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The Impact of Proportional Transaction Costs on Systematically Generated Portfolios*

Johannes Ruf[†] and Kangjiana Xie[‡]

Abstract. The effect of proportional transaction costs on systematically generated portfolios is studied empirically. The performance of several portfolios (the index tracking portfolio, the equally-weighted portfolio, the entropy-weighted portfolio, and the diversity-weighted portfolio) in the presence of dividends and transaction costs is examined under different configurations involving the trading frequency, constituent list size, and renewing frequency. All portfolios outperform the index tracking portfolio in the absence of transaction costs. This outperformance is statistically significant for daily and weekly traded portfolios but not for monthly traded portfolios. However, when proportional transaction costs of 0.5% are imposed, most portfolios no longer outperform the market. Some exceptional cases include the entropy-weighted and the diversity-weighted portfolios under specific configurations. The only statistical significant difference appears for the relative underperformance of the equally-weighted portfolio.

Key words. Diversity-weighted portfolio; equally-weighted portfolio; functionally generated portfolio; portfolio analysis; Stochastic Portfolio Theory; transaction cost

AMS subject classifications. 91G10

1. Introduction. Although often neglected in portfolio analysis for sake of simplicity, transaction costs matter significantly for portfolio performance. Even small proportional transaction costs can have a large negative effect, especially when trades are made to rebalance the portfolio in a relatively high frequency. Hence, one should at least test the performance of a given portfolio when transaction costs are imposed, even if transaction costs are not explicitly taken into account while constructing the portfolio.

In this paper, we examine the effects of imposing transaction costs on systematically generated portfolios, in particular, functionally generated portfolios. Such portfolios play a significant role in Stochastic Portfolio Theory; see [7]. [23] and [13] demonstrate empirically that functionally generated portfolios outperform the market portfolio in the absence of transaction costs. To explore whether or to what extent this result still holds when transaction costs are imposed, we empirically examine the performance of portfolios (the index tracking portfolio, the equally-weighted portfolio, the entropy-weighted portfolio, and the diversity-weighted portfolio) under different configurations relating to trading frequency, transaction cost rate, constituent list size, and renewing frequency.

[16] are among the first to study the impact of proportional transaction costs in portfolio choice. We refer to [12] and [17] for an overview of the transaction cost literature that evolved afterwards. Most of this literature focuses on the case of one risky asset only. For a discussion

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of transaction costs in the presence of several risky assets, we refer to [18], [4], and [22]. An empirical analysis of the effects of transaction costs is provided in [25], [1], [21], and [20]. We follow up on this research by providing a systematic analysis of the impact of transaction costs on functionally generated portfolios.

When backtesting the portfolios with historical data, the index tracking portfolio is used as benchmark. In the absence of transaction costs, the equally-weighted, the entropy-weighted, and the diversity-weighted portfolios outperform the index tracking portfolio. The outperformance is statistically significant for daily and weekly traded portfolios but not for monthly traded portfolios. In particular, the equally-weighted portfolio performs better than any other portfolio under the same configuration. When proportional transaction costs of 0.5% are imposed, however, the equally-weighted portfolio underperforms all other portfolios. The entropy-weighted and the diversity-weighted portfolios still outperform the benchmark but not significantly under appropriate trading frequencies and constituents list sizes with yearly excess returns around 1bp to 4bp.

The following is an outline of this paper. In section 2, we propose a framework of backtesting portfolio performance in the presence of transaction costs. In particular, we incorporate proportional transaction costs when rebalancing a portfolio in subsection 2.1 and provide some practical considerations and details when backtesting portfolio performance in subsection 2.2. In section 3, we empirically examine the performance of several different portfolios under various configurations. The conclusions follow in section 4.

2. Backtesting in the presence of transaction costs.

2.1. Incorporating transaction costs into wealth dynamics. We shall study the performance of long-only stock portfolios that are rebalanced discretely. The market is not assumed to be frictionless; transaction costs are imposed when we trade in the market to rebalance the portfolios. The portfolios are constructed in such a way that their weights match given target weights after paying transaction costs. This construction is more rigid than the one in [11], for example, where the portfolio weights may deviate from the target weights.

To be more specific, consider a market with $d \geq 2$ stocks. Denote the amount of currency invested in each stock by $\psi(\cdot) = (\psi_1(\cdot), \dots, \psi_d(\cdot))'$ and the total amount invested in a portfolio by $V(\cdot) = \sum_{i=1}^d \psi_i(\cdot) \geq 0$. Furthermore, denote the portfolio weights by $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))'$. Note that $\psi_i(\cdot) = \pi_i(\cdot)V(\cdot)$, for all $i \in \{1, \dots, d\}$.

Assume that trading stocks involves proportional transaction costs at a time-invariant rate tc^b (tc^s), with $0 \leq tc^b, tc^s < 1$ for buying (selling) a stock. This means that the sale of one unit of currency of a stock nets only $(1 - tc^s)$ units of currency in cash, while buying one unit of currency of a stock costs $(1 + tc^b)$ units of currency.

Let us now consider how to trade the stocks in order to match the target weights when transaction costs are imposed. To begin, let us focus on trading at a specific time t . When rebalancing the portfolio at time t , we know the wealth $\psi(t-)$ invested in each stock and hence the total wealth of the portfolio $V(t-) = \sum_{i=1}^d \psi_i(t-)$ (exclusive of dividends). We also know the dividends paid at time $t-$, their total denoted by $D(t-) \geq 0$.

Given target weights π , we require $\pi(t) = \pi$ after the portfolio is rebalanced at time t .

78 After trading, the wealth $\psi(t)$ invested in each stock in the portfolio satisfies

$$79 \quad (2.1) \quad \psi_j(t) = \pi_j(t) \sum_{i=1}^d \psi_i(t), \quad j \in \{1, \dots, d\}.$$

80 We provide details about how to compute $\psi(t)$ later in this subsection.

81 As the portfolio needs to be self-financing, the amount of currency used to buy extra
82 stocks should be exactly the amount of currency obtained from selling redundant stocks plus
83 the dividends if there are any. This yields

$$84 \quad (2.2) \quad (1 + \text{tc}^b) \sum_{i=1}^d (\psi_i(t) - \psi_i(t-))^+ = (1 - \text{tc}^s) \sum_{i=1}^d (\psi_i(t-) - \psi_i(t))^+ + D(t-).$$

85 The total transaction costs imposed from trading stocks at time t are computed by

$$86 \quad (2.3) \quad \text{TC}(t) = \text{tc}^b \sum_{i=1}^d (\psi_i(t) - \psi_i(t-))^+ + \text{tc}^s \sum_{i=1}^d (\psi_i(t-) - \psi_i(t))^+.$$

87 Therefore, the total wealth of the portfolio at time t , given by $V(t) = \sum_{i=1}^d \psi_i(t)$, satisfies

$$88 \quad V(t) = V(t-) + D(t-) - \text{TC}(t).$$

89 **Method of computing $\psi(t)$.** In the following, we propose a method to compute $\psi(t)$,
90 given $\psi(t-)$, $D(t-)$, and the target weights π . Throughout this section, we assume

$$91 \quad V(t-) > 0, \quad D(t-) \geq 0, \quad \sum_{i=1}^d \pi_i = 1, \quad \pi_j \geq 0, \quad \text{and} \quad \psi_j(t-) \geq 0,$$

92 for all $j \in \{1, \dots, d\}$.

93 To begin with, (2.1) implies that $\psi(t)$ is of the form

$$94 \quad (2.4) \quad \psi_j(t) = cV(t-)\pi_j(t), \quad j \in \{1, \dots, d\},$$

95 for some $c > 0$. Note that if the market is frictionless, i.e., if $\text{tc}^b = \text{tc}^s = 0$, and if there
96 are no dividends paid at time $t-$, i.e., if $D(t-) = 0$, then $V(t) = V(t-)$ and $c = 1$. When
97 transaction costs are imposed, we shall use the constraint (2.2) to determine c .

98 To make headway, define

$$99 \quad (2.5) \quad \hat{D} = \frac{D(t-) + (1 - \text{tc}^s) \sum_{i=1}^d \psi_i(t-) \mathbf{1}_{\pi_i(t)=0}}{V(t-)}$$

100 and

$$101 \quad c_j = \frac{\pi_j(t-)}{\pi_j(t)} \mathbf{1}_{\pi_j(t)>0}, \quad j \in \{1, \dots, d\}.$$

102 Then dividing both sides of (2.2) by $V(t-)$ yields

$$103 \quad (2.6) \quad \left(1 + tc^b\right) \sum_{i=1}^d (c - c_i)^+ \pi_i(t) = (1 - tc^s) \sum_{i=1}^d (c_i - c)^+ \pi_i(t) + \widehat{D}.$$

104 Note that the LHS of (2.6) is a continuous function of c and strictly increasing from 0
 105 to ∞ , as c changes from $\min_{i \in \{1, \dots, d\}} c_i$ to ∞ . Moreover, the RHS of (2.6) is a continuous
 106 function of c strictly decreasing from ∞ to $\widehat{D} \geq 0$, as c changes from $-\infty$ to $\max_{i \in \{1, \dots, d\}} c_i$,
 107 and equals \widehat{D} afterwards, as c changes from $\max_{i \in \{1, \dots, d\}} c_i$ to ∞ . Hence, both sides of (2.6)
 108 as functions of c must intersect at some unique point, i.e., a unique solution exists for (2.6).
 109 To proceed, define

$$110 \quad (2.7) \quad \widehat{D}_j = \left(1 + tc^b\right) \sum_{i=1}^d (c_j - c_i)^+ \pi_i(t) - (1 - tc^s) \sum_{i=1}^d (c_i - c_j)^+ \pi_i(t), \quad j \in \{1, \dots, d\}.$$

111 We are now ready to provide an expression for the unknown constant c .

112 **Proposition 2.1.** *Recall that (2.5) and (2.7) imply $\widehat{D} \geq 0$ and $\min_{i \in \{1, \dots, d\}} \widehat{D}_i \leq 0$. Hence,*

$$113 \quad (2.8) \quad j = \arg \max_{i \in \{1, \dots, d\}} \left\{ \widehat{D}_i; \widehat{D}_i \leq \widehat{D} \right\}$$

114 *is well-defined. Then*

$$115 \quad (2.9) \quad c = \frac{\left(1 + tc^b\right) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i \leq c_j} + (1 - tc^s) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i > c_j} + \widehat{D}}{\left(1 + tc^b\right) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i \leq c_j} + (1 - tc^s) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i > c_j}}$$

116 *solves (2.6) uniquely.*

117 The proof of [Proposition 2.1](#) is given in [Appendix A](#).

118 **Remark 2.2.** In practice, we can apply both numerical and analytical methods to find the
 119 constant c . As suggested by (2.6), to find c numerically, we can simply search for the minimum
 120 of the function

$$121 \quad c \mapsto \left| \left(1 + tc^b\right) \sum_{i=1}^d (c - c_i)^+ \pi_i(t) - (1 - tc^s) \sum_{i=1}^d (c_i - c)^+ \pi_i(t) - \widehat{D} \right|.$$

122 Alternatively, by determining the index j given by (2.8), we can apply [Proposition 2.1](#) to
 123 compute c analytically.

124 If the analytical approach is implemented, we can speed up the algorithm by making the
 125 following observations. We expect the value of c not to be far away from 1, which is precisely
 126 the value in the case of no transaction costs and no dividends. As suggested by the proof of
 127 [Proposition 2.1](#), the family $(\widehat{D}_i)_{i \in \{1, \dots, d\}}$ has the same ranking as $(c_i)_{i \in \{1, \dots, d\}}$. Therefore, we
 128 proceed by ranking all c_i 's in ascending order and comparing \widehat{D}_k with \widehat{D} , where

$$129 \quad k = \arg \max_{i \in \{1, \dots, d\}} \{c_i; c_i \leq 1\}.$$

130 If $\widehat{D}_k = \widehat{D}$, then $j = k$ and we are done. If $\widehat{D}_k > \widehat{D}$, then we repeatedly compute \widehat{D}_i
 131 corresponding to a smaller $c_i < c_k$ each time until we find the exact index j . If $\widehat{D}_k < \widehat{D}$, then
 132 we simply go the other way around.

133 **Proposition 2.1** is applied to determine the constant c used in (2.4) in order to compute
 134 $\psi(t)$. Note that, in this subsection, we take $\psi(t-)$ and $D(t-)$ as given. In the next subsection,
 135 we discuss how to compute $\psi(t-)$ and $D(t-)$ from the data.

136 **2.2. Practical considerations.** For the preparation of the empirical study in the next
 137 section, we now introduce the method used to backtest the portfolio performance. To begin
 138 with, assume that we are given the total market capitalizations $S(\cdot) = (S_1(\cdot), \dots, S_d(\cdot))'$ and
 139 the daily returns $r(\cdot) = (r_1(\cdot), \dots, r_d(\cdot))'$ for all stocks. Assume that there are in total N
 140 days. For all $l \in \{1, \dots, N\}$, let t_l denote the end of day l , at which the end of day total
 141 market capitalizations and the daily returns for day l are available. Moreover, if we trade on
 142 day l , then we call day l a trading day and the trade is made at time t_l .

143 Now focus on a specific trading day l with $l \in \{1, \dots, N\}$ and fix $i \in \{1, \dots, d\}$ for the
 144 moment. In subsection 2.1, given $\psi(t_l-)$ and $D(t_l-)$, as well as the target weights specified by
 145 the corresponding portfolio at time t_l , we have shown how to compute $\psi(t_l)$. In the following,
 146 we show how to obtain $\psi(t_l-)$ and $D(t_l-)$.

147 The daily return $r_i(t_l)$ includes the dividends of stock i if there are any. We decompose
 148 the daily return $r_i(t_l)$ into two parts: the dividend yield $r_i^D(t_l)$ and the realised rate $r_i^R(t_l)$.
 149 The dividend yield $r_i^D(t_l)$ is computed as

$$150 \quad (2.10) \quad r_i^D(t_l) = \max \left\{ 1 + r_i(t_l) - \frac{S_i(t_l)}{S_i(t_{l-1})}, 0 \right\}$$

151 and yields the amount of dividends received at time t_l for each unit of currency invested in
 152 stock i at time t_{l-1} ¹. The realised rate $r_i^R(t_l)$ is computed as

$$153 \quad r_i^R(t_l) = r_i(t_l) - r_i^D(t_l)$$

154 and yields the units of currency held in stock i at time t_l for each unit of currency invested in
 155 stock i at time t_{l-1} .

156 The maximum is used in (2.10) to make sure that the dividend yield is nonnegative.
 157 Indeed, occasionally the data may suggest $S_i(t_{l-1})(1 + r_i(t_l)) < S_i(t_l)$. This can happen, for
 158 example, when company i issues extra stocks at time t_l . In this case, we simply assume that
 159 there are no dividends paid at time t_l .

160 A special situation requires us to pay extra attention. A few times, some stock i is delisted
 161 from the market at time t_l , for example, due to bankruptcy or merger. In this case, we still
 162 have data for $r_i(t_l)$, but not for $S_i(t_l)$. To deal with this situation, we assume that there are
 163 no dividends paid in stock i at time t_l . As a result, we have $r_i^D(t_l) = 0$ and $r_i^R(t_l) = r_i(t_l)$ for

¹The dividends computed from the dividend yield r^D contain not only the actual stock dividends, but also other corporate actions. For example, AT&T, which dominated the telephone market for most of the 20th century, was broken up into eight smaller companies in 1984. This led to a significant drop in the stock price. In our analysis below, we assume that the investor obtained cash in exchange (instead of stocks in the newly established companies).

164 such stock i . To close the position in stock i , we assume that one needs to pay transaction
165 costs.

166 Without loss of generality, assume that there are $n \geq 1$ days (including the trading day
167 l) involved since the last trading day, i.e., the last trading day before l is $l - n$. For all
168 $k \in \{l - n + 1, \dots, l\}$, we compute $r^D(t_k)$ and $r^R(t_k)$ as above. In particular, if some stock
169 i in the portfolio is delisted from the market at time t_u , for some $u \in \{l - n + 1, \dots, l - 1\}$,
170 then we set $r_i^R(t_v) = r_i^D(t_v) = 0$, for all $v \in \{u + 1, \dots, l\}$.

171 Then given $\psi(t_{l-n})$, we compute

$$172 \quad \psi_i(t_{l-}) = \psi_i(t_{l-n}) \prod_{k=l-n+1}^l (1 + r_i^R(t_k)), \quad i \in \{1, \dots, d\}.$$

173 Since all dividends paid between two consecutive trading days are only reinvested at time t_l ,
174 the total dividends available for reinvesting are computed by

$$175 \quad D(t_{l-}) = \sum_{i=1}^d \psi_i(t_{l-n}) \sum_{k=l-n+1}^l r_i^D(t_k) \prod_{u=l-n+1}^{k-1} (1 + r_i^R(t_u)).$$

176 **3. Examples and empirical results.** In this section, we analyze the performance of several
177 portfolios empirically. The target weights are expressed in terms of the market weights $\mu(\cdot) =$
178 $(\mu_1(\cdot), \dots, \mu_d(\cdot))'$ with components

$$179 \quad \mu_j(\cdot) = \frac{S_j(\cdot)}{\sum_{i=1}^d S_i(\cdot)}, \quad j \in \{1, \dots, d\}.$$

180 We shall consider the largest d stocks. We will vary the number d between 100 and 500.
181 The constituent list (the list of the top d stocks) is renewed either monthly or quarterly.
182 Whenever we renew the constituent list, we keep the d stocks with the largest total market
183 capitalizations at that time. We trade only these d stocks afterwards until we renew the
184 constituent list again. If any of these stocks stops to exist in the market due to any reason, we
185 simply invest in the remaining stocks without adding a new stock to the list before we renew
186 it next time. Note that renewing the constituent list implies trading to replace the old top
187 d stocks with the new top d stocks. We trade with a specific frequency, which can be either
188 daily, weekly, or monthly. For research on optimal trading frequency, we refer to [6].

189 At time t_0 , we take the transaction costs due to initializing a portfolio as sunk cost, i.e.,
190 we set $\text{TC}(t_0) = 0$. Moreover, we start a portfolio with initial wealth $V(t_0) = 1000$. Note that
191 unless otherwise mentioned, the logarithmic scale is used when plotting $V(\cdot)$ and $\text{TC}(\cdot)$ for the
192 purpose of better interpretability. To simplify the analysis, we impose a uniform transaction
193 cost rate tc on both buying and selling the stocks, i.e., we set $\text{tc}^b = \text{tc}^s = \text{tc}$.

194 For each example, we provide tables with the yearly returns², the excess returns (relative
195 to the corresponding index tracking portfolio), the standard deviations of the yearly returns,

²The t -statistics of yearly returns of all portfolios considered in this section range from 3.29 to 4.98. Since they are all significant, we shall omit these numbers in the tables below.

196 the Sharpe ratios³, and the average ratio of the yearly transaction costs to the beginning of
 197 year portfolio wealth of the portfolios.

198 **Data source.** The data of the total market capitalizations $S(\cdot)$ and the daily returns $r(\cdot)$
 199 is downloaded from the CRSP US Stock Database⁴. This database contains the traded stocks
 200 on all major US exchanges. More precisely, we focus on ordinary common stocks⁵. The data
 201 starts January 2nd, 1962 and ends December 30th, 2016.

202 The total market capitalizations are computed by multiplying the numbers of outstanding
 203 shares with the share prices, and are essential in determining the target weights. The daily
 204 returns include dividends but also delisting returns in case stocks get delisted (for example,
 205 the recovery rate in case a traded firm goes bankrupt).

206 **3.1. Index tracking portfolio.** In this subsection, we introduce the index tracking port-
 207 folio. This portfolio is used to benchmark the performance of other portfolios studied in the
 208 following subsections. The index tracking portfolio has target weights

$$209 \quad \pi_j(\cdot) = \mu_j(\cdot), \quad j \in \{1, \dots, d\}.$$

210 Note that this portfolio is rebalanced only when the constituent list changes or when dividends
 211 are reinvested.

212 The index tracking portfolio includes the effects of paying transaction costs and reinvesting
 213 dividends. In contrast, the capitalization index with wealth process

$$214 \quad \sum_{i=1}^d S_i(\cdot) \times \frac{1000}{\sum_{i=1}^d S_i(t_0)}$$

215 does not take transaction costs and dividends into consideration.

216 In the following, we examine the performance of the index tracking portfolio under different
 217 trading frequencies, renewing frequencies, as well as constituent list sizes d , when there are
 218 no transaction costs, i.e., when $tc = 0$, and when $tc = 0.5\%$ and $tc = 1\%$, respectively. These
 219 numbers are consistent with the transaction cost estimates in [25], [15], [19], and [10].

220 **Varying the trading frequency.** We fix the constituent list size $d = 100$ and use monthly
 221 renewing frequency. Table 1 shows the performance of the index tracking portfolio and the
 222 corresponding capitalization index under daily, weekly, and monthly trading frequencies, re-
 223 spectively. Note that the capitalization index does not depend on the trading frequency. As
 224 expected, with the same trading frequency, the portfolio performs worse under a larger trans-
 225 action cost rate tc . In addition, the portfolio outperforms the corresponding index, which
 226 implies that the dividends paid exceed the transaction costs imposed even if $tc = 1\%$.

227 **Varying the renewing frequency.** Still fixing the constituent list size $d = 100$, we now
 228 use daily trading frequency and vary the renewing frequency between monthly and quarterly

³To compute the Sharpe ratios of the portfolios and the indices, the one-year U.S. Treasury yields are used. The data of these yields can be downloaded from <https://www.federalreserve.gov>.

⁴See <http://www.crsp.com/products/research-products/crsp-us-stock-databases> for details.

⁵Those stocks in CRSP which have ‘Share Code’ 10, 11, or 12.

Table 1

Yearly returns (YR) in percentage, standard deviations of yearly returns (Std), Sharpe ratios (SR), and the average ratio of the yearly transaction costs to the beginning of year portfolio wealth (TR) in percentage of the index tracking portfolio (IT) and the corresponding capitalization index (CI) under different trading frequencies, renewing frequencies, constitute list sizes, and transaction cost rates tc. The first superscripts d , w , and m indicate daily, weekly, and monthly trading frequencies, respectively, and the second superscripts M and Q correspond to monthly and quarterly renewing frequencies, respectively. The first subscripts s and l indicate $d = 100$ and $d = 500$, respectively, and the second subscript x corresponds to $tc = x\%$.

	CI_s^M	$IT_{s,0}^{d,M}$	$IT_{s,0.5}^{d,M}$	$IT_{s,1}^{d,M}$	$IT_{s,0}^{w,M}$	$IT_{s,0.5}^{w,M}$	$IT_{s,1}^{w,M}$	$IT_{s,0}^{m,M}$	$IT_{s,0.5}^{m,M}$	$IT_{s,1}^{m,M}$
YR	8.84	10.30	10.09	9.89	10.30	10.10	9.90	10.27	10.08	9.89
Std	16.59	16.87	16.84	16.81	16.88	16.85	16.82	16.88	16.86	16.83
SR	0.22	0.30	0.29	0.28	0.30	0.29	0.28	0.30	0.29	0.28
TR			0.21	0.42		0.20	0.40		0.19	0.38
	CI_s^Q	$IT_{s,0}^{d,Q}$	$IT_{s,0.5}^{d,Q}$	$IT_{s,1}^{d,Q}$	CI_l^M	$IT_{l,0}^{d,M}$	$IT_{l,0.5}^{d,M}$	$IT_{l,1}^{d,M}$		
YR	8.82	10.34	10.20	10.06	9.01	10.83	10.71	10.59		
Std	16.44	16.83	16.81	16.79	16.15	16.61	16.60	16.58		
SR	0.22	0.31	0.30	0.29	0.24	0.34	0.33	0.33		
TR			0.15	0.29			0.14	0.27		

229 frequencies, respectively. As shown in Table 1, under the same transaction cost rate tc, the less
 230 frequently the constituent list is renewed, the better the portfolio performs. As trades are made
 231 when we renew the constituent list, renewing more frequently will impose larger transaction
 232 costs, which impacts the performance of the portfolio to a higher degree. Additionally, the
 233 more frequently the constituent list is renewed, the more sensitive the portfolio is to a larger
 234 transaction cost rate tc.

235 **Varying the constituent list size d .** With daily trading and monthly renewing frequencies,
 236 we now backtest the performance of the index tracking portfolio under different constituent
 237 list sizes d . As shown in Table 1, the portfolio outperforms the corresponding index even with
 238 transaction cost rate $tc = 1\%$. The more stocks the constituent list contains, the better the
 239 portfolio performs.

240 **3.2. Equally-weighted portfolio.** This subsection examines the equally-weighted portfolio
 241 (see [3] and [26] for a discussion of this portfolio in the context of defined contribution plans,
 242 and [5] for a careful study of its properties). Here, the target weights are given by

$$243 \quad \pi_j(\cdot) = \frac{1}{d}, \quad j \in \{1, \dots, d\}.$$

244 For each portfolio with a specific trading frequency, a specific renewing frequency, and
 245 a specific constituent list size d , we examine its performance when there are no transaction
 246 costs, i.e., when $tc = 0$, and when $tc = 0.5\%$ and $tc = 1\%$, respectively. As shown in
 247 the following, the equally-weighted portfolio outperforms the corresponding index tracking
 248 portfolio when there are no transaction costs. This well-behaved performance of the equally-
 249 weighted portfolio within a frictionless market is popular in the academic literature. However,
 250 the equally-weighted portfolio is very sensitive to transaction costs. Its performance is strongly
 251 compromised even with a small transaction cost rate $tc = 0.5\%$.

252 **Varying the trading frequency.** Let us fix $d = 100$ and apply monthly renewing frequency.
 253 **Table 2** summarises the performance of the equally-weighted portfolio under different trading
 254 frequencies and transaction cost rates tc . When there are no transaction costs, i.e., when
 255 $tc = 0$, the equally-weighted portfolio outperforms the corresponding index tracking portfolio
 256 under all three different trading frequencies. A similar observation is also provided in [2]. In
 257 addition, the more frequently the portfolio is traded, the better it performs. Trading more
 258 frequently also allows to reinvest the dividends faster, which helps to enhance the portfolio
 259 performance.

Table 2

Yearly returns (YR) and excess returns (ER) with respect to the index tracking portfolio shown in Table 1 in percentage (t-statistics in brackets), standard deviations of yearly returns (Std), Sharpe ratios (SR), and the average ratio of the yearly transaction costs to the beginning of year portfolio wealth (TR) in percentage of the equally-weighted portfolio (EW) under different trading frequencies, renewing frequencies, constitute list sizes, and transaction cost rates tc . The superscripts and subscripts have the same meaning as in Table 1.

	$EW_{s,0}^{d,M}$	$EW_{s,0.5}^{d,M}$	$EW_{s,1}^{d,M}$	$EW_{s,0}^{w,M}$	$EW_{s,0.5}^{w,M}$	$EW_{s,1}^{w,M}$	$EW_{s,0}^{m,M}$	$EW_{s,0.5}^{m,M}$	$EW_{s,1}^{m,M}$
YR	11.10	9.19	7.31	10.94	9.82	8.72	10.53	9.81	9.10
ER	0.79	-0.91	-2.58	0.64	-0.27	-1.18	0.26	-0.27	-0.79
	[2.35]	[-2.77]	[-7.80]	[1.92]	[-0.83]	[-3.60]	[0.84]	[-0.88]	[-2.60]
Std	16.83	16.65	16.48	16.93	16.81	16.69	17.00	16.91	16.83
SR	0.35	0.24	0.13	0.34	0.28	0.21	0.31	0.27	0.23
TR		1.81	3.58		1.06	2.10		0.68	1.36
	$EW_{s,0}^{d,Q}$	$EW_{s,0.5}^{d,Q}$	$EW_{s,1}^{d,Q}$	$EW_{l,0}^{d,M}$	$EW_{l,0.5}^{d,M}$	$EW_{l,1}^{d,M}$			
YR	11.21	9.47	7.76	12.52	10.46	8.43			
ER	0.86	-0.73	-2.30	1.70	-0.25	-2.16			
	[2.34]	[-2.06]	[-6.56]	[3.08]	[-0.47]	[-4.07]			
Std	16.82	16.65	16.50	17.07	16.90	16.74			
SR	0.36	0.26	0.16	0.43	0.31	0.19			
TR		1.64	3.25		1.94	3.85			

260 When transaction costs are imposed, **Table 2** suggests that under the same transaction
 261 cost rate tc , the more frequently the portfolio is traded, the larger the decrease in portfo-
 262 lio performance is. The performance of the equally-weighted portfolio is strongly affected
 263 by transaction costs. Even with $tc = 0.5\%$, the corresponding index tracking portfolio out-
 264 performs the equally-weighted portfolio. However, slowing down trading helps to reduce the
 265 influence of transaction costs. Indeed, the performance of the monthly traded equally-weighted
 266 portfolio when $tc = 1\%$ is similar to that of the daily traded one when $tc = 0.5\%$.

267 **Varying the renewing frequency.** Now, with $d = 100$, and daily trading frequency, we
 268 examine the performance of the equally-weighted portfolio under monthly and quarterly re-
 269 newing frequencies, respectively. As shown in **Table 2**, under the same transaction cost rate
 270 tc , the less frequently the constituent list is renewed, the better the portfolio performs. With
 271 $tc = 0.5\%$, the equally-weighted portfolio already performs worse than the corresponding in-
 272 dex tracking portfolio. In particular, the portfolio with a more frequent renewing frequency
 273 is more sensitive to transaction costs. As studied in more detail in **subsection 3.4**, the reason
 274 behind these observations is that trading on renewing days incurs extremely large transaction
 275 costs compared with trading on other days when the constituent list is not renewed. These

276 large transaction costs paid on renewing days strongly impact the portfolio performance.

277 **Varying the market size d .** With daily trading and monthly renewing frequencies, [Table 2](#)
 278 summarises the performance of the equally-weighted portfolio under different constituent list
 279 sizes d . The more stocks the constituent list contains, the better the portfolio performs under
 280 the same transaction cost rate t_c . Again, its performance is reduced by transaction costs.
 281 Even with $d = 500$ and $t_c = 0.5\%$, the equally-weighted portfolio performs worse than the
 282 corresponding index tracking portfolio. In addition, the portfolio with a larger constituent
 283 list size d is not necessarily more sensitive to transaction costs.

284 **Sensitivity of the Sharpe ratio.** We now study the sensitivity of the Sharpe ratio with re-
 285 spect to the transaction cost rate t_c . Specifically, we compute the Sharpe ratios of the monthly
 286 traded equally-weighted and index tracking portfolio for $t_c \in \{0, 0.01\%, 0.02\%, \dots, 0.5\%\}$. As
 287 plotted in [Figure 1](#), the Sharpe ratios of both the equally-weighted and the index tracking port-
 288 folio decrease as t_c becomes larger. On the left hand side of the intersection when $t_c < 0.22\%$,
 289 the equally-weighted portfolio has a higher Sharpe ratio. On the right hand side of the inter-
 290 section when $t_c > 0.22\%$, the inverse situation holds. This indicates that the equally-weighted
 291 portfolio depends more on transaction costs than the index tracking portfolio.

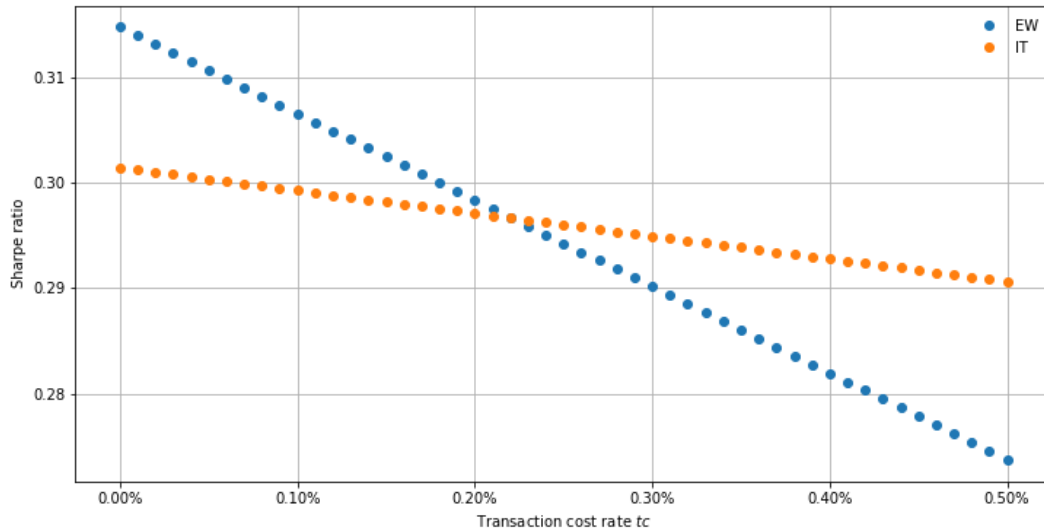


Figure 1. Sharpe ratios of the equally-weighted portfolio (EW) and the index tracking portfolio (IT) under different transaction cost rates t_c with $d = 100$, monthly trading frequency, and monthly renewing frequency.

292 Moreover, as shown in [Figure 1](#), the Sharpe ratio is roughly affine in the transaction cost
 293 rate. As the standard deviations of yearly returns remain relatively stable for each portfolio,
 294 the average yearly return is also roughly affine in transaction cost rate. This observation is
 295 consistent with the value of yearly returns reported in all tables, regardless of the portfolio
 296 considered. In particular, the slope of the line, when multiplied by the negative of the standard
 297 deviation of the portfolio yearly return, is an approximation of the portfolio turnover, as
 298 suggested below by [Remark 3.1](#).

299 *Remark 3.1.* Consider a single period from time 0 to time 1 and let tc_1 and tc_2 be two
 300 different transaction cost rates. Then, given the initial wealth $V(0)$ of a portfolio at time 0,
 301 we have

$$302 \quad r_1 - r_2 \approx \frac{V(1) - TC_1 - V(0)}{V(0)} - \frac{V(1) - TC_2 - V(0)}{V(0)} \approx \frac{(tc_2 - tc_1)TV}{V(0)} = (tc_2 - tc_1)\text{Turnover},$$

303 where r_1 and r_2 are the net returns of the portfolio from time 0 to time 1 with tc_1 and tc_2 ,
 304 respectively, $V(1)$ is the portfolio wealth at time 1 if there are no transaction costs, and TV
 305 is the trading volume of the portfolio. Therefore, we have

$$306 \quad \frac{SR_1 - SR_2}{tc_1 - tc_2} \approx \frac{r_1 - r_2}{\sigma(tc_1 - tc_2)} \approx -\frac{\text{Turnover}}{\sigma},$$

307 where SR_1 and SR_2 are the Sharpe ratios of the portfolio with tc_1 and tc_2 , respectively, and
 308 σ is the standard deviation of the portfolio return.

309 **3.3. Entropy-weighted portfolio.** In this subsection, we consider the entropy-weighted
 310 portfolio (see Section 2.3 in [7] and Example 5.3 in [14]), which relies on target weights

$$311 \quad \pi_j(\cdot) = \frac{\mu_j(\cdot) \log \mu_j(\cdot)}{\sum_{i=1}^d \mu_i(\cdot) \log \mu_i(\cdot)}, \quad j \in \{1, \dots, d\}.$$

312 In the following, we examine the performance of the entropy-weighted portfolio under
 313 specific configurations when there are no transaction costs, i.e., when $tc = 0$, and when
 314 $tc = 0.5\%$. The performance of the entropy-weighted portfolio is less sensitive to transaction
 315 costs and is better when $tc = 0.5\%$, compared with that of the equally-weighted portfolio.

316 **Varying the trading frequency.** As before, when backtesting the portfolio under different
 317 trading frequencies, we set the constituent list size $d = 100$ and apply monthly renewing fre-
 318 quency. Table 3 summarises the performance of the entropy-weighted portfolio under different
 319 trading frequencies. Compared with the equally-weighted portfolio summarised in Table 2,
 320 the entropy-weighted portfolio performs worse (but still outperforms the corresponding index
 321 tracking portfolio) when there are no transaction costs, i.e., when $tc = 0$. However, oppo-
 322 site to the equally-weighted portfolio, the weekly and the monthly traded entropy-weighted
 323 portfolio still outperforms the corresponding index tracking portfolio when $tc = 0.5\%$.

324 Over a large time horizon, the loss in the portfolio wealth resulting from paying transac-
 325 tion costs is usually higher than the cumulative transaction costs imposed. Indeed, paying
 326 transaction costs not only takes money out of the portfolio, but also deprives the opportunity
 327 for making potential gains.

328 **Varying the renewing frequency.** With $d = 100$ and daily trading frequency, we now
 329 examine the performance of the entropy-weighted portfolio applying monthly and quarterly
 330 renewing frequencies, respectively. As shown in Table 3, similar to the equally-weighted
 331 portfolio, the less frequently the constituent list is renewed, the better the entropy-weighted
 332 portfolio performs. When transaction costs are imposed, its performance depends more on
 333 the renewing frequency. However, compared with the equally-weighted portfolio summarised
 334 in Table 2, the performance of the entropy-weighted portfolio is less sensitive to transaction
 335 costs under the same renewing frequency.

Table 3

Yearly returns (YR) and excess returns (ER) with respect to the index tracking portfolio shown in Table 1 in percentage (t -statistics in brackets), standard deviations of yearly returns (Std), Sharpe ratios (SR), and the average ratio of the yearly transaction costs to the beginning of year portfolio wealth (TR) in percentage of the entropy-weighted portfolio (ET) and the corresponding index tracking portfolio (IT) under different trading frequencies, renewing frequencies, constitute list sizes, and transaction cost rates tc . The superscripts and subscripts have the same meaning as in Table 1.

	$ET_{s,0}^{d,M}$	$ET_{s,0.5}^{d,M}$	$ET_{s,0}^{w,M}$	$ET_{s,0.5}^{w,M}$	$ET_{s,0}^{m,M}$	$ET_{s,0.5}^{m,M}$	$ET_{s,0}^{d,Q}$	$ET_{s,0.5}^{d,Q}$	$ET_{l,0}^{d,M}$	$ET_{l,0.5}^{d,M}$
YR	10.53	9.97	10.50	10.12	10.40	10.11	10.58	10.11	11.16	10.75
ER	0.23	-0.12	0.21	0.02	0.14	0.03	0.24	-0.09	0.33	0.04
	[2.03]	[-1.08]	[1.82]	[0.21]	[1.24]	[0.29]	[2.08]	[-0.78]	[2.51]	[0.27]
Std	16.90	16.83	16.92	16.88	16.94	16.90	16.86	16.81	16.66	16.62
SR	0.32	0.28	0.31	0.29	0.31	0.29	0.32	0.29	0.36	0.34
TR		0.53		0.36		0.28		0.45		0.39

336 **Varying the market size d .** Applying daily trading and monthly renewing frequencies, we
 337 backtest the entropy-weighted portfolio under different constituent list sizes d ($= 100$ and 500 ,
 338 respectively), as shown in Table 3. Similar to the equally-weighted and the index tracking
 339 portfolio, the more stocks the constituent list contains, the better the entropy-weighted port-
 340 folio performs. Compared with the equally-weighted portfolio, the entropy-weighted portfolio
 341 with the same d depends less on transaction costs. In particular, with $d = 500$ and $tc = 0.5\%$,
 342 the entropy-weighted portfolio still outperforms the corresponding index tracking portfolio.

343 **3.4. Diversity-weighted portfolio and smoothing transaction costs.** One portfolio that
 344 draws much attention in Stochastic Portfolio Theory is the so-called diversity-weighted port-
 345 folio generated from the “measure of diversity”

$$346 \quad G_p(x) = \left(\sum_{i=1}^d x_i^p \right)^{1/p}, \quad x \in \left\{ (y_1, \dots, y_d)' \in [0, 1]^d; \sum_{i=1}^d y_i = 1 \right\},$$

347 for some fixed $p \in (0, 1)$. Without changing the relative ranking of the stocks, the function
 348 $G_p(\cdot)$ generates portfolio weights smaller (larger) than the corresponding market weights for
 349 stocks with large (small) market weights. This diversification property of G_p is closely re-
 350 lated to the implementation of relative arbitrage portfolios; see Section 7 in [9] for details.
 351 Section 6.3 in [7] provides a theoretical approximation of the diversity-weighted portfolio
 352 turnover. An empirical study of this portfolio using S&P 500 market data can be found in [8]
 353 and Chapter 7 of [7], as well as in Example 5 of [23].

354 In the following, we examine the performance of this portfolio and illustrate the tradeoff
 355 between trading with a higher frequency and paying transaction costs. To achieve this, we
 356 shall replace the market weights by a smoothed version, given by

$$357 \quad \bar{\mu}(\cdot) = \alpha\mu(\cdot) + (1 - \alpha)\Lambda(\cdot)$$

358 with $\alpha \in (0, 1)$. Here, the moving average process $\Lambda(\cdot) = (\Lambda_1(\cdot), \dots, \Lambda_d(\cdot))'$ is given by

$$359 \quad \Lambda_j(\cdot) = \begin{cases} \frac{1}{\delta} \int_0^\cdot \mu_j(t) dt + \frac{1}{\delta} \int_{-\delta}^0 \mu_j(0) dt & \text{on } [0, \delta) \\ \frac{1}{\delta} \int_{-\delta}^\cdot \mu_j(t) dt & \text{on } [\delta, \infty) \end{cases}, \quad j \in \{1, \dots, d\},$$

Table 4

Yearly returns (YR) and excess returns (ER) with respect to the index tracking portfolio (IT) summarised here and in Table 1 in percentage (t-statistics in brackets), standard deviations of yearly returns (Std), Sharpe ratios (SR), and the average ratio of the yearly transaction costs to the beginning of year portfolio wealth (TR) in percentage of the diversity-weighted portfolio (DW) under different trading frequencies, convexity weights α , and transaction cost rates tc with $d = 100$ and quarterly renewing frequency. The superscripts and subscripts have the same meaning as in Table 1.

	$IT_{s,0}^{w,Q}$	$IT_{s,0.5}^{w,Q}$	$IT_{s,1}^{w,Q}$	α	$DW_{s,0}^{d,Q}$	$DW_{s,0.5}^{d,Q}$	$DW_{s,1}^{d,Q}$	$DW_{s,0}^{w,Q}$	$DW_{s,0.5}^{w,Q}$	$DW_{s,1}^{w,Q}$
YR	10.34	10.20	10.06	0.2	10.36	10.20	10.03	10.36	10.20	10.05
				0.6	10.43	10.18	9.93	10.42	10.23	10.03
				1	10.54	10.11	9.68	10.51	10.24	9.96
ER				0.2	0.02	0.00	-0.03	0.02	0.00	-0.01
					[1.35]	[-0.23]	[-1.78]	[1.30]	[0.24]	[-0.78]
				0.6	0.09	-0.02	-0.13	0.09	0.03	-0.03
					[1.74]	[-0.37]	[-2.49]	[1.60]	[0.55]	[-0.51]
				1	0.20	-0.09	-0.38	0.18	0.04	-0.10
					[2.12]	[-1.03]	[-4.19]	[1.90]	[0.41]	[-1.10]
Std	16.85	16.83	16.81	0.2	16.84	16.81	16.79	16.85	16.83	16.80
				0.6	16.84	16.81	16.77	16.86	16.83	16.80
				1	16.84	16.79	16.74	16.87	16.83	16.79
SR	0.31	0.30	0.29	0.2	0.31	0.30	0.29	0.31	0.30	0.29
				0.6	0.31	0.30	0.28	0.31	0.30	0.29
				1	0.32	0.29	0.27	0.32	0.30	0.28
TR		0.14	0.28	0.2		0.16	0.32		0.15	0.29
				0.6		0.24	0.48		0.18	0.37
				1		0.41	0.81		0.26	0.52

360 for a fixed constant $\delta > 0$. This moving average process $\Lambda(\cdot)$ is also included in the portfolio
361 generating function studied in [24]. Then the target weights are given by

$$362 \quad \pi_j(\cdot) = \mu_j(\cdot) \left(\Xi_j(\cdot) - \sum_{i=1}^d \mu_i(\cdot) \Xi_i(\cdot) + 1 \right), \quad j \in \{1, \dots, d\},$$

363 where

$$364 \quad \Xi_j(\cdot) = \frac{\alpha (\bar{\mu}_j(\cdot))^{p-1}}{\sum_{i=1}^d (\bar{\mu}_i(\cdot))^p}, \quad j \in \{1, \dots, d\}.$$

365 To backtest the portfolio, we fix $d = 100$, the renewing frequency to be quarterly, and
366 the “diversity degree” $p = 0.8$. Moreover, we compute the moving average process $\Lambda(\cdot)$ using
367 a one-year window. To be more specific, with daily trading frequency, we set $\delta = 250$; with
368 weekly trading frequency, we set $\delta = 52$. To compute $\Lambda(\cdot)$ under weekly trading frequency,
369 we only use market weights μ 's on the days when transactions are made.

370 **Varying the convexity weight α and the trading frequency.** In Table 4, we summarise
371 the wealth processes of the diversity-weighted and the corresponding index tracking portfolio
372 under both daily and weekly trading frequencies and with three different choices for the
373 convexity weight α , when there are no transaction costs, i.e., when $tc = 0$, and when $tc = 0.5\%$
374 and $tc = 1\%$, respectively.

375 We first consider the case when there are no transaction costs. Everything else equal, the
 376 daily traded diversity-weighted portfolio performs similarly to the weekly traded portfolio.
 377 Under either trading frequency, the smaller the convexity weight α is, the worse the port-
 378 folio performs. Generating the portfolio with a smaller α is somewhat alike to trading less
 379 frequently, as it assigns less weights on the volatile term $\mu(\cdot)$ and more weights on the stable
 380 term $\Lambda(\cdot)$ when constructing $\bar{\mu}(\cdot)$, and thus makes $\bar{\mu}(\cdot)$ less volatile.

381 Next, we consider the case with transaction costs. Under either daily or weekly trading
 382 frequency, a smaller convexity weight α tends to improve the portfolio performance when the
 383 transaction cost rate tc becomes larger. This can be useful, since decreasing α partially cancels
 384 out the effect of transaction costs. Moreover, when $tc = 1\%$, the daily traded portfolio with
 385 $\alpha = 0.2$ performs similarly as the weekly traded portfolio with $\alpha = 0.6$. This indicates that,
 386 instead of trading less frequently in order to avoid paying transaction costs, one can adjust
 387 the convexity weight α to reach a more favourable balance between trading frequently and
 388 paying transaction costs.

389 **4. Conclusion.** In this paper, we empirically study the impact of proportional transaction
 390 costs on systemically generated portfolios. Given a target portfolio, we provide a scheme to
 391 backtest the portfolio using total market capitalization and daily stock return time series. Im-
 392 plementing this scheme, we examine the performance of several portfolios (the index tracking
 393 portfolio, the equally-weighted portfolio, the entropy-weighted portfolio, and the diversity-
 394 weighted portfolio), assuming various transaction cost rates, trading frequencies, portfolio
 395 constituent list sizes, and renewing frequencies.

396 As expected, everything else equal, a portfolio performs worse as transaction costs are
 397 higher and the portfolio renewing frequency of the underlying constituent list is higher. In
 398 the absence of transaction costs, trading under a higher frequency leads to better portfolio
 399 performance. However, in the presence of transaction costs, implementing a higher trading
 400 frequency can also result in larger transaction costs and reduce the portfolio performance
 401 significantly. Hence, trading under an appropriate frequency is necessary in practice. In
 402 addition, with or without transaction costs, a more diversified portfolio containing more stocks
 403 usually performs better.

404 The empirical results indicate that the equally-weighted portfolio performs well relative
 405 to the index tracking portfolio when there are no transaction costs. However, the perfor-
 406 mance of the equally-weighted portfolio is very sensitive to transaction costs. Although the
 407 entropy-weighted portfolio performs a bit worse than the equally-weighted portfolio (but still
 408 outperforms the index tracking portfolio) when there are no transaction costs, its performance
 409 depends much less on transaction costs, compared to the equally-weighted portfolio.

410 **Appendix A. Proof of Proposition 2.1.**

411 *Proof.* By the definition of \widehat{D}_j given in (2.7) and by some basic computations, (2.9) is
 412 equivalent to

$$413 \quad c = c_j + \frac{\widehat{D} - \widehat{D}_j}{(1 + tc^b) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i \leq c_j} + (1 - tc^s) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i > c_j}},$$

414 which implies $\mathbf{1}_{c_i \leq c} \geq \mathbf{1}_{c_i \leq c_j}$, for all $i \in \{1, \dots, d\}$.

415 In the case $\max_{i \in \{1, \dots, d\}} \widehat{D}_i \leq \widehat{D}$, we have $\mathbf{1}_{c_i \leq c_j} = 1$, hence $\mathbf{1}_{c_i \leq c} \leq \mathbf{1}_{c_i \leq c_j}$, for all $i \in$
 416 $\{1, \dots, d\}$. In the case $\max_{i \in \{1, \dots, d\}} \widehat{D}_i > \widehat{D}$, define

$$417 \quad j' = \arg \min_{i \in \{1, \dots, d\}} \left\{ \widehat{D}_i; \widehat{D}_i > \widehat{D} \right\}.$$

418 Then (2.9) is equivalent to

$$419 \quad c = \frac{(1 + \text{tc}^b) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i < c_{j'}} + (1 - \text{tc}^s) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i \geq c_{j'}} + \widehat{D}}{(1 + \text{tc}^b) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i < c_{j'}} + (1 - \text{tc}^s) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i \geq c_{j'}}}$$

$$= c_{j'} + \frac{\widehat{D} - \widehat{D}_{j'}}{(1 + \text{tc}^b) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i < c_{j'}} + (1 - \text{tc}^s) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i \geq c_{j'}}},$$

420 which implies $\mathbf{1}_{c_i > c} \geq \mathbf{1}_{c_i > c_j}$, for all $i \in \{1, \dots, d\}$. All in all, we have shown $\mathbf{1}_{c_i \leq c} = \mathbf{1}_{c_i \leq c_j}$,
 421 for all $i \in \{1, \dots, d\}$.

422 Define next

$$423 \quad \Pi^b = (1 + \text{tc}^b) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i \leq c_j}, \quad \Pi^s = (1 - \text{tc}^s) \sum_{i=1}^d \pi_i(t) \mathbf{1}_{c_i > c_j},$$

$$424 \quad \bar{\Pi}^b = (1 + \text{tc}^b) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i \leq c_j}, \quad \bar{\Pi}^s = (1 - \text{tc}^s) \sum_{i=1}^d c_i \pi_i(t) \mathbf{1}_{c_i > c_j}.$$

426 Hence, after inserting c by (2.9) into (2.6), the LHS of (2.6) becomes

$$427 \quad \text{LHS} = c \Pi^b - \bar{\Pi}^b = \frac{\Pi^b \bar{\Pi}^s - \Pi^s \bar{\Pi}^b + \Pi^b \widehat{D}}{\Pi^b + \Pi^s},$$

428 and the RHS of (2.6) becomes

$$429 \quad \text{RHS} = \bar{\Pi}^s - c \Pi^s + \widehat{D} = \frac{\Pi^b \bar{\Pi}^s - \Pi^s \bar{\Pi}^b - \Pi^s \widehat{D}}{\Pi^b + \Pi^s} + \widehat{D} = \text{LHS}.$$

430 Therefore, c defined by (2.9) indeed solves (2.6). ■

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