Hyperelastic finite deformation analysis with the unsymmetric finite element method containing homogeneous solutions of linear elasticity

Zhi Li[1, 3], Song Cen[1, 4 *], Junbin Huang[2], Chen-Feng Li[3]

[1] *Department of Engineering Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*

[2] *Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

[3] *Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Swansea SA1 8EN, UK*

[4] *AML, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*

Abstract

A recent unsymmetric 4-node, 8-DOF plane finite element US-ATFQ4 is generalized to hyperelastic finite deformation analysis. Since the trial functions of US-ATFQ4 contain the homogenous closed analytical solutions of governing equations for linear elasticity, the key of the proposed strategy is how to deal with these linear *analytical trial functions* (ATFs) during the hyperelastic finite deformation analysis. Assuming that the ATFs can properly work in each increment, an algorithm for updating the deformation gradient interpolated by ATFs is designed. Furthermore, the update of the corresponding ATFs referred to current configuration is discussed with regard to the hyperelastic material model, and a specified model, neo-Hookean model, is employed to verify the present formulation of US-ATFQ4 for hyperelastic finite deformation analysis. Various examples show that the present formulation not only remain the high accuracy and mesh distortion tolerance in the geometrically nonlinear problems, but also possess excellent performance in the compressible or quasi-incompressible hyperelastic finite deformation problems where the strain is large.

* Corresponding to: Song Cen, Department of Engineering Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China. E-mail: censong@tsinghua.edu.cn

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1. Introduction

How to correctly simulate the incompressible limit state is one of the most challenging issues for finite element methods, in which conventional low-order element models often perform very poorly in those problems involving nearly incompressible materials [1-3]. Therefore, researchers have to make great efforts to develop the advanced low-order elements that are free of volumetric locking. In earlier time, within the geometrically linear range, Pian *et al*. [4] and Simo *et al.* [5] proposed the assumed stress formulations and the enhanced assumed strain methods, respectively, that can well solve the nearly incompressible problems of linear elasticity. Actually, robust low-order elements may be more significant for nonlinear and large deformation analysis. Related developments can be found in various literatures, such as the mixed variational methods developed by Simo *et al*. [6], the mixed u/p formulations proposed by Sussman and Bathe [7], the geometric nonlinearity extension of the enhanced assumed strain elements of Simo *et al.* [8]; the F-bar method proposed by de Souza Neto *et al* [9, 10], and so on. Recently, new approaches for hyperelastic materials with regard to finite element analysis can still be found in various literartures. For example, Schröder et al. proposed a new mixed finite element based on a modified Hu-Washizu principle, which shows lower mesh distortion sensitivity than the F-bar method [11]; Müller *et al*. studied a Least-Squares mixed variational formulation for hyperelastic deformations based on approximating stresses and displacements [12]; Wulfinghoff *et al*. proposed a hybrid discontinuous Galerkin quadrilateral element formulation which is free of shear and volumetric locking for hyperelastic finite deformation analysis [13]; Hollenstein *et al*. proposed a Macro-Cosserat Point Element for isotropic and anisotropic hyperelastic materials [14]; Gültekin *et al*. systematically studied an three-field Hu-Washizu mixed finite element formulation for anisotropic hyperelastic materials [15]; and others work including the study of stability can be found in references [16-19].

The unsymmetric finite element method originally proposed by Rajendran *et al*. also exhibits some advantages to solve the incompressible problems as well as improve the distortion tolerance of elements [20-22] in both geometrically linear and nonlinear situations. However, these elements still possess some inherent defects, such as direction dependence, interpolation failure and ineffectiveness for low-order elements, so that they are not suitable for the practical applications [23]. By introducing the analytical trial function (ATF) method and the generalized conforming technique [24], Cen *et al*. proposed a series of new unsymmetric elements [25-31] that can overcome all original defects and greatly improve precisions. The unsymmetric finite element method employs two different sets of interpolation (test function and trial function) for displacement fields, just like the Petrov-Galerkin formulations. Similar to the Trefftz methods [32], the ATF method used in the new unsymmetric finite element method employs the homogenous solutions of governing equations of linear elasticity as the trial functions for finite element discretization. Due to the merits of these techniques, the resulting models possess high precisions as well as avoid many locking problems. For example, a recent low-order unsymmetric 4-node, 8-DOF plane solid element, denoted by US-ATFQ4 [26], exhibits better performance than most existing 4-node plane element models. It can strictly pass both the constant stress/strain (first-order) patch test and second-order patch test for pure bending, which has been proved impossible for other symmetric 4-node, 8-DOF plane element [33, 34], and is free of volume locking and other tricky problems. The key for the success is the employment of the homogenous solutions of governing equations of linear elasticity as the trial functions.

However, some researchers believe that those finite element models, which employ the solutions of governing equations of linear elasticity as trial functions, will be limited to the applications of linear elastic situations [35]. Actually, these *analytical trial functions* (ATFs) can work well in the geometrically nonlinear problems if an appropriate update strategy is adopted, as shown in reference [29]. This fact also verifies that the proposed assumption that the ATFs can properly work in each increment. On the other hand, it should be noted that the strategy proposed in reference [29] is only limited to the geometrically nonlinear problems where the strain is still small, in which the material parameters remain constant during the update procedure for the corresponding ATFs. As for hyperelasticity problems with large strains, both geometric and material nonlinearity must be considered simultaneously. The geometrically nonlinear scheme for small strain problem cannot be directly and easily generalized to hyperelasticity problem with large strains, because the material parameters in ATFs are not constants any more.

In present paper, a new strategy is proposed to solve the hyperelastic finite deformation problems where strain is large, and the linear low-order unsymmetric plane element US-ATFQ4 is generalized to hyperelastic finite deformation analysis according to the proposed strategy. The paper is organized as follows. At the beginning of Section 2.1, the computation for the internal nodal force vector of the unsymmetric element US-ATFQ4 is reviewed. Then, an algorithm for updating the deformation gradient, which is interpolated by ATFs, is designed in Section 2.1.1. Furthermore, the update of the corresponding ATFs referred to current configuration is discussed in Section 2.1.2. With regard to the hyperelastic material model, a specified model, neo-Hookean model, is used to illustrate and verify the present formulation in Section 2.1.3. The numerical implementation is introduced in Section 2.2. In the following Section 3, various examples are tested to evaluate the present formulation, which show that the present formulation not only remain the high accuracy and mesh distortion tolerance in the geometrically nonlinear problems, but also possess excellent performance in the compressible or quasi-incompressible hyperelastic finite deformation problems where the strain is large.

Note: in order to clarify the notation types that appear in following sections, the tensors are denoted only by bold alphabets, while the matrices and vectors in finite element formulations are denoted by bold alphabets together with $[\ ]$ and $\{\ \}$, respectively.

2. The formulation of unsymmetric elements for hyperelastic finite deformation

2.1 The internal nodal force vector computation of the unsymmetric element US-ATFQ4

As shown in Figure 1, a body experiences a large deformation motion. The equilibrium conditions of a system of finite elements at time t+Δt can be expressed as:

$$\left\{{}^{t+\Delta t}\mathbf{F}_{\text{ext}}\right\}-\left\{{}^{t+\Delta t}\mathbf{F}_{\text{int}}\right\}=0\,, \tag{1}$$

where $\left\{{}^{t+\Delta t}\mathbf{F}_{\text{ext}}\right\}$ and $\left\{{}^{t+\Delta t}\mathbf{F}_{\text{int}}\right\}$ are the element external and internal nodal force vector, respectively. The general approach to this nonlinear equation is an incremental step-by-step solution, in which the solutions for the discrete time t are known and the solutions for the discrete time t+Δt need to be determined. Then Equation (1) can be written as:

$$\left[{}^{t}\mathbf{K}_{T}\right]\left\{\Delta\mathbf{q}\right\}=\left\{{}^{t+\Delta t}\mathbf{F}_{\text{ext}}\right\}-\left\{{}^{t}\mathbf{F}_{\text{int}}\right\}, \tag{2}$$

where $\left\{\Delta\mathbf{q}\right\}$ is the incremental nodal displacement vector; $\left[{}^{t}\mathbf{K}_{T}\right]$ is the tangent stiffness matrix and defined by:

$$\left[{}^{t}\mathbf{K}_{T}\right]=\frac{\partial\left(\left\{{}^{t+\Delta t}\mathbf{F}_{\text{ext}}\right\}-\left\{{}^{t}\mathbf{F}_{\text{int}}\right\}\right)}{\partial\left\{{}^{t}\mathbf{q}\right\}}, \tag{3}$$

which means the tangent stiffness matrix is the derivative of the right-hand-side vector of Equation (2) with respect to the nodal displacement vector $\left\{{}^{t}\mathbf{q}\right\}$. Generally, $\left\{{}^{t+\Delta t}\mathbf{F}_{\text{ext}}\right\}$ is assumed to be independent of the deformation, and the Newton-Raphson iteration scheme for the solution of Equation (1)-(3) is necessary. In order to obtain the accurate and reasonable solutions of Equation (1), the proper and effective approximations of the internal nodal force vector are essential.

In the nonlinear formulation of the unsymmetric elements, the internal nodal force vector at time t can be written as [29]:

$$\left\{{}_{t}^{t}\mathbf{F}_{\text{int}}\right\}=\int_{{}^{t}V_{e}}\left[{}_{t}^{t}\bar{\mathbf{B}}_{L}\right]^{\mathrm{T}}\left\{{}^{t}\hat{\boldsymbol{\sigma}}\right\}\mathrm{d}^{t}V, \tag{4}$$

where the left subscript t of $\left\{{}_{t}^{t}\mathbf{F}_{\text{int}}\right\}$ means it is referred to the configuration at time t; $\left\{{}^{t}\hat{\boldsymbol{\sigma}}\right\}$ is the Voigt notation of Cauchy stress tensor ${}^{t}\hat{\boldsymbol{\sigma}}$ at time t, which is calculated by the displacement field interpolated by ATFs; $\left[{}_{t}^{t}\bar{\mathbf{B}}_{L}\right]$ is the linear strain-displacement transformation matrix and defined by the conventional isoparametric shape functions:

$$\left[{}_{t}^{t}\bar{\mathbf{B}}_{L}\right]=\begin{bmatrix}\bar{N}_{1,{}^{t}x} & 0 & \ldots & \bar{N}_{4,{}^{t}x} & 0\\ 0 & \bar{N}_{1,{}^{t}y} & \ldots & 0 & \bar{N}_{4,{}^{t}y}\\ \bar{N}_{1,{}^{t}y} & \bar{N}_{1,{}^{t}x} & \ldots & \bar{N}_{4,{}^{t}y} & \bar{N}_{4,{}^{t}x}\end{bmatrix}, \tag{5}$$

with

$$\bar{N}_{I}=\frac{1}{4}\left(1+\xi_{I}\xi\right)\left(1+\eta_{I}\eta\right),\qquad\left(I=1,2,3,4\right), \tag{6}$$

and $\left(\xi_{I},\eta_{I}\right)$ are the nodal isoparametric coordinates.

2.1.1 An algorithm for calculating the deformation gradient interpolated by the ATFs

For general nonlinear problems, the Cauchy stresses can be calculated by the deformation gradient by means of constitutive models. In the case of hyperelastic materials, the Cauchy stress tensor ${}^{t}\hat{\boldsymbol{\sigma}}$ at time t can be treated as a function of ${}^{t}\mathbf{F}$ and expressed by:

$$ {}^{t}\hat{\boldsymbol{\sigma}} = f\left({}^{t}\mathbf{F}\right). \tag{7} $$

in which ${}^{t}\mathbf{F}$ is the deformation gradient tensor at time t. So, how to calculate the deformation gradient using the ATFs is the key work of present study. However, the whole displacement field cannot be interpolated straightly by the analytical trial functions [29]. The strategy we proposed is to interpolate the incremental displacement field $\{\Delta\mathbf{u}\}$ with the assumption that the analytical trial functions can properly work in each increment, and we have:

$$ \{\Delta\mathbf{u}\} = \begin{Bmatrix} \Delta u_x \\ \Delta u_y \end{Bmatrix} = [\mathbf{P}]\{\boldsymbol{\alpha}\} = \begin{bmatrix} 1 & 0 & {}^{t+\Delta t/2}x & 0 & {}^{t+\Delta t/2}y & 0 & {}^{t+\Delta t/2}U_7 & {}^{t+\Delta t/2}U_8 \\ 0 & 1 & 0 & {}^{t+\Delta t/2}x & 0 & {}^{t+\Delta t/2}y & {}^{t+\Delta t/2}V_7 & {}^{t+\Delta t/2}V_8 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_8 \end{Bmatrix}, \tag{8} $$

where α_i (i=1~8) are eight undetermined coefficients; ${}^{t+\Delta t/2}U_7$, ${}^{t+\Delta t/2}V_7$, ${}^{t+\Delta t/2}U_8$ and ${}^{t+\Delta t/2}V_8$ are the linear displacement solutions for plane pure bending in arbitrary direction and in terms of the second form of quadrilateral area coordinates (QACM-II) (S, T) [26] (see Appendix A) at time t+Δt/2, and their detailed expressions are as follows:

$$ \begin{aligned} {}^{t+\Delta t/2}U_7 = \frac{3}{16A^3}\{ & [\bar{c}_1^{\,2}\bar{c}_2(4A-\bar{b}_2\bar{c}_1)\hat{C}_{11} - \bar{b}_1^{\,3}\bar{b}_2^{\,2}\hat{C}_{22} + 16\bar{b}_1A^2\hat{C}_{12} - \bar{b}_1\bar{b}_2^{\,2}\bar{c}_1^{\,2}(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) \\ & -16\bar{c}_1A^2\hat{C}_{16} + \bar{b}_2^{\,2}\bar{c}_1^{\,3}(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_1^{\,2}\bar{b}_2^{\,2}\bar{c}_1(\hat{C}_{26}+\hat{C}_{62})]\,{}^{t+\Delta t/2}S^2 + [2\bar{b}_2\bar{c}_1^{\,4}\hat{C}_{11} + 2\bar{b}_1^{\,4}\bar{b}_2\hat{C}_{22} \\ & +2\bar{b}_1^{\,2}\bar{b}_2\bar{c}_1^{\,2}(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 2\bar{b}_1\bar{b}_2\bar{c}_1^{\,3}(\hat{C}_{16}+\hat{C}_{61}) - 2\bar{b}_1^{\,3}\bar{b}_2\bar{c}_1(\hat{C}_{26}+\hat{C}_{62})]\,{}^{t+\Delta t/2}S\,{}^{t+\Delta t/2}T \\ & +[-\bar{b}_1\bar{c}_1^{\,4}\hat{C}_{11} - \bar{b}_1^{\,5}\hat{C}_{22} - \bar{b}_1^{\,3}\bar{c}_1^{\,2}(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) + \bar{b}_1^{\,2}\bar{c}_1^{\,3}(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_1^{\,4}\bar{c}_1(\hat{C}_{26}+\hat{C}_{62})]\,{}^{t+\Delta t/2}T^2\} \end{aligned}, \tag{9} $$

$$ \begin{aligned} {}^{t+\Delta t/2}V_7 = \frac{3}{16A^3}\{ & [-\bar{c}_1^{\,3}\bar{c}_2^{\,2}\hat{C}_{11} - \bar{b}_1^{\,2}\bar{b}_2(4A+\bar{b}_1\bar{c}_2)\hat{C}_{22} + 16\bar{c}_1A^2\hat{C}_{21} - \bar{b}_1^{\,2}\bar{c}_1\bar{c}_2^{\,2}(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) \\ & -16\bar{b}_1A^2\hat{C}_{26} + \bar{b}_1^{\,3}\bar{c}_2^{\,2}(\hat{C}_{26}+\hat{C}_{62}) + \bar{b}_1\bar{c}_1^{\,2}\bar{c}_2^{\,2}(\hat{C}_{16}+\hat{C}_{61})]\,{}^{t+\Delta t/2}S^2 + [2\bar{c}_1^{\,4}\bar{c}_2\hat{C}_{11} + 2\bar{b}_1^{\,4}\bar{c}_2\hat{C}_{22} \\ & +2\bar{b}_1^{\,2}\bar{c}_1^{\,2}\bar{c}_2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 2\bar{b}_1\bar{c}_1^{\,3}\bar{c}_2(\hat{C}_{16}+\hat{C}_{61}) - 2\bar{b}_1^{\,3}\bar{c}_1\bar{c}_2(\hat{C}_{26}+\hat{C}_{62})]\,{}^{t+\Delta t/2}S\,{}^{t+\Delta t/2}T \\ & +[-\bar{c}_1^{\,5}\hat{C}_{11} - \bar{b}_1^{\,4}\bar{c}_1\hat{C}_{22} - \bar{b}_1^{\,2}\bar{c}_1^{\,3}(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) + \bar{b}_1\bar{c}_1^{\,4}(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_1^{\,3}\bar{c}_1^{\,2}(\hat{C}_{26}+\hat{C}_{62})]\,{}^{t+\Delta t/2}T^2\} \end{aligned}, \tag{10} $$

$$
\begin{aligned}
{}^{t+\Delta t/2}U_8 = \frac{3}{16A^3}\{[-\bar{b}_2\bar{c}_2^4\hat{C}_{11} - \bar{b}_2^5\hat{C}_{22} - \bar{b}_2^3\bar{c}_2^2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) + \bar{b}_2^2\bar{c}_2^3(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_2^4\bar{c}_2(\hat{C}_{26} \\
+\hat{C}_{62})]^{t+\Delta t/2}S^2 + [2\bar{b}_1\bar{c}_2^4\hat{C}_{11} + 2\bar{b}_1\bar{b}_2^4\hat{C}_{22} + 2\bar{b}_1\bar{b}_2^2\bar{c}_2^2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 2\bar{b}_1\bar{b}_2\bar{c}_2^3(\hat{C}_{16}+\hat{C}_{61}) \\
-2\bar{b}_1\bar{b}_2^3\bar{c}_2(\hat{C}_{26}+\hat{C}_{62})]^{t+\Delta t/2}S^{\,t+\Delta t/2}T + [-\bar{c}_1\bar{c}_2^2(4A+\bar{b}_1\bar{c}_2)\hat{C}_{11} - \bar{b}_1^2\bar{b}_2^3\hat{C}_{22} + 16\bar{b}_2A^2\hat{C}_{12} \\
-\bar{b}_1^2\bar{b}_2\bar{c}_2^2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 16\bar{c}_2A^2\hat{C}_{16} + \bar{b}_1^2\bar{c}_2^3(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_1^2\bar{b}_2^2\bar{c}_2(\hat{C}_{26}+\hat{C}_{62})]^{t+\Delta t/2}T^2\}
\end{aligned}
\quad , \tag{11}
$$

$$
\begin{aligned}
{}^{t+\Delta t/2}V_8 = \frac{3}{16A^3}\{[-\bar{c}_2^5\hat{C}_{11} - \bar{b}_2^4\bar{c}_2\hat{C}_{22} - \bar{b}_2^2\bar{c}_2^3(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) + \bar{b}_2\bar{c}_2^4(\hat{C}_{16}+\hat{C}_{61}) + \bar{b}_2^3\bar{c}_2^2(\hat{C}_{26} \\
+\hat{C}_{62})]^{t+\Delta t/2}S^2 + [2\bar{c}_1\bar{c}_2^4\hat{C}_{11} + 2\bar{b}_2^4\bar{c}_1\hat{C}_{22} + 2\bar{b}_2^2\bar{c}_1\bar{c}_2^2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 2\bar{b}_2\bar{c}_1\bar{c}_2^3(\hat{C}_{16}+\hat{C}_{61}) \\
-2\bar{b}_2^3\bar{c}_1\bar{c}_2(\hat{C}_{26}+\hat{C}_{62})]^{t+\Delta t/2}S^{\,t+\Delta t/2}T + [-\bar{c}_1^2\bar{c}_2^3\hat{C}_{11} + \bar{b}_1\bar{b}_2^2(4A-\bar{b}_2\bar{c}_1)\hat{C}_{22} + 16\bar{c}_2A^2\hat{C}_{21} \\
-\bar{b}_2^2\bar{c}_1^2\bar{c}_2(\hat{C}_{12}+\hat{C}_{21}+\hat{C}_{66}) - 16\bar{b}_2A^2\hat{C}_{26} + \bar{b}_2^3\bar{c}_1^2(\hat{C}_{26}+\hat{C}_{62}) + \bar{b}_2\bar{c}_1^2\bar{c}_2^2(\hat{C}_{16}+\hat{C}_{61})]^{t+\Delta t/2}T^2\}
\end{aligned}
\quad , \tag{12}
$$

where, the parameters $\hat{C}_{11}$, $\hat{C}_{12}\ldots$ are referred to Equation (19); other parameters can be found in Appendix A. The derivations of expressions of ${}^0U_7$, ${}^0V_7$, ${}^0U_8$ and ${}^0V_8$ referred to the initial configuration are given in reference [26].

Substitution of nodal coordinates and nodal displacement increments into Equation (8) yields:

$$
\{\Delta\mathbf{u}\} = \left[{}^{t+\Delta t/2}\hat{\mathbf{N}}\right]\{\Delta\mathbf{q}\}, \tag{13}
$$

which means the ATFs of configuration at time $t+\Delta t/2$ are used to interpolate the incremental displacement field.

Then, as shown in Figure 1, the incremental deformation gradient matrix is defined as:

$$
\left[{}^{t+\Delta t}\mathbf{f}\right] = \frac{\partial\left\{{}^{t+\Delta t}\mathbf{x}\right\}}{\partial\left\{{}^{t}\mathbf{x}\right\}} = [\mathbf{I}] + \frac{\partial\{\Delta\mathbf{u}\}}{\partial\left\{{}^{t}\mathbf{x}\right\}}, \tag{14}
$$

where $[\mathbf{I}]$ is the identity matrix. Substitution of Equation (13) into Equation (14) yields:

$$
\left[{}^{t+\Delta t}\mathbf{f}\right] = [\mathbf{I}] + \frac{\partial\left[{}^{t+\Delta t/2}\hat{\mathbf{N}}\right]}{\partial\left\{{}^{t}\mathbf{x}\right\}}\{\Delta\mathbf{q}\}. \tag{15}
$$

After some trivial manipulations, Equation (15) can be rewritten as:

$$
\left[{}^{t+\Delta t}\mathbf{f}\right] = [\mathbf{I}] + \left[\frac{\partial\left[{}^{t+\Delta t/2}\hat{\mathbf{N}}\right]}{\partial\left\{{}^{t+\Delta t/2}\mathbf{x}\right\}}\{\Delta\mathbf{q}\}\right]\left[\mathbf{I} - \frac{1}{2}\frac{\partial\left[{}^{t+\Delta t/2}\hat{\mathbf{N}}\right]}{\partial\left\{{}^{t+\Delta t/2}\mathbf{x}\right\}}\{\Delta\mathbf{q}\}\right]^{-1}. \tag{16}
$$

Consequently, the deformation gradient matrix at time $t+\Delta t$ can be calculated by multiplying the incremental deformation gradient matrix by the deformation gradient matrix at time t :

$$
\left[{}^{t+\Delta t}\mathbf{F}\right] = \left[{}^{t+\Delta t}\mathbf{f}\right]\left[{}^{t}\mathbf{F}\right]. \tag{17}
$$

Thus, based on Equation (16) and (17), an algorithm of calculating the deformation gradient interpolated by the ATFs is proposed.

2.1.2 The update of the corresponding analytical trial functions (ATFs)

In the linear elastic element US-ATFQ4 formulation, the following strain-stress relation is used to calculate the analytical solutions of strains [26]:

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \left[\hat{\mathbf{C}}\right]\{\boldsymbol{\sigma}\} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{16} \\ \hat{C}_{21} & \hat{C}_{22} & \hat{C}_{26} \\ \hat{C}_{61} & \hat{C}_{62} & \hat{C}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \tag{18}$$

where $\left[\hat{\mathbf{C}}\right]$ is the elasticity matrix of compliances. Then, the linear displacement solutions (${}^{0}U_{7}$, ${}^{0}V_{7}$, ${}^{0}U_{8}$ and ${}^{0}V_{8}$) can be obtained, which are used to form the ATFs. However, in the case of finite strain problems, the following equation is proposed to calculate the corresponding analytical trial functions at time *t*:

$$\begin{Bmatrix} {}^{t}D_{11} \\ {}^{t}D_{22} \\ 2\,{}^{t}D_{12} \end{Bmatrix} = \begin{bmatrix} {}^{t}\hat{C}_{11} & {}^{t}\hat{C}_{12} & {}^{t}\hat{C}_{16} \\ {}^{t}\hat{C}_{21} & {}^{t}\hat{C}_{22} & {}^{t}\hat{C}_{26} \\ {}^{t}\hat{C}_{61} & {}^{t}\hat{C}_{62} & {}^{t}\hat{C}_{66} \end{bmatrix} \begin{Bmatrix} {}^{t}\tau_{11}^{\nabla c} \\ {}^{t}\tau_{22}^{\nabla c} \\ {}^{t}\tau_{12}^{\nabla c} \end{Bmatrix}, \tag{19}$$

where ${}^{t}D_{ij}$ and ${}^{t}\tau_{ij}^{\nabla c}$ ($i, j = 1, 2$) are components of the rate-of-deformation tensor and Lie derivative of Kirchhoff stress tensor, respectively. Then, we have

$${}^{t}\boldsymbol{\tau}^{\nabla c} = {}^{t}\mathbf{C} : {}^{t}\mathbf{D}. \tag{20}$$

Therefore, the shape function formed by ATFs is also the function of ${}^{t}\mathbf{C}$, and can be expressed by:

$${}^{t}\hat{\mathbf{N}} = g\left({}^{t}\mathbf{x}, {}^{t}\mathbf{C}\right), \tag{21}$$

where ${}^{t}\mathbf{C}$ is the fourth-order tensor of elastic moduli which are assumed to be constant for the small strain case, including the geometrically nonlinear problems [29]. While in the case of finite strain problems, ${}^{t}\mathbf{C}$ should be consistent to the spatial elasticity tensor (the fourth elasticity tensor) [36, 37], i.e.

$$ {}^{t}\mathbf{C} \doteq {}^{t}\mathbf{C}^{\tau}, \tag{22} $$

where the spatial elasticity tensor ${}^{t}\mathbf{C}^{\tau}$ is the relation between the Lie derivative of Kirchhoff stress tensor $L_v\boldsymbol{\tau}$ and the rate-of-deformation tensor $\mathbf{D}$ with the definition:

$$ L_v\,{}^{t}\boldsymbol{\tau} \equiv {}^{t}\boldsymbol{\tau}^{\nabla c} = {}^{t}\mathbf{C}^{\tau} : {}^{t}\mathbf{D}. \tag{23} $$

Consequently, Equation (21) can be rewritten as:

$$ {}^{t}\hat{\mathbf{N}} = g\left({}^{t}\mathbf{x},\, {}^{t}\mathbf{C}^{\tau}\right). \tag{24} $$

2.1.3 Hyperelastic matrial model example: neo-Hookean material

For the hyperelastic materials, it is easy to derive the spatial elasticity tensor from the specified hyperelastic models. Here, the neo-Hookean material model is chose to illustrate the calculation of Equation (24). The stored energy function for a compressible neo-Hookean material is given as follow:

$$ {}^{t}W = \lambda \frac{{}^{t}J^{2}-1}{4} - \left(\frac{\lambda}{2}+G\right)\ln {}^{t}J + \frac{1}{2}G\left(tr\,{}^{t}\boldsymbol{C}-3\right), \tag{25} $$

where λ and G are the Lamé constants of the linearized theory; ${}^{t}\boldsymbol{C}$ is the right Cauchy–Green deformation tensor and ${}^{t}J = \det\left[{}^{t}\mathbf{F}\right]$. Then, the spatial elasticity tensor can be obtained by pushing forward the material elasticity tensor ${}^{t}\mathbf{C}^{SE}$:

$$ {}^{t}\mathbf{C}^{SE} = 4\frac{\partial^{2}\,{}^{t}W}{\partial\,{}^{t}\boldsymbol{C}\,\partial\,{}^{t}\boldsymbol{C}}, \tag{26} $$

$$ {}^{t}C^{\tau}_{ijkl} = {}^{t}F_{im}\,{}^{t}F_{jn}\,{}^{t}F_{kp}\,{}^{t}F_{lq}\,{}^{t}C^{SE}_{mnpq}, \tag{27} $$

$$ {}^{t}C^{\tau}_{ijkl} = \lambda\,{}^{t}J^{2}\delta_{ij}\delta_{kl} + \mu\left[1+\frac{\lambda}{2\mu}\left(1-{}^{t}J^{2}\right)\right]\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right). \tag{28} $$

And the Cauchy stress tensor is given by

$$ {}^{t}\hat{\boldsymbol{\sigma}} = \lambda\frac{{}^{t}J^{2}-1}{2\,{}^{t}J}\mathbf{I} + G\left({}^{t}\mathbf{F}\cdot{}^{t}\mathbf{F}^{T}-\mathbf{I}\right) \tag{29} $$

Based on the major symmetry and minor symmetries of the spatial elasticity tensor, it is convenient to calculate the ATFs using its Voigt notation based on Equation (19)-(21). Finally, the proposed computational procedures of internal nodal force vector are schematically illustrated in Box 1.

2.2 Numerical implementation

As described in Section 2.1, the tangent stiffness $\left[{}^{t}\mathbf{K}_{T}\right]$ matrix is the derivative of the right-hand-side vector of Equation (2) with respect to the nodal displacement vector $\left\{{}^{t}\mathbf{q}\right\}$. Owing to the complicated form of the internal nodal force vector of element US-ATFQ4, the derivation of tangent stiffness matrix is completed with the help of the automatic differentiation program AceGen developed by Korelc [38].

The implementation of present formulation of element US-ATFQ4 for hyperelastic finite deformation is via the user element subroutine (UEL) of commercial software SIMULIA Abaqus [39]. The computation flowchart is given in Figure 3 in reference [29]. The postprocessings of results are also completed in Abaqus with some instructions in reference [40]. All terms of the element formulation are evaluated by using a 2×2 Gauss integration scheme. And the incremental-iterative Newton-Raphson scheme is employed to solving the nonlinear problems.

3. Numerical examples

Six examples are tested to evaluate the performance of the present formulation of US-ATFQ4 for the hyperelastic finite deformation problems. Results obtained by some other 2D 4-node solid elements, as listed below, are also given form comparison.

- CPE4H: the 4-node bilinear, hybrid plane strain element with constant pressure in Abaqus [39].
- CPE4IH: the 4-node bilinear, incompatible and hybrid plane strain element with linear pressure in Abaqus [39].
- CPE8H: the 8-node biquadratic, hybrid plane strain element with linear pressure in Abaqus [39].
- Q1: the 4-node bilinear standard isoparametric plane element [41].

- Q1/P0: the 4-node mean dilatation plane element proposed by Simo *et al*. [6].
- Q1/E4: the 4-node enhanced strain plane elements with four enhanced strain modes proposed by Simo *et al*. [8].
- Q1/Fbar: the 4-node bilinear plane element with the F-bar technique proposed by de Souza Neto *et al*. [9].
- Q1SP: the 4-node enhanced strain plane elements with stabilization technique proposed by Reese *et al*. [41]

Note: a plane strain state with is assumed in all the following examples.

3.1 Cook's membrane problem.

This example is a standard test frequently employed to evaluate the element performance in bending dominated linear elastic problems. In the case of hyperelastic finite deformation, it is also a popular numerical example to test the element performance in compressible and quasi-incompressible situations [8, 9, 41]. The geometric parameters and boundary conditions are depicted in Figure 2. Here, a neo-Hookean material model is used with the strain energy function given by Equation (25). The material parameters $\lambda = 400888.8$, $G = 80.1938$, which represents a quasi-incompressible situation. As shown in Figure 3, the vertical displacement of Point A is calculated using various numbers of elements (2×2, 4×4, 8×8…) and with 20 equal increments. It can be seen that element US-ATFQ4 presents relatively better results even using the coarsest mesh (2×2) and possesses the best convergence, which means the performances of the present formulation are superior to those of other elements that are developed specially for the quasi-incompressible situations, such as Q1/E4 [8], Q1/Fbar [9], and Abaqus element CPE4H [39].

Furthermore, two distorted coarse mesh cases are given in Figure 4 to test the mesh distortion tolerance of the present formulation. The loading force F is increased to 384 to enlarge the deformation, and the reference solutions of vertical displacement of point A is obtained using a fine mesh (64×64) of CPE4IH, which is consistent with that of reference [42]. As shown in Table I, the present element US-ATFQ4 can give much better results those obtained by CPE4H and CPE4IH using both distorted coarse meshes under very large loading force. On the other hand, the sensitivity test for the present formulation with respect to the total number of increments (NINC) is given in Figure 5, which shows that the present formulation are insensitive to the total number of increments as long as NINC is enough large (such as 10 in present example). This insensitivity is unexpected considering the assumption we proposed in Section 2.1.1 and Equation (8), and it indicates that the present formulation for the hyperelastic finite deformations are not only accurate but also robust.

3.2 Angle frame problem

The following example is often employed to test the element performances in geometrically nonlinear problems where strain is still small [29]. Here, as shown in Figure 6, it is modified as the problem involving both geometric and material nonlinearity, in which the quasi-incompressible neo-Hookean material model is adopted. The material parameters $\lambda = 5.0\times10^{10}$, $G = 1.0\times10^{7}$, and the load F is increased to 3.0×10^{5}. Actually, the modified example will experience a moderate strain state after a large rotation. Two coarse mesh cases given in Figure 6 are considered: regular and distorted mesh with 9 elements. The reference results are obtained by using element CPE4H with a fine mesh (304 rectangular elements). The horizontal displacement of point A and the vertical displacement of point B with respect to the loading process are given in Figure 7.

It can be found that the results obtained by both the present element US-ATFQ4 and element CPE4IH can agree very well with the reference results using the regular mesh. However, for distorted mesh case, CPE4IH cannot present satisfied results, especially during the large rotation state. But US-ATFQ4 presents a relative insensitive performance with respect to the distorted mesh. The reason why US-ATFQ4 cannot give accurate results of point B in Figure 7 (d) mainly comes from the geometry discretization error due to the coarse mesh. On the other hand, both CPE4H and Q1/Fbar cannot provide good results using both mesh cases.

3.3 Hyperelastic beam problems

3.3.1 Slender cantilever beam

This example is also often used for testing element performance in the geometrically nonlinear analysis. From reference [41], it can be found that those elements behaving well for Cook's membrane problem may perform poorly in extreme bending situations, such as the bending of a slender cantilever beam. So, it is necessary to evaluate the element performance by employing this example. As shown in Figure 8, the geometric parameters and boundary conditions for a slender cantilever beam are given. Here, the compressible neo-Hookean material model with $\lambda = 24000$ and $G = 6000$ is considered. In order to compare with the results given in [41], five regular mesh cases: 1×10, 2×20, 4×40, 8×80 and 16×160 meshes, are used. The convergence curves for the vertical displacement of point A are plotted in Figure 9. It can be seen that the present element US-ATFQ4 performs as well as element Q1/E4 [8], and better than Q1SP [41]. In fact, almost the same results can also be obtained by the element US-ATFQ4 even when only single layer elements are allocated along the beam height. Furthermore, it should be noted that the performance of present formulation is also consistent with the high-performance geometrically nonlinear element US-ATFQ4 proposed in reference [29], especially for bending behaviors.

3.3.2 Curved beam

As shown in Figure 10, a curved beam subjected to a concentrated force with clamped ends is considered. In this example, the beam is made of the Mooney-Rivlin material with the following stored energy function:

$$ {}^{t}W = C_1\left({}^{t}\bar{I}_1 - 3\right) + C_2\left({}^{t}\bar{I}_2 - 3\right) + \frac{K}{2}\left(\ln {}^{t}J\right)^2 , \tag{30} $$

with

$$ {}^{t}\bar{I}_1 = \mathrm{tr}\, {}^{t}\boldsymbol{B}_{\mathrm{iso}},\ {}^{t}\bar{I}_2 = \frac{1}{2}\left(\left({}^{t}\bar{I}_1\right)^2 - tr\left({}^{t}\boldsymbol{B}_{\mathrm{iso}}^2\right)\right), \tag{31} $$

$$ {}^{t}\boldsymbol{B}_{\mathrm{iso}} = \left(\det {}^{t}\mathbf{F}\right)^{-\frac{2}{3}} {}^{t}\mathbf{F} \cdot {}^{t}\mathbf{F}^{\mathrm{T}} , \tag{32} $$

and material parameters $C_1 = 10, C_2 = 30, K = 400000$, which can represent a quasi-incompressible situation. One coarse mesh (1×10 elements) is employed to evaluate the performance of present formulation, and the reference results are obtained by using CPE4IH with a fine mesh (4×40 elements). The load-displacement curves of point A are plotted in Figure 11 (a), and the final deformation shape using present formulation is given in Figure 11 (b). It can be concluded that the present element US-ATFQ4 shows a highly accurate and robust performance in this hyperelastic beam problems in which another material model is used, and much better than the element CPE4H. Since the shapes of deformed elements are still regular, the element CPE4IH performs as well as present element US-ATFQ4.

3.4 Compression tests

It has been observed that the instability problems will arise in the enhanced strain elements in the large compression situation [43, 44]. Therefore, it is necessary to evaluate the stability of the present formulation using large compression examples.

3.4.1 Symmetric compression

The geometry and boundary conditions of a solid block are given in Figure 12. Owing to the symmetry, only one half of the model is considered. The block is made of quasi-incompressible neo-Hookean material model, and its material parameters are the same as those given in Section 3.1. Two unstructured coarse meshes and two structured fine meshes given in Figure 13 are employed. The convergence curves for three different load cases (P=200, P=400 and P=600) are shown in Figure 14, which shows that both the US-ATFQ4 and CPE4H exhibit excellent performances. Furthermore, the final deformation shapes of present formulation using mesh II and mesh III are also given in Figures 15 and 16 in comparison with those of CPE4H under the last two load cases (P=400 and P=600).

Although the vertical displacements of point A obtained by present formulation and CPE4H under three different load cases are almost same, a few different shapes of deformed elements can be found under the largest load case (P=600), as shown in Figure 15 (b) (d) and Figure 16 (b) (d). In the case of largest load case, the compression of point A is around 65%, where the shapes of few upper layer elements become non-convex shapes using the present element US-ATFQ4, however, the shapes of all CPE4H elements maintain the convex shapes. This phenomenon indicates that the US-ATFQ4 elements using present formulation seem to be more flexible and can work well even in extremely distorted mesh, which is also consistent with the highly distortion-tolerant performance of its linear formulation [26]. Some other enhanced strain elements, such as Q1SP, also lose their convex shapes in above situation, but the results of these elements are undesirable. For example, the upper middle node moves up in relation to the neighbouring nodes, as reported in [41]. On the contrary, the deformed non-convex shapes do not affect the accuracy and stability of the present formulation, as shown in Figure 14.

Furthermore, we try to figure out why few elements become non-convex shapes using the present formulation in the case of largest load case. First, the vertical displacement of point B in mesh III, as shown in Figure 13, is tracked. The reference value -2.302 is obtained by using CPE4H (the element CPE4IH cannot work properly in this example due to the seriously distorted mesh) with a 100×100 fine mesh, and the results using present formulation and CPE4H with mesh III are -2.193 and -2.412, respectively, which means that both vertical displacement of point B obtained by present formulation and CPE4H are not accurate enough. It is interesting that, the reference value is just at the mid between the values obtained by above two element formulations. So, one of the reasons that causes the non-convex element shapes is that the convergence path of the present formulation is opposite to that of CPE4H. In addition, it should be noted that there is no any stabilization technique used in the present formulation.

3.4.2 Unsymmetric compression

As shown in Figure 17, another indentation of a rubber block is considered. This example has been employed in references [10] to investigate the low-order element formulations for elastic large strain problems under high compressive strains. Here, the material parameters are the same as the previous example, which enforce a quasi-incompressible situation. One 5×8 regular mesh is employed, and the final deformations obtained by US-ATFQ4, CPE4H and CPE4IH are given in Figure 18. It can be remarked that, the present formulation does not exhibit the instability problems that have been associated with many enhance strain elements under high compression without corresponding stabilization techniques [10, 43, 44], while CPE4IH seems to show the spurious hourglass patterns and failed to converge at around 0.50 step time (the whole step time is 1). In addition, a refined mesh (10×16) is also used to evaluate the performances of element US-ATFQ4 and CPE4H. Unfortunately, CPE4H failed to converge at around 0.79 step time in this refined mesh, while US-ATFQ4 still works, as shown in Figure 19, which indicates that US-ATFQ4 is more robust when the elements in mesh are severely distorted.

4. Conclusions

In this paper, the strategy for hyperelastic finite deformation analysis with the unsymmetric finite element method containing homogeneous solutions of linear elasticity is proposed, and a new nonlinear low-order unsymmetric plane element US-ATFQ4 is developed to illustrate it. Based on the assumption that the analytical trial functions (ATFs) can properly work in each increment, which has been verified in the small strain geometrically nonlinear analysis [29], an algorithm for updating the deformation gradient which is interpolated by ATFs is designed. The present work can be called the first attempt for the materially nonlinear analysis. Furthermore, the update of the corresponding ATFs referred to current configuration is discussed with regard to the hyperelastic material model, and a specified model, neo-Hookean model, is used to illustrate and verify the present formulation of US-ATFQ4 for hyperelastic finite deformation analysis.

Various examples are employed to test and evaluate the performance of the present formulation. In the first four examples with the neo-Hookean and Mooney-Rivlin material model, it can be concluded that the present formulation is superior to many existing advanced elements in compressible and quasi-incompressible hyperelastic problems, with respect to accuracy, robust and mesh distortion tolerance. Two compression tests show that the present formulation without any stabilization technique does not show the instability problems under high compression. Although the non-convex shapes are observed during the extremely high compression in comparison with Abaqus element CPE4H, the results of present formulation seem to be unaffected by these non-convex shapes.

The present hyperelastic finite deformation (large strain) analysis with the unsymmetric finite element method containing homogeneous solutions of linear elasticity is indeed a brand-new strategy. First, as far as authors know, there is no such formulation for the hyperelastic analysis before where the closed-form solutions of linear elasticity are employed. Second, the extension of the unsymmetric finite elements based on ATFs to hyperelastic analysis (material non-linearity) is not straightforward, although the formulation for geometric non-linearity (material linearity) has been derived before [29]. The updating algorithm for geometric non-linearity given by paper [29] is simple and only limited to the small strain state where material linearity is still considered. But the strategy for element containing linear ATFs in hyperelastic analysis is quite different. Therefore, an algorithm for updating the deformation gradient interpolated by ATFs is designed in present paper, which is also valid for geometric non-linearity (material linearity) analysis. Furthermore, the Equation (24) is proposed to update the corresponding ATFs in hyperelastic analysis. The present work shows the ability and advantage of the unsymmetric elements based on ATFs within the hyperelastic finite deformation situation. The analysis of 3D and inelastic problem is the next goal using the unsymmetric elements based on ATFs. The application into the elastic-plastic finite deformation problems will be reported in the near future.

APPENDIX A. THE SECOND FORM OF QUADRILATERAL AREA COORDINATES (QACM-II) [26]

As shown in Figure A, M$_i$ (i=1,2,3,4) are the mid-side points of element edges $\overline{23}$, $\overline{34}$, $\overline{41}$ and $\overline{12}$, respectively. Then, the position of an arbitrary point P within the quadrilateral element $\overline{1234}$ can be uniquely specified by the area coordinates S and T (QACM-II), which are defined as:

$$S = 4\frac{\Omega_1}{A},\ T = 4\frac{\Omega_2}{A}, \tag{A.1}$$

where A is the area of the quadrilateral element; Ω_1 and Ω_2 are the *generalized areas* of ΔPM_2M_4 and ΔPM_3M_1, respectively. The values of *generalized areas* Ω_1 and Ω_2 can be both positive and negative: for ΔPM_2M_4 (or ΔPM_3M_1), if the permutation order of points P, M$_2$ and M$_4$ (or P, M$_3$ and M$_1$) is anticlockwise, a positive Ω_1 (or Ω_2) should be taken; otherwise, Ω_1 (or Ω_2) should be negative.

Two shape parameters $\bar{g}_1$ and $\bar{g}_2$ are defined here as:

$$\begin{cases} \bar{g}_1 = \dfrac{A_{\Delta 123} - A_{\Delta 124}}{A} \\ \bar{g}_2 = \dfrac{A_{\Delta 234} - A_{\Delta 123}}{A} = \dfrac{A - A_{\Delta 124} - A_{\Delta 123}}{A} \end{cases}, \tag{A.2}$$

in which $A_{\Delta 123}$, $A_{\Delta 124}$ and $A_{\Delta 234}$ are the areas of Δ123, Δ124 and Δ234, respectively. Different values of these shape parameters mean different shapes of a quadrangle. Thus, the local coordinates of the corner nodes and mid-side points can be written as:

$$\begin{array}{ll} \text{node}1: (S_1, T_1) = (-1+\bar{g}_2, -1+\bar{g}_1); & \text{node}2: (S_2, T_2) = (1-\bar{g}_2, 1-\bar{g}_1); \\ \text{node}3: (S_3, T_3) = (1+\bar{g}_2, 1+\bar{g}_1); & \text{node}4: (S_4, T_4) = (-1-\bar{g}_2, -1-\bar{g}_1); \\ \text{M}_1: (1, 0); & \text{M}_2: (0, 1); \\ \text{M}_3: (-1, 0); & \text{M}_4: (0, -1). \end{array} \tag{A.3}$$

Above coordinate values are only small modifications for isoparametric coordinates:

$$\begin{cases} S = \xi + \bar{g}_2 \xi\eta \\ T = \eta + \bar{g}_1 \xi\eta \end{cases}. \tag{A.4}$$

And the relationship between QACM-II and the Cartesian coordinates is

$$\begin{cases} S = \frac{1}{A}\left[(a_3 - a_1) + (b_3 - b_1)x + (c_3 - c_1)y\right] + \bar{g}_1 = \frac{1}{A}\left[\bar{a}_1 + \bar{b}_1 x + \bar{c}_1 y\right] + \bar{g}_1 \\ T = \frac{1}{A}\left[(a_4 - a_2) + (b_4 - b_2)x + (c_4 - c_2)y\right] + \bar{g}_2 = \frac{1}{A}\left[\bar{a}_2 + \bar{b}_2 x + \bar{c}_2 y\right] + \bar{g}_2 \end{cases}, \qquad \text{(A.5)}$$

where

$$\begin{cases} \bar{a}_1 = a_3 - a_1, \quad \bar{b}_1 = b_3 - b_1, \quad \bar{c}_1 = c_3 - c_1, \\ \bar{a}_2 = a_4 - a_2, \quad \bar{b}_2 = b_4 - b_2, \quad \bar{c}_2 = c_4 - c_2, \end{cases} \qquad \text{(A.6)}$$

$$\begin{gathered} a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j, \\ (i = 1,2,3,4; \quad j = 2,3,4,1; \quad k = 3,4,1,2) \end{gathered} \qquad \text{(A.7)}$$

in which (x_i, y_i) (i=1, 2, 3, 4) are the Cartesian coordinates of the four corner nodes.

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