{-z}$; see, for example, [55]. Cantor bouquets are loosely a union of curves for which a cross section gives a Cantor set. See Figure 1.1 and Figure 1.2 for two figures illustrating how these sets look.

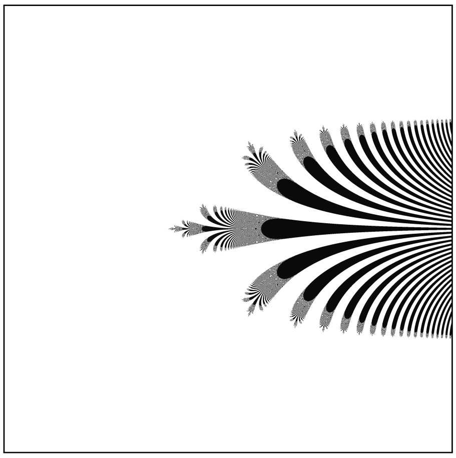

Figure 1.2: The escaping set of $\frac{1}{4}\exp(z)$ in black. The Fatou set, an attracting basin, is in white.

1.2 THE ESCAPING SET

The escaping set

$$I(f) = \{z : f^n(z) \to \infty\}$$

has a critical role in the study of transcendental entire and meromorphic functions. This set was first investigated in earnest by Eremenko [26], who proved many important foundational properties of the escaping set. A significant result in [26] is that the escaping set is always non-empty. In fact, there always exist points that lie in both the escaping set and the Julia set. The proof that the escaping set is non-empty makes use of Wiman-Valiron theory, which describes the behavior near points where the maximum modulus is attained; see Section 1.6.1 for more details. Further, Eremenko showed that the boundary of the escaping set is the Julia set. This gives an explicit link between the escaping set and the Julia set. For many functions, the Julia set is the closure of the escaping set. In particular, Eremenko and Lyubich [27] showed that this is true for all functions in the class $\mathcal{B}$; see the end of Section 1.4. An example of the escaping set for the exponential function is shown in Figure 1.2. A detailed description of the properties of the escaping set for the exponential function is given in [25].

Eremenko also showed that the closure of the escaping set has no bounded components. This leads to arguably the most famous conjecture in the iteration of transcendental entire functions, Eremenko's conjecture. He conjectured first that $I(f)$ cannot have bounded components. Second, that $I(f)$ cannot have bounded path-connected components, that is every point $z \in I(f)$ can be joined to ∞ by a curve in $I(f)$. One can see that the second conjecture is stronger than the first. Rottenfusser, Rückert, Rempe, and Schleicher [63] show that the answer to the second stronger conjecture is negative and in fact in a dramatic way. They construct an entire function whose Julia set has only bounded path-connected components. Moreover, the functions they construct are in the class $\mathcal{B}$. The first conjecture, though, is still open.

1.3 RATES OF ESCAPE

Major progress on Eremenko's first conjecture has been made by considering different rates of escape. A key breakthrough was the following from [60].

Theorem 1.3.1 (Rippon and Stallard, 2005)**.** *If* f *is a transcendental entire function, then* $I(f)$ *has at least one unbounded component.*

This theorem was proved by studying the so called *fast escaping set*, $A(f)$.

1.3.1 *Fast escape*

The fast escaping set was first introduced by Bergweiler and Hinkkanen [13] and can be defined by

$$A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$$

where

$$A_R(f) = \{z : |f^n(z)| \geqslant M^n(R) \text{ for } n \in \mathbb{N}\}$$

as in [60]. Here,

$$M(R) = \max_{|z|=R} |f(z)| \quad \text{for } R > 0$$

denotes the maximum modulus of f at radius R (also denoted $M(R, f)$) and R is such that $M(r) > r$ for all $r \geqslant R$. Note that $M^n(R)$ denotes iteration of $M(R)$.

In a sense, points in $A(f)$ escape as fast possible, as the orbit of a point in this set always beats the maximum modulus after some number of iterations. For many functions, $J(f)$ is a Cantor bouquet for which all the curves belong to $A(f)$, apart from some of the endpoints (see [55]). An example of such a function is $f(z) = \frac{1}{4}e^z$, see Figure 1.2.

The fast escaping set also has many interesting properties. For example, $J(f) \cap A(f) \neq \emptyset$ and $J(f) = \partial A(f)$. Note that Eremenko's original proof in [26] that $I(f)$ is non-empty, in fact, shows that $J(f) \cap A(f)$ is non-empty. Further, in order to prove Theorem 1.3.1, Rippon and Stallard show that all the components of $A(f)$ are unbounded. This leads to the natural question of whether $A(f)$ is the same as $I(f)$, as, while implausible, it would mean a positive answer to Eremenko's conjecture. However, there exist points which escape arbitrarily slowly and are never in the fast escaping set.

1.3.2 *Slow escape*

Motivated by their work on the fast escaping set, Rippon and Stallard [59] proved the following theorem showing there always exist points in the Julia set that escape as slowly as desired.

Theorem 1.3.2. *Let f be a transcendental meromorphic function. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exist*

$$\zeta \in I(f) \cap J(f) \text{ and } N \in \mathbb{N}$$

such that

$$|f^n(\zeta)| \leqslant a_n, \text{ for } n \geqslant N.$$

In Chapter 2, we generalize this result to show that there exist points that escape arbitrarily slowly in a particular type of domain, called a direct tract, provided we have some restrictions on the geometry of this domain.

Rippon and Stallard [59] further show that a two sided slow escape result is possible if there are restrictions on the minimum modulus.

Theorem 1.3.3. *Let f be a transcendental meromorphic function with a finite number of poles. Then f has the property that, for all positive sequences (a_n) such that $a_n \to \infty$ as $n \to \infty$ and $a_{n+1} = O(M(a_n, f))$ as $n \to \infty$, there exist $\zeta \in J(f)$ and $C > 1$ such that*

$$a_n \leqslant |f^n(\zeta)| \leqslant Ca_n, \quad \textit{for } n \in \mathbb{N},$$

if and only if there are positive constants c, d, and r_0 such that $d > 1$ and

$$\textit{for all } r \geqslant r_0 \textit{ there exists } \rho \in (r, dr) \textit{ such that } m(\rho, f) \leqslant c,$$

where

$$m(r, f) = \min_{|z|=r} |f(z)|, \quad \textit{for } r > 0,$$

denotes the minimum modulus.

They also introduce the notion of the *slow escaping set*,

$$L(f) = \{z \in I(f) : \limsup_{n\to\infty} \frac{1}{n} \log |f^n(z)| < \infty\}.$$

This set has many interesting properties similar to the escaping set. In particular:

- $L(f)$ is completely invariant under f,
- $L(f)$ is dense in $J(f)$, and
- $\partial L(f) = J(f)$.

It should be noted that there is a whole alphabet of other sets based on different rates of escape that have been studied, such as the quite fast escaping set $Q(f)$ studied, for example, in [62] and the set of points that escape faster than iterating a polynomial, $Z(f)$, studied in [56] and introduced in [45]. This idea was generalized even further by Rippon and Stallard [61] to consider sets of points whose orbit lies within a prescribed sequence of annuli, leading to the notion of an *annular itinerary*.

Slow escape has also been studied in other contexts. In particular, by Nicks [49] for quasiregular mappings and by Warren [75] for quasimeromorphic maps.

1.4 DIRECT TRACTS

The name 'tract' to describe a domain on which a function tends to an asymptotic value, say a, inside the domain was first introduced by Valiron [74], who introduced the 'tract of determination a' based on what he called paths of determination. However, these ideas had been studied before, with Boutroux [20] calling the collection of asymptotic paths a 'langue'. These ideas were also studied in different settings with Maclane [43] describing tracts as nested components of $\{z : |f(z)| > R\}$ for functions f defined in the unit disc.

In this section, we first give a more detailed discussion of singularities of the inverse, as well as define direct and logarithmic tracts.

1.4.1 *Singularities of the inverse*

We recall the classification of the singularities of the inverse function due to Iversen [37] as well as the definition of a tract, following terminology found in [16]. Let f be an entire function and consider $a \in \widehat{\mathbb{C}}$. For $R > 0$, let U_R be a component of $f^{-1}(D(a, R))$ (where $D(a, R)$ is the open disc centered at a with radius R with respect to the spherical metric) chosen so that $R_1 < R_2$ implies that $U_{R_1} \subset U_{R_2}$. Then either $\bigcap_R U_R = \{z\}$ for some unique $z \in \mathbb{C}$ or $\bigcap_R U_R = \emptyset$.

In the first case we have that $a = f(z)$ and we have the following two possibilities:

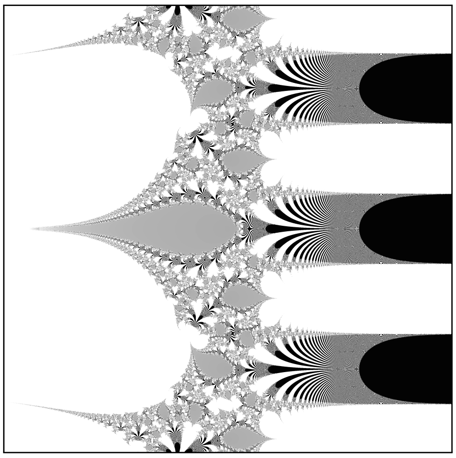

Figure 1.3: The Julia set of $\frac{e^{2z}-1}{e^z-1/z}$. Points in black escape to infinity in a logarithmic tract. The attracting basin of zero is drawn in gray. The remaining points are drawn in white and contain infinitely many Baker domains.

- $f'(z) \neq 0$ and we call a an *ordinary point*.
- $f'(z) = 0$ and we call a a *critical value* of f. We call z a *critical point* of f.

In the second case, f has a *transcendental singularity* over a. The transcendental singularity is called *direct* if $f(z) \neq a$ for all $z \in U_R$, for some $R > 0$. Otherwise it is *indirect*. Further, a direct singularity is called *logarithmic* if $f : U_R \to D(a, R) \setminus \{a\}$ is a universal covering. For example, the exponential function has a logarithmic singularity over 0 and a logarithmic singularity over ∞, while the tangent function has ∞ only as an indirect singularity. The Julia set of a less trivial example is illustrated in Figure 1.3. Note that these U_R are called *tracts* (or a tract over a) for f. More generally, we use the following definition of a tract, as we often restrict f solely to the tract, and do not require that f is entire or even defined outside of it.

Definition 1.4.1. Let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves and suppose that the complement of D is unbounded. Further, let f be a complex valued function whose domain of definition contains the closure $\overline{D}$ of D. Then D is called a *direct tract* of f if f is holomorphic in D, continuous in $\overline{D}$, and if there exists $R > 0$

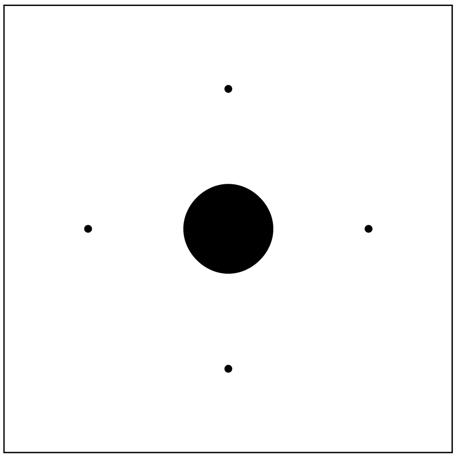

Figure 1.4: The direct tract of $\sin(z)\sinh(z)$ in white for a fixed boundary value.

such that $|f(z)| = R$ for $z \in \partial D$ while $|f(z)| > R$ for $z \in D$. If, in addition, the restriction $f : D \to \{z \in \mathbb{C} : |z| > R\}$ is a universal covering, then D is called a *logarithmic tract*. We further call the value R the *boundary value* of the direct tract.

Every transcendental entire function has a direct tract. Moreover, all direct tracts of a function in the Eremenko-Lyubich class $\mathcal{B}$ are logarithmic tracts, for sufficiently large R in the above. Recall from Section 1.1.2 that the much studied class $\mathcal{B}$ consists of those transcendental entire functions for which the set of critical and asymptotic values is bounded.

The geometry of these direct tracts can be varied and quite wild. In general, direct tracts need not be simply connected and they need not have an unbounded boundary component, instead possibly comprising the plane with holes removed around zeros of the function. For example, the function $\sin(z)\sinh(z)$ in Figure 1.4 has a multiply connected direct tract which is not logarithmic and in fact has no unbounded boundary component.

1.4.2 *Logarithmic tracts*

For many purposes, logarithmic tracts are easier to work with than general direct tracts. Many important results are known for functions with logarithmic tracts. Barański, Karpińska, and

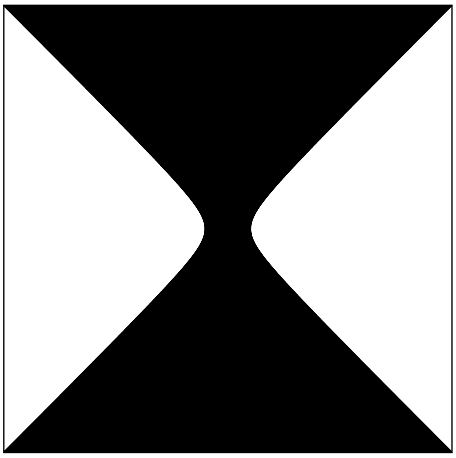

Figure 1.5: The direct tracts of $\exp(z^2)$ in white with boundary value 2.

Zdunik [6] showed that, if a meromorphic function f has a logarithmic tract, then the Hausdorff dimension of the Julia set of f is strictly greater than 1. Rottenfusser, Rückert, Rempe, and Schleicher [63], as well as Bergweiler, Rippon, and Stallard [16], proved results on the structure of the escaping set for functions with a logarithmic tract. Also, one can construct orbits of points that escape slower than any given sequence within a logarithmic tract, as we describe in Chapter 2.

We study logarithmic tracts in detail in Chapter 3 and give geometric conditions on a given tract to be logarithmic and to contain logarithmic tracts. In particular, we show that a simply connected direct tract bounded by a single curve which tends to ∞ at both ends is a logarithmic tract. An example of this is illustrated in Figure 1.5 which shows a function with two logarithmic tracts.

The main tool to study logarithmic tracts is the logarithmic transform, first introduced by Eremenko and Lyubich [27]. Let D be a logarithmic tract, f be holomorphic in D, and suppose that $f(D) = \mathbb{C} \setminus \overline{D(0,1)}$ with $f(0) \in D(0,1)$. Then we can lift via the exponential function in order to obtain a map from $\log D$ to a right half-plane. That is, the following diagram commutes.

$$\begin{tikzcd}
\log D \arrow[r, "F"] \arrow[d, "e^t"'] & H \arrow[d, "e^t"] \\
D \arrow[r, "f"] & \mathbb{C} \setminus \overline{D(0,1)}
\end{tikzcd}$$

Here $\exp(F(t)) = f(\exp(t))$ for $t \in \log D$, F is a conformal isomorphism, and $H = \{z : \operatorname{Re}(z) > 0\}$. We call this map F the *logarithmic transform* of f. Eremenko and Lyubich [27] used this logarithmic transform in order to prove the following useful expansion estimate. Note that in the following lemma we have normalized the function f as above.

Lemma 1.4.2. *Let D be a logarithmic tract of f. Then, for $z \in D$, we have*

$$\left|\frac{zf'(z)}{f(z)}\right| \geqslant \frac{1}{4\pi} \log|f(z)|.$$

This gives more knowledge of the behavior of an entire function in a logarithmic tract than for an arbitrary direct tract and has been a major tool in studying properties of entire functions with logarithmic tracts. For example, this result was used by Eremenko and Lyubich [27] to show that $I(f) \subset J(f)$ for $f \in \mathcal{B}$.

1.5 HAUSDORFF DIMENSION AND THE PRESSURE FUNCTION

The *Hausdorff dimension* (sometimes known as the Hausdorff–Besicovitch dimension) was first introduced by Felix Hausdorff in order to assign a dimension to fractal sets that better describe them than an integer dimension. Loosely speaking, the Hausdorff dimension measures how well a set can be covered by discs. For a basic introduction to the Hausdorff dimension and its properties, see [28] and [19]. For a more in depth discussion of the Hausdorff dimension related to transcendental entire and meromorphic functions, see [69].

1.5.1 *Hausdorff dimension definition*

Following [28], let K and U_i for $i \in \mathbb{N}$ be non-empty subsets of $\mathbb{R}^n$. If $K \subset \cup_{i=1}^{\infty} U_i$ with $0 < |U_i| \leqslant \delta$ for each i, then we call $\{U_i\}$ a δ-*cover* of K. We then define

$$\mathcal{H}^s_\delta(K) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } K\right\},$$

where $\delta > 0$ and s is a non-negative number. We then decrease the possible covers of K by letting δ tend to 0. We call

$$\mathcal{H}^s(K) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(K)$$

the *s-dimensional Hausdorff measure* of K.

As s increases there is a threshold where $\mathcal{H}^s(K)$ changes from being infinite to taking the value 0. The value of s at which this change takes place is called the *Hausdorff dimension* of K and is defined to be

$$\dim(K) = \inf\{s : \mathcal{H}^s(K) = 0\}.$$

1.5.2 *Hausdorff dimension of the Julia set of entire functions*

The Hausdorff dimension of the Julia set, $\dim J(f)$, of transcendental entire and meromorphic functions has been widely studied. Misiurewicz [47] proved that the Julia set of the exponential function is the entire plane, and hence $\dim J(f) = 2$ in this case. Major work was done by McMullen [45], who gave examples for which $\dim J(f) \cap I(f) = 2$ but the Julia set is not the entire plane. It was shown by Baker [3] that the Julia set of every transcendental entire function contains a continuum and hence $\dim J(f) \geqslant 1$. It was further shown by Stallard [68] that for each $d \in (1, 2)$ there exists a transcendental entire function for which $\dim J(f) = d$.

More is known if one restricts to a specific class of entire functions. Many results have been proven for functions in the Eremenko–Lyubich class $\mathcal{B}$. For example, if f is a function of finite order in the class $\mathcal{B}$, then $I(f)$ ($\subset J(f)$) has Hausdorff dimension 2; see [4] and [65]. Stallard [67] proved that for all entire functions in the Eremenko–Lyubich class $\mathcal{B}$, $\dim J(f) > 1$. This was then improved by Barański, Karpińska, and Zdunik [6] who showed that, for a meromorphic function with a logarithmic tract, the Hausdorff dimension of the set of points in the Julia set with bounded orbit is strictly greater than one. This contrasts with Stallard's proof which made use of escaping points.

Until recently it was an open question as to whether there exists an entire function f for which the Hausdorff dimension of the Julia set of f is equal to 1. However, Bishop [18] has constructed an entire function f with $\dim J(f) = 1$.

The Hausdorff dimension of Julia sets of even simple entire functions can have several surprising and paradoxical properties. Recall that for λe^z where $0 < \lambda < 1/e$, the Julia set is a Cantor Bouquet. Karpińska ([38] and [39]) proved that for these functions, the Hausdorff dimension of the Julia set is 2, although, when one removes the endpoints of the Cantor bouquet, the Hausdorff dimension becomes 1. In a sense, all the dimension is contained within the endpoints.

The Hausdorff dimension of points with bounded orbit has not been studied as much as that of escaping points. The main results are due to Barański, Karpińska, and Zdunik [6] (as described above) and Bergweiler [11]. Bergweiler proved that the Hausdorff dimension of the set of points with bounded orbits in the Julia set of an entire function is positive. He further showed this is best possible and there exist transcendental entire functions for which the Hausdorff dimension of the points that have a bounded orbit can be arbitrarily small.

1.5.3 *The pressure function*

Tools from thermodynamic formalism have been applied with great success in the study of dimensions related to entire functions. In many cases it is possible to construct an *iterated function system* of a family of contractions for which the invariant set is part of the Julia set.

A family of contractions $S_i : D \to D$, $1 \leqslant i \leqslant m$, is called an iterated function system, where D is a closed subset of $\mathbb{R}^n$. It is known that for such a system there exists a unique non-empty compact set $F \subset D$ such that $F = \bigcup_{i=1}^{m} S_i(F)$. This set is known as the invariant set for the system. For an introduction to iterated function systems and their properties, see [28]. In the context of

Julia sets, the contractions are taken to be branches of inverse functions.

Bowen's formula says that the Hausdorff dimension of the invariant set F is the unique zero of what is known as the *pressure function*

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\|^{-t}$$

where $g^n = S_{i_1} \circ \ldots \circ S_{i_n}$ for $S_{i_k} \in \{S_i : 1 \leqslant i \leqslant m\}$. See [51] and [64] for an introduction to the pressure function in a general context including the case for rational functions.

Major work by Kotus, Rempe-Gillen, Urbański, Zdunik, and others (see, for example [73], [53] and [42]) has gone into generalizing these tools for use in the case of transcendental entire and meromorphic maps. In fact, Bowen's formula holds for a specific subset of the Julia set, the *radial Julia set*. The radial Julia set, for certain functions, consists of those points in the Julia set which don't escape to infinity, see for example [54]. In Chapter 4, we use the notion of pressure in order to obtain a new result on the Hausdorff dimension of the Julia set for a large class of entire functions whose direct tracts satisfy certain properties.

1.6 TECHNIQUES AND TOOLS IN COMPLEX ANALYSIS

Many different techniques from complex analysis are used throughout this thesis in order to prove the main theorems. Not only are tools from complex analysis used in complex dynamics, but results in complex dynamics can lead to major results in complex analysis. For example, Wiman–Valiron theory was first used in complex dynamics in order to prove fundamental properties of the escaping set. This theory was then improved in work motivated by complex dynamics.

1.6.1 *Wiman–Valiron theory*

Wiman-Valiron theory was introduced by Anders Wiman [77] and began as a tool to study entire functions near points where the function attains the maximum modulus. The theory describes how well a function may be approximated to look like the largest terms in its power series. Wiman [77] obtained a fundamental result on the asymptotic behavior of an entire function near these points. He showed that for an entire function f, $\tau > 1/2$, and arbitrarily large r,

$$f(z) = (1 + o(1))(\frac{z}{z_r})^{n(r,f)} f(z_r),$$

for $z \in D(z_r, r/n(r,f)^\tau)$, $|z_r| = r$, and $|f(z_r)| = M(r,f)$. Here the power $n(r,f)$ is known as the *central index* and corresponds to the maximal term in the power series of f. These tools were then used to prove important covering results by Valiron [74]. Macintyre [44] in his theory of flat regions of entire functions, showed that the central index can be replaced by

$$a(r,f) = \frac{rM'(r,f)}{M(r,f)}$$

in the above. Here, if $\nu : \mathbb{C} \to [0,\infty)$ is a non-constant subharmonic function, then we define the function

$$B(r,\nu) = \max_{|z|=r} \nu(z)$$

which is increasing and tends to ∞ as r tends to ∞. So, we again define

$$a(r,\nu) = \frac{dB(r,\nu)}{d\log r} = rB'(r,\nu) = \frac{rM'(r,f)}{M(r,f)}.$$

This quantity exists except for possibly a countable set of r values and is non-decreasing. In our case, we further define

$$\nu(z) = \log\frac{|f(z)|}{R}, \tag{1.6.1}$$

for $z \in D$ and $\nu(z) = 0$ elsewhere. Bergweiler, Rippon and Stallard [16] showed that the above estimate holds in any direct tract with the following theorem. In the following, we say that a set E has *finite logarithmic measure* if $\int_E dt/t < \infty$ for $E \subset [1,\infty)$.

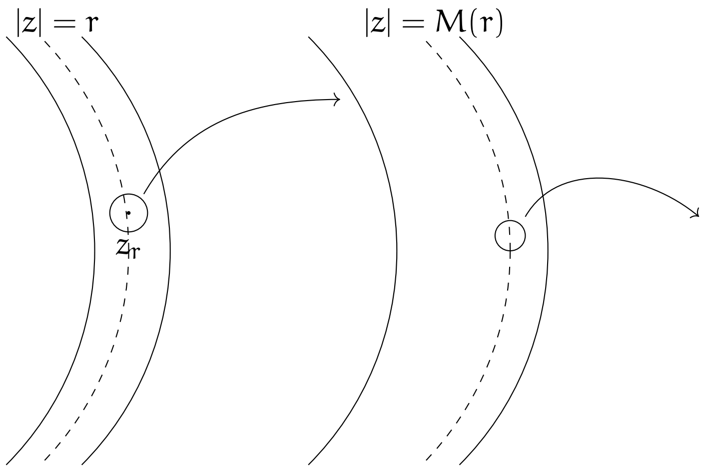

Figure 1.6: Construction of a point in $I(f)$.

Theorem 1.6.1 ([16], Theorem 2.2). *Let* D *be a direct tract of* f *and let* $\tau > \frac{1}{2}$. *Let* ν *be defined as in* (1.6.1) *and let* z_r *be a point satisfying* $|z_r| = r$ *and* $\nu(z_r) = B(r, \nu)$. *Then there exists a set* $E \subset [1, \infty)$ *of finite logarithmic measure such that if* $r \in [1, \infty) \setminus E$, *then* $D(z_r, r/a(r, \nu)^{\tau}) \subset D$. *Moreover,*

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,\nu)} f(z_r), \quad \textit{for } z \in D(z_r, r/a(r, \nu)^{\tau}),$$

as $r \to \infty$, $r \notin E$.

This result gives the existence of a large disc wholly contained inside a direct tract. In Chapter 4, we show a much larger disc is possible for many tracts with an unbounded boundary component. Wiman–Valiron theory has had a major impact on the iteration of entire functions. Eremenko [26] first proved the existence of points that escape to infinity, that is that $I(f)$ is nonempty, using Wiman–Valiron theory. His construction involves showing that a disc covers a large annulus which itself contains another disc and generating a sequence of domains that map onto each other tending to infinity; see Figure 1.6. The existence of these discs is explicitly given by the tools involved in Wiman–Valiron theory. In fact, this proof gives points in the fast escaping set.

Another important result of Wiman–Valiron theory concerns the derivative near points where the maximum modulus is attained. If $k \in \mathbb{N}$, then it follows from Theorem 1.6.1 that

$$f^{(k)}(z) \sim \left(\frac{a(r,\nu)}{z}\right)^{k} \left(\frac{z}{z_r}\right)^{a(r,\nu)} f(z_r), \quad \text{for } z \in D\left(z_r, \frac{r}{a(r,\nu)^{\tau}}\right)$$

as $r \to \infty$ for r outside some exceptional set. This asymptotic estimate on the derivative has been applied to differential equations.

1.6.2 *Hyperbolic distance*

The hyperbolic distance plays an important part in the tools we use to study direct tracts, especially in Chapter 2.

Let $\mathbb{D}$ be the unit disc. The *hyperbolic distance* on $\mathbb{D}$ is defined by

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{z_1}^{z_2} \frac{|dz|}{1 - |z|^2}$$

where this infimum is taken over all smooth curves γ joining z_1 to z_2 in $\mathbb{D}$. Now, the *hyperbolic density* on $\mathbb{D}$ is

$$\sigma_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2}.$$

Note that in some cases one defines the hyperbolic distance and the hyperbolic density with a 2 in the numerator, as this has implications on the curvature.

If $D \neq \mathbb{C}$ is a simply connected domain, then we can take a conformal map $\phi : D \to \mathbb{D}$ and set

$$\rho_D(z_1, z_2) = \rho_{\mathbb{D}}(\phi(z_1), \phi(z_2)).$$

For a general hyperbolic Riemann surface or hyperbolic domain D, we can consider the hyperbolic density $\sigma_D(z)$ which is obtained by use of the universal cover of D; see [1]. Then, the length of a curve in D is given by

$$l(\gamma) = \int_\gamma \sigma_D(z)|dz|$$

and the hyperbolic distance between two points in D is given by taking the infimum over those curves joining these two points. More simply, the hyperbolic distance between two points becomes large as one moves towards the boundary of the domain. A more detailed discussion of the hyperbolic distance and hyperbolic geometry is given in both Keen and Lakic [40], and Beardon and Minda [7].

We will also need to make use of the contraction property of the hyperbolic metric when one domain, or more generally a Riemann surface, is mapped into another. This theorem is commonly known as Pick's theorem; see [22, Theorem I.4.1].

Theorem 1.6.2. *Suppose* f *maps a hyperbolic Riemann surface* R *holomorphically into a hyperbolic surface* S. *Then*

$$\rho_S(f(z_1), f(z_2)) \leqslant \rho_R(z_1, z_2), \ z_1, z_2 \in R,$$

with strict inequality unless f *lifts to a Möbius transformation mapping* $\mathbb{D}$ *onto* $\mathbb{D}$.

The hyperbolic metric also has a bound on the complex plane missing out at least two points, which we use in Chapter 2. This bound can be found in [22] and in the following more precise form as a consequence of a theorem of Landau in [35, Theorem 9.13].

Theorem 1.6.3. *We have*

$$\sigma_{\mathbb{C}\setminus\{0,1\}}(z) \geqslant \frac{1}{2|z|(|\log|z|| + A_0)}$$

with $A_0 = 10\pi$.

1.6.3 *Harmonic measure*

Now we move on to defining the harmonic measure and introducing some results which will be useful in bounding values of a function defined in the direct tract. Let u be a real valued function in a domain D in $\mathbb{C}$ where u has continuous second order partial derivatives. If

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for all $z \in D$, then u is called *harmonic* in D. We note that if f is analytic in D, then $\log|f|$ is harmonic in $D \setminus \{z : f(z) = 0\}$.

Especially important when studying the level curves, level set, and direct tracts of a transcendental entire or meromorphic function is knowledge of the maximum of a harmonic function. This brings us to the maximum modulus principle for harmonic functions; see, for example, [1].

Theorem 1.6.4. *Let u be harmonic in a domain D and $K \subset D$ be compact. Then,*

$$u(z) \leqslant \sup_{\zeta \in \partial K} u(\zeta), \textit{ for } z \in K,$$

with equality if and only if u is constant.

The next two basic results are fundamental theorems concerning harmonic functions in the disc and may be found in [32]. In the following, λ is the Lebesgue linear measure normalised to give length 1 to the bounding circle. The first theorem gives a way of expressing a harmonic function in terms of its boundary values using what is known as the *Poisson kernel*.

Theorem 1.6.5. *Let u be harmonic in a domain D containing $\overline{D(z_0, r)}$. Then*

$$u(z) = \int_{|\zeta - z_0| = r} P_\zeta(z) u(\zeta) d\lambda(\zeta)$$

for $z \in D(z_0, r)$*, where*

$$P_\zeta(z) = \frac{r^2 - |z - z_0|^2}{|z - \zeta|^2}.$$

The second theorem is a specialized case of Theorem 1.6.5. This theorem is commonly known as the mean value equality, which states that the value of a harmonic function at the center of a disc is the integral of the function around the boundary of the disc.

Theorem 1.6.6. *Let* u *be harmonic in a domain containing* $D(z_0, r)$*. Then*

$$u(z_0) = \int\limits_{|\zeta - z_0| = r} u(\zeta) d\lambda(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

Next, let D be a bounded, open, simply-connected plane set whose boundary consists of a finite number of analytic arcs and $E \subseteq \partial D$ be such that E', the boundary of E with respect to ∂D, is null with respect to D, that is, has measure 0; see [32]. Set

$$U(\zeta) = \begin{cases} 1 & \zeta \in E \\ 0 & \zeta \in \partial D \setminus E. \end{cases}$$

Then U is continuous at each point of $\partial D \setminus E'$ and thus there is a unique function $\omega(z, E, D)$ which is harmonic and bounded in D such that

$$\lim_{z \to \zeta} \omega(z, E, D) = U(\zeta), \quad \zeta \in \partial D \setminus E'.$$

The function $\omega(z, E, D)$ is called the *harmonic measure* of E with respect to D at z. The harmonic measure can also be thought of as the probability of a random walk starting at z hitting the boundary of D in the set E.

We now state two estimates related to the harmonic measure which we will make use of in Chapter 4. First, following [16], fix a domain D and let $C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$ be a circle which intersects D. Then, we denote by $r\theta(a, r)$

the linear measure of the intersection of D and $C(a, r)$ and let $r\theta^*(a, r) = r\theta(a, r)$ if $C(a, r) \not\subset D$. Otherwise, $r\theta^*(a, r) = \infty$. One important estimate on the harmonic measure comes from [71, pp 112] and relates the harmonic measure to the linear measure of the intersection of a domain and a circle.

Lemma 1.6.7. *Let $D \subset \mathbb{C}$ be a domain, $a \in D$ and $r > 0$. Let V be the component of $D \cap D(a, r)$ that contains a and let $\Gamma = \partial V \cap C(a, r)$. For $0 < \kappa < 1$ we then have*

$$\omega(a, \Gamma, V) \leqslant \frac{3}{\sqrt{1-\kappa}} \exp\left(-\pi \int_0^{\kappa r} \frac{dt}{t\theta^*(a, t)}\right).$$

Finally, we state the two constants theorem from [16, Lemma 10.2] (see also [52, Theorem 4.3.7]).

Lemma 1.6.8. *Let V be a bounded domain with piecewise smooth boundary. Let Σ be a subset of ∂V consisting of finitely many boundary arcs and let m, M be real constants with $m < M$. Suppose that $u : \overline{V} \to [-\infty, \infty)$ is continuous in $\overline{V}$ and subharmonic in V. Suppose also that $u(z) \leqslant M$ for all $z \in \overline{V}$ and that $u(z) \leqslant m$ for $z \in \Sigma$. Then*

$$u(z) \leqslant \omega(z, \Sigma, V)m + (1 - \omega(z, \Sigma, V))M.$$

1.7 STRUCTURE OF THE THESIS

This thesis studies properties of entire functions with direct tracts. It is divided in the following way.

First, in Chapter 2, we study the iterates of points within a direct tract. We show that for many functions, there exist points in the Julia set that escape arbitrarily slowly in a direct tract. This is accomplished by first proving several covering lemmas. The first covering lemma and results deal with general logarithmic tracts. These results include showing that there exist points that escape as slowly as desired, as well as giving the ability to further control orbits to tend in modulus to any reasonable given sequence. We then generalize to direct tracts with a well controlled boundary. Using techniques involving the hyperbolic distance we obtain a covering result that enables us to prove the existence of points that escape to infinity arbitrarily slowly.

In Chapter 3, we investigate the properties of logarithmic tracts. First, we show that a direct tract bounded by a single curve is a logarithmic tract. We then give geometric conditions for a direct tract to contain logarithmic tracts and asymptotic values, as well as count the number of critical points. Finally, we use these results to give an example of a function in the class $\mathcal{B}$ with infinitely many direct singularities, but no logarithmic singularities over any finite value.

In Chapter 4, a new Wiman–Valiron type estimate is given, showing that for many functions there exists a disc inside a direct tract that is much larger than that given by Wiman-Valiron theory. We are further able to obtain an estimate for the function inside this larger disc. This estimate agrees with the Wiman–Valiron estimate on the Wiman–Valiron disc, but is weaker outside this disc. We use this new machinery in order to prove that the Hausdorff dimension of the points in the Julia set with bounded orbit is strictly greater than 1 for these functions. We then give examples to illustrate these new results.

Finally, in Chapter 5, we look at possible future directions in which this work could be taken.

2

SLOW ESCAPE IN TRACTS

2.1 INTRODUCTION

This chapter concerns points which escape to infinity within a direct tract D. Such points were first studied by Bergweiler, Rippon, and Stallard [16], who considered fast escaping points in a direct tract and defined

$$A(f, D, \rho) = \{z \in D : f^n(z) \in D \text{ and } |f^n(z)| \geqslant M_D^n(\rho) \text{ for } n \in \mathbb{N}\},$$

where

$$M_D(\rho) = \max_{|z|=\rho,\ z \in D} |f(z)|.$$

They proved that $A(f, D, \rho) \neq \emptyset$ and that all components of $A(f, D, \rho)$ are unbounded. (The definition of a tract is given in Section 1.4.)

Here we consider points which escape to infinity *slowly* within a direct tract. Recall (see Theorem 1.3.2) that Rippon and Stallard [59, Theorem 1] showed that, for any transcendental meromorphic function, there are always escaping points that are not fast escaping. In fact, they showed that there exist points in $J(f)$ that escape arbitrarily slowly. They further proved a two-sided slow escape result, Theorem 1.3.3 ([59, Theorem 2]), showing that, for many transcendental meromorphic functions, the orbit of a slow escaping point can be controlled to lie between two constant multiples of a specified sequence.

The aim of this chapter is to generalize these two results to prove the existence of points which escape arbitrarily slowly within certain tracts. If there is only one tract, then this result follows directly from Rippon and Stallard's theorem. However, if there is more than one tract, then all we know from their theorem is that there is a point which escapes suitably slowly within the union of these tracts. We begin by proving the existence of points which escape arbitrarily slowly within any prescribed *logarithmic* tract.

Theorem 2.1.1. *Let* f *be a transcendental entire function with a logarithmic tract* D*. Then, given any positive sequence* (a_n) *such that* $a_n \to \infty$ *as* $n \to \infty$*, there exists*

$$\zeta \in I(f) \cap J(f) \cap \overline{D} \text{ and } N \in \mathbb{N}$$

such that

$$f^n(\zeta) \in \overline{D}, \text{ for } n \geqslant 1,$$

and

$$|f^n(\zeta)| \leqslant a_n, \text{ for } n \geqslant N.$$

We also prove a two-sided slow escape result for logarithmic tracts. Note that this result is stronger than Rippon and Stallard's two-sided slow escape result in that the modulus of the iterates can be controlled to both lie between a given sequence and any given constant multiple of that sequence, and further converge to the given sequence.

Theorem 2.1.2. *Let* f *be a transcendental entire function with a logarithmic tract* D*. Then, given any positive sequence* (a_n) *such that* $a_n \to \infty$ *as* $n \to \infty$ *and satisfying* $a_{n+1} = O(M_D(a_n))$ *as* $n \to \infty$ *there exists*

$$\zeta \in J(f) \cap \overline{D} \text{ and } N \in \mathbb{N},$$

such that

$$f^n(\zeta) \in \overline{D}, \text{ for } n \geqslant 1,$$

and

$$1 \leqslant \frac{|f^n(\zeta)|}{a_n} \leqslant 1 + o(1), \text{ for } n \geqslant N.$$

It follows from the proofs that both Theorem 2.1.1 and Theorem 2.1.2 readily generalize to a prescribed orbit through a finite number of tracts. More care must be taken with an infinite number of tracts. However, with a few minor restrictions on the growth of the given sequence compared to the location of the tracts, these two results still follow from the proofs of Theorem 2.1.1 and Theorem 2.1.2.

The proofs of the main results on the existence of points that escape arbitrarily slowly in [59] rely on certain annulus covering results. However, the techniques used there do not readily generalize to the case of a tract. The proofs of Theorem 2.1.1 and Theorem 2.1.2 instead rely on an annulus covering result obtained by using a derivative estimate within a logarithmic tract due to Eremenko and Lyubich [27]. In the case where a tract is not logarithmic, this derivative estimate need not hold and in Section 2.4 we show that if the boundary of the tract is suitably well behaved in a certain precise sense, then we can prove a covering result by using the harmonic measure of a component of the boundary. We say that such tracts have 'bounded geometry with respect to harmonic measure' (see Definition 2.4.2) and the covering result enables us to prove the following.

Theorem 2.1.3. *Let f be a transcendental entire function and let D be a tract of f with bounded geometry with respect to harmonic measure. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exists*

$$\zeta \in I(f) \cap J(f) \cap \overline{D} \text{ and } N \in \mathbb{N}$$

such that

$$f^n(\zeta) \in \overline{D}, \textit{ for } n \geqslant 1,$$

and

$$|f^n(\zeta)| \leqslant a_n, \textit{ for } n \geqslant N.$$

The organization of this chapter is the following. Section 2.2 is devoted to giving a preliminary result on constructing slow escaping points in general. Section 2.3 focuses on proving Theorem 2.1.1 and Theorem 2.1.2. Section 2.4 focuses on proving Theorem 2.1.3. Finally, Section 2.5 gives two examples of tracts to which our results can be applied.

2.2 CONSTRUCTING SLOW ESCAPING POINTS

In order to prove our results on slow escaping points, the following basic topological lemma (see, for example, [59, Lemma 1]) is used to obtain an orbit passing through a sequence of specified compact sets.

Lemma 2.2.1. *Let E_n, $n \geqslant 0$, be a sequence of nonempty compact sets in $\mathbb{C}$ and $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a continuous function such that*

$$f(E_n) \supset E_{n+1}, \text{ for } n \geqslant 0.$$

Then there exists ζ such that $f^n(\zeta) \in E_n$, for $n \geqslant 0$.

This lemma enables us to give the following general recipe for constructing slow escaping points. Note that we will only apply Theorem 2.2.2 with $m_j = n_j = j$ for $j \in \mathbb{N}$.

Theorem 2.2.2. *Let f be a transcendental entire function and D be an unbounded domain. Suppose there exists a sequence of distinct bounded open sets $\Sigma_n \subset D$ such that $\min\{|z| : z \in \overline{\Sigma}_n\} \to \infty$ as $n \to \infty$ and, for each $n \in \mathbb{N}$,*

$$f(\overline{\Sigma}_n) \supset \overline{\Sigma}_{n+1}. \tag{2.2.1}$$

Further suppose that there exist increasing sequences (n_j) and (m_j) such that,

$$f(\overline{\Sigma}_{n_j}) \supset \overline{\Sigma}_{m_j}, \tag{2.2.2}$$

with $0 \leqslant m_j \leqslant n_j$, $j \in \mathbb{N}$, and $m_j \to \infty$ as $j \to \infty$. Then, given any positive sequence (a_n) such that $a_n \to \infty$ as $n \to \infty$, there exists

$$\zeta \in I(f) \cap J(f) \cap \overline{D} \text{ and } N \in \mathbb{N},$$

such that

$$f^n(\zeta) \in \overline{D}, \text{ for } n \geqslant 1,$$

and

$$|f^n(\zeta)| \leqslant a_n, \text{ for } n \geqslant N.$$

Proof. We will choose a new sequence of open sets from (Σ_n) where we repeat some blocks of the sequence of sets Σ_n suitably

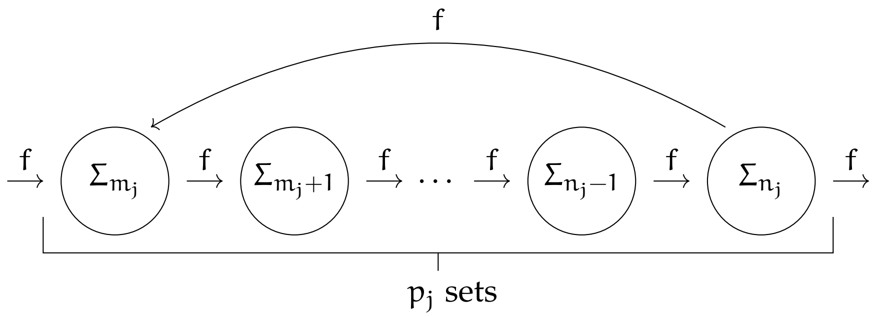

Figure 2.1: Construction of the sets in Theorem 2.2.2.

often in order to hold up the rate of escape of an orbit that passes through each of these sets in turn. The jth block is illustrated in Figure 2.1, where the cycle $\Sigma_{n_j}, \Sigma_{m_j}, \ldots, \Sigma_{n_j-1}$ is repeated q_j times.

Now, consider the sequence (Q_j) with $Q_0 = 0$ and with $Q_j = q_1p_1 + \ldots + q_jp_j$, for $j \geqslant 1$. Here $p_j = n_j - m_j + 1$ is the length of the jth repeated block we will use and q_j, the number of repeats of the jth block, will be chosen to give a desired rate of escape. Next, define

$$E_k = \begin{cases} \overline{\Sigma}_k & \text{for } 0 \leqslant k \leqslant n_0, \\ \overline{\Sigma}_{k-Q_{j-1}} & \text{for } n_{j-1} + Q_{j-1} \leqslant k \leqslant n_j + Q_{j-1},\ j \geqslant 1, \\ \overline{\Sigma}_{m_j+i} & \begin{array}{l} \text{for } n_j + Q_{j-1} < k < n_j + Q_j, \\ \text{and } i \equiv k - (n_j + Q_{j-1} + 1) \pmod{p_j},\ 0 \leqslant i < p_j. \end{array} \end{cases} \tag{2.2.3}$$

Therefore, by (2.2.1) and (2.2.2), $f(E_k) \supset E_{k+1}$ for $k \geqslant 0$. So, by Lemma 2.2.1 there exists $\zeta \in E_0$ such that

$$f^k(\zeta) \in E_k, \text{ for } k \in \mathbb{N}, \tag{2.2.4}$$

and hence

$$\zeta \in I(f) \cap \overline{D} \text{ and } f^n(\zeta) \in \overline{D} \text{ for } n \in \mathbb{N}.$$

Now, note that we can assume (a_n) is an increasing sequence. Choose a sequence $N_j \to \infty$ such that

$$\max\{|z| : z \in \overline{\Sigma}_n, n \leqslant n_j\} \leqslant a_{N_j}, \text{ for } j \in \mathbb{N}.$$

Further, choose (q_j) such that $n_{j-1} + Q_{j-1} \geqslant N_j$ for j sufficiently large. Then, by (2.2.3) and (2.2.4),

$$|f^k(\zeta)| \leqslant a_{N_j} \leqslant a_{n_{j-1}+Q_{j-1}} \leqslant a_k,$$

for $n_{j-1} + Q_{j-1} \leqslant k < n_j + Q_j$, and j sufficiently large.

Finally, we show that we can ensure that $\zeta \in J(f)$. Suppose that $E_n \subset F(f)$ for some $n \in \mathbb{N}$. Then $E_n \subset I(f)$ by normality, since there exists a point $\zeta \in E_n \cap I(f)$. However, by using a repeated block and Lemma 2.2.1, there also exists a point whose orbit remains bounded, which gives a contradiction. Hence E_n meets $J(f)$ for all n. Further, since $J(f)$ is completely invariant,

$$f(E_n \cap J(f)) \supset E_{n+1} \cap J(f), \text{ for } n \in \mathbb{N}.$$

Hence, by Lemma 2.2.1 we can choose a point $\zeta \in I(f) \cap J(f) \cap \overline{D}$ for which $f^n(\zeta) \in \overline{D}$, for $n \geqslant 1$, and $|f^n(\zeta)| \leqslant a_n$, for n sufficiently large. □

Note that the only point in the proof where we use that f is a transcendental entire function is in the final step, showing that we can choose ζ to be in the Julia set.

2.3 SLOW ESCAPE IN LOGARITHMIC TRACTS

In this section, we first prove an annulus covering result based on the derivative estimate due to Eremenko and Lyubich [27] in a logarithmic tract, which we gave in Lemma 1.4.2. This then allows us to prove our result on slow escaping points in a logarithmic tract, Theorem 2.1.1, and our two-sided slow escape result, Theorem 2.1.2, by constructing a sequence of compact sets and applying Theorem 2.2.2.

We use Lemma 1.4.2 to estimate the lengths of the images of sections of level curves of f (that is, connected components of $\{z : |f(z)| = R\}$) in $\overline{D}$, and so obtain an annulus covering result. We denote the open annulus $\{z : r < |z| < R\}$ by $A(r, R)$. Note that in the following lemma we have normalized the function f as in Lemma 1.4.2.

Lemma 2.3.1. *Let D be a logarithmic tract as in Section 1.4.2, $c > 1$, and r_0 be sufficiently large that $M_D(r_0) > \exp(\frac{8\pi^2 c}{c-1})$. If $\Sigma = A(r_0, cr_0) \cap D$, then*

$$f(\Sigma) \supset \bar{A}\left(\exp\left(\frac{8\pi^2 c}{c-1}\right), M_D(r_0)\right).$$

Proof. Consider a connected component of $\{z : |f(z)| = R\}$ which lies in $\overline{D}$, and choose a segment $\sigma = \sigma(R)$ of this level curve such that $\sigma \subset \Sigma$, and σ meets both $\{z : |z| = r_0\}$ and $\{z : |z| = cr_0\}$. We will have level curves that fulfill this for all $R \in [1, M_D(r_0)]$, not necessarily in the same component of Σ. Further, denote by $l(\sigma)$ the length of the curve σ and consider the image of σ under f. Then, by Lemma 1.4.2,

$$\begin{aligned}
l(f(\sigma)) &= \int_\sigma |f'(z)|\,|dz| \\
&\geqslant \int_\sigma \frac{1}{4\pi}\left|\frac{f(z)}{z}\right| \log|f(z)|\,|dz| \\
&\geqslant \int_\sigma \frac{1}{4\pi}\frac{R}{cr_0} \log R\,|dz| \\
&= \frac{1}{4\pi}\frac{R}{cr_0} l(\sigma)\log R \\
&\geqslant \frac{1}{4\pi}\frac{R}{cr_0}(cr_0 - r_0)\log R.
\end{aligned}$$

Since f has no critical points on σ, because D is a logarithmic tract, we deduce that $f(\sigma)$ covers the circle of radius R provided that $\frac{1}{4\pi}\frac{R}{cr_0}(cr_0 - r_0)\log R \geqslant 2\pi R$. This holds if we take $R \geqslant \exp(\frac{8\pi^2 c}{c-1})$. Therefore,

$$f(\Sigma) \supset \bar{A}\left(\exp\left(\frac{8\pi^2 c}{c-1}\right), M_D(r_0)\right),$$

as required. □

We are now ready to prove our slow escaping result within a logarithmic tract of a transcendental entire function, Theorem 2.1.1. We will use a version of Lemma 2.3.1 with $c = 2$ in order to construct a sequence of annuli intersected with our tract and then apply Theorem 2.2.2 in order to obtain an orbit that escapes suitably slowly.

Proof of Theorem 2.1.1. Take $r_0 > e^{16\pi^2}$ sufficiently large that $M_D(r) \geqslant 4r$ for $r \geqslant r_0$. Then we can apply Lemma 2.3.1 with $c = 2$ to $\Sigma_0 = A(r_0, 2r_0) \cap D$ to deduce that there exists $r_1 \geqslant 2r_0$ such that

$$f(\Sigma_0) \supset \overline{A}(e^{16\pi^2}, M_D(|r_0|)) \supset \overline{A(r_1, 2r_1) \cap D} = \overline{\Sigma}_1.$$

Further,

$$f(\Sigma_0) \supset \overline{A(r_0, 2r_0) \cap D} = \overline{\Sigma}_0.$$

Repeating this process we obtain a sequence $r_n \to \infty$ and a sequence of open sets, Σ_n, such that

$$f(\overline{\Sigma}_n) \supset \overline{\Sigma}_n \cup \overline{\Sigma}_{n+1}, \quad \text{for } n \geqslant 0.$$

Applying Theorem 2.2.2, we obtain the desired result. □

We can also use Lemma 2.3.1 to prove Theorem 2.1.2, our two-sided slow escape result. To accomplish this, we use the following consequence of a convexity property of $\log M_D(r)$.

Lemma 2.3.2. *Let D be a direct tract. Then there exists $R > 0$ such that, for all $r > R$ and all $c > 1$,*

$$M_D(r^c) \geqslant M_D(r)^c,$$

and thus, for $C > 1$,

$$\lim_{r \to \infty} \frac{M_D(Cr)}{M_D(r)} = \infty.$$

The proof is similar to that of [58, Lemma 2.2] using the convexity of $\log M_D(r)$ with respect to $\log r$ and that $\frac{\log M_D(r)}{\log r} \to \infty$ as $r \to \infty$ [16, Theorem 2.1].

Proof of Theorem 2.1.2. Let $C > 1$ and let (a_n) be the given positive sequence which satisfies $a_n \to \infty$ as $n \to \infty$ and

$$a_{n+1} \leqslant KM_D(a_n), \text{ for } n \geqslant 0, \tag{2.3.1}$$

and for some constant $K > 0$. Take $c \in (1, C)$ and choose $N \in \mathbb{N}$ so large that

$$a_n > \exp(\frac{8\pi^2 c}{c-1}),\ M_D(\frac{C}{c}a_n) > Ca_n, \text{ and } \frac{M_D(Ca_n/c)}{M_D(a_n)} > CK, \tag{2.3.2}$$

for $n \geqslant N$. This is possible since $\frac{M_D(r)}{r} \to \infty$, and $\frac{M_D(Cr/c)}{M_D(r)} \to \infty$ as $r \to \infty$ by Lemma 2.3.2. These conditions will allow us to apply Lemma 2.3.1.

Define $\Sigma_0 = A\left(\frac{C}{c}a_N, Ca_N\right) \cap D$. Note we may assume that $A(a_n, Ca_n) \cap D \neq \emptyset$ for $n \geqslant N$. Applying Lemma 2.3.1 with $r_0 = \frac{C}{c}a_N$, we obtain, by (2.3.2),

$$f(\Sigma_0) \supset \overline{A}\left(\exp\left(\frac{8\pi^2 c}{c-1}\right), M_D\left(\frac{C}{c}a_N\right)\right) \tag{2.3.3}$$

$$\supset \overline{A\left(a_N, Ca_N\right) \cap D} \tag{2.3.4}$$

$$\supset \overline{\Sigma}_1, \tag{2.3.5}$$

where $\Sigma_1 = A\left(\frac{C}{c}a_N, Ca_N\right) \cap D = \Sigma_0$. Let $\Sigma_n = \Sigma_0$ for all $n = 2, \ldots, N$, so that $f(\Sigma_n) \supset \overline{\Sigma}_{n+1}$ for $n < N$, as in (2.3.3). By Lemma 2.3.1, (2.3.3), and (2.3.1),

$$\begin{aligned} f(\Sigma_N) &\supset \overline{A}\left(\exp\left(\frac{8\pi^2 c}{c-1}\right), M_D\left(\frac{C}{c}a_N\right)\right) \\ &\supset \overline{A(a_{N+1}, Ca_{N+1}) \cap D} \\ &\supset \overline{\Sigma}_{N+1}, \end{aligned}$$

where $\Sigma_{N+1} = A\left(\frac{C}{c}a_{N+1}, Ca_{N+1}\right) \cap D$.

We now apply this argument repeatedly for all $n \geqslant N$ to obtain a sequence of sets $\Sigma_n = A(\frac{C}{c}a_n, Ca_n) \cap D$, for $n \geqslant N$, such that

$$f(\overline{\Sigma}_n) \supset \overline{A(a_{n+1}, Ca_{n+1})} \supset \overline{\Sigma}_{n+1}, \text{ for } n \geqslant 0.$$

By Lemma 2.2.1 there exists a point $\zeta \in \overline{\Sigma}_0$ such that

$$f^n(\zeta) \in \overline{\Sigma}_n, \text{ for } n \geqslant 0.$$

Therefore, there exists a point $\zeta \in \overline{D}$ such that

$$a_n \leqslant |f^n(\zeta)| \leqslant Ca_n,$$

for all $n \geqslant N$.

Next, we show that we can also choose $\zeta \in J(f)$. Since f is bounded on a curve going to ∞, f has no unbounded, multiply connected Fatou components (by [3]), so all the components of $J(f)$ are unbounded (see, for example, [41, Theorem 1]). The image of D contains an unbounded connected set in $J(f)$ and so, by complete invariance, $J(f)$ will meet any annulus with sufficiently large radius intersected with D. Therefore, $\overline{\Sigma}_n$ meets $J(f)$ for all n sufficiently large and so we can choose $\zeta \in \overline{D} \cap J(f)$ such that $a_n \leqslant |f^n(\zeta)| \leqslant Ca_n$, for all n sufficiently large.

Finally, we can infer Theorem 2.1.2 by modifying the above proof by choosing $\Sigma_n = A(\frac{C_j}{c_j}a_n, C_j a_n) \cap D$ for $N_{j-1} < a_n \leqslant N_j$, where $C_j \to 1$ as $N_j \to \infty$, and $c_j \in (1, C_j)$, for $j \geqslant 1$.

□

At each stage in the proofs of Theorem 2.1.1 and Theorem 2.1.2 we may choose a suitable annular intersection with any logarithmic tract as long as it is covered by the previous set. It then follows that Theorem 2.1.1 and Theorem 2.1.2 hold for a prescribed orbit through logarithmic tracts for which the covering holds.

2.4 SLOW ESCAPE IN MORE GENERAL TRACTS

In Section 2.3 we showed that we can obtain points that escape arbitrarily slowly in a logarithmic tract. We can construct points that escape arbitrarily slowly within a more general direct tract, provided that the boundary of the tract is sufficiently well behaved. First, we prove an annulus covering result based on the hyperbolic metric and then we use this to obtain another annulus covering lemma giving conditions on the harmonic measure and function value. Finally, we apply this covering by estimating some function values of points in the tract compared to the hyperbolic distance between them.

The proof of this first lemma uses the contraction property of the hyperbolic metric, which we discussed in Section 1.6.2. Recall that we denote the hyperbolic density at a point z in a domain, or more generally a hyperbolic Riemann surface, Σ by $\sigma_\Sigma(z)$ and the hyperbolic distance between two points z_1 and z_2 on Σ by $\rho_\Sigma(z_1, z_2)$. The result below is a stronger version of a theorem of Bergweiler, Rippon, and Stallard [17, Theorem 3.3].

Lemma 2.4.1. *Let Σ be a hyperbolic Riemann surface. For a given $K > 1$, if $f : \Sigma \to \mathbb{C} \setminus \{0\}$ is holomorphic, then for all $z_1, z_2 \in \Sigma$ such that*

$$\rho_\Sigma(z_1, z_2) < \frac{1}{2}\log\left(1 + \frac{\log K}{10\pi}\right) \quad \textit{and} \quad |f(z_2)| \geqslant K|f(z_1)|$$

we have

$$f(\Sigma) \supset \bar{A}(|f(z_1)|, |f(z_2)|).$$

Proof. Suppose that $\rho_\Sigma(z_1, z_2) < \lambda$ and $|f(z_2)| \geqslant K|f(z_1)|$ for some value of λ to be chosen. Suppose also for a contradiction that there exists some point $w_0 \in \bar{A}(|f(z_1)|, |f(z_2)|) \setminus f(\Sigma)$. By Pick's Theorem (Theorem 1.6.2),

$$\begin{aligned} \rho_\Sigma(z_1, z_2) &\geqslant \rho_{f(\Sigma)}(f(z_1), f(z_2)) \\ &\geqslant \rho_{\mathbb{C}\setminus\{0, w_0\}}(f(z_1), f(z_2)) \\ &= \rho_{\mathbb{C}\setminus\{0,1\}}(f(z_1)/w_0, f(z_2)/w_0). \end{aligned}$$

Let γ be a hyperbolic geodesic in $\mathbb{C} \setminus \{0, 1\}$ from $t_1 = f(z_1)/w_0$ to $t_2 = f(z_2)/w_0$. There exists a segment γ' of γ joining the point t_1' to t_2', where $|t_2'| = K|t_1'|$ and $1 \in A(|t_1'|, |t_2'|)$. This choice is possible since $|f(z_2)| \geqslant K|f(z_1)|$. Hence we have, $t_1', t_2' \in \bar{A}(1/K, K)$.

From Theorem 1.6.3 ([35, Theorem 9.13]), the density of the hyperbolic metric on the domain $\mathbb{C} \setminus \{0, 1\}$ is bounded below by $1/(2|w|(|\log |w|| + 10\pi))$. Hence,

$$\begin{aligned}
\rho_\Sigma(z_1, z_2) &\geqslant \rho_{\mathbb{C}\setminus\{0,1\}}(t_1', t_2') \\
&= \int_{\gamma'} \sigma_{\mathbb{C}\setminus\{0,1\}}(z)|dz| \\
&\geqslant \int_{t_1'}^{t_2'} \frac{|dz|}{2|z|(|\log |z|| + 10\pi)} \\
&\geqslant \int_{|t_1'|}^{|t_2'|} \frac{dr}{2r(|\log r| + 10\pi)} \\
&= \int_{|t_1'|}^{1} \frac{dr}{2r(10\pi - \log r)} + \int_{1}^{K|t_1'|} \frac{dr}{2r(10\pi + \log r)} \\
&= -\frac{1}{2}\log\left(1 - \frac{\log r}{10\pi}\right)\Big|_{|t_1'|}^{1} + \frac{1}{2}\log\left(1 + \frac{\log r}{10\pi}\right)\Big|_{1}^{K|t_1'|} \\
&= \frac{1}{2}\log\left(\left(1 - \frac{\log |t_1'|}{10\pi}\right)\left(1 + \frac{\log K|t_1'|}{10\pi}\right)\right) \\
&= \frac{1}{2}\log\left(1 + \frac{\log K}{10\pi} - \frac{\log |t_1'| \log K|t_1'|}{100\pi^2}\right) \\
&\geqslant \frac{1}{2}\log\left(1 + \frac{\log K}{10\pi}\right), \text{ since } 1/K \leqslant |t_1'| \leqslant 1.
\end{aligned}$$

So, if we set $\lambda = \frac{1}{2}\log\left(1 + \frac{\log K}{10\pi}\right)$, then we reach a contradiction to our initial assumption that $\rho_\Sigma(z_1, z_2) < \lambda$. □

Now, we apply Lemma 2.4.1 to an annulus or bounded domain intersected with a tract for which we can continue applying Lemma 2.4.1 to obtain a slow escape result. This leads us to impose a few extra conditions on the tracts we consider, which we now define.

Definition 2.4.2. Let D be a direct tract of a function f, where $|f(z)| = 1$ on ∂D, for which there exists a sequence Σ_n of quadrilaterals in D tending to ∞ each of which contains a point z_n such that $|f(z_n)| > \max\{|z| : z \in \Sigma_{n+1}\}$ and the harmonic measure

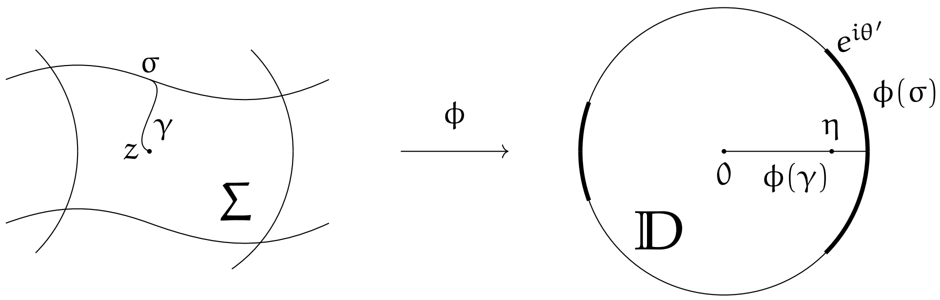

Figure 2.2: Illustration of the map ϕ in Lemma 2.4.3.

in Σ_n at z_n of some connected component of $\partial\Sigma_n \cap \partial D$, σ_n say, is uniformly bounded from below by some positive value. Then, D is said to have *bounded geometry with respect to harmonic measure*.

Using the assumptions of Definition 2.4.2, consider a general quadrilateral Σ_n, called Σ for simplicity, with its associated point $z \in \Sigma$ and the associated set $\sigma \subset \partial\Sigma \cap \partial D$. Then there exists a hyperbolic geodesic γ joining z to σ so that σ is invariant under hyperbolic reflection in γ. Consider the Riemann map $\phi : \Sigma \to \mathbb{D}$ such that $\phi(z) = 0$ and $\phi(\gamma)$ is the interval $[0, 1)$, as illustrated in Figure 2.2. Then 1 is the midpoint of the arc $\phi(\sigma)$. Let $\theta' \in (0, \pi)$ be the infimal angle of $\theta \in (0, \pi)$ for which $|f(\phi^{-1}(e^{i\theta}))| \neq 1$ and take $\eta \in (\cos\theta', 1)$.

In the following $P_\theta(\eta) = (1-\eta^2)/|\eta - e^{i\theta}|^2$ denotes the Poisson kernel of $\mathbb{D}$ with singularity at $e^{i\theta}$.

Lemma 2.4.3. *Let* f, Σ, σ, ϕ, θ', *and* $\eta \in (\cos\theta', 1)$ *be as above and such that*

$$\log|f(\phi^{-1}(0))| > \frac{20\pi\eta}{(1 - P_{\theta'}(\eta))(1-\eta)}. \tag{2.4.1}$$

Then,

$$f(\Sigma) \supset \bar{A}(|f(\phi^{-1}(\eta))|, |f(\phi^{-1}(0))|).$$

Proof. Consider $u(z) = \log|f(\phi^{-1}(z))|$. We estimate the value of the function u at η. Since u is harmonic in $\mathbb{D}$ and vanishes on $\phi(\sigma)$,

$$\begin{aligned} u(\eta) &= \frac{1}{2\pi}\int_0^{2\pi} P_\theta(\eta)u(e^{i\theta})d\theta \\ &\leqslant \frac{P_{\theta'}(\eta)}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\theta \\ &= P_{\theta'}(\eta)u(0), \end{aligned}$$

by the definition of θ' and the mean value theorem. Hence,

$$u(0) - u(\eta) \geqslant (1 - P_{\theta'}(\eta))u(0).$$

Now we let $\log K = (1 - P_{\theta'}(\eta))u(0)$. Then $K > 1$ since $P_{\theta'}(\eta) < 1$. Therefore, by the formula for the hyperbolic distance in [35, page 688],

$$\begin{aligned} \rho_{\mathbb{D}}(0,\eta) = \frac{1}{2}\log\left(\frac{1+\eta}{1-\eta}\right) &< \frac{1}{2}\log\left(1 + \frac{\log K}{10\pi}\right) \\ \iff \left(\frac{1+\eta}{1-\eta}\right) &< 1 + \frac{\log K}{10\pi} \\ \iff 10\pi\left(\frac{2\eta}{1-\eta}\right) &< \log K \\ \iff 10\pi\left(\frac{2\eta}{1-\eta}\right) &< (1 - P_{\theta'}(\eta))u(0) \\ \iff u(0) &> \frac{20\pi\eta}{(1 - P_{\theta'}(\eta))(1-\eta)}. \end{aligned}$$

Hence, by (2.4.1), the conditions of Lemma 2.4.1 are satisfied and the result follows. □

We now apply the previous lemmas in order to prove Theorem 2.1.3. We first estimate the function value inside the tract compared to the hyperbolic distance. We then apply Lemma 2.4.3 and construct a sequence of domains to which we can apply Theorem 2.2.2.

Proof of Theorem 2.1.3. Without loss of generality we can let a_n be any increasing positive sequence such that $a_n \to \infty$ as $n \to \infty$. Let D be a direct tract of f with bounded geometry with respect to harmonic measure. Consider a quadrilateral Σ_n and the point $z_n \in \Sigma_n$ as in Definition 2.4.2. Then by Definition 2.4.2 there exists ε independent of n, σ_n, and θ_n' such that $\theta_n' \geqslant \varepsilon > 0$ for all $n \geqslant 0$. Let $\eta_n \in (\cos\theta_n', 1)$ be chosen so that

$$P_{\theta_n'}(\eta_n) \leqslant \frac{C(\varepsilon)}{\log|f(z_n)|}, \tag{2.4.2}$$

where $C(\varepsilon)$ is a constant, to be chosen, that depends solely on ε. To apply Lemma 2.4.3, we further need to choose η_n such that

$$\log|f(z_n)| > \frac{20\pi\eta_n}{(1-P_{\theta_n'}(\eta_n))(1-\eta_n)}. \tag{2.4.3}$$

So, we are choosing η_n very close to 1 and $P_{\theta_n'}(\eta_n)$ close to 0 for sufficiently large $|f(z_n)|$. We want (2.4.2) and (2.4.3) not to conflict.

We have by definition that $P_{\theta_n'}(\eta_n) = (1-\eta_n^2)/|\eta_n - e^{i\theta_n'}|^2$, so we want both

$$(1-\eta_n)\log|f(z_n)| \leqslant \frac{C(\varepsilon)}{1+\eta_n}|\eta_n - e^{i\theta_n'}|^2$$

and

$$(1-\eta_n)\log|f(z_n)| > \frac{20\pi\eta_n}{1-P_{\theta_n'}(\eta_n)} \approx 20\pi,$$

from (2.4.2) and (2.4.3), to be true. First, since $\cos\theta_n' < \eta_n < 1$, we can observe that

$$|\eta_n - e^{i\theta_n'}| > \sin\theta_n' \geqslant \sin\varepsilon.$$

So, it is sufficient to choose $\eta_n \in (\cos\theta_n', 1)$ such that

$$40\pi \leqslant (1-\eta_n)\log|f(z_n)| \leqslant \frac{1}{2}C(\varepsilon)\sin^2\varepsilon.$$

This choice is possible if we have both $C(\varepsilon) = 160\pi/\sin^2\varepsilon$ and $\log|f(z_n)| \geqslant 40\pi/(1-\cos\varepsilon)$, for $n \geqslant N$ say (for example, take $\eta_n = 1 - 40\pi/\log|f(z_n)|$).

Hence, by Lemma 2.4.3 and Definition 2.4.2,

$$f(\Sigma_n) \supset \overline{\Sigma}_{n+1}, \text{ for } n \geqslant N.$$

Further, we may assume that

$$f(\Sigma_n) \supset \overline{\Sigma}_n, \text{ for } n \geqslant N,$$

since we have $\log|f(\phi^{-1}(\eta_n))| \leqslant P_{\theta_n'}(\eta_n)\log|f(z_n)| < C(\varepsilon)$, by (2.4.2) and the reasoning at the start of the proof of Lemma 2.4.3.

Relabeling, we obtain a sequence of domains of the form Σ_n, such that

$$f(\overline{\Sigma}_n) \supset \overline{\Sigma}_n \cup \overline{\Sigma}_{n+1}, \quad \text{for } n \geqslant 0.$$

Applying Theorem 2.2.2, we obtain the desired result. □

2.5 EXAMPLES

In this section, we give two concrete examples which demonstrate both the kinds of tracts that exist and some methods with which to construct points that escape arbitrarily slowly in these tracts. First, we give an example of a function to which we can apply Lemma 2.4.1 to construct slow escaping points in a direct tract with no logarithmic singularities using a similar method to that in Theorem 2.1.3. In the next chapter, we will show the reciprocal of this function is in the class $\mathcal{B}$, thus giving an example of a function in the class $\mathcal{B}$ with a direct singularity over a finite value, but no logarithmic singularity over any finite value. This example is the reciprocal of the entire function studied in [12]; for an illustration of the tracts of this function, see [12, Figure 1].

Example 2.5.1. Consider the entire function

$$f(z) = \exp(-g(z)), \text{ where } g(z) = \sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}.$$

Then,

1. no direct tract D of f contains any logarithmic tracts, and
2. there exists $z \in D$ such that $f^n(z) \to \infty$ arbitrarily slowly to any given direct singularity of f.

First, we recall from Bergweiler and Eremenko [12, Section 6] that $\exp(g(z))$ has uncountably many direct singularities over 0, but no logarithmic singularity over any finite value. So, f has infinitely many direct singularities over ∞, but no logarithmic singularity over ∞, and hence no tract of f contains any logarithmic tracts. In order to prove that $\exp(g(z))$ has infinitely many direct singularities over 0, but no logarithmic singularity over any finite value, they show that there is an infinite binary tree in $\mathbb{C}$ for which every unbounded path on the tree is an asymptotic path along which $\operatorname{Re} g(z) \to -\infty$ as $z \to \infty$. The direct singularities correspond to different branches of this infinite tree. Note that as any direct tract of f contains multiple branches of this infinite tree, it can not be logarithmic. We shall use the estimates they give in obtaining this result in order to prove (2).

We now introduce the following notation and results from [12, Section 6]. We fix an ε with $0 < \varepsilon \leqslant \frac{1}{8}$ and set $r_n = (1+\varepsilon)2^{n+1}$ and $r_n' = (1-2\varepsilon)2^{n+2}$ for $n \in \mathbb{N}$. Then for $j \in \{0, 1, \cdots, 2^n - 1\}$ we define the sets

$$A_{j,n} = \left\{ r \exp\left(\frac{2\pi i j}{2^n}\right) : r \geqslant r_n \right\}$$

and

$$B_{j,n} = \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n}\right) : r_n \leqslant r \leqslant r_n' \right\}.$$

From [12], we have that both $\operatorname{Re} g(z) > 2^{2^n}$ for $z \in A_{j,n}$ and $\operatorname{Re} g(z) < -2^{2^n}$ for $z \in B_{j,n}$. Further, $\arg g(re^{i\theta})$ is an increasing function of θ, for $r_n \leqslant r \leqslant r_n'$, and it increases by $2^n 2\pi$ as θ increases by 2π.

We show that we have slow escaping points in the tracts of this function and can control the orbits of these points as they escape. To do this we need to further estimate the function in order to apply Lemma 2.4.1.

Fix j and n as above, and consider the annular sector, $\Sigma_{j,n}$, about $B_{j,n}$, bounded on the sides by $A_{j,n}$ to $A_{j+1,n}$ and from r_n to r'_n. That is,

$$\Sigma_{j,n} = \left\{ re^{i\theta} : r_n \leqslant r \leqslant r'_n \text{ and } \frac{2\pi j}{2^n} \leqslant \theta \leqslant \frac{2\pi(j+1)}{2^n} \right\}.$$

This domain will meet exactly one connected component of the tract intersected with $\overline{A}(r_n, r'_n)$, by a counting argument, since $\arg g(re^{i\theta})$ is an increasing function of θ, and it increases by $2^n 2\pi$ as θ increases by 2π. Let $z = r\exp\left(\frac{\pi i}{2^{n+2}} + \frac{2\pi i j}{2^n}\right)$ for $r_n \leqslant r \leqslant r'_n$ and $s = r/2^n \in [2(1+\varepsilon), 4(1-2\varepsilon)]$. Then we can evaluate the following terms of $g(z)$:

$$n\text{th term: } s^{2^n}(\cos(\pi/4) + i\sin(\pi/4)) = s^{2^n}(1/\sqrt{2} + i/\sqrt{2})$$

$$(n+1)\text{st term: } (s/2)^{2^{n+1}}(\cos(\pi/2) + i\sin(\pi/2)) = (s/2)^{2^{n+1}} i$$

$$(n+2)\text{nd term: } (s/4)^{2^{n+2}}(-1) = -(s/4)^{2^{n+2}}$$

$$(n+3)\text{rd term: } (s/8)^{2^{n+3}}.$$

So, we have that, for z as above and where C is some absolute constant,

$$\begin{aligned}
|f(z)| &\leqslant \exp\left(s^{2^{n-1}}(n-1)2^{2^{n-1}} - \frac{s^{2^n}}{\sqrt{2}} - \left(\frac{s}{4}\right)^{2^{n+2}} - \left(\frac{s}{8}\right)^{2^{n+3}} - \dots \right) \\
&\leqslant \exp\left(s^{2^{n-1}} \left((n-1)2^{2^{n-1}} - s^{2^{n-1}}/\sqrt{2} \right) - C \right) \\
&\leqslant \exp\left(s^{2^{n-1}} 2^{2^{n-1}} \left((n-1) - \frac{(1+\varepsilon)^{2^{n-1}}}{\sqrt{2}} \right) - C \right) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

We now choose two points in $\Sigma_{j,n}$:

$$z_1 = \frac{r_n + r'_n}{2} \exp\left(\frac{\pi i}{2^{n+2}} + \frac{2\pi i j}{2^n} \right) \text{ and}$$

$$z_2 = \frac{r_n + r'_n}{2} \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} \right).$$

Then, $\rho_{\Sigma_{j,n}}(z_1, z_2)$ is bounded by some absolute constant. Since $z_2 \in B_{j,n}$, we deduce that, for $n \geqslant N$ say, we have

$$\rho_{\Sigma_{j,n}}(z_1, z_2) < \frac{1}{2}\log\left(1 + \frac{1}{10\pi}\log\frac{|f(z_2)|}{|f(z_1)|}\right).$$

So, by Lemma 2.4.1, $f(\Sigma_{j,n}) \supset \bar{A}(|f(z_1)|, |f(z_2)|) \supset \Sigma_{j,n} \cup \Sigma_{j,n+1}$ for $n \geqslant N$ and so we can apply Theorem 2.2.2 to obtain points that escape as slowly as we wish through any sequence $(\Sigma_{j_n,n})_{n \geqslant N}$, where $j_n \in \{0, 1, \dots, 2^n - 1\}$, and in particular to any direct singularity.

Next, we give an example to show how we can sometimes obtain slow escaping points in a tract that is not simply connected. As the direct tract is not simply connected, we can not apply Theorem 2.1.1 or Theorem 2.1.2. Instead, we use Theorem 2.1.3 and then give an alternative method involving the zeros of the function. Note that the latter method relies on the existence of large critical values.

Example 2.5.2. Consider the function

$$f(z) = e^{z^2}\cos z.$$

Then f has two multiply connected non-logarithmic tracts (shown in Figure 2.3), each of which contains orbits of points that escape arbitrarily slowly.

First, f has zeros at $(2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$. Call the rightmost tract D, where $|f(z)| = 1$ on ∂D. As e^{z^2} has two tracts that are quadrants with one symmetrically containing the positive real axis and the other the negative real axis, and e^{z^2} grows much faster than $\cos|z|$, it is natural to assume the shape of the tract of f will be similar away from the zeros of $\cos(z)$. In fact, it is easy to check that ∂D consists of a Jordan curve passing through ∞ and lying in $\{x + iy : x \geqslant 0, x \geqslant |y|\}$ together with infinitely many bounded Jordan curves surrounding the zeros at $(2n+1)\frac{\pi}{2}$.

Now, let $\Sigma_n = A\left((2n+1)\frac{\pi}{2}, (2n+3)\frac{\pi}{2}\right) \cap D \cap \mathbb{H}$, for $n \geqslant 0$, and where $\mathbb{H}$ denotes the upper half-plane. Each Σ_n is a simply connected domain that contains $z_n = (2n+2)\frac{\pi}{2}\exp(\frac{i\pi}{8})$. This

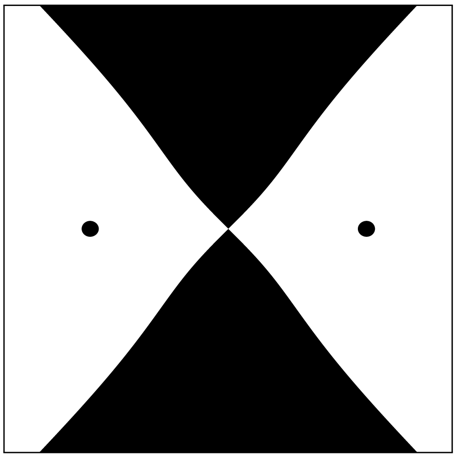

Figure 2.3: The tracts of $e^{z^2}\cos(z)$ with boundary value 1 in white, showing holes around the zeros at $\pm\frac{\pi}{2}$.

choice of z_n ensures that $|f(z_n)| \geqslant (2n+5)\frac{\pi}{2}$. It is not difficult to see that the harmonic measure in Σ_n at z_n of the largest boundary component of $\Sigma_n \cap \partial D$ is uniformly bounded from below with respect to n. This can be seen by viewing Σ_n as a quadrilateral with one side on the boundary of the tract and the ratio of the sides uniformly bounded. Therefore, we can apply Theorem 2.1.3 to obtain slow escaping points in the tract D.

Finally, we outline another method to obtain slow escaping points for this function in the tract D. The critical points of f satisfy $z = \frac{1}{2}\tan(z)$. These occur for large positive z at $z = \frac{(2n+1)\pi}{2} - \varepsilon_n$, where $\varepsilon_n > 0$. For such a point z, we have that $|\cos(z)| = \sin(\varepsilon_n) \approx \varepsilon_n$. We also have

$$\frac{(2n+1)\pi}{2} - \varepsilon_n = \frac{1}{2}\tan\left(\frac{(2n+1)\pi}{2} - \varepsilon_n\right)$$

and

$$\left|\tan\left(\frac{(2n+1)\pi}{2} - \varepsilon_n\right)\right| = \frac{\cos(\varepsilon_n)}{\sin(\varepsilon_n)} \approx \frac{1-\frac{1}{2}\varepsilon_n^2}{\varepsilon_n} \approx \frac{1}{\varepsilon_n}.$$

Hence, $\varepsilon_n \approx \frac{1}{(2n+1)\pi}$. Therefore the critical values of f are approximately

$$\frac{1}{(2n+1)\pi}\exp\left(\frac{(2n+1)^2\pi^2}{4}\right).$$

The images of the level curves around a zero form circles for function values up to at least the critical value of the nearest critical point. By estimating f on a quadrilateral containing such a critical point and a zero, we see that the domain $A\left(2n\frac{\pi}{2}, (2n+2)\frac{\pi}{2}\right) \cap D$ contains all of these level curves and that the critical value of f in this domain has modulus much larger than $(2n+4)\frac{\pi}{2}$. Hence, the conditions for Theorem 2.2.2 are satisfied and we again obtain points that escape arbitrarily slowly in D.

3

IDENTIFYING LOGARITHMIC TRACTS

3.1 INTRODUCTION

As discussed in Chapter 1, logarithmic tracts have much stronger properties than general direct tracts and they have found many applications in complex dynamics. It is therefore useful to be able to identify which direct tracts are logarithmic.

In this chapter, we assume that direct tracts D have the property that all unbounded curves in ∂D tend to ∞ at both ends. We call such direct tracts *regular direct tracts*. Note that any direct tract of a meromorphic function is a regular direct tract, as well as any subtract $\{z : |f(z)| > R'\}$, for $R' > R$, of a general direct tract with boundary value R. In fact, we are not aware of any direct tracts that are not regular.

In this chapter, we give a sufficient condition for a regular direct tract to contain logarithmic tracts. We start with a simple geometric condition for a direct tract to be logarithmic, which perhaps surprisingly does not seem to have been stated previously.

Theorem 3.1.1. *Let D be a regular direct tract whose boundary is an unbounded simple curve. Then D is logarithmic.*

While regular direct tracts bounded by a single curve are logarithmic, logarithmic tracts need not be bounded by a single curve. The function e^{e^z} has direct tracts which are horizontal strips when the boundary value of the direct tract considered is $R = 1$. No critical points lie in these direct tracts and the asymptotic values do not lie in the image of the tracts, so these direct tracts are logarithmic. However, with the additional assumption that there are no asymptotic paths in a logarithmic tract D with finite asymptotic values, then the following converse to Theorem 3.1.1 is true.

Theorem 3.1.2. *Let* D *be a regular logarithmic tract containing no asymptotic paths with finite asymptotic values. Then* D *is bounded by a single unbounded curve. Further, if* D *is a logarithmic tract with boundary value* R*, then for all* $R' > R$, $\{z \in D : |f(z)| > R'\}$ *is a logarithmic tract bounded by a simple curve.*

Note that, in the first part of Theorem 3.1.2, D need not be a simple curve. For example, the function ze^z with boundary value 1 has a logarithmic tract with a boundary that self intersects.

In the case where a direct tract is bounded by more than one curve, and possibly by infinitely many curves, we give a sufficient condition for the direct tract to *contain* at least one logarithmic tract. In particular, if a simply connected direct tract is bounded by finitely many curves, we show that the direct tract contains only logarithmic tracts and asymptotic paths with asymptotic values of modulus equal to the boundary value of the direct tract. Further, in the case where there are finitely many boundary curves of a simply connected direct tract, there can be only finitely many critical points in the direct tract, at most $m-1$, where m is the number of logarithmic tracts contained in it. In order to better describe this situation, we define an access to a point, as is done in [5], and a channel of a tract, a new concept based on choosing part of a tract that contains only one access to ∞. Note that, only the concept of an access to *infinity* will be used.

Definition 3.1.3. Let U be a simply connected domain in $\mathbb{C}$. Fix a point $z_0 \in U$. A homotopy class of curves $\gamma : [0,1] \to \hat{\mathbb{C}}$ such that $\gamma([0,1)) \subset U$, $\gamma(0) = z_0$, and $\gamma(t) \to \infty$ as $t \to 1$ is called an *access* from U to ∞.

Definition 3.1.4. Let D be a direct tract. An unbounded simply connected component G of $\{z \in D : |z| > r\}$ for some $r > 0$ is called a *channel* of D if there exists exactly one access to ∞ in G.

Note that such a channel must be bounded by a single unbounded simple curve.

Theorem 3.1.5. *Let* D *be a regular direct tract of* f*, whose boundary includes at least one, and possibly infinitely many, distinct unbounded* *simple curves, with* $|f(z)| = R$*, for* $z \in \partial D$*. Then, for any channel in* D*, either*

- *the channel contains a logarithmic tract, or*
- f *has the same finite asymptotic value of modulus* R *along all paths to infinity in the channel.*

Moreover, if ∂D *consists of* m *distinct unbounded simple curves, then* D *contains at least one logarithmic tract and at most* $m - 1$ *critical points according to multiplicity.*

Note that the tract D need not be simply connected in order to satisfy the hypotheses of Theorem 3.1.5.

There are many entire functions with direct tracts which, while not logarithmic, do in fact contain a logarithmic tract, as in Theorem 3.1.5. This containment corresponds to an access to a logarithmic singularity in a direct tract. Simple examples of this are given in Example 3.3.1 and Example 3.3.2. As a more complicated example, Bergweiler and Eremenko [12] constructed an entire function with infinitely many direct, but no logarithmic singularities over any finite value; see Example 3.3.3. In Chapter 2, we showed that there exist points that escape arbitrarily slowly in the direct tracts of the reciprocal of this function. Using Theorem 3.1.1 we show that Bergweiler and Eremenko's function is in fact in the class $\mathcal{B}$, thus giving an example of a function in the class $\mathcal{B}$ with infinitely many direct singularities, but no logarithmic singularities over any finite value, which was not previously known.

The organization of this chapter is the following. Section 3.2 will be devoted to the proofs of Theorem 3.1.1, Theorem 3.1.2, and Theorem 3.1.5. Section 3.3 contains three examples to illustrate applications of these results.

3.2 PROOFS OF THEOREM 3.1.1, 3.1.2, AND 3.1.5

The proof of Theorem 3.1.1, which states that regular direct tracts bounded by a simple curve are logarithmic, is straightforward and similar to the proof of [12, Theorem 5], though the statement

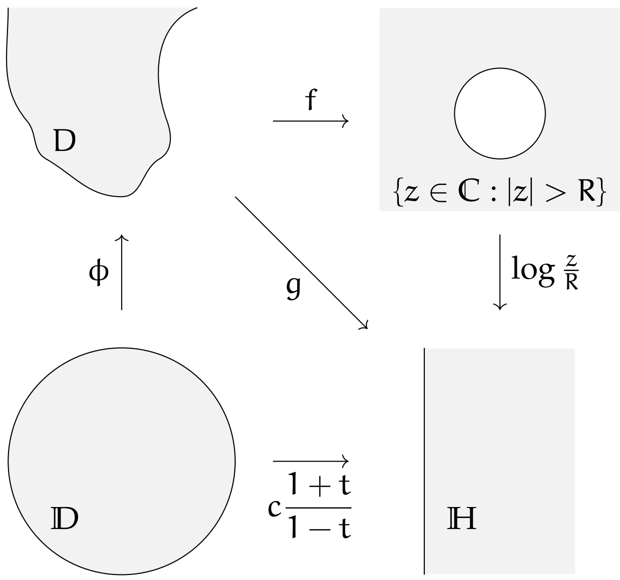

Figure 3.1: Construction in the proof of Theorem 3.1.1

of [12, Theorem 5] is somewhat different. We include this proof for completeness.

Proof of Theorem 3.1.1. Let $\phi : \mathbb{D} \to D$ be a Riemann map, where $\mathbb{D}$ denotes the open unit disc. The following construction is illustrated in Figure 3.1. The set D is a Jordan domain in the Riemann sphere with boundary $\partial D \cup \{\infty\}$, so ϕ extends continuously and one-to-one to $\partial \mathbb{D}$, by Carathéodory's Theorem ([23] and [32, Theorem I.3.1]), and without loss of generality $\phi(1) = \infty$. (This is where we use the assumption that ∂D is a simple curve tending to ∞ at both ends.) So,

$$u(t) = \log \frac{|f(\phi(t))|}{R}, \text{ for } t \in \mathbb{D},$$

is a positive harmonic function in $\mathbb{D}$ with $u(t) = 0$ if $t \in \partial\mathbb{D} \setminus \{1\}$. Therefore u is a positive multiple of the Poisson kernel in $\mathbb{D}$ with singularity at 1. For a discussion of positive harmonic functions, and the Poisson kernel and its properties, see [32] or [36], for example. Hence,

$$u(t) = c \operatorname{Re}\left(\frac{1+t}{1-t}\right), \text{ where } c > 0.$$

Now we can define an analytic branch g of $\log f/R$ in D by the monodromy theorem, since D is simply connected and any local branch g of $\log f/R$ can be analytically continued along any path in D. Then,

$$g(\phi(t)) = \log \frac{f(\phi(t))}{R}, \text{ for } t \in \mathbb{D},$$

is analytic in $\mathbb{D}$, with

$$\operatorname{Re} g(\phi(t)) = \log \frac{|f(\phi(t))|}{R} = c \operatorname{Re}\left(\frac{1+t}{1-t}\right).$$

Hence, for some constant $\theta \in \mathbb{R}$, we have

$$\begin{aligned} g(\phi(t)) = c\frac{1+t}{1-t} + i\theta \implies f(\phi(t)) &= Re^{i\theta} \exp\left(c\frac{1+t}{1-t}\right), \\ &\quad \text{for } t \in \mathbb{D}, \\ \implies f(z) &= Re^{i\theta} \exp\left(c\frac{1+\phi^{-1}(z)}{1-\phi^{-1}(z)}\right), \\ &\quad \text{for } z \in D. \end{aligned} \tag{3.2.1}$$

It follows immediately that f has no critical points in D. Also, there are no asymptotic paths in D with finite asymptotic values, as the exponential function has none in $\mathbb{H} = \{z : \operatorname{Re} z > 0\}$. Indeed, if $\gamma \to \infty$ in D, then $\phi^{-1}(z) \to 1$ along γ. So, we have that $(1+\phi^{-1}(z))/(1-\phi^{-1}(z)) \to \infty$ in $\mathbb{H}$ as $z \to \infty$ for $z \in \gamma$. Hence, by (3.2.1), $f(z)$ cannot tend to a finite limit as $z \to \infty$ for $z \in \gamma$. $\square$

Now, we prove Theorem 3.1.2, giving a converse to Theorem 3.1.1.

Proof of Theorem 3.1.2. Let D be a regular logarithmic tract with boundary value R. Then the function $g(z) = \log f(z)$ is a univalent map from D onto $H_R = \{t : \operatorname{Re} t > \log R\}$, with a univalent inverse function $h : H_R \to D$.

Now, D is simply connected and ∂D consists of piecewise smooth curves. Thus, on the Riemann sphere, the boundary of D is the curve $\partial D \cup \{\infty\}$ (not necessarily a simple curve). Therefore, by a version of Carathéodory's Theorem [50, Theorem 2.1], the univalent function h extends continuously to $\partial H_R \cup \{\infty\}$, with values in $\partial D \cup \{\infty\}$. If there exists $t_0 \in \partial H_R$ such that $h(t_0) = \infty$, then the function $g = h^{-1}$ has asymptotic value t_0 in D. So, by our assumption, $h(t)$ must be finite for all $t \in \partial H_R$. Hence, $h(\partial H_R) = \partial D$ is a single unbounded curve, possibly with self-intersection.

The final part of the theorem follows immediately from the above discussion of g and h, since

$$\{z \in D : |f(z)| = R'\} = h(L_{R'}),$$

where $L_{R'} = \{t : \operatorname{Re} t = \log R'\}$ and $R' > R$. □

The proof of Theorem 3.1.5 uses similar machinery, but is rather more complicated.

Proof of Theorem 3.1.5. Consider a regular direct tract D bounded by more than one unbounded curve with $|f(z)| = R$, for $z \in \partial D$. If it exists, choose some channel, G, of D. Then either $|f(z)|$ will be unbounded or bounded within this channel. Let $\phi : \mathbb{D} \to G$ be a Riemann map with $\phi(z) \to \infty$ as $z \to 1$ for $z \in \mathbb{D}$. This is possible by the definition of a channel. Then, ϕ extends continuously and one-to-one to $\partial\mathbb{D}$ by Carathéodory's Theorem, and once again $\phi(1) = \infty$.

Let E be the subset of ∂G where $|f(z)| \neq R$. Then,

$$u(t) = \log \frac{|f(\phi(t))|}{R}, \text{ for } t \in \mathbb{D},$$

is a positive harmonic function in $\mathbb{D}$. Also, the set $\phi^{-1}(E)$ is contained in a closed arc of $\partial\mathbb{D}$ which does not contain 1. Note that $\phi^{-1}(E)$ may be disconnected.

First, assume f is unbounded in G and denote by $P(t, \zeta)$ the Poisson kernel in $\mathbb{D}$ with singularity at ζ. Since f is unbounded

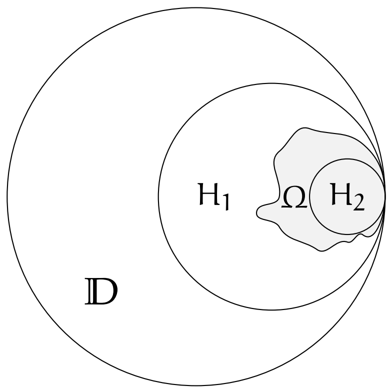

Figure 3.2: $H_2 \subset \Omega \subset H_1 \subset \mathbb{D}$

in G, it follows from the Poisson integral formula (see [36, Theorem 1.16]) that

$$u(t) = cP(t,1) + \int_{\phi^{-1}(E)} P(t,\zeta)u(\zeta)d\lambda(\zeta), \text{ for } t \in \mathbb{D}, \tag{3.2.2}$$

where $c > 0$ and $\lambda(\zeta)$ is the normalized Lebesgue measure on $\partial\mathbb{D}$. Now, choose $\epsilon > 0$ and $\delta > 0$ such that

$$\int_{\phi^{-1}(E)} P(t,\zeta)u(\zeta)d\lambda(\zeta) < \epsilon, \tag{3.2.3}$$

for $|t-1| < \delta$ and $t \in \mathbb{D}$. Then, choose $R_2 > R_1 > 0$ such that the horodiscs

$$H_j = \{t \in \mathbb{D} : cP(t,1) > R_j\}, \text{ for } j = 1,2,$$

lie inside $D(1,\delta)$ and $R_2 > R_1 + 2\epsilon$. So, for $t \in H_2$, we have $u(t) > cP(t,1) > R_2$. Let Ω be the component of $\{t \in \mathbb{D} : u(t) > R_2\}$ that contains H_2. For $t \in \Omega$, $u(t) > R_2$, so

$$cP(t,1) > R_2 - \int_{\phi^{-1}(E)} P(t,\zeta)u(\zeta)d\lambda(\zeta).$$

Hence, by (3.2.2) and (3.2.3), $cP(t,1) > R_2 - \epsilon > R_1$, for $t \in \Omega$. Therefore, $H_2 \subset \Omega \subset H_1$, as shown in Figure 3.2.

We next claim that Ω is bounded by a single curve if R_2 is sufficiently large. To prove this, consider the Riemann map $\psi : \mathbb{D} \to \mathbb{H}$ given by $\psi(t) = \frac{1+t}{1-t}$. Then, consider the positive harmonic function U on $\mathbb{H}$ defined by

$$
\begin{aligned}
U(x+iy) &= u(\psi^{-1}(x+iy)) \\
&= cP(\psi^{-1}(x+iy), 1) \\
&+ \int_{\psi(\phi^{-1}(E))} P(\psi^{-1}(x+iy), \psi^{-1}(s))u(\psi^{-1}(s))\frac{ds}{1+s^2} \\
&= cx + x \int_{\psi(\phi^{-1}(E))} \frac{U(s)ds}{x^2+(y-s)^2}, \text{ for } x+iy \in \mathbb{H}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{\partial U(x+iy)}{\partial x} &= c + \int_{\psi(\phi^{-1}(E))} \frac{U(s)ds}{x^2+(y-s)^2} \\
&+ x\frac{\partial}{\partial x} \int_{\psi(\phi^{-1}(E))} \frac{U(s)ds}{x^2+(y-s)^2} \\
&\geqslant c - \int_{\psi(\phi^{-1}(E))} \frac{2x^2U(s)}{(x^2+(y-s)^2)^2}ds \\
&\geqslant c - \frac{2}{x^2} \int_{\psi(\phi^{-1}(E))} U(s)ds \\
&> 0
\end{aligned}
$$

for x sufficiently large and $y \in \mathbb{R}$, since $\psi(\phi^{-1}(E))$ is contained in a bounded interval of the imaginary axis and $U(s)$ is bounded on $\psi(\phi^{-1}(E))$. So, $\frac{\partial U(x+iy)}{\partial x} > 0$ in the half-plane $\psi(H_1)$ for sufficiently small δ. Therefore, $U(x+iy)$ is monotonic with respect to x in the half-plane $\psi(H_1)$ for any fixed y. Hence $\{x+iy : U(x+iy) > R_2\}$ is bounded by a single simple unbounded curve, and so $\Omega = \psi^{-1}(\{x+iy : U(x+iy) > R_2\})$ is a Jordan domain bounded by a simple curve. Thus, by Theorem 3.1.1, $\phi(\Omega)$ is a logarithmic tract in G.

Now, assume that f is bounded in G. Then there exists K such that $u(t) < K$ for t in a neighborhood of the boundary singularity at 1. On the boundary of $\mathbb{D}$, $u(t) \equiv 0$ for t in a neighborhood of 1 except possibly at 1. So, by the extended maximum principle [36, Theorem 5.16], u has boundary value 0 at 1. We want to show that $f(\phi(t)) \to \alpha$ as $t \to 1$, where $|\alpha| = R$. Since $u \equiv 0$ on $\partial\mathbb{D}$ in a neighborhood of 1, we deduce, by the reflection principle [36, Example 1, p. 35], that we can find a neighborhood, N say, of 1 in $\mathbb{C}$ to which u extends harmonically. Therefore, there exists a complex conjugate v of u so that $u + iv$ is analytic on this neighborhood. Let

$$g(z) = R\exp(u(\phi^{-1}(z)) + iv(\phi^{-1}(z))), \text{ for } z \in \phi(N \cap \mathbb{D}).$$

Then, g is analytic on $\phi(N \cap \mathbb{D}) \subset G$ and $|g(z)| = |f(z)|$ in $\phi(N \cap \mathbb{D})$. Non-constant analytic maps are open, so $g(z) = cf(z)$ in $\phi(N \cap \mathbb{D})$, where $|c| = 1$. Therefore, $\arg f(z)$ tends to a finite limit as $z \to \infty$ in $\phi(N \cap \mathbb{D})$, and hence in G, since any sequence tending to infinity in G is eventually contained in $\phi(N \cap \mathbb{D})$. Therefore, there exists α such that $f(z) \to \alpha$ as $z \to \infty$ in G, with $|\alpha| = R$.

Finally, if D is bounded by m distinct unbounded simple curves, then following the method in the proof of Theorem 3.1.1, we again define the positive harmonic function

$$u(t) = \log \frac{|f(\phi(t))|}{R}, \text{ for } t \in \mathbb{D},$$

where $\phi : \mathbb{D} \to D$ is a conformal map and each access to infinity in D corresponds under ϕ to a family of paths tending to a point on $\partial\mathbb{D}$ along which u tends to either 0 or ∞. Suppose there exist $n \leqslant m$ accesses to infinity on which f is unbounded. Then, there exist n points $\zeta_1, \dots, \zeta_n \in \partial\mathbb{D}$ such that u is a positive harmonic function in $\mathbb{D}$ with $u(t) = 0$, for $t \in \partial\mathbb{D} \setminus \{\zeta_1, \dots, \zeta_n\}$. Therefore u is a sum of positive multiples of the Poisson kernel in $\mathbb{D}$ with singularities at $\zeta_1, \dots, \zeta_n$. Hence,

$$u(t) = \sum_{k=1}^{n} c_k \operatorname{Re}\left(\frac{\zeta_k + t}{\zeta_k - t}\right), \text{ where } c_k > 0, \text{ for } k = 1, \dots, n.$$

Now, again, we can define an analytic branch g of $\log f/R$ in D by the monodromy theorem, since D is simply connected and any local branch g of $\log f/R$ can be analytically continued along any path in D. Then,

$$g(\phi(t)) = \log \frac{f(\phi(t))}{R}, \text{ for } t \in \mathbb{D},$$

is analytic in $\mathbb{D}$, with

$$\operatorname{Re} g(\phi(t)) = \log \frac{|f(\phi(t))|}{R} = \sum_{k=1}^{n} c_k \operatorname{Re}\left(\frac{\zeta_k + t}{\zeta_k - t}\right).$$

Hence, for some constant $\theta \in \mathbb{R}$, we have

$$\begin{aligned} g(\phi(t)) &= \sum_{k=1}^{n} c_k \frac{\zeta_k + t}{\zeta_k - t} + i\theta \\ \implies f(\phi(t)) &= Re^{i\theta} \exp\left(\sum_{k=1}^{n} c_k \frac{\zeta_k + t}{\zeta_k - t}\right), \text{ for } t \in \mathbb{D}. \end{aligned}$$

The critical points of $f(\phi(t))$ for $t \in \mathbb{D}$ are the solutions of

$$\sum_{k=1}^{n} c_k \frac{2\zeta_k}{(\zeta_k - t)^2} = 0,$$

for which there are at most $2n - 2$ solutions in $\mathbb{C}$. Further, by the Cauchy-Riemann equations, critical points of $f(\phi(t))$ occur if and only if $|\nabla u(t)| = 0$, where ∇u denotes the gradient of u. For a point $t \in \mathbb{D}$, noting that $\zeta_k = 1/\overline{\zeta}_k$ gives that

$$\operatorname{Re}\left(\frac{\zeta_k + t}{\zeta_k - t}\right) = -\operatorname{Re}\left(\frac{\zeta_k + 1/\overline{t}}{\zeta_k - 1/\overline{t}}\right), \text{ for } k = 1, \dots, n.$$

Hence, each Poisson kernel is 'symmetric' under reflection in the unit circle and so a sum of Poisson kernels is as well. Further, no solutions of $|\nabla u(t)| = 0$ lie on $\partial\mathbb{D}$, by the behavior of the Poisson kernel near $\partial\mathbb{D}$. So, the solutions of $|\nabla u(t)| = 0$ occur in pairs and are symmetric with respect to $\partial\mathbb{D}$. Therefore, $\mathbb{D}$ contains at most $n - 1$ critical points of $f(\phi(t))$ and hence D contains at most $n - 1$ critical points of f. □

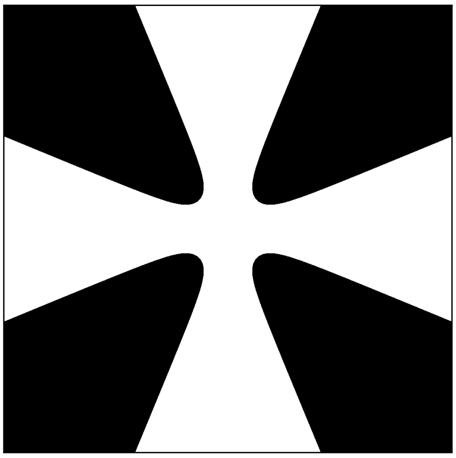

Figure 3.3: The tract of $2\exp(z^4)$ with boundary value 1 in white with its complement in black, for $|\operatorname{Re} z| \leqslant 5$ and $|\operatorname{Im} z| \leqslant 5$.

3.3 EXAMPLES

In this section, we give three examples to show the kinds of direct tracts that can exist, and to which we can apply Theorem 3.1.1 and Theorem 3.1.5. First, to illustrate Theorem 3.1.5, we give a simple example of a transcendental entire function with a simply connected direct tract bounded by finitely many boundary curves.

Example 3.3.1. Consider $G(z) = 2\exp(z^4)$. Note that G is in the class $\mathcal{B}$.

First, $G(z) \to \infty$ as $z \to \infty$ along the real and imaginary axes, and $G(z) \to 0$ as $z \to \infty$ along the rays with angle an odd multiple of $\pi/4$. With a boundary value of $R = 1$, as in Figure 3.3, a neighborhood about the origin is contained in the direct tract. Further, on the lines with angle an odd multiple of $\pi/8$ the modulus of G is 2. Hence, G has one direct tract bounded by four unbounded simple curves. By Theorem 3.1.5, G contains at most three critical points, and in fact, contains a single critical point of multiplicity 3 at 0. The direct tract of G has four channels on which f is unbounded and hence contains four logarithmic tracts.

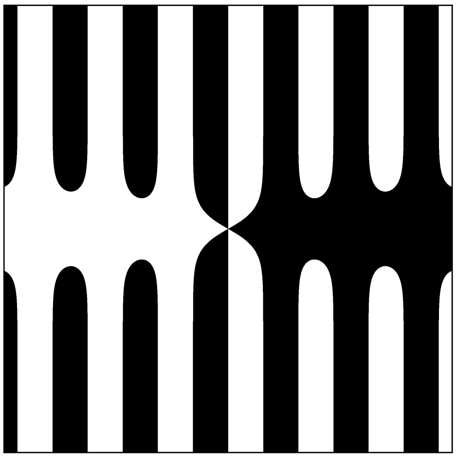

Figure 3.4: The tracts of $\exp(\sin(z) - z)$ with boundary value 1 in white with their complement in black, for $|\operatorname{Re} z| \leqslant 20$ and $|\operatorname{Im} z| \leqslant 20$.

Next, we illustrate Theorem 3.1.5 by giving an example of a direct non-logarithmic tract with infinitely many logarithmic tracts inside it.

Example 3.3.2. Consider $f(z) = \exp(\sin(z) - z)$. Note that f is not in the class $\mathcal{B}$ as it has critical values at $e^{2k\pi}$ for all $k \in \mathbb{Z}$.

First, $f(z) \to \infty$ as $z \to \infty$ along the negative real axis and $f(z) \to \infty$ as $z \to \infty$ along translates of the imaginary axis by $\frac{\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$. Also, $f(z) \to 0$ as $z \to \infty$ along the positive real axis and along translates of the imaginary axis by $\frac{3\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$. So, f will have infinitely many direct tracts in the right half-plane and one direct tract in the left half-plane, with $|f(z)| = R$ on ∂D for some suitable $R > 0$. See Figure 3.4, where $R = 1$. Further, f has no zeros and its only finite asymptotic value is 0, along the positive real axis and translates of the imaginary axis by $\frac{3\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$. Hence, by Theorem 3.1.5, the direct tract in the left half-plane contains infinitely many logarithmic tracts, each corresponding to channels about translates of the positive and negative imaginary axes by $\frac{\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$. In contrast, the tracts in the right half-plane are themselves logarithmic tracts in their own right.

Finally, we use our results to show that an entire function constructed by Bergweiler and Eremenko [12] to have no logarithmic singularities over any finite value is, moreover, in the class $\mathcal{B}$. Note that the reciprocal of this function (studied in Example 2.5.1) is an example of a function with a direct tract which, while simply connected, does not contain any channels.

Example 3.3.3. Consider the entire function

$$h(z) = \exp(g(z)), \text{ where } g(z) = \sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}.$$

Then, h has infinitely many direct singularities, but no logarithmic singularity over any finite value, and is in the class $\mathcal{B}$.

The first two statements are proved in [12], so it remains to check that h is in the class $\mathcal{B}$. In order to see this, we will show that every direct tract (over ∞) for some fixed boundary value is bounded by a single curve, and so all the direct tracts are logarithmic by Theorem 3.1.1. This implies that no critical points lie in these tracts and no asymptotic paths for finite asymptotic values, and so the singular values of h form a bounded set. We then conclude that h is in the class $\mathcal{B}$.

Next, we introduce some notation from [12, Section 6]. Fix ε with $0 < \varepsilon \leqslant \frac{1}{8}$ and set $r_n = (1+\varepsilon)2^{n+1}$ and $r_n' = (1-2\varepsilon)2^{n+2}$ for $n \in \mathbb{N}$. Then for $j \in \{0, 1, \cdots, 2^n - 1\}$ we define the sets

$$B_{j,n} = \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n}\right) : r_n \leqslant r \leqslant r_n' \right\}$$

and

$$C_{j,n}^{\pm} = \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} \pm \frac{r - r_n'}{r_{n+1} - r_n'} \frac{\pi i}{2^{n+1}}\right) : r_n' \leqslant r \leqslant r_{n+1} \right\}.$$

Bergweiler and Eremenko [12, Section 6] show that every unbounded simple path starting at 0 and lying in the infinite tree,

$$T = [-ir_1, ir_1] \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^n - 1} (B_{j,n} \cup C_{j,n}^{\pm}),$$

is an asymptotic curve along which the function tends to 0. Further,

$$\operatorname{Re} g(z) < -2^{2^n} \text{ for } z \in B_{j,n} \cup C_{j,n}^{\pm}.$$

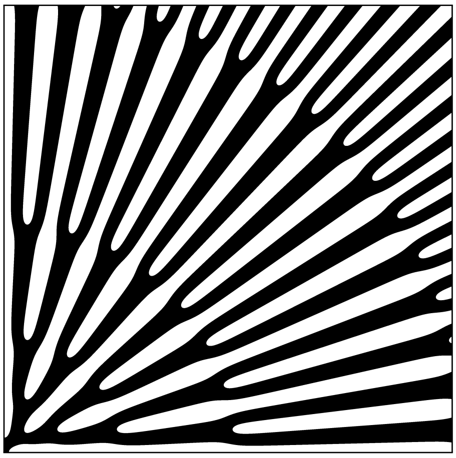

Figure 3.5: The direct tracts of h over ∞ in white with the direct tract over 0 in black, for $0 \leqslant \operatorname{Re} z \leqslant 250$ and $0 \leqslant \operatorname{Im} z \leqslant 250$.

So, if some direct tract (over ∞) with a sufficiently large boundary value was not bounded by a single curve, then at least one of the following three possibilities would occur:

1. there would be another direct tract over a finite value,
2. the tract would have to cross the infinite tree T,
3. there would be zeros of the function h.

The first case is shown not to happen in [12, Section 6] by proving that $\arg g(re^{i\theta})$ is an increasing function of θ, that it increases by $2^n 2\pi$ as θ increases by 2π, and then using a counting argument to show that all the direct tracts are accounted for and there can be no others. We can assume that the tract does not cross the infinite tree T, since $h(z)$ is bounded on T by a value smaller than the tract boundary value. Finally, the last case cannot happen since the exponential function omits 0. Therefore, each direct tract (over ∞) is bounded by a single curve, so by Theorem 3.1.1 we can deduce that $h \in \mathcal{B}$.

4

DIMENSIONS OF JULIA SETS IN TRACTS

4.1 INTRODUCTION

In this chapter we consider the question of when the Julia set of a transcendental meromorphic function has Hausdorff dimension greater than one. Recall from Section 1.5.2 that Barański, Karpińska, and Zdunik [6] proved that this is the case whenever a transcendental meromorphic function f has a logarithmic tract, in fact showing that for such functions $\dim J(f) \cap K(f) > 1$, where $K(f)$ is the set of points of bounded orbit.

This raised the question as to how generally the result of Barański, Karpińska, and Zdunik holds. In [11], Bergweiler showed that there exist transcendental entire functions for which $\dim K(f)$ can be arbitrarily small.

It was recently shown by Bishop [18] that there exists a transcendental entire function f for which $\dim J(f) = 1$. This example has multiply connected wandering domains, and so its direct tract does not have an unbounded boundary component. This suggests the question of whether Barański, Karpińska, and Zdunik's result can be generalized to show that, for a transcendental meromorphic function f with a direct tract with an unbounded boundary component, we have that $\dim J(f) > 1$. We prove this in several situations, and specifically in the case where the direct tract is simply connected and there is an additional restriction on the singular values associated with the direct tract.

Theorem 4.1.1. *Let f be a transcendental meromorphic function with a simply connected direct tract D. Suppose that there exists $\lambda > 1$ such that for arbitrarily large r there exists an annulus $A(r/\lambda, \lambda r)$ containing no singular values of the restriction of f to D. Then* $\dim J(f) \cap K(f) > 1$.

Our proof of Theorem 4.1.1 partially follows the approach in Barański, Karpińska, and Zdunik [6], with some significant differences. The hypotheses in [6] specify a logarithmic tract, for which one can make use of the logarithmic transform and the resulting expansion estimate in the tract. However, in our case, we no longer have this expansion estimate throughout the direct tract. Instead, our proof uses Wiman–Valiron theory, discussed in Section 1.6.1. We use tools developed by Bergweiler, Rippon, and Stallard [16] for general direct tracts, and apply them to direct tracts with certain properties. We show that for such direct tracts a larger disc than that given by Wiman–Valiron theory lies in the direct tract, and on this disc an estimate of the function holds which is somewhat weaker than the Wiman–Valiron estimate. While this estimate on the new disc is weaker than the Wiman–Valiron estimate, it enables us to cover a much larger annulus than does the Wiman–Valiron estimate many times over. These results are of independent interest.

The chapter is organized as follows. Section 4.2 discusses the relationship between direct tracts and the logarithmic transform, giving an expansion estimate that can be viewed as a local version of Lemma 1.4.2. In Section 4.3, we give our results on the size of enlarged Wiman–Valiron discs. We then use these results in Section 4.4 in order to prove a theorem on the Hausdorff dimension of certain Julia sets, of which Theorem 4.1.1 is a special case. Finally, in Section 4.5 we give two nontrivial examples of transcendental entire functions to which we can apply our results.

4.2 DIRECT TRACTS AND THE LOGARITHMIC TRANSFORM

As discussed in Section 1.4.2, a major tool used in connection with logarithmic tracts is the logarithmic transform, which was first studied in the context of entire functions by Eremenko and Lyubich [27]. We now introduce the logarithmic transform, F, in the setting of a direct tract with an unbounded boundary component. While the logarithmic transform has been studied in many papers on transcendental entire dynamics, it has almost always been applied to a logarithmic tract.

Let f be a meromorphic function with a direct tract D with an unbounded complementary component; any simply connected direct tract must have such a complementary component. By performing a translation if necessary, assume further that 0 is contained in an unbounded complementary component of D. Then, denote by F the logarithmic transform of f; that is, $\exp \circ F = f \circ \exp$. Note that F is periodic with period $2\pi i$ and maps each component of $\log D$ into a right half-plane. Unlike for a logarithmic tract, F is not necessarily univalent on each component of $\log D$. However, as there exists an unbounded complementary component, one may still lift f by the branches of the logarithm.

Applying the Koebe $1/4$-theorem, we are able to obtain an estimate on the function in the direct tract on a domain on which the function is univalent. Lemma 4.2.1 is a modification of the expansion estimate in Lemma 1.4.2 for a logarithmic tract proved by Eremenko and Lyubich [27]. The method of proof is similar to the approach of Rippon and Stallard in [57, Lemma 2.5].

Lemma 4.2.1. *Let f be a transcendental meromorphic function with a direct tract D, and let $\lambda > 1$. Suppose, for some $r > 0$, that the restriction of f to D has no singular values in $A(r/\lambda, \lambda r)$ and $\gamma \subset D$ is a simple unbounded curve on which $|f(z)| = r$. Then,*

1. *the function $\Phi(z) = \log f(z)$, for $z \in D$, is univalent on a simply connected domain Ω such that $\gamma \subset \Omega \subset D$, and*

$$\Phi(\Omega) = S = \{w : \log(r/\lambda) < \operatorname{Re} w < \log \lambda r\};$$

2. *the function $F(w) = \Phi(e^w) = \log f(e^w)$ maps each component of $\log \Omega$ univalently onto S;*

3. *any analytic branch H of $F^{-1} : S \to \log D$ satisfies*

$$|H'(w)| \leqslant \frac{4\pi}{\operatorname{dist}(w, \partial S)}, \quad \textit{for } w \in S, \tag{4.2.1}$$

or equivalently

$$\left|\frac{zf'(z)}{f(z)}\right| \geqslant \begin{cases} \frac{1}{4\pi}\log|\lambda r/f(z)|, & \textit{for } |f(z)| \geqslant r \\ \frac{1}{4\pi}\log|\lambda f(z)/r|, & \textit{for } |f(z)| \leqslant r, \end{cases} \tag{4.2.2}$$

for $z \in \Omega$.

Proof. Take a base point $z_0 \in \gamma$ and let $w_0 = \log f(z_0)$. Then, let Ψ denote the inverse branch of $z \mapsto \Phi(z) = \log f(z)$ such that $\Psi(w_0) = z_0$. By our hypothesis on the singular values of f, the branch Ψ can be analytically continued along every curve in S starting from w_0, with image values in D. Since S is simply connected, we deduce from the monodromy theorem that Ψ extends to be analytic throughout S, and $\Omega := \Psi(S) \subset D$.

Two cases can then arise [48, p. 283]: either Ψ is univalent in S or Ψ is periodic in S, with period $2\pi i m$, where $m \in \mathbb{N}$. In the latter case, however, $\Psi(S)$ must be bounded, which is impossible, since $\gamma \subset \Psi(S) = \Omega$.

Now, we use the fact that $H = \log \Psi$, for some branch of the logarithm, and hence H maps S univalently onto a domain that contains no disc of radius greater than π. Therefore, (4.2.1) holds by the Koebe $1/4$-theorem, and (4.2.2) follows by the change of variables $z = e^w$. □

4.3 WIMAN–VALIRON THEORY IN TRACTS

In this section, we show that a larger disc than given by Theorem 1.6.1 is possible inside a simply connected direct tract. In a general direct tract this is not possible, as shown by an example due to Bergweiler [9]. Note that, Bergweiler's example has a direct tract with no unbounded boundary component and the direct tract is the plane with neighborhoods of the zeros removed.

We use the notation from Section 1.6.1. If $\nu : \mathbb{C} \to [0, \infty)$ is a non-constant subharmonic function, then we set

$$B(r, \nu) = \max_{|z|=r} \nu(z),$$

and

$$a(r, v) = \frac{dB(r, v)}{d \log r} = rB'(r, v).$$

First, we recall a standard estimate on $a(r, v)$; see [16, Lemma 6.10].

Lemma 4.3.1. *Let $v : \mathbb{C} \to [-\infty, \infty)$ be subharmonic and let $\epsilon > 0$. Then there exists a set $E \subset [1, \infty)$ of finite logarithmic measure such that*

$$a(r, v) \leqslant B(r, v)^{1+\epsilon}$$

for $r \geqslant 1$, $r \notin E$.

Next, we state a more general version of [16, Lemma 11.2], giving a slightly different estimate of $B(r, v)$ for a general subharmonic function which is suited for our purpose. This statement is extracted from the inequalities given in the proof of [16, Lemma 11.2] (in which the function $\Phi(x) = B(r, v)$ and $\Phi'(x) = a(r, v)$, with $r = e^x$).

Lemma 4.3.2. *Let $v \to [-\infty, \infty)$ be subharmonic and let $\beta > \alpha > 0$. Then there exists a set $E \subset [1, \infty)$ of finite logarithmic measure such that*

$$B(s, v) \leqslant B(r, v) + a(r, v) \log \frac{s}{r} + a(r, v)^{1-\alpha-\beta},$$

for $\left|\log \frac{s}{r}\right| \leqslant 1/a(r, v)^{\beta}$, $r \notin E$, uniformly as $r \to \infty$.

We shall use Lemma 4.3.2 to prove a generalization of Theorem 1.6.1, which provides a much larger disc in a direct tract that satisfies certain extra conditions, at the expense of less control on the function as we move further out from the original Wiman–Valiron disc. For the remainder of this chapter, let D be a direct tract with boundary value R. We define

$$v(z) = \log \frac{|f(z)|}{R}, \tag{4.3.1}$$

for $z \in D$ and $v(z) = 0$ elsewhere. In this case,

$$B(r, v) = \max_{|z|=r} \log \frac{|f(z)|}{R}.$$

First, if there exists a direct tract surrounding a sparse collection of zeros, we show that a larger disc exists inside the direct tract than given by Theorem 1.6.1. Note that this hypothesis is satisfied by a simply connected direct tract.

Theorem 4.3.3. *Let* f *be a meromorphic function with a direct tract* D *with an unbounded boundary component. Let* ν *be defined as in* (4.3.1), *and let* $z_r \in D$ *be a point for which* $|z_r| = r$ *and* $\nu(z_r) = B(r, \nu)$.

Suppose that there exists $\lambda > 1$ *and a set* $E \subset [1, \infty)$ *of finite logarithmic measure such that, for* $r \in [1, \infty) \setminus E$, *we have that* $z_r \in L_r$, *a simply connected component of* $A(r/\lambda, \lambda r) \cap D$. *Also, let* $\frac{1}{2} > \tau > 0$. *Then, for* $r \in [1, \infty) \setminus E'$, *where* E' *has finite logarithmic measure, there exists* $D(z_r, r/a(r, \nu)^\tau) \subset D$.

Proof. Choose α and β such that $0 < \alpha < \beta < \tau < 1/2$ and $1 - \alpha - \beta = \sqrt{1 - 2\tau} = \xi(\tau)$, say. Let E' be the union of E and the exceptional sets in Lemma 4.3.1 and Lemma 4.3.2 for these values of α and β, where $\epsilon = \beta$. Further, set $\rho = 2ra(r, \nu)^{-\tau}$, where r is large enough that $a(r, \nu) \neq 0$. Consider the subharmonic function

$$u(z) = \nu(z) - B(r, \nu) - a(r, \nu) \log \frac{|z|}{r} \leqslant a(r, \nu)^{\xi(\tau)}, \qquad (4.3.2)$$

for $r \notin E'$ and $z \in \overline{D}(z_r, 512\rho)$ by Lemma 4.3.2, and the fact that for $z \in \overline{D}(z_r, 512\rho)$,

$$\left| \frac{z - z_r}{z_r} \right| \leqslant \frac{512\rho}{r} = \frac{1024}{a(r, \nu)^\tau} = o(1) \text{ as } r \to \infty,$$

since $\lim_{r \to \infty} a(r, \nu) = \infty$, and so, for large r,

$$\left| \log \frac{|z|}{r} \right| = \left| \log \left| 1 + \frac{z - z_r}{z_r} \right| \right| \leqslant 2 \left| \frac{z - z_r}{z_r} \right| \leqslant \frac{2048}{a(r, \nu)^\tau} \leqslant \frac{1}{a(r, \nu)^\beta}. \qquad (4.3.3)$$

Now, we show that $D(z_r, r/a(r, \nu)^\tau) \subset D$ for $r \notin E'$ sufficiently large. Following [16, p. 395–396], suppose not. First, by assumption, $D(z_r, 256\rho)$ does not contain any bounded complementary components, since the exceptional set of r has finite logarithmic length and L_r is assumed simply connected. Then, since D has an unbounded boundary component, there exists an unbounded component of the complement of D that intersects $\partial D(z_r, t)$ for $\rho \leqslant t \leqslant 256\rho$. Let V be the component of $D \cap D(z_r, 512\rho)$ that contains z_r and let $\Gamma = \partial V \cap \partial D(z_r, 512\rho)$. Note that V is simply connected, as L_r is simply connected. Further, in this case, let $t\theta^*(z_r, t)$ be the linear measure of the intersection of $\partial D(z_r, t)$ with D. Then, $\theta^*(z_r, t) \leqslant 2\pi$ for $\rho \leqslant t \leqslant 256\rho$, since $\partial D(z_r, t)$ is not wholly contained in D. Hence, by a result of Tsuji concerning bounds on harmonic measure (see [71, p. 112]),

$$\begin{aligned}
\omega(z_r, \Gamma, V) &\leqslant 3\sqrt{2}\exp\left(-\pi \int_\rho^{256\rho} \frac{dt}{t\theta^*(z_r, t)}\right) \\
&\leqslant 3\sqrt{2}\exp\left(-\frac{1}{2}\int_\rho^{256\rho} \frac{dt}{t}\right) \\
&= \frac{3\sqrt{2}}{2^4} \\
&< \frac{1}{2},
\end{aligned}$$

where $\omega(z_r, \Gamma, V)$ denotes the harmonic measure of Γ at z_r in V. Thus, for $\Sigma = \partial V \setminus \Gamma$ we have

$$\omega(z_r, \Sigma, V) = 1 - \omega(z_r, \Gamma, V) > \frac{1}{2}.$$

Next, for $z \in \Gamma$, we have $v(z) = 0$, since z is on the boundary of D, so from (4.3.2), (4.3.3), and Lemma 4.3.1 with $\epsilon = \beta$, if $r \notin E'$ is sufficiently large, then

$$\begin{aligned}
u(z) &= -B(r, \nu) - a(r, \nu)\log\frac{|z|}{r} \\
&\leqslant -B(r, \nu) + a(r, \nu)^{1-\beta} \\
&\leqslant -B(r, \nu) + B(r, \nu)^{(1-\beta)(1+\epsilon)} \\
&\leqslant -\frac{1}{2}B(r, \nu).
\end{aligned}$$

Hence, by (4.3.2) and the above, we can apply the two constant theorem (see for example [52, Theorem 4.3.7]) for large $r \notin E'$ and obtain

$$u(z_r) \leqslant -\frac{1}{2}\omega(z_r, \Sigma, V)B(r, \nu) + a(r, \nu)^{\xi(\tau)}(1 - \omega(z_r, \Sigma, V))$$

$$\leqslant -\frac{1}{4}B(r,\nu) + a(r,\nu)^{\xi(\tau)}$$
$$\leqslant -\frac{1}{4}B(r,\nu) + B(r,\nu)^{\xi(\tau)(1+\epsilon)}$$
$$\leqslant -\frac{1}{8}B(r,\nu),$$

which gives a contradiction, since $u(z_r) = 0$, while $B(r,\nu) \to \infty$ as $r \to \infty$. Hence, $D(z_r, r/a(r,\nu)^{\tau}) \subset D$ for $r \notin E'$ sufficiently large. □

We now show that, given a disc of the form $D(z_r, r/a(r,\nu)^{\tau})$, for $0 < \tau < 1/2$, inside our tract, we can obtain an estimate for the function f inside that disc. This estimate includes the case of the original Wiman–Valiron estimate when $\tau > 1/2$. A larger disc than that given by Wiman–Valiron theory introduces a worse error to the estimate the larger the new disc is. Note that for the following theorem, we do not insist on any additional hypotheses on the tract, such as being simply connected; we just require the existence of a suitably large disc inside the direct tract. Further, note that the function $\xi(\tau) = \sqrt{1-2\tau}$ is chosen so that $\xi(\tau) \to 1$ as $\tau \to 0$ and $\xi(\tau) \to 0$ as $\tau \to 1/2$, while also $\xi(\tau) < 1-\tau$. These properties will be important in this and subsequent proofs.

Theorem 4.3.4. *Let f be a meromorphic function with a direct tract D and $\tau > 0$. Further, let ν be defined as in (4.3.1), and let $z_r \in D$ be a point for which $|z_r| = r$ and $\nu(z_r) = B(r,\nu)$. There exists a set $E \subset [1,\infty)$ of finite logarithmic measure, so that if a disc $D(z_r, r/a(r,\nu)^{\tau}) \subset D$ for $r \notin E$ sufficiently large, then there exists an analytic function g in $D(z_r, r/a(r,\nu)^{\tau})$ such that*

$$\log f(z) = \log f(z_r) + a(r,\nu)\log\frac{z}{z_r} + g(z), \quad \textit{for } z \in D(z_r, r/a(r,\nu)^{\tau}),$$

where

$$g(z) = \begin{cases} O(a(r,\nu)^{\xi(\tau)}) & \textit{for } z \in D(z_r, r/a(r,\nu)^{\tau}) \textit{ and } \tau < 1/2, \\ o(1) & \textit{for } z \in D(z_r, r/a(r,\nu)^{\tau}) \textit{ and } \tau > 1/2, \end{cases}$$

and $\xi(\tau) = \sqrt{1-2\tau}$, as $r \to \infty$, $r \notin E$.

Proof. Set

$$g(z) = \log \frac{f(z)}{f(z_r)} - a(r,\nu) \log \frac{z}{z_r}, \tag{4.3.4}$$

where the branches of the logarithms are chosen so that $g(z_r) = 0$. By the Borel–Carathéodory inequality [74, p. 20],

$$\max_{|z-z_r|\leqslant t} |g(z)| \leqslant 4 \max_{|z-z_r|\leqslant 2t} \operatorname{Re} g(z) \leqslant 4a(r,\nu)^{\xi(\tau)}, \tag{4.3.5}$$

for $0 < t < \rho/2$ and $r \notin E$, by Lemma 4.3.2 with α and β chosen so that $\beta > \alpha > 0$ and $1 - \alpha - \beta = \xi(\tau)$.

Thus, by (4.3.4), for $z \in D(z_r, ra(r,\nu)^{-\tau})$ and $r \notin E$,

$$\log f(z) = \log f(z_r) + a(r,\nu) \log \frac{z}{z_r} + g(z),$$
$$\text{for } z \in D(z_r, r/a(r,\nu)^{\tau}),$$

where $g(z) = O(a(r,\nu)^{\xi(\tau)})$ for $\tau < 1/2$ by (4.3.5) and $g(z) = o(1)$ inside $D(z_r, r/a(r,\nu)^{\tau})$ for $\tau > 1/2$ by Theorem 1.6.1. □

Note that in the above result, the case $0 < \tau < 1/2$ is new. Further, for $\tau = 1/2$ it is natural to ask what the error, g, in the Wiman–Valiron estimate is and whether it is possibly $O(1)$. However, this proof does not show it.

We now use Theorem 4.3.4 in order to obtain an estimate on the size of the image of this new disc. First, we label three sets where we choose the principal branch of the logarithm; that is the branch corresponding to the principal value of the argument. Let

$$S_r = \left\{ w : |\operatorname{Re} w - \log r| \leqslant \frac{1/2}{a(r,\nu)^{\tau}}, |\operatorname{Im} w - \arg z_r| \leqslant \frac{1/2}{a(r,\nu)^{\tau}} \right\}. \tag{4.3.6}$$

Further, let

$$Q = \left\{ w : \begin{array}{l} |\operatorname{Re} w - \log|f(z_r)|| < \frac{1}{8} a(r,\nu)^{1-\tau}, \\ |\operatorname{Im} w - \arg f(z_r)| < \frac{1}{8} a(r,\nu)^{1-\tau} \end{array} \right\}, \tag{4.3.7}$$

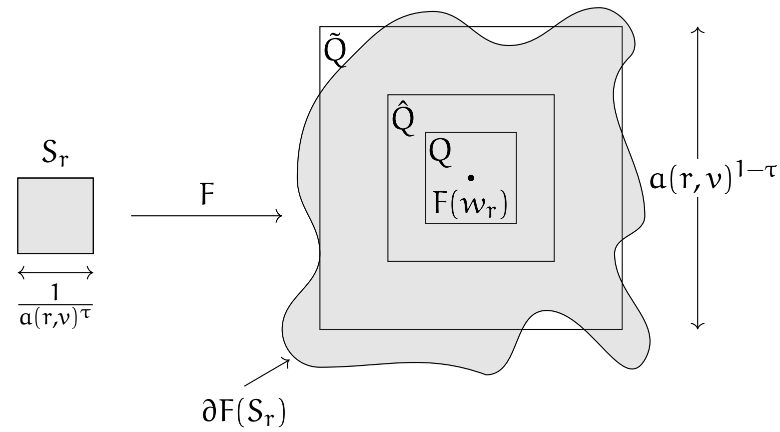

Figure 4.1: A sketch of the argument in the proof of Theorem 4.3.5.

and

$$\hat{Q} = \left\{ w : \begin{array}{l} |\operatorname{Re} w - \log|f(z_r)|| < \frac{1}{4} a(r,v)^{1-\tau}, \\ |\operatorname{Im} w - \arg f(z_r)| < \frac{1}{4} a(r,v)^{1-\tau} \end{array} \right\}. \tag{4.3.8}$$

Theorem 4.3.5. *Let* f *be a meromorphic function with a direct tract* D *with an unbounded boundary component and* τ, ν, z_r, *and* E *be as in Theorem* 4.3.4. *Consider the logarithmic transform* F *of* f. *If, for* $r \in [1, \infty) \setminus E$, *there exists* $D(z_r, r/a(r,\nu)^{\tau}) \subset D$, *then* $S_r \subset \log D(z_r, r/a(r,\nu)^{\tau})$ *is mapped univalently by* F *and*

$$F(S_r) \supset \hat{Q} \supset Q.$$

Proof. Let $w_r = \log z_r + i \arg z_r$, where we choose the principal branch of the logarithm and recall that F is the logarithmic transform of f. From Theorem 4.3.4,

$$F(w) = F(w_r) + a(r,\nu)(w - w_r) + O(a(r,\nu)^{\xi(\tau)}), \tag{4.3.9}$$

for $w \in \log D(z_r, r/a(r,\nu)^{\tau})$ and as $r \to \infty$.

The domain

$$S_r = \left\{ w : |\operatorname{Re} w - \log r| \leqslant \frac{1/2}{a(r,\nu)^{\tau}}, |\operatorname{Im} w - \arg z_r| \leqslant \frac{1/2}{a(r,\nu)^{\tau}} \right\}$$
$$\subset \log D(z_r, r/a(r,\nu)^{\tau})$$

is a square of side length $1/a(r,\nu)^{\tau}$ centered at w_r and mapped by

$$w \mapsto F(w_r) + a(r,\nu)(w - w_r)$$

to a square, $\tilde{Q}$, of side $a(r,\nu)^{1-\tau}$ centered at $F(w_r)$. Since

$$F(w) = F(w_r) + a(r,\nu)(w - w_r) + O(a(r,\nu)^{\xi(\tau)}),$$

the image of ∂S_r under F is a closed curve lying entirely outside of a square of side

$$a(r,\nu)^{1-\tau} - O(a(r,\nu)^{\xi(\tau)})$$

centered at $F(w_r)$ for large enough r. This closed curve winds exactly once around $F(w_r)$, since $1 - \tau > \sqrt{1 - 2\tau} = \xi(\tau)$. Hence, by the argument principle, F is univalent on S_r and $F(S_r)$ contains a square of side length greater than $\frac{1}{2}a(r,\nu)^{1-\tau}$. □

4.4 HAUSDORFF DIMENSION

In this section, we show that the set of points of bounded orbit in the Julia set of a meromorphic map with a direct tract with conditions on the singular values, zeros, and the boundary of the tract has Hausdorff dimension strictly greater than one. Theorem 4.1.1 follows from Theorem 4.4.1, below, together with Theorem 4.3.3, replacing λ by $\lambda/2$ in Theorem 4.4.1 and using the fact that the exceptional set has finite logarithmic length. Recall that ν is defined as in (4.3.1), and z_r is a point for which $|z_r| = r$ and $\nu(z_r) = B(r,\nu)$.

Theorem 4.4.1. *Let f be a meromorphic map with a direct tract D with an unbounded boundary component. Fix $\lambda > 1$ and $\frac{1}{2} > \tau > 0$, and let E be the set of finite logarithmic measure in Theorem 4.3.4. Suppose that for arbitrarily large $r \in [1,\infty) \setminus E$ there exists an annulus $A(r/\lambda, \lambda r)$ such that both*

1. $D(z_r, r/a(r,\nu)^{\tau}) \subset A(r/\lambda, \lambda r) \cap D$, *and*

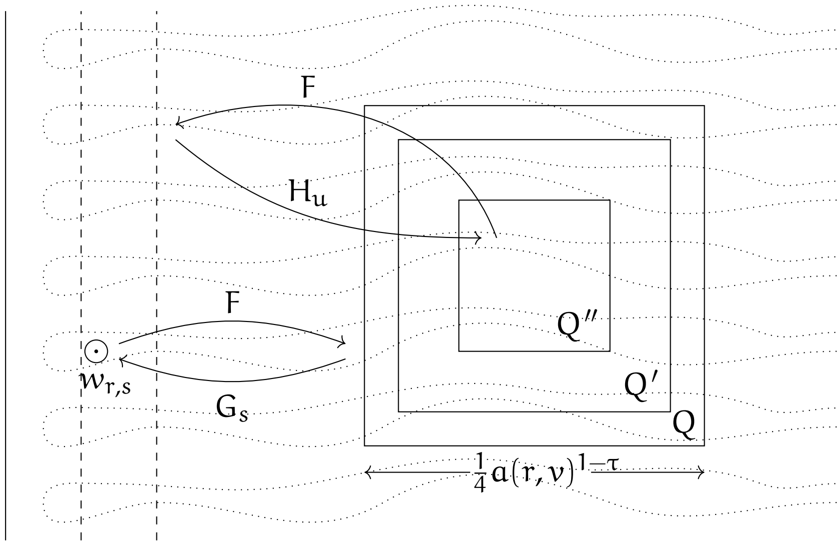

Figure 4.2: Construction of J_B in Theorem 4.4.1.

2. $A(r/\lambda, \lambda r)$ *contains no singular values of the restriction of* f *to* D.

Then $\dim J(f) \cap K(f) > 1$.

Proof. First, let E be the set of finite logarithmic measure in Theorem 4.3.4. Fix $\frac{1}{2} > \tau > 0$ and $\lambda > 1$. Then, take $r \in [1, \infty) \setminus E$ so large that 1 and 2 are satisfied. We use the sets S_r, Q, and $\hat{Q}$ introduced in (4.3.6), (4.3.7), and (4.3.8) before the statement of Theorem 4.3.5. By Theorem 4.3.5, the image of $S_r \subset D(z_r, r/a(r,\nu)^\tau)$ under $\log f$ covers the squares Q and $\hat{Q}$.

Following [6] we will construct an iterated function system based on taking inverse branches of F^2 that map Q to Q, where F is the $2\pi i$ periodic logarithmic transform of f which satisfies $\exp \circ F = f \circ \exp$; see Figure 4.2. We show that the dimension of the attractor of this system is strictly greater than 1. The method for constructing this iterated function system is, however, somewhat different from that of Barański, Karpińska, and Zdunik, and we make use of the tools from Wiman–Valiron theory developed in Section 4.3.

Step 1. We use our results from Section 4.3 to construct branches of F^{-1} from Q to $\log D(z_r, r/a(r,\nu)^\tau)$. We now define $w_r = \log z_r + i \arg z_r$, where arg is the principal argument. We let $w_{r,s} = w_{r,0} + 2\pi i s$, for $s \in \mathbb{Z}$, where $w_{r,0} = w_r$, and let $S_{r,s} = S_r + 2\pi i s$, for $s \in \mathbb{Z}$. Further, let G_s be the branch of F^{-1} satisfying $G_s(F(w_{r,0})) = w_{r,s}$. Then, $G_s : \hat{Q} \to S_{r,s}$ is univalent, since F is univalent on $S_{r,s}$ and $F(S_{r,s}) \supset \hat{Q} \supset Q$, by Theorem 4.3.5. By (4.3.9) and by Cauchy's estimate

$$F'(w) = a(r,\nu) + O(a(r,\nu)^{\xi(\tau)+\tau}) = a(r,\nu)(1+o(1)),$$

for $w \in S_{r,s}$, $F(w) \in Q$, and $s \in \mathbb{Z}$, as $r \to \infty$. Hence there exists an absolute constant $K > 0$ such that

$$\max_{w \in Q} |(G_s)'(w)| < \frac{K}{a(r,\nu)}, \text{ for } s \in \mathbb{Z}. \tag{4.4.1}$$

Note that $F'(w_{r,s}) = a(r,\nu)$, and so

$$|(G_s)'(F(w_{r,s}))| = \frac{1}{a(r,\nu)}, \text{ for } s \in \mathbb{Z}. \tag{4.4.2}$$

Step 2. Next, using similar methods to [6], we construct another family of branches of F^{-1}. First, choose r so large that $\max_{w \in l} \operatorname{Re} F(w) > \log r$, where $l = \{w : \operatorname{Re} w = \log r\}$. This is possible since

$$\frac{\max_{w \in l} \operatorname{Re} F(w)}{\log r} = \frac{B(r,\nu)}{\log r} \to \infty$$

as $r \to \infty$. Since D has an unbounded boundary component with boundary value R, any domain of the form $\{z \in D : |f(z)| > R'\}$, for $R' > R$, must also have an unbounded boundary component. Thus we can choose an unbounded curve l_0, say, in $\log D$ on which $\operatorname{Re} F(w) = \log r$ and $l_0 \cap \{w : \operatorname{Re} w < \log r\} \neq \emptyset$. Then, by Lemma 4.2.1 part 2, $F(l_0) = l$. Let $\zeta_0 \in l_0$ with $F(\zeta_0) \in l$ and let $\zeta_u = \zeta_0 + 2\pi i u$, for $u \in \mathbb{Z}$. Finally, let H_u be the branch of F^{-1} that maps $F(\zeta_0)$ to ζ_u, as given by Lemma 4.2.1 part 2.

Since there exist no singular values of f restricted to D that lie in $A(r/\lambda, \lambda r)$, we can obtain the following estimate on the derivative of H_u by applying Lemma 4.2.1 part 3 for $w \in \{w : \log(r/\sqrt{\lambda}) < \operatorname{Re} w < \log \sqrt{\lambda} r\}$:

$$\max_{w \in S_{r,s}} |(H_u)'(w)| < \frac{4\pi}{\log(\lambda r) - \log(\sqrt{\lambda} r)} = \frac{4\pi}{\log \sqrt{\lambda}}. \tag{4.4.3}$$

Step 3. Now, we estimate the diameter of $Q_{u,s} = H_u \circ G_s(Q)$, a second preimage of Q. Recall that $G_s(Q) \subset S_{r,s}$, so by (4.4.1) and (4.4.3),

$$\max_{w \in Q} |(H_u \circ G_s)'(w)| < \frac{4\pi K}{a(r,\nu) \log \sqrt{\lambda}}.$$

So, there exists $d \in (0,1)$ such that

$$\operatorname{diam} Q_{u,s} < \max_{w \in Q} |(H_u \circ G_s)'(w)| \operatorname{diam} Q < \frac{4\pi\sqrt{2} K a(r,\nu)^{1-\tau}}{4a(r,\nu) \log \sqrt{\lambda}} < d, \tag{4.4.4}$$

for r sufficiently large.

Let

$$Q' = \left\{ w : \begin{array}{l} |\operatorname{Re} w - \log |f(z_r)|| < \frac{1}{8} a(r,\nu)^{1-\tau} - d, \\ |\operatorname{Im} w - \arg f(z_r)| < \frac{1}{8} a(r,\nu)^{1-\tau} - d \end{array} \right\}$$

and

$$Q'' = \left\{ w : \begin{array}{l} |\operatorname{Re} w - \log |f(z_r)|| < \frac{1}{16} a(r,\nu)^{1-\tau}, \\ |\operatorname{Im} w - \arg f(z_r)| < \frac{1}{16} a(r,\nu)^{1-\tau} \end{array} \right\},$$

where d satisfies (4.4.4).

Step 4. We now consider level curves of F that meet Q''. Let l_u be the image under H_u of $l = \{w : \operatorname{Re} w = \log r\}$, that is a translate of the curve l_0 introduced in Step 2. Denote by $l_{u,Q'}$ the intersection of l_u and Q'. From Step 2, we know that l_u is unbounded, and so if $l_{u,Q'} \cap Q'' \neq \emptyset$, then

$$\operatorname{length}(l_{u,Q'}) \geqslant \frac{1}{16} a(r,\nu)^{1-\tau} - d. \tag{4.4.5}$$

Note that since $l_0 \cap l \neq \emptyset$ (from Step 2) there are at least $a(r,\nu)^{1-\tau}/16\pi$ values of u such that $l_u \cap Q'' \neq \emptyset$.

Step 5. Next, we obtain an estimate on $\sum_u \sum_s |(H_u \circ G_s)'(w)|$ for $w \in Q$, and u and s such that $H_u \circ G_s(Q) = Q_{u,s} \subset Q$. Now, $l_{u,Q'} \subset H_u(l)$. Since the map H_u is univalent on the strip $\{w : \log(r/\lambda) < \operatorname{Re} w < \log(\lambda r)\}$, it follows from the Koebe distortion theorem (applied to a covering of l by discs of a uniform size) that there exists a constant $C_2 > 0$ such that

$$\operatorname{length}(l_{u,Q'}) < C_2 \sum_s |(H_u)'(w_{r,s})|, \tag{4.4.6}$$

for any u such that $l_{u,Q'} \cap Q'' \neq \emptyset$, summing over all s such that $H_u(w_{r,s}) \in l_{u,Q'}$. Note that if $H_u(w_{r,s}) \in l_{u,Q'}$, then $Q_{u,s} \subset Q$ by (4.4.4).

The distortion of $H_u \circ G_s$ is uniformly bounded on Q, since $G_s : \hat{Q} \to S_{r,s}$ is univalent on $\hat{Q} \supset Q$, and H_u is univalent on $\{w : \log(r/\lambda) < \operatorname{Re} w < \log(\lambda r)\}$. Therefore, there exists a constant $C > 0$ such that

$$|(H_u \circ G_s)'(w)| > \frac{1}{C}|(H_u \circ G_s)'(F(w_{r,s}))|, \quad \text{for } w \in Q.$$

So, by (4.4.2), (4.4.5), and (4.4.6), for $w \in Q$, and u and s such that $Q_{u,s} \subset Q$,

$$\begin{aligned}
\sum_s |(H_u \circ G_s)'(w)| &> \frac{1}{C} \sum_s |(H_u \circ G_s)'(F(w_{r,s}))| \\
&= \frac{1}{Ca(r,\nu)} \sum_s |(H_u)'(w_{r,s})| \\
&> \frac{\frac{1}{16}a(r,\nu)^{1-\tau} - d}{CC_2 a(r,\nu)} \\
&> \frac{a(r,\nu)^{(1-\tau)}}{C_3 a(r,\nu)},
\end{aligned}$$

for r sufficiently large, where $C_3 > 0$ is some constant. As we noted at the end of Step 4, there exist at least $a(r,\nu)^{1-\tau}/16\pi$ curves l_u that meet Q''. Hence, for $w \in Q$, and u and s such that $Q_{u,s} \subset Q$,

$$\sum_u \sum_s |(H_u \circ G_s)'(w)| > \frac{a(r,\nu)^{2(1-\tau)}}{C_3 16\pi a(r,\nu)}. \tag{4.4.7}$$

Step 6. Following [6, p. 622 - 623], we obtain a conformal iterated function system from the family of maps $H_u \circ G_s : Q \to Q$. The sets $H_u \circ G_s(Q) = Q_{u,s}$ are pairwise disjoint and the system gives a maximal compact invariant set, J_B, say. Note that by construction $F^2(J_B) = J_B$ and $F(J_B) \subset \cup_{s \in \mathbb{Z}} D_s$. Further, we have that $\lim_{n\to\infty} |(F^n)'(w)| = \infty$ for every $w \in J_B$ by (4.4.1) and (4.4.3).

Recall from Section 1.5.3 that the Hausdorff dimension of J_B is the unique zero of the pressure function

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\|^t,$$

where $g^n = g_{i_1} \circ \cdots \circ g_{i_n}$, with $g_{i_j} = H_u \circ G_s$ for u and s such that $Q_{u,s} \subset Q$. Further recall that the pressure function is strictly decreasing, so in order to prove that $\dim J_B > 1$ it is sufficient to show that $P(1) > 0$. For an introduction to the pressure function see [21] and [42].

Now, estimating the pressure using (4.4.7),

$$\begin{aligned} P(1) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\| \\ &\geqslant \lim_{n\to\infty} \frac{1}{n} \log \left(\inf_{w \in Q} \sum_{(u,s)} |(H_u \circ G_s)'(w)| \right)^n \\ &\geqslant \log \left(\frac{a(r,v)^{2(1-\tau)}}{C_3 16\pi a(r,v)} \right) \\ &> 0, \end{aligned}$$

for r sufficiently large, since, by hypothesis, $\tau < 1/2$. This implies that the Hausdorff dimension of the invariant set J_B that arises from this system is greater than 1.

Let $X = \exp(J_B \cup F(J_B))$. As $\exp \circ F = f \circ \exp$ and $J_B \cup F(J_B)$ is F-invariant, X is f-invariant. Further, the exponential function is a smooth covering map, so $\dim X = \dim J_B \cup F(J_B) > 1$. Finally, we have that $\lim_{n\to\infty} |(F^n)'(w)| = \infty$ for $w \in J_B$ by (4.4.1) and (4.4.3). Therefore, $\lim_{n\to\infty} |(f^n)'(z)| = \infty$ for $z \in X$. However, $f^n(X)$ is bounded, so by normality, $X \subset J(f)$. Therefore, $\dim J(f) \cap K(f) > 1$. $\square$

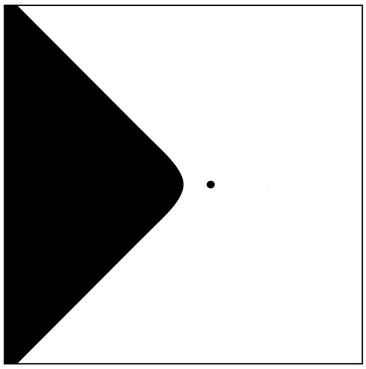

Figure 4.3: The direct tract of $\cos(z)\exp(z)$ in white with boundary value 1 and zeros at $\pi/2 + k\pi$ for $k \in \mathbb{Z}$. Its boundary includes the point 0.

4.5 EXAMPLES

In this section, we give two examples in order to illustrate the application of our results. First, we consider a function f with one direct tract, which is not simply connected. However, this function satisfies the hypotheses of Theorem 4.4.1 and so the Hausdorff dimension of $J(f) \cap K(f)$ is strictly greater than one.

Example 4.5.1. Let

$$f(z) = \cos(z)\exp(z).$$

Then $\dim J(f) \cap K(f) > 1$.

Consider the direct tract D of f with boundary value 1; see Figure 4.3. Then D contains $\{iy : y \neq 0\}$. Note that the only zeros of f are real and the only finite asymptotic value of f is 0, by the Denjoy-Carleman-Ahlfors theorem [48, Section XI.4], since f has order 1 and is symmetric in the real axis. We show that the points where f attains its maximum in the upper half-plane lie asymptotic to the line $y = x$ and that D contains a disc satisfying Theorem 4.4.1.

We use the following notation and results of Tyler [72] on maximum modulus curves; that is curves on which $|f(z)| = M(|z|)$. First, let

$$A(z) = \frac{zf'(z)}{f(z)} \text{ and } B(z) = zA'(z).$$

Further, let

$$b(r) = \frac{d^2 \log M(r)}{d(\log r)^2}.$$

Then, by [72, p. 2562] we have the following two properties:

1. $|f(z)| = M(r) \implies A(z)$ is real and positive, for $|z| = r$,

2. $|f(z)| = M(r) \implies B(x) = b(r) \geqslant 0$, for $x = r$.

For $f(z) = \cos(z)\exp(z)$, we have that $A(z) = z(1 - \tan z)$. Now, $A(z)$ is real and positive either in various intervals on the real line or asymptotically close to the line $y = x$, since $\tan(z) \to i$ as $\operatorname{Im} z \to \infty$.

By considering $B(x)$, we show that there are no large maximum modulus points on the positive real axis. First,

$$B(x) = x(1 - \tan x - x\sec^2 x) = x(1 - \tan x - x(1 + \tan^2 x)).$$

Hence, for $x > 0$,

$$B(x) \geqslant 0 \iff x \leqslant \frac{1 - \tan x}{1 + \tan^2 x} \iff x \leqslant 0.43\ldots,$$

approximately. Therefore, there are no maximum modulus points on the positive real axis greater than 0.44. So, the points satisfying both 1 and 2 in the upper half-plane must lie on an unbounded curve in the upper half-plane asymptotically close to the line $y = x$.

Now, all the zeros of f lie on the real axis. Since,

$$|f(x+iy)| = e^x(\cos^2(x) + \sinh^2(y))^{\frac{1}{2}} \geqslant e^x \sinh(y) > 1, \quad \text{for } x > 0 \text{ and } |y| \geqslant 1,$$

the direct tract D contains $\{x + iy : x > 0, |y| \geqslant 1\}$.

Hence for $\tau > 0$ there exists a disc of radius $r^{1-\tau}$ with center on the line $y = x$ inside a tract of f. Further, the critical values of f grow like the exponential function; that is, the modulus of the critical values of f are $\exp(\pi k + \pi/4)/\sqrt{2}$ for $k \in \mathbb{Z}$. So, there exists λ and an annulus $A(r/\lambda, \lambda r)$ which contains no singular values of the restriction of f to D for arbitrarily large r. Hence the conditions of Theorem 4.4.1 are satisfied, so we deduce from Theorem 4.4.1 that $\dim J(f) \cap K(f) > 1$.

Next, we give an example with a direct tract with no logarithmic singularities. This example is the reciprocal of the entire function studied in [12]; for an illustration of the tracts of this function, see [12, Figure 1].

Example 4.5.2. Consider the entire function

$$f(z) = \exp(-g(z)), \text{ where } g(z) = \sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}.$$

Then $\dim J(f) \cap K(f) > 1$.

We will use several key results about f from [12, p.254-258]. The first is the existence of an infinite tree on which f is very large. Introducing some notation, fix $0 < \varepsilon \leqslant 1/8$ and set $r_n = (1+\varepsilon)2^{n+1}$ and $r'_n = (1-2\varepsilon)2^{n+2}$ for $n \in \mathbb{N}$. Then, for $j \in \{0, 1, \dots, 2^n - 1\}$, let

$$B_{j,n} = \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n}\right) : r_n \leqslant r \leqslant r'_n \right\}$$

and

$$C^{\pm}_{j,n} = \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} \pm \frac{r - r'_n}{r_{n+1} - r'_n} \frac{\pi i}{2^{n+1}}\right) : r'_n \leqslant r \leqslant r_{n+1} \right\}.$$

We then obtain an infinite binary tree

$$T = [-ir_1, ir_1] \cup \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^n-1} (B_{j,n} \cup C^{\pm}_{j,n}).$$

If n is large enough, then $\operatorname{Re} g(z) < -2^{2^n}$, for $z \in B_{j,n} \cup C^{\pm}_{j,n}$, $j = 0, 1, \dots, 2^n - 1$.

The second key result in [12] is that if $r_n \leqslant |z| \leqslant r'_n$, then

$$g(z) = (1 + \eta(z)) \left(\frac{z}{2^n}\right)^{2^n},$$

where $\eta(z) \to 0$ as $n \to \infty$.

Let $\rho_n = (1+2\varepsilon)2^{n+1}$ and $\rho'_n = (1-3\varepsilon)2^{n+2}$. The authors then show a third key result that

$$\left|zg'(z) - 2^n g(z)\right| < 1/2|g(z)|,$$

for $\rho_n \leqslant |z| \leqslant \rho'_n$.

The final key result we need is that for $\rho_n \leqslant r \leqslant \rho'_n$, we have that $\arg g(re^{i\theta})$ is an increasing function of θ and increases by $2^n 2\pi$ as θ increases by 2π. The authors conclude that for $\rho_n \leqslant r \leqslant \rho'_n$ the circle $\{z : |z| = r\}$ contains exactly 2^n arcs where $\operatorname{Re} g(re^{i\theta}) < \log \rho$ for r and n sufficiently large that $\{z : |z| = r\} \cap D \neq \emptyset$ and $\log \rho > -2^{2^n}$.

Let D be a direct tract of f with boundary value $R > 0$. As f has no zeros, D is simply connected. Further, from [76, Example 3.3], f has a bounded set of asymptotic values, and we can assume they all lie in $D(0, R)$. Hence, it remains to check that the critical values of f are suitably well separated in order to satisfy condition 2 in Theorem 4.4.1. We deduce this by using the key results above discussed in [12], as well as an application of Rouché's theorem and properties of level curves.

First, from the third key result above $g'(z) \neq 0$ in $\overline{A(\rho_n, \rho'_n)}$ for any n and so any critical points of f must lie in $\cup_{n\in\mathbb{N}} A(\rho'_{n-1}, \rho_n)$. Now, since

$$\left|zg'(z) - 2^n g(z)\right| < 1/2|g(z)|,$$

for $\rho_n \leqslant |z| \leqslant \rho'_n$, by Rouché's theorem $g(z)$ and $zg'(z)$ have the same number of zeros in $D(0, \rho_n)$. Further, $g(z)$ has 2^n zeros in $D(0, \rho_n)$ by the second key result and another application of Rouché's theorem. Therefore, f has exactly $2^n - 1$ critical points in $D(0, \rho_n)$ and hence 2^{n-1} critical points in $A(\rho'_{n-1}, \rho_n)$, for $n \geqslant 1$. All such critical points of f, apart from 0, must lie in components of $A(\rho'_{n-1}, \rho_n) \cap D$, for some $n \geqslant 1$. From each such critical point there must originate at least 4 unbounded level curves which go to ∞ through $A(\rho_n, \rho'_n) \cap D$. By the fourth key result above, exactly 2 of these curves can pass through each component of $A(\rho_n, \rho'_n) \cap D$, and the unions of these level curves must meet the tree T. Hence, the modulus of the level curves must be at least the minimum modulus of f on the tree T in $A(\rho'_{n-1}, \rho_n) \cap D$. Therefore, the modulus of each of the critical values of the critical points in $A(\rho'_{n-1}, \rho_n) \cap D$ is at least

$$\min\{|f(z)| : z \in T, |z| = \rho'_{n-1}\} \geqslant \exp(2^{2^{n-1}}), \tag{4.5.1}$$

for $n \geqslant 1$, by the first key result and since $r_{n-1} < \rho'_{n-1} < r'_{n-1}$.

We claim that for any $k > 1$ there exist arbitrarily large r such that the annulus $A(r, kr)$ contains no critical values of f. To prove this, first note that, for all $n \geqslant 1$, the function f has $2^n - 1$ critical points in $D(0, \rho_n)$. Hence, by (4.5.1) (with $n-1$ replaced by n), there are at most $2^n - 1$ critical values of f in $A(\exp(2^{2^{n-1}}), \exp(2^{2^n}))$. Now, the number of annuli of the form $A(\exp(2^{2^{n-1}})k^m, \exp(2^{2^n})k^{m+1})$ that lie in $A(\exp(2^{2^{n-1}}), \exp(2^{2^n}))$ is at least $2^{2^{n-1}}/\log k$, because

$$\exp(2^{2^n}) \geqslant \exp(2^{2^{n-1}})k^{2^{2^{n-1}}/\log k}.$$

This proves the claim, since at most $2^n - 1$ of these annuli can contain a critical value of f. Therefore, we may apply Theorem 4.4.1 and obtain that $\dim J(f) \cap K(f) > 1$.

Remark 4.5.3. Bergweiler and Karpińska [14, Theorem 1.1] show that if f is a transcendental entire function that satisfies a certain regularity condition, then $\dim J(f) \cap I(f) = 2$. This result applies to our Example 4.5.1, but not to Example 4.5.2 since the regularity condition in [14] implies that f has finite order [14, p. 533], which is not the case for Example 4.5.2 by a theorem of Pólya [34, Theorem 2.9].

5 FURTHER WORK

In this last chapter, we discuss some further work and open questions that the contents of this thesis naturally lead to. This chapter is divided into questions on escaping points and their prescribed orbits, questions on the Hausdorff dimension of points with certain orbits, and questions on Wiman–Valiron type discs in direct tracts.

5.1 ITERATION IN TRACTS

The first questions come from Chapter 2. Following Rippon and Stallard's original results on slow escaping points [59], we proved in Theorem 2.1.1 that there exist points that escape arbitrarily slowly in a given logarithmic tract and then in Theorem 2.1.3 we showed that these points exist provided there are certain geometric conditions on the given direct tract, namely that the tract has bounded geometry with respect to harmonic measure. This motivates the first question.

Question 5.1.1. Can the results in Theorem 2.1.1 and Theorem 2.1.3 be generalized to an arbitrary direct tract?

The proofs that there exist points that escape arbitrarily slowly in both a logarithmic tract (Theorem 2.1.1) and the direct tracts with nice geometric properties (Theorem 2.1.3) rely crucially on annulus covering results (Lemma 2.3.1 and Lemma 2.4.1). In general, new techniques will be needed in order to fully resolve this question.

In Chapter 2 we also proved a two-sided result on slow escape (Theorem 2.1.2) for logarithmic tracts. This motivates the second question.

Question 5.1.2. Can the results in Theorem 2.1.2 be generalized to an arbitrary direct tract?

To start with, one can possibly generalize Theorem 2.1.2 to more general direct tracts similar to those with bounded geometry with respect to harmonic measure which we considered in Theorem 2.1.3. While Theorem 2.1.2 requires even more control on what is covered by our annulus covering results, we note that an argument similar to that given in the proof of Theorem 2.1.3 would yield a result akin to the two-sided slow escape result in Theorem 2.1.2. The difference is how the sequence of domains in the definition of bounded geometry with respect to harmonic measure (Definition 2.4.2) is chosen. Given a sequence as in Theorem 2.1.2, one would then need to choose direct tracts with properties based on the given sequence.

Next, in [61], Rippon and Stallard give the notion of an annular itinerary. A point is given an annular itinerary which describes the orbit of this point in relation to a sequence of annuli that partition the plane. There exist many types of prescribed annular itineraries, and they describe when these annular itineraries can or can not be realized. This leads to the question of whether such results apply for direct tracts.

Question 5.1.3. Can one obtain annular itineraries as in [61] in direct tracts?

Note that the covering lemma used to prove the existence of points that escape arbitrarily slowly in logarithmic tracts, Lemma 2.3.1, should allow one to prove results analogous to those in [61] for logarithmic tracts.

5.2 HAUSDORFF DIMENSION

The techniques in Chapter 2 give an infinite set of points that escape slower than a given sequence. Bergweiler and Peter [15] have shown that for transcendental entire functions in the class $\mathcal{B}$, the Hausdorff dimension of this set is greater than or equal to one. Sixsmith [66] studied slow escaping points for functions in the exponential family, that is functions of the form λe^z for $\lambda \in \mathbb{C} \setminus \{0\}$. He proved that the set of points with a certain two sided slow escape condition have Hausdorff dimension equal to one. He further looked at the points that escape moderately slowly and showed that their dimension is equal to one. This motivates the next question related to the Hausdorff dimension of the set of points that both escape slower than a given sequence, as well as the dimension of the set of points whose orbits are bounded by two prescribed sequences.

Question 5.2.1. What is the Hausdorff dimension of the set of points that escape slower than a certain sequence in a direct tract? What can be said for a slow escaping set determined by a two-sided condition?

Currently, the only known examples of entire functions, f, for which $\dim J(f) = 1$ have multiply connected wandering domains, namely the example given by Bishop [18] and examples based on it. We note that if a function has multiply connected wandering domains, then it has only one direct tract that has no unbounded complementary components. Further, Bergweiler has given examples in [11] for which the dimension of $K(f)$ is arbitrarily small. These functions have no direct tracts with an unbounded complementary component. On the other hand, it is known that $\dim J(f) \cap K(f) > 1$ for functions in the class $\mathcal{B}$ [6]. So, it seems natural to ask the following.

Question 5.2.2. Let f be a meromorphic function with a direct tract with an unbounded complementary component. Is $\dim J(f) \cap K(f) > 1$?

We gave a partial result towards this in Chapter 4. However, in our proof, we are limited by assumptions on the singular values of f and the existence of a large enough Wiman–Valiron type disc on which we can apply our results.

5.3 DISCS IN TRACTS

Bergweiler [9] shows that the size of a Wiman–Valiron disc is, in fact, slightly larger than the estimate given in [16]. He also gives examples in [9] to show this new bound is sharp, proving that any larger disc would contain zeros for the considered function. These examples, though, all have no direct tracts with unbounded complementary components. This suggests the next question, which has connections with the previous question.

Question 5.3.1. Let D be a direct tract of a function f and let $z_r \in D$ be a point at which $|f|$ takes its maximum modulus on $\{z \in D : |z| = r\}$. What is the largest disc centered at z_r and contained in D if certain conditions are imposed on D? In particular, what is the largest disc if D is logarithmic?

Most likely similar techniques to those in [9] would give a larger disc in a logarithmic tract than the size in Chapter 4.

Finally, inside a Wiman–Valiron disc, the Wiman–Valiron estimate on the size of f holds as in [16], that is for $\tau > 1/2$ in Theorem 4.3.4. In Theorem 4.3.4, we gave an estimate on the error in a larger disc for $\tau < 1/2$, but not for $\tau = 1/2$.

Question 5.3.2. What estimate can we give for the error g in Theorem 4.3.4 for $\tau = 1/2$?

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NOTATION

(a_n)	Sequence given by $a_0, a_1, a_2, \ldots$
$A(f)$	Fast escaping set of f (see Section 1.3)
$A(r, R)$	Open annulus $\{z : r < \|z\| < R\}$
$\mathcal{B}$	The Eremenko-Lyubich class (see Section 1.1.2)
$\mathbb{C}$	Complex plane
$\mathbb{D}$	Unit disc
$D(a, r)$	Disc centered at a of radius r
$\dim U$	Hausdorff dimension of U
f^{-1}	Inverse of f
f^n	The nth iterate of f
$F(f)$	Fatou set of f (see Section 1.1 and Section 1.1.3)
$\mathbb{H}$	Right half-plane $\{z : \operatorname{Re} z > 0\}$
H_R	Right half-plane $\{z : \operatorname{Re} z > R\}$
$I(f)$	Escaping set of f (see Section 1.2)
$\mathbb{N}$	Set of non-negative integers
$J(f)$	Julia set of f (see Section 1.1 and Section 1.1.4)
$K(f)$	Set of points of bounded orbit under f
$L(f)$	Slow escaping set of f
$M(r, f)$	Maximum modulus of f on the circle of radius r, also $M(r)$
$m(r, f)$	Minimum modulus of f on the circle of radius r
$M_D(r, f)$	Maximum modulus of f on the circle of radius r inside the direct tract D, also $M_D(r)$
$\overline{U}$	Closure of U in $\mathbb{C}$
$P_\zeta(z)$	Poisson kernel (see Section 1.6.3)
$\rho_D(z_1, z_2)$	Hyperbolic distance between z_1 and z_2 in D (see Section 1.6.2)

$\mathcal{S}$	The Speiser class (see Section 1.1.2)
$\operatorname{sing}(f^{-1})$	Singularities of the inverse function (see Section 1.1.2)
σ_D	Hyperbolic density on D (see Section 1.6.2)
$\omega(z, E, D)$	Harmonic measure of E with respect to D at z (see Section 1.6.3)
∂U	Boundary of U
$\mathbb{Z}$	Set of integers