CONDENSED MATTER PHYSICS

Minimal quantum viscosity from fundamental physical constants

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Viscosity of fluids is strongly system dependent, varies across many orders of magnitude, and depends on molecular interactions and structure in a complex way not amenable to first-principles theories. Despite the variations and theoretical difficulties, we find a new quantity setting the minimal kinematic viscosity of fluids: $\nu_m = \frac{1}{4\pi}\frac{\hbar}{\sqrt{m_e m}}$, where m_e and m are electron and molecule masses. We subsequently introduce a new property, the "elementary" viscosity ι with the lower bound set by fundamental physical constants and notably involving the proton-to-electron mass ratio: $\iota_m = \frac{\hbar}{4\pi}\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}}$, where m_p is the proton mass. We discuss the connection of our result to the bound found by Kovtun, Son, and Starinets in strongly interacting field theories.

INTRODUCTION

Several important physical properties can be expressed in terms of fundamental physical constants including, for example, the Bohr radius, the Rydberg energy, and fine structure constant (*1*, *2*). These properties serve as a basis for atomic units and importantly set the scale of energy and length: The Bohr radius gives a characteristic interatomic distance in condensed matter phases on the order of angstroms, and the Rydberg energy gives a characteristic binding energy on the order of several electron volts. It is interesting to see whether a transport property such as viscosity or diffusion can be similarly expressed in terms of fundamental constants, setting their characteristic scale. Here, we find a quantum quantity setting the minimal kinematic viscosity of fluids, ν_m, as

$$\nu_m = \frac{1}{4\pi}\frac{\hbar}{\sqrt{m_e m}} \tag{1}$$

where m_e is the electron mass and m is the mass of the molecule set by the nucleon mass. For atomic hydrogen with the mass given by the proton mass m_p, ν_m is defined by fundamental constants only.

We subsequently introduce a new property: the "elementary" viscosity $\iota = \nu_m m$ with the lower bound ι_m set by fundamental physical constants as

$$\iota_m = \frac{\hbar}{4\pi}\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}} \tag{2}$$

which is on the order of $\hbar$.

Equation 2 interestingly involves the proton-to-electron mass ratio, one of few dimensionless combinations of fundamental constants of importance in a variety of areas, including formation of stars, ordered molecular structures, and life-supporting environment (*2*).

We recall that viscosity of fluids, η, varies in a wide range, from about 10^{-6} Pa·s for the normal component of He to 10^{13} Pa·s in viscous liquids approaching liquid-glass transition. η strongly depends on temperature and pressure. η is additionally strongly system dependent and is governed by the activation energy barrier for molecular rearrangements, U, which, in turn, is related to the intermolecular interactions and structure. This relationship is complicated in general, and no universal way to predict U and η from first principles exists [tractable theoretical models describe the dilute gas limit of fluids where perturbation theory applies, but not dense liquids of interest here (*3*)]. This is appreciated outside the realm of condensed matter physics: The difficulty of calculating the viscosity of water was compared to the problem of calculating the fundamental constants themselves (*4*). As far as thermodynamic properties of liquids are concerned, the absence of a small parameter due to the combination of strong interactions and the absence of small oscillations is considered to disallow a possibility of calculating liquid thermodynamic properties in general form (*5*). For example, theoretical calculation and understanding liquid energy and heat capacity have remained a longstanding problem (*6*), which started to lift only recently when new understanding of phonons in liquids came in (*7*). In view of these problems related to liquid theory, the existence of universal ν_m (Eq. 1) and ι_m (Eq. 2) is notable.

We note that viscosity is mostly considered as a classical quantity. At the same time, it is governed by molecular interactions set by quantum effects. Hence, we can suppose that there is a characteristic value of viscosity-related quantities involving $\hbar$, as in Eqs. 1 and 2.

In addition to condensed matter, the universal lower bound of viscosity is important in high-energy physics and strongly interacting quantum field theory. Using the duality between strongly interacting field theories and gravity models, Kovtun, Son, and Starinets (KSS) (*8*) have found a universal ratio between fluid viscosity and volume density of entropy s as

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \tag{3}$$

This result has generated an ample interest from a theoretical perspective and from the point of view of understanding the properties of quark-gluon plasma and its viscosity in particular. Relations of this result to a wider range of systems, and more general effects have been of subsequent interest, including Planckian dissipation [see, e.g., (*9*–*14*) for review]. KSS have conjectured that $\frac{\eta}{s}$ has a lower bound that more generally follows from strongly coupled quantum field theories: $\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$ and found that the bound is about 25 times smaller than the viscosity minima in familiar liquids such as H_2O and N_2. This raises a question of how ordinary liquids are different from high-energy hydrodynamic models. We will see that an important difference is the presence of the ultraviolet (UV) cutoff in condensed matter, setting the viscosity minima.

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RESULTS AND DISCUSSION

Kinematic viscosity

We start with recalling the origin of viscosity minima shown in Fig. 1A where we collected available experimental η (*15*) for several noble (Ar, Ne, and He), molecular (H_2, N_2, CO_2, CH_4, O_2, and CO), and network (H_2O) fluids. For some fluids, we show the viscosity minimum at two pressures. The low pressure was chosen to be far above the critical pressure so that the viscosity minimum is not affected by near-critical anomalies. The highest pressure was chosen to (i) make the pressure range considered as wide as possible and (ii) be low enough to see the viscosity minima in the temperature range available experimentally.

In the liquid-like regime of molecular dynamics at low temperature, η decreases with temperature as

$$\eta = \eta_0 \exp\left(\frac{U}{T}\right) \tag{4}$$

where η_0 is a prefactor and U can be temperature dependent, resulting in the super-Arrhenius temperature dependence.

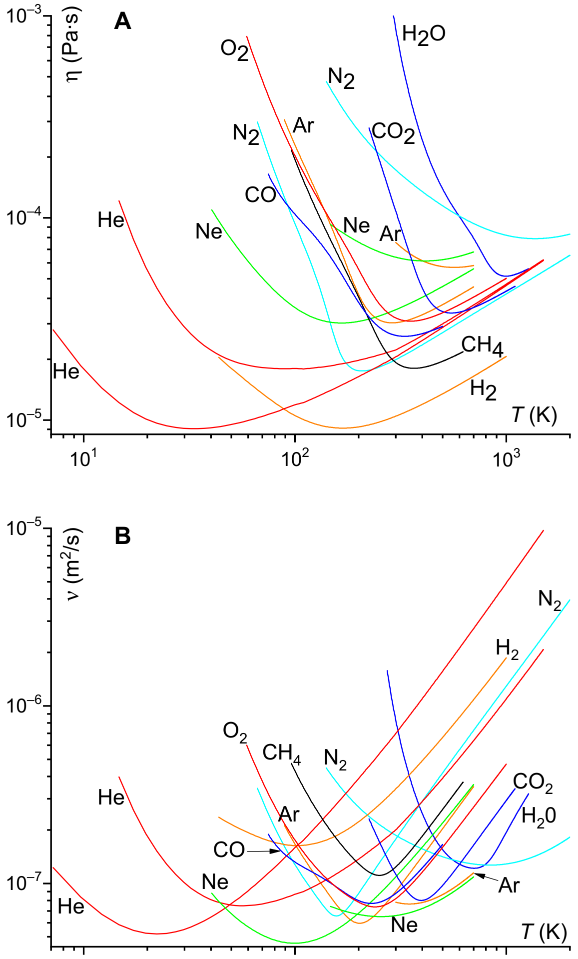

Fig. 1. Viscosity and kinematic viscosity of fluids. Experimental viscosity η (**A**) and kinematic viscosity ν (**B**) of noble, molecular, and network liquids (*15*) showing minima. η for H_2, H_2O, and CH_4 are shown for pressure P = 50, 100, and 20 MPa, respectively. η for He, Ne, Ar, and N_2 are shown at two pressures each: 20 and 100 MPa for He, 50 and 300 MPa for Ne, 20 and 100 MPa for Ar, and 10 and 500 MPa for N_2. The minimum at higher pressure is above the minimum at lower pressure for each fluid.

In the gas-like regime of molecular dynamics, η is

$$\eta = \frac{1}{3}\rho v L \tag{5}$$

where ρ is density, v is average particle velocity, and L is the particle mean free path.

For gases, $L \propto \frac{1}{\rho}$ and $\eta \propto v \propto \sqrt{T}$ (*3*). Hence, η increases with temperature without bound, although new effects such as ionization start operating at higher temperature and change the system properties and η.

Equations 4 and 5 imply that η should have a minimum as seen in Fig. 1A. Before calculating η at the minimum, it is useful to qualify the terms "liquid-like" and "gas-like" regimes of molecular dynamics and the conditions at which the minima are seen. Molecular motion in low-temperature liquids combines solid-like oscillations around quasi-equilibrium positions and diffusive jumps to new positions, enabling liquid flow. These jumps are due to temperature-induced molecular rearrangements over an energy barrier set by the interaction with other molecules, resulting in Eq. 4. The jumps are characterized by liquid relaxation time, τ, the average time between the molecular jumps, which is related to η by the Maxwell relationship $\eta = G\tau$, where G is the high-frequency shear modulus (*16*). τ decreases with temperature in the same way as Eq. 4 and is bound by the elementary vibration period, commonly approximated by the Debye vibration period in the Debye model, τ_D. At this point, the oscillatory component of molecular motion is lost, and molecules start moving in a purely diffusive manner. At high temperature and/or low density, molecules gain enough energy to move distance L without collisions. In this gas-like regime, the fluid viscosity can be calculated by assuming that a molecule's momentum is unchanged between collisions, resulting in Eq. 5.

If the temperature is increased at pressure below the critical point, then the system crosses the boiling line and undergoes the liquid-gas transition. As a result, η undergoes a sharp change at the phase transition, rather than showing a smooth minimum as in Fig. 1A. To avoid the effects related to the phase transition, we need to consider the supercritical state. Here, the Frenkel line (*7*, *17*, *18*) formalizes the qualitative change of molecular dynamics from combined oscillatory and diffusive, where η is given by the activation behavior (Eq. 4), to purely diffusive, where η follows the gas-like behavior (Eq. 5). The location of the minima of η slightly depends on the path taken on the phase diagram. As a result, the minimum of η may deviate from the Frenkel line (FL) depending on the path (*7*).

We now calculate viscosity at the minimum, η_m. There are two ways in which η_m can be evaluated: by taking the low-temperature limit of the gas-like viscosity (Eq. 5) or by taking the high-temperature limit of the liquid-like viscosity given by the Maxwell relation $\eta = G\tau$. We start from the high-temperature gas-like dynamics, corresponding to the hydrodynamic regime considered in high-energy physics calculations (*8*) and consider how $\eta = \rho v L$ changes with temperature decrease (we drop the factor $\frac{1}{3}$ in Eq. 5 since our calculation evaluates the order of magnitude of viscosity minimum as discussed below). η and the mean free path L decrease on lowering the temperature but, differently from scale-invariant quantum field theories, L is bound by a UV cutoff in condensed matter systems: interparticle separation a where intermolecular interactions become appreciable [a similar effect is related to the mean free path of quasi-particles limited by the interatomic separation at the Ioffe-Regel crossover or the phonon mean free path limited by the atomistic scale (*19*, *20*)]. From this point on, L has no room to decrease further. Instead, the system enters the liquid-like regime where η starts increasing on further temperature decrease according to Eq. 4 because the diffusive motion of molecules crosses over to thermally activated as discussed earlier. Therefore, the minimum of η approximately corresponds to $L \approx a$. When L becomes comparable to a, v can be evaluated as $v = \frac{a}{\tau_D}$ because the time for a molecule to move distance a in this regime is given by the characteristic time scale set by τ_D. Setting $L = a$, $v = \frac{a}{\tau_D} = \frac{1}{2\pi}\omega_D a$, where ω_D is Debye frequency, and $\rho \approx \frac{m}{a^3}$ gives η_m as

$$\eta_m = \frac{1}{2\pi}\frac{m\,\omega_D}{a} \tag{6}$$

We note that Eq. 5 applies in the regime where L is larger than a, and in this sense, our evaluation of viscosity minimum is an order-of-magnitude estimation, as are our other results below. In this regard, we note that theoretical models can only describe viscosity in a dilute gas limit where perturbation theory applies (*3*), but not in the regime where L is comparable to a and where the energy of intermolecular interaction is comparable to the kinetic energy. In view of theoretical issues as well as many orders of magnitude by which η can vary, we consider our evaluation meaningful. In addition to being informative, an order-of-magnitude evaluation is perhaps unavoidable if a complicated property such as viscosity is to be expressed in terms of fundamental constants only.

η_m in Eq. 6 approximately matches the result obtained by approaching the viscosity minimum from low temperature where η is given by Eq. 4 and considering the Maxwell relationship $\eta = G\tau$. η and τ decrease with temperature according to Eq. 4, but this decrease is bound from below because τ starts approaching the shortest time scale in the system given by the Debye vibration period, τ_D. From this point on, τ has no room to decrease further. Instead, the system enters the gas-like regime where η starts increasing with temperature according to Eq. 5 because the thermally activated motion of molecules crosses over to diffusive as discussed earlier. Therefore, the minimum of η can be approximately evaluated from $\tau \approx \tau_D$. G can be estimated as $G = \rho c^2$, where $c \approx \frac{a}{\tau_D}$ is the transverse speed of sound. Then, $\eta_m = G\tau_D = \rho\frac{a^2}{\tau_D} = \frac{1}{2\pi}\frac{m\omega_D}{a}$ as in Eq. 6, where we used $\rho = \frac{m}{a^3}$ as before.

Before calculating the kinematic viscosity, we first see how well Eq. 6 estimates the minima of η. Taking the typical values of a=3 to 6 Å, $\frac{\omega_D}{2\pi}$ on the order of 1 THz and atomic weights 2 to 40 for liquids in Fig. 1A, we find η_m in the range 10^{-5} to 10^{-4} Pa·s, providing an order of magnitude estimation of η_m consistent with Fig. 1A. We also observe that high pressure reduces a and increases ω_D. Equation 6 predicts that η_m increases as a result, in agreement with the experimental behavior in Fig. 1A.

The viscosity minima of strongly bonded metallic liquids were not measured because of their high critical points; however, we note that high-temperature η is close to 10^{-3} Pa·s for Fe (2000 K), Zn (1100 K), Bi (1050 K) (*21*), Hg (573 K), and Pb (1173 K) and is expected to be close to η at the minima. This is larger than η_m in Fig. 1A and is consistent with Eq. 6 predicting that η_m increases with molecular mass ($m\,\omega_D \propto \sqrt{m}$) and decreases with a (a is smaller in metallic systems as compared with noble and molecular ones in Fig. 1A).

As discussed above, the minimum of viscosity is ultimately related to the UV cutoff in condensed matter such as interparticle spacing a or characteristic time scale τ_D. This cutoff is absent in common scale-free field theories used in high-energy physics [see, however, (*22*) where the cutoff is discussed]. Below, we relate the UV cutoff to fundamental physical constants.

We now consider the kinematic viscosity ν shown in Fig. 1B. ν also describes momentum diffusivity, analogous to thermal diffusivity involved in heat transfer, and gives the diffusion constant in the gas-like regime of molecular dynamics (*16*). Another benefit of considering ν is that it makes the link to the high-energy result (Eq. 3) where η is divided by the volume density of entropy. Using $\nu = \frac{\eta}{\rho} = vL$, $v = \frac{1}{2\pi}a\,\omega_D$, and $L = a$ as before gives the minimal value of ν, ν_m, as

$$\nu_m = \frac{1}{2\pi}\omega_D a^2 \tag{7}$$

An expression similar to Eq. 7 was heuristically obtained for thermal diffusivity and interpreted as the random walk of heat transfer consisting of jumps distance a with a certain frequency (*23*).

We now recall that the properties defining the UV cutoff in condensed matter can be expressed in terms of fundamental physical constants. Two quantities of interest are Bohr radius, a_B, setting the characteristic scale of interparticle separation on the order of angstroms

$$a_B = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \tag{8}$$

and the Rydberg energy, $E_R = \frac{e^2}{8\pi\epsilon_0 a_B}$ (*1*), setting the characteristic scale for the cohesive energy in condensed matter phases on the order of several electron volts

$$E_R = \frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2} \tag{9}$$

where e and m_e are electron charge and mass.

We now recall the known ratio between the characteristic phonon energy $\hbar\omega_D$ and the cohesive energy E, $\frac{\hbar\omega_D}{E}$. Approximating $\hbar\omega_D$ as $\hbar\left(\frac{E}{ma^2}\right)^{\frac{1}{2}}$, taking the ratio $\frac{\hbar\omega_D}{E}$ and using $a = a_B$ from Eq. 8 and $E = E_R$ from Eq. 9 gives, up to a factor close to 1

$$\frac{\hbar\omega_D}{E} = \left(\frac{m_e}{m}\right)^{\frac{1}{2}} \tag{10}$$

We note that the factor $\left(\frac{m_e}{m}\right)^{\frac{1}{2}}$ also appears in the ratio of sound and melting velocity (*14*). Combining Eqs. 7 and 10 gives

$$\nu_m = \frac{1}{2\pi}\frac{Ea^2}{\hbar}\left(\frac{m_e}{m}\right)^{\frac{1}{2}} \tag{11}$$

a and E in Eq. 11 are set by their characteristic scales a_B and E_R as discussed earlier. Using $a = a_B$ from Eq. 8 and $E = E_R$ from Eq. 9 in Eq. 11 gives a remarkably simple ν_m as

$$\nu_m = \frac{1}{4\pi}\frac{\hbar}{\sqrt{m_e m}} \tag{12}$$

Equation 12 is one of the main results of this paper.

The same result for ν_m in Eq. 12 can be obtained without explicitly using a_B and E_R in Eq. 11. The cohesive energy, or the characteristic energy of electromagnetic interaction, is

$$E = \frac{\hbar^2}{2m_e a^2} \tag{13}$$

Using Eq. 13 in Eq. 11 gives Eq. 12.

Another way to derive Eq. 12 is to consider the "characteristic" viscosity η^* (*24*)

$$\eta^* = \frac{(Em)^{\frac{1}{2}}}{a^2} \tag{14}$$

η^* is used to describe scaling of viscosity on the phase diagram. For example, the ratio between viscosity and η^* is the same for systems described by the same interaction potential in equivalent points of the phase diagram. For systems described by the Lennard-Jones potential, the experimental and calculated viscosity near the triple point and close to the melting line is about three times larger than η^* (*24*, *25*). Near the critical point, η^* is about four times larger than viscosity near the critical point and is expected to give the right order of magnitude of viscosity at the minimum at moderate pressure. The kinematic viscosity corresponding to Eq. 14 is

$$\frac{\eta^*}{\rho} = \frac{E^{\frac{1}{2}} a}{m^{\frac{1}{2}}} \tag{15}$$

Using $a = a_B$ from Eq. 8 and $E = E_R$ from Eq. 9 in Eq. 15 gives the same result as Eq. 12 up to a constant factor on the order of unity. As before, we can also use Eq. 13 in Eq. 15 to get the same result.

We now analyze Eq. 12 and its implications. ν_m contains $\hbar$ and electron and molecule masses only. m characterizes the molecules involved in viscous flow. m_e characterizes electrons setting the intermolecular interactions.

m in Eq. 12 is $m = Am_p$, where A is the atomic weight and m_p is the proton mass. The inverse square root dependence $\nu_m \propto \frac{1}{\sqrt{A}}$ interestingly implies that, for different liquids, ν_m varies by a factor of about 10 only.

Setting $m = m_p$ ($A = 1$) for H in Eq. 12 (similarly to Eqs. 8 and 9 derived for the H atom) gives the fundamental kinematic viscosity in terms of $\hbar$, m_e, and m_p as

$$\nu_f = \frac{1}{4\pi}\frac{\hbar}{\sqrt{m_e m_p}} \tag{16}$$

on the order of 10^{-7} m^2/s.

The quantum origin of ν_m, signified by $\hbar$ in Eq. 12, is due to the quantum nature of interparticle interactions. We note that in the Eyring theory, the viscosity prefactor η_0 in Eq. 4 also contains $\hbar$ (*26*). This follows from assuming that the frequency of molecular oscillation in a single minimum, ω_0 (attempt frequency) is set by the frequency of excited phonons as $\hbar\omega_0 = k_B T$. In later works, the prefactor was mostly treated as a fitting parameter, but its quantum nature was not examined further.

In Table 1 we compare ν_m calculated according to Eq. 12 to the experimental ν_m (*15*) for all liquids shown in Fig. 1. The ratio between experimental and predicted ν_m is in the range of about 0.5 to 3. For the lightest liquid, H_2, experimental ν_m is close to the theoretical fundamental viscosity (Eq. 16). We therefore find that Eq. 12 predicts the right order of magnitude of ν_m.

Table 1. Calculated and experimental ν_m.

	ν_m (calc.)	ν_m (exp.)
	$\times 10^8$ m^2/s	$\times 10^8$ m^2/s
Ar (20 MPa)	3.4	5.9
Ar (100 MPa)	3.4	7.7
Ne (50 MPa)	4.8	4.6
Ne (300 MPa)	4.8	6.5
He (20 MPa)	10.7	5.2
He (100 MPa)	10.7	7.5
N_2 (10 MPa)	4.1	6.5
N_2 (500 MPa)	4.1	12.7
H_2 (50 MPa)	15.2	16.3
O_2 (30 MPa)	3.8	7.4
H_2O (100 MPa)	5.1	12.1
CO_2 (30 MPa)	3.2	8.0
CH_4 (20 MPa)	5.4	11.0
CO (30 MPa)	4.1	7.7

We observe that ν_m increases with pressure in Table 1, similarly to η_m in Fig. 1. However, pressure dependence is not accounted in ν_m in Eq. 12 since Eq. 12 is derived in the approximation involving Eqs. 8 to 11, which do not account for the pressure dependence of ω_D and E.

We make three further remarks regarding the comparison in Table 1. First, the important term in Eq. 12 is the combination of fundamental constants $\frac{\hbar}{\sqrt{m_e m}}$, which sets the characteristic scale of the minimal kinematic viscosity, whereas the numerical factor in Eq. 12 may be affected by the approximations used and mentioned earlier. Second, Eqs. 8 to 10 assume valence electrons directly involved in bonding and hence strongly bonded systems, including metallic, covalent, and ionic liquids. Their viscosity in the supercritical state is unavailable because of high critical points. The available data in Fig. 1 and Table 1 include weakly bonded systems such as noble, molecular, and hydrogen-bonded fluids. Although bonding in these systems is also electromagnetic in origin, weak dipole and van der Waals interactions result in smaller E and, consequently, smaller η as compared with strongly bonded ones, with the viscosity of hydrogen-bonded fluids lying in between (*27*). However, ν_m in Eq. 11 and η in Eq. 15 contain factors Ea^2 and $E^{\frac{1}{2}}a$, respectively. $E^{\frac{1}{2}}$ is 3 to 10 times smaller and a is 2 to 4 times larger in weakly bonded as compared with strongly bonded systems (*27*). This results in a weak dependence of ν_m on bonding type, and the order-of-magnitude evaluation (Eq. 12) is unaffected as Table 1 shows. Third, Eq. 12 for strongly bonded fluids serves as a prediction for future experimental work.

Elementary viscosity

Equation 16 gives the maximal value of the minimum of kinematic viscosity for H. It is interesting to ask what viscosity-related quantity has an absolute minimum. We introduce a new quantity: the elementary viscosity ι ("iota") defined as the product of η_m and elementary volume a^3: $\iota = \eta_m a^3$ or, equivalently, as $\iota = \nu_m m$. Using Eq. 12, ι is

$$\iota = \frac{\hbar}{4\pi}\left(\frac{m}{m_e}\right)^{\frac{1}{2}} \tag{17}$$

which has the lower bound, ι_m, for $m = m_p$ in H

$$\iota_m = \frac{\hbar}{4\pi}\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}} \tag{18}$$

and is on the order of ℏ.

Equation 18 notably involves the proton-to-electron mass ratio, one of few dimensionless combinations of fundamental constants of general importance (*2*).

In Fig. 2 (A and B), we show the product νm in the units of ℏ for two lightest liquids, H_2 and He, for which the minimum of νm, $\nu_m m = \iota$, should be the closest to the lower bound (Eq. 18). We calculate νm using the experimental data (*15*) and show it above and below the critical pressure P_c. For He, the temperature range is above the superfluid transition (we do not consider superfluidity in this work).

The liquid-gas phase transition results in sharp changes of viscosity below P_c. For H_2, the minimum of νm is kinked as a result and decreases with pressure up to P_c. This is followed by the minimum becoming smooth and increasing above P_c. We observe that the smooth minimum just above the critical point (where our derivation of η_m and ν_m, assuming a noninterrupted variation of viscosity, applies) is very close to the minimum at P_c. For He, the minimum similarly increases with pressure in the supercritical region and weakly varies below P_c.

The smallest value of νm, $\iota = \nu_m m$, in Fig. 2 (A and B) is in the range (1.5 to 3.5)ℏ for He and H_2. This is consistent with the estimation of the lower bound of ι, ι_m in Eq. 18. Given that νm varies four to six orders of magnitude in Fig. 2, the agreement with our result (Eq. 18) is notable.

We also show νm for common H_2O in Fig. 2C as a reference and include the triple and critical point in the pressure range. The qualitative behavior of νm is similar to that of H_2, with ι of about 30ℏ.

Our lower bound for ι is consistent with the uncertainty principle. As discussed earlier, the minimum of ν can be evaluated as $\nu_m = va$, corresponding to $\iota = mva = pa$, where p is particle momentum. According to the uncertainty principle applied to a particle localized in the region set by a, $\iota \geq \hbar$. This is consistent with our bound ι_m in Eq. 18, although a more general Eq. 17 gives a stronger bound, which increases for heavier molecules.

We now return to the high-energy physics result and the finding of KSS (*8*) that their lower bound (Eq. 3) is about 25 times smaller than in liquid H_2O and N_2. We consider, more generally, the ratio between η and the volume density $d = \frac{qN}{V}$ of any intensive quantity $Q = qN$, where N is the number of particles. Then, $\frac{\eta}{d} = \frac{\nu m}{q}$. If Q is entropy $S = qN$ and q_0 is q corresponding to ν_m, the experimental data (*15*) show that the minimum of $\frac{\eta}{d}$ or $\frac{\nu m}{q}$, R_m, is close to $\frac{\nu_m m}{q_0} = \frac{\iota}{q_0}$ because of slow temperature variation of entropy. Then, R_m is conveniently written in terms of ι using Eq. 17 as

$$R_m = \frac{\iota}{q_0} = \frac{\hbar}{4\pi q_0}\left(\frac{m}{m_e}\right)^{\frac{1}{2}} \tag{19}$$

We observe that, compared to the KSS bound (Eq. 3), Eq. 19 contains an additional factor, $\left(\frac{m}{m_e}\right)^{\frac{1}{2}}$. For H_2O and N_2 considered by KSS, $\left(\frac{m}{m_e}\right)^{\frac{1}{2}}$ is 182 and 227. Using $q_0 = 8.7k_B$ for H_2O and $14.6k_B$ for N_2 (*15*) at pressures considered (*8*), Eq. 19 predicts that the ratio $\frac{\eta}{s}$ in these liquids exceeds the bound (Eq. 3) by a factor of 16 to 21, in an order-of magnitude agreement with the KSS finding and the ability of Eq. 12 to predict experimental ν_m within an order of magnitude.

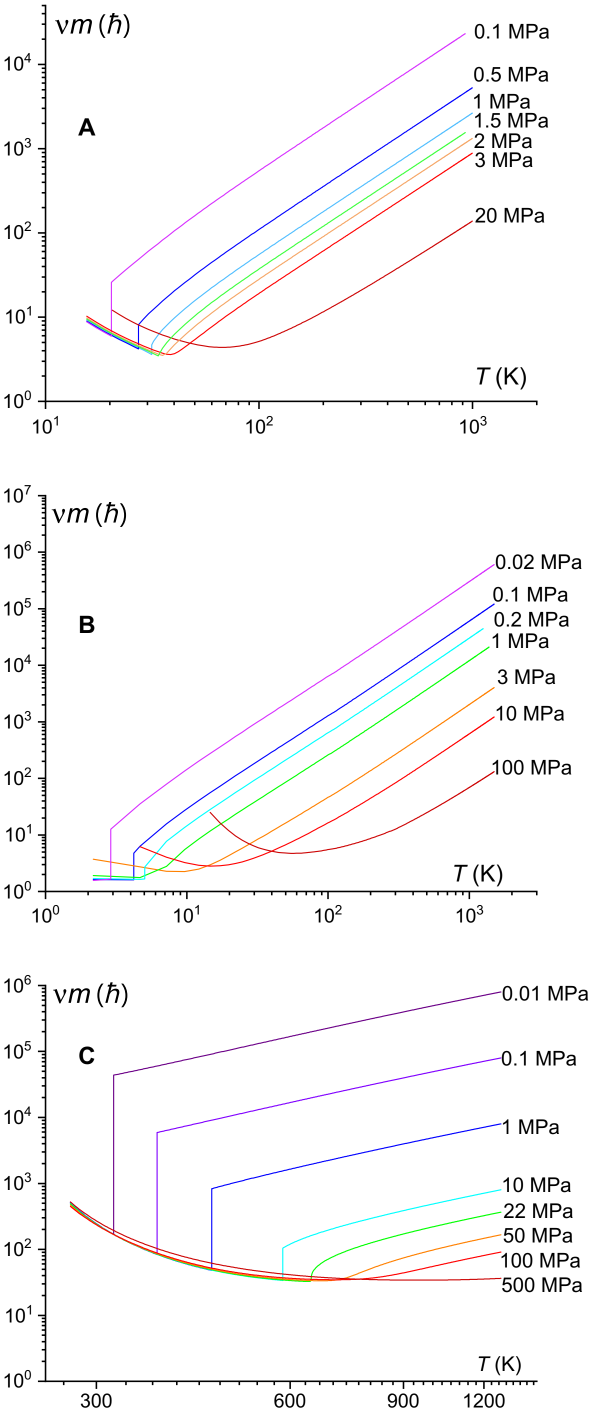

Fig. 2. Elementary viscosity of fluids. νm calculated from experimental kinematic viscosity (*15*) for H_2 (**A**), He (**B**), and H_2O (**C**) below and above the critical pressure P_c. P_c = 1.3 MPa for H_2, 0.23 MPa for He, and 22 MPa for H_2O. The smallest value of νm, ι, is consistent the lower bound (Eq. 18).

CONCLUSIONS

In summary, we have found a new quantum quantity corresponding to the minimum of kinematic viscosity, an interesting result in view of wide variation of viscosity across different systems and external parameters as well as complexity of theoretical description. A related result is the new elementary viscosity ι with the lower bound set by fundamental physical constants and involving the proton-to-electron mass ratio.

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