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Some Reviews on Various Definitions of a Random Conjugate Space together with Various Kinds of Boundedness of a Random Linear Functional

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Abstract A detailed review on the three stages of work in the formative course of the theory of random conjugate spaces is given. in particular the connections and essential distinctions among the three stages of work are pointed out while the relationships among strongly bounded, topologically bounded and almost surely bounded random linear functionals are also given. Finally some shortages currently available in the study of linear operators defined on probabilistic normed spaces are also pointed out.

Key words probabilistic normed spaces; E-norm spaces; random normed spaces; random normed modules; strongly bounded random linear functionals; topologically bounded random linear functionals; almost surely bounded random linear functionals; random conjugate spaces
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1 Introduction and Background

In [1], the first author of this paper redefined random normed spaces and further introduced random normed modules and random inner product modules. Based on these basic notions, Guo, in [1], defined the random conjugate space of a random normed space to be the random normed module consisting of all almost surely bounded random linear functionals defined on the random normed space. A series of recent work on the theory of random conjugate spaces and its applications[2-7] have shown that this definition of a random conjugate space not only provides a proper framework for previous results of random conjugate spaces but also have overcome all serious shortcomings of all previous definitions of a random conjugate space, and thus we regard it as the definitive definition of a random conjugate space.

However, the formative course of this definitive definition is long, intermittent and closely related to many topics from the theory of probabilistic normed spaces, random functional analysis and random metric theory[8-10]. Chronologically, we can divide the formative course into the following three stages: the first is Sultanbekov′s work on strongly bounded random linear functionals in spaces of strongly measurable functions[11] (see also Section 2 of this paper); the second is Zhu′s work on almost surely bounded random linear functionals under the framework of an E-norm space[12] (see also Section 3 of this paper); the third is Guo′s work on random conjugate spaces under the framework of a random normed space[13,14] together with a series of

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Guo′s further work[1, 4-7] (see also Sections 4 and 5 of this paper).

The purpose of this paper is to give some reviews on the above stated each stage of work so that one can make clear the substantial distinctions among the three stages of work together with some basic concepts presented at each stage. In particular, Section 6 of this paper gives complete relationships among strongly bounded, topologically bounded and almost surely bounded random linear functionals, where we also discuss the reasonability of the definitive definition of a random conjugate space from the angle of the operator space theory.

To make precise our reviews on the above subject, we first recall some necessary notions from the theory of probabilistic metric spaces and random functional analysis.

Throughout this paper, K always denotes the scalar field of all real numbers (briefly, R) or of all complex numbers (briefly, C). $D^+ = \{F: (-\infty, +\infty) \to [0, 1] \mid F$ is left continuous, nondecreasing, $F(0) = 0$, $\lim_{x\to+\infty} F(x) = 1\}$, namely the set of all regular distance distribution functions (see [8]).

A two-place function $T: [0, 1] \times [0, 1] \to [0, 1]$ is called a weak t-norm if it satisfies the following three conditions: 1) $T(a, b) = T(b, a)\ \forall a, b \in [0, 1]$; 2) $T(a, b) \leq T(c, d)\ \forall a \leq c, b \leq d$; 3) $T(1, 0) = 0$. A weak t-norm T is called a t-norm if it also satisfies the following two conditions: 4) $T(1, a) = a\ \forall a \in [0, 1]$; 5) $T(a, T(b, c)) = T(T(a, b), c)\ \forall a, b, c \in [0, 1]$.

Clearly, $T_{\max}(a, b) = 1$ if $a \cdot b > 0$ and 0 otherwise, is the greatest weak t-norm in all weak t-norms. Min defined by $\mathrm{Min}(a, b) = a \wedge b\ \forall a, b \in [0, 1]$, and W defined by $W(a, b) = \max(a + b - 1, 0)\ \forall a, b \in [0, 1]$, are both t-norms.

Definition 1.1[8] A triple $(S, \mathscr{F}, T)$ is called a Menger probabilistic normed space (briefly a Menger-PN space) over K if S is a linear space over K, T is a weak t-norm and $\mathscr{F}: S \to D^+$ is a mapping such that the following hold:

(PN-1) $F_p(t) = \chi_0(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$ $(\forall t \in R)$ if $p = \theta$ (the null in S);

(PN-2) $F_{\alpha p}(t) = F_p\left(\dfrac{t}{|\alpha|}\right)\ \forall t \in R$, $\alpha \in K$ and $\alpha \neq 0$, and $p \in S$;

(PN-3) $F_p = \chi_0$ implies $p = \theta$;

(PN-4) $F_{p+q}(x + y) \geq T(F_p(x), F_q(y))\ \forall p, q \in S$, $x, y \in R$.

Here F_p stands for $\mathscr{F}(p)$, called the probabilistic norm of the vector p in S. If $(S, \mathscr{F}, T)$ satisfies only (PN-1), (PN-2) and (PN-4) above then it is called a probabilistic pseudonormed space (briefly, a Menger-PPN space).

Let $(S, \mathscr{F}, T)$ be a Menger-PN space. Denote the set of all such weak t-norms T_1 for which (PN-4) holds by $\mathscr{T}$, define $\tilde{T}: [0, 1] \times [0, 1] \to [0, 1]$ by $\tilde{T}(a, b) = \sup\{T_1(a, b) \mid T_1 \in \mathscr{T}\}\ \forall a, b \in [0, 1]$. Then $\tilde{T}$ still satisfies (PN-4) and is also a weak t-norm. $\tilde{T}$ is called the greatest weak t-norm of $(S, \mathscr{F}, T)$[15, 16].

Definition 1.2[8] Let $(S, \mathscr{F}, T)$ be a Menger-PPN space such that $T(1, a) \geq a\ \forall a \in [0, 1]$. Given an element p in S, let $\tilde{p} = \{q \in S \mid F_{p-q} = \chi_0\}$, $\tilde{S} = \{\tilde{p} \mid p \in S\}$ and $\tilde{\mathscr{F}}: \tilde{S} \to D^+$ be defined by $\tilde{F}_{\tilde{p}} = F_p$ for any p in S. Then $(\tilde{S}, \tilde{\mathscr{F}}, T)$ is a Menger-PN space, called the quotient space of $(S, \mathscr{F}, T)$.

Remark 1.1 Let $(S, \mathcal{F}, T)$ be the same as in Def. 1.1, $p, q \in S$ be such that $F_{p-q} = \varepsilon_0$. Then $F_p(t) = F_q(t)\ \forall t \in R$. In fact, we need only to check $F_p(t) = F_q(t)\ \forall t > 0$. Given a positive number $t > 0$, and $0 < x < t$, then (PN-4) yields $F_p(t) \geq T(F_{p-q}(x), F_q(t-x)) = T(1, F_q(t-x)) \geq F_q(t-x)\ \forall x \in (0, t)$, and hence $F_p(t) \geq F_q(t)\ \forall t > 0$ since F_q is left continuous. Similarly $F_q(t) \geq F_p(t)\ \forall t > 0$, thus $F_p = F_q$, this $\tilde{\mathcal{F}}$ is well defined. Obviously, all t-norms T satisfy the condition $T(1, a) = a$, and hence also $T(1, a) \geq a\ \forall a \in [0, 1]$.

Proposition 1.1[15, 16] Let $(S, \mathcal{F}, T)$ be a Menger-PN space over K with its greatest weak t-norm $\tilde T$. For each $r \in (0, 1)$, define $N_r(\cdot): S \to [0, +\infty)$ by $N_r(p) = F_p^{\wedge}(r)$, where $F_p^{\wedge}(r) = \sup\{t \geq 0 \mid F_p(t) < r\}$. Then we have the following statements:

1) $(S, \{N_r(\cdot)\}_{r\in(0,1)})$ is a pseudonormed linear space iff $\sup\{\tilde T(a, a) \mid 0 < a < 1\} = 1$;

2) $(S, \{N_r(\cdot)\}_{r\in(0,1)})$ is a B_0-type space (namely each $N_r(\cdot)$ is a semi-norm on S) iff $\tilde T \geq \mathrm{Min}$, namely $\tilde T(a, b) \geq a \wedge b\ \forall a, b \in [0, 1]$.

3) there exists a norm $\|\cdot\|$ on S such that $N_r(p) = \|p\|$ for all $r \in (0, 1)$ and $p \in S$ iff $\tilde T = T_{\max}$, namely $\tilde T(a, b) = 1$ if $a \cdot b > 0$, and 0 otherwise.

Remark 1.2 According to LaSalle[17], that the above $(S, \{N_r(\cdot)\}_{r\in(0,1)})$ is a pseudonormed linear space over K means it satisfies the following conditions:

1) $N_r(p) = 0$ for all $r \in (0, 1)$ iff $p = \theta$;

2) $N_r(\lambda p) = |\lambda| N_r(p)$ for all $\lambda \in K$, $p \in S$ and all $r \in (0, 1)$;

3) for each $r \in (0, 1)$ there exists $t \in (0, 1)$ such that $N_r(p + q) \leq N_t(p) + N_t(q)$ for all $p, q \in S$.

Since for any fixed p in S, $N_r(p)$ is nondecreasing in r, as shown in [17], such pseudonormed linear spaces as $(S, \{N_r(\cdot)\}_{r\in(0,1)})$ exactly give all metrizable linear topological spaces, and thus give all metrizable locally convex spaces when $\tilde T \geq \mathrm{M}$ in. Proposition 1.1 first occurred in [16] in the above form together with a brief proof, it is a slight improvement of the corresponding results in [15].

Proposition 1.2[8] Let $(S, \mathcal{F}, T)$ be a Menger-PN space such that its greatest weak t-norm $\tilde T$ satisfies $\sup\{\tilde T(a, a) \mid 0 < a < 1\} = 1$. Given $\varepsilon > 0$, $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{p \in S \mid F_p(\varepsilon) > 1 - \lambda\}$, then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base at θ of some metrizable linear topology for S, called the (ε, λ)-linear topology for S. Form now on, for a given Menger-PN space, we say it is a linear topological space iff its linear topology is exactly the above (ε, λ)-linear topology.

Remark 1.3 In fact, in Proposition 1.2, the (ε, λ)-metrizable linear topology is exactly that determined by the pseudonorms $\{N_r(\cdot) \mid r \in (0, 1)\}$ as in [17]. It is obvious that a set A of $(S, \mathcal{F}, T)$ is topologically bounded (namely A can be absorbed by any neighborhood of θ) iff A is probabilistically bounded in terms of [8] (namely $D_A(\cdot): R \to [0, 1]$ defined by $D_A(t) = \sup\limits_{x<t} \inf\limits_{p\in A} F_p(x)$ for all $t \in R$, belongs to D^+), this is also equivalent to $\sup\limits_{a\in A} N_r(a) < +\infty$ for all $r \in (0, 1)$.

Definition 1.2 Let $(S_1, \mathcal{F}^1, T^1)$ and $(S_2, \mathcal{F}^2, T^2)$ be any two Menger-PN spaces over the same scalar field K. Then a linear operator $Q: S_1 \to S_2$ is called strongly bounded if there exists a positive number M such that $F^2_{Q(p)}(t) \geq F^1_p\left(\dfrac{t}{M}\right)$ for all p in S_1 and all $t \in R$; is called topologically bounded (if $\sup_{0<a<1} T^1(a, a) = 1$ and $\sup_{0<a<1} T^2(a, a) = 1$) if Q maps every topologically bounded set of S^1 to a topologically bounded set of S^2.

In this sequel of this paper, all the greatest weak t-norms $\tilde{T}$ of Menger-PN spaces appearing in this paper are assumed to satisfy the condition $\sup_{0<a<1} \tilde{T}(a, a) = 1$. Then it is easy to check that a strongly bounded linear operator is topolotically bounded, and that a linear operator is topologically bounded iff it is continuous.

Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be any two normed spaces over the same scalar field K. Define $\mathscr{F}^1: S_1 \to D^+$ and $\mathscr{F}^2: S_2 \to D^+$ by $F^1_p(t) = \chi_0(t - \|p\|_1)\ \forall p \in S_1, t \in R$; $F^2_p(t) = \chi_0(t - \|p\|_2)\ \forall p \in S_2, t \in R$. Here χ_0 is the same as in Definition 1.1, then $(S_1, \mathscr{F}^1, \mathrm{Min})$ and $(S_2, \mathscr{F}^2, \mathrm{Min})$ are both Menger-PN spaces, and it is obvious that their (ε, λ)-linear topologies are exactly the ordinary norm-topologies of $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ respectively. It is also clear that at this time a strongly bounded linear operator Q from $(S_1, \mathscr{F}^1, \mathrm{Min})$ to $(S_2, \mathscr{F}^2, \mathrm{Min})$ satisfies $F^2_{Q(p)}(t) \geq F^1_p\left(\frac{t}{M}\right)$ for all $t \in R$, $p \in S_1$ for some $M > 0$ iff $\|Q(p)\|_2 \leq M\|p\|_1$ for all $p \in S_1$, this is again iff Q is topologically bounded. Generally speaking, a topologically bounded linear operator is not necessarily strongly bounded, see Section 6 of this paper.

Although the topological boundedness of linear operators is strictly weaker than the strong boundedness of linear operators, the set of all topologically bounded linear operators from a Menger-PN space to another forms a linear space under the ordinary scalar multiplication and addition operation of linear operators. However, the similar conclusion, generally, no longer holds for strongly bounded linear operators, a not very dificult counterexample on this respect is left to the reader. We give only the following affirmative result.

Proposition 1.3 Let $(S_1, \mathscr{F}^1, T^1)$ and $(S_2, \mathscr{F}^2, T^2)$ be any two Menger-PN spaces over the same scalar field K, and $T^2 \geq \mathrm{Min}$. Then the set of all strongly bounded linear operators form S_1 to S_2 forms a linear space over K.

Proof Let $\{N^1_r(\cdot)\}_{r\in(0,1)}$ and $\{N^2_r(\cdot)\}_{r\in(0,1)}$ correspond to $\mathscr{F}^1$ and $\mathscr{F}^2$ respectively as in Prop. 1.1. Since $T^2 \geq \mathrm{Min}$, then for each $r \in (0, 1)$ $N^2_r(\cdot)$ is a seminorm on S_2 by Prop. 1.1. We need only to check $Q_1 + Q_2$ is still strongly bounded if both Q_1 and Q_2 are strongly bounded linear operators from S_1 to S_2. Suppose M_1 and M_2 are positive such that $F^2_{Q_1(p)}(t) \geq F^1_p\left(\frac{t}{M_1}\right)$, $F^2_{Q_2(p)}(t) \geq F^1_p\left(\frac{t}{M_2}\right)$ for all $t \in R$ and all $p \in S_1$; equivalently $N^2_r(Q_1(p)) \leq M_1 \cdot N^1_r(p)$, and $N^2_r(Q_2(p)) \leq M_2 \cdot N^1_r(p)$ for all $r \in (0, 1)$ and all $p \in S_1$.

Since $N^2_r((Q_1 + Q_2)(p)) \leq N^2_r(Q_1(p)) + N^2_r(Q_2(p)) \leq M_1 \cdot N^1_r(p) + M_2 \cdot N^1_r(p) = (M_1 + M_2) \cdot N^1_r(p)$ for all $p \in S_1$ and all $r \in (0, 1)$, then equivalently we can have:

$$F^2_{(Q_1+Q_2)(p)}(t) \geq F^1_p\left(\frac{t}{M_1 + M_2}\right) \text{ for all } t \in R \text{ and all } p \in S_1.$$

$Q_1 + Q_2$ is, clearly, also linear, and thus $Q_1 + Q_2$ is still a strongly bounded linear operator.

This completes the proof.

Proposition 1.4 Let $(S, \mathscr{F}^1, T^1)$ and $(S_2, \mathscr{F}^2, T^2)$ be two Menger-PN spaces over K.

Then we have the following:

1) if $T^1 \geq M$ in, then there exists a nonzero continuous linear operator from S_1 to S_2;

2) if $T^2 \geq M$ in and there doesn't exist a nonzero continuous linear functional on $(S_1, \mathcal{F}, T^1)$, then it is impossible that there exists a nonzero continuous linear operator from $(S_1, \mathcal{F}, T^1)$ to $(S_2, \mathcal{F}, T^2)$.

Proof 1) Since $T^1 \geq M$ in, then $(S_1, \mathcal{F}, T^1)$ must be locally convex, of course, there exists a nonzero continuous linear functional f on S_1. Let q be any nonzero element in S_2, then $Q: S_1 \to S_2$ defined by $Q(p) = f(p) \cdot q$ for all $p \in S_1$, is clearly a nonzero continuous linear operator from S_1 to S_2.

2) Suppose there exists a nonzero continuous linear operator Q from S_1 to S_2, namely $Q(p) \neq \theta$ for some p in S_1. Since $T^2 \geq M$ in, $(S_2, \mathcal{F}, T^2)$ must be locally convex, then by Hahn-Banach theorem there must be some nonzero continuous linear functional f on S_2 such that $f(Q(p)) \neq 0$. Define $F: S_1 \to K$ by $F(\tilde{p}) = (f \circ Q)(\tilde{p})\ \forall\, \tilde{p} \in S_1$, then it is obvious that F is a nonzero continuous linear functional on S_1, this is a contradiction to the hypothesis on 2).

This completes the proof.

Remark 1.4 Proposition 1.4 shows there are not necessarily nontrivial continuous (equivalently, topologically bounded) linear operators between Menger-PN spaces. Despite this fact, many scholars have discussed the so-called operator space problems under the hypothesis that $T^2 \geq M$ in, we hope these scholars can take seriously this fact.

E-norm spaces and semi-norm-generated spaces are frequently employed in this paper, before they are cited, let us first recall some basic notions. Throughout the rest of this paper, $(K, \mathcal{A}, \_)$ always denotes a given probability space unless otherwise stated.

Definition 1.3 Let $(B, \|\cdot\|)$ be a normed space. A mapping $V: (K, \mathcal{A}, \_) \to (B, \|\cdot\|)$ is called an $\mathcal{A}$-random element (also called an $\mathcal{A}$-generalized random variable) if $V^{-1}(G) = \{k \in K \mid V(k) \in G\} \in \mathcal{A}$ for all open set G of B (an $\mathcal{A}$-random element is often simply said to be a random element if no other e-algebras than $\mathcal{A}$ are considered)[9,10]; An $\mathcal{A}$-random element $V: (K, \mathcal{A}, \_) \to B$ is called simple if V takes only finitely many values in B, furthermore a mapping from $(K, \mathcal{A}, \_)$ to $(B, \|\cdot\|)$ is called an $\mathcal{A}$-random variable if it is the pointwise limit of a sequence of simple $\mathcal{A}$-random elements[18]. A mapping from $(K, \mathcal{A}, \_)$ to $(B, \|\cdot\|)$ is called a strongly measurable function if it is the pointwise limit almost everywhere of a sequence of simple $\mathcal{A}$-random elements[19].

Remark 1.5 It is well known from [9, 10] that a mapping is $\mathcal{A}$-random variable iff it is $\mathcal{A}$-random element and its range is separable. It is easy to see that the notion of a strongly measurable function amounts to that of a _-measurable function introduced in [20]. For any two $\mathcal{A}$-random variables $V_1, V_2: (K, \mathcal{A}, \_) \to (B, \|\cdot\|)$, it is well known from [9] that $V_1 + V_2$ is still an $\mathcal{A}$-random variable, and hence $\|V_1 + V_2\|$ defined by $\|V_1 + V_2\|(k) = \|V_1(k) + V_2(k)\|$ for all k in K is a nonnegative $\mathcal{A}$-random variable. However, when V_1, V_2 are only $\mathcal{A}$-random elements, $V_1 + V_2$ is not necessarily a $\mathcal{A}$-random element, and even $\|V_1 + V_2\|$ is not necessarily $\mathcal{A}$-measurable either (See [9]). Finally it is also obvious that every strongly measurable function must be _-equivalent to an $\mathcal{A}$-random variable.

Definition 1.4[8, 21] An ordered pair $(S, \mathcal{F})$ is called an E-norm space with base $(\Omega, \mathcal{A}, \mu)$ and target $(B, \|\cdot\|)$ (where $(B, \|\cdot\|)$ is a normed space over K) if S is a linear space over K of mappings from $(\Omega, \mathcal{A}, \mu)$ to $(B, \|\cdot\|)$ under the ordinary pointwise addition and scalar multiplication, and if $\mathcal{F}$ is a mapping from S to D^+ such that the following hold:

(EN-1) for each $p \in S$, $\|p\|: \Omega \to [0, +\infty)$ defined by $\|p\|(\omega) = \|p(\omega)\|$ for all ω in Ω, is a nonnegative $\mathcal{A}$-random variable;

(EN-2) $F_p(t) = \mu(\{\omega \in \Omega \mid \|p(\omega)\| < t\})$ for all $t \in R$ and all $p \in S$.

Furthermore, if $F_p = H$ implies $p(\omega) = \theta$ for all $\omega \in \Omega$ (where θ is the null of B), then $(S, \mathcal{F})$ is called a canonical E-norm space.

Remark 1.6 Let $(S, \mathcal{F})$ be an E-norm space with base $(\Omega, \mathcal{A}, \mu)$ and target $(B, \|\cdot\|)$. Then $(S, \mathcal{F}, W)$ is a Menger-PPN space[8], where $W(a, b) = \max(a + b - 1, 0)\ \forall a, b \in [0, 1]$, furthermore if $(S, \mathcal{F})$ is canonical then $(S, \mathcal{F}, W)$ is a Menger-PN space. It should also be pointed out that the null of S is the mapping taking the constant value θ (the null of B), and thus for any two elements p and q in S, $p = q$ means $p(\omega) = q(\omega)$ for all ω in Ω. That $(S, \mathcal{F})$ is canonical means it is impossible that p and q are simultaneously contained in S if p and q are only equal almost surely but not identical, this is rather stringent.

Definition 1.5[8, 21] An ordered pair $(S, \mathcal{F})$ is called a semi-norm-generated space over K with base $(\Omega, \mathcal{A}, \mu)$ if S is a linear space over K, $\mathcal{F}$ is a mapping from S to D^+, and there is a seminorm $\|\cdot\|_\omega$ on S for each ω in Ω such that the following hold:

1) $\|p\|_\omega = 0$ for all ω in Ω iff $p = \theta$ (the null in S);

2) for each $p \in S$, $\|p\|_\omega$ is a nonnegative $\mathcal{A}$-measurable function of ω;

3) $F_p(t) = \mu(\{\omega \in \Omega \mid \|p\|_\omega < t\})$ for all $t \in R$ and all $p \in S$.

Furthermore if $F_p = H$ implies $p = \theta$, then $(S, \mathcal{F})$ is called separated.

Let $(S, \mathcal{F})$ is a semi-norm-generated space, then $(S, \mathcal{F}, W)$ is a Menger-PPN space. If $(S, \mathcal{F})$ is separated, then $(S, \mathcal{F}, W)$ is a Menger-PN space. Now, suppose $(S, \mathcal{F})$ is an E-norm space with base $(\Omega, \mathcal{A}, \mu)$ and target $(B, \|\cdot\|)$, define $\|\cdot\|_\omega: S \to [0, +\infty)$ by $\|p\|_\omega = \|p(\omega)\|$ for all ω in Ω and all $p \in S$, then $\{\|\cdot\|_\omega \mid \omega \in \Omega\}$ is a family of seminorms on S and satisfies Definition 1.5, and hence $(S, \mathcal{F})$ becomes a semi-norm-generated space. The following proposition shows a semi-norm-generated space can, essentially, also be regarded as an E-norm space.

Proposition 1.5[21] A Menger-PPN space $(S, \mathcal{F}, T)$ is an E-norm space iff it is isomorphically isometric to a semi-norm-generated space, where isometric means probabilistic-norm-preserving.

2 Some Reviews on Sultanbekov's Work on Random Functionals in Spaces of Strongly Measurable Functions

Let $(B, \|\cdot\|)$ be a normed space over K, denote by $L^0(\Omega, B)$ the linear space of all random variables from $(\Omega, \mathcal{A}, \mu)$ to $(B, \|\cdot\|)$, and by $L(\Omega, B)$ the linear space of all equivalence classes of the elements in $L^0(\Omega, B)$.

Define $\mathcal{F}^0: L^0(\Omega, B) \to D^+$ by $F_p^0(t) = \mu(\{\omega \in \Omega \mid \|p(\omega)\| < t\})$ for all $p \in L^0(\Omega, B)$

and all $t \in R$, then $(L^0(\Omega, B), \mathscr{F}^0)$ is an E-norm space with base $(\Omega, \mathscr{A}, \mu)$ and target $(B, \|\cdot\|)$, namely all B-valued random variable generated E-norm space in terms of [8]. Clearly, $(L^0(\Omega, B), \mathscr{F}^0)$ is merely an E-norm space but not necessarily canonical. By Remark 1.1, $(L^0(\Omega, B), \mathscr{F}^0)$ admits a quotient space $(S, \mathscr{F})$, it is obvious that S is exactly $L(\Omega, B)$, and hence $(L(\Omega, B), \mathscr{F})$ is a Menger-PN space under the t-norm W, when $B = K$, we simply write $(L(\mu, K), \mathscr{F}^+)$ for $(L(\Omega, B), \mathscr{F})$.

As spaces of equivalence classes, our $(L(\Omega, B), \mathscr{F})$ is identical with the space of all equivalence classes of the strongly measurable functions empolyed in [11].

Definition 2.1[11] Let $(B, \|\cdot\|)$ be a real normed space. A linear operator f from $L(\Omega, B)$ to $L(\Omega, R)$ is called a random linear functional. A strongly bounded linear operator f from $(L(\Omega, B), \mathscr{F})$ to $(L(\Omega, R), \mathscr{F}^+)$ is called a strongly bounded random linear functional on $L(\Omega, B)$, namely there exists some positive number M such that $F^1_{f(p)}(t) \geq F_p\left(\frac{t}{M}\right)$ for all $p \in L(\Omega, B)$ and all $t \in R$, $\|f\| = \inf\{M > 0 \mid F^1_{f(p)}(t) \geq F_p\left(\frac{t}{M}\right)$ for all $p \in L(\Omega, B)$ and all $t \in R\}$ is called the norm of f. Denote by $L(\Omega, B)'$ the set of all strongly bounded random linear functionals on $L(\Omega, B)$.

Sultanbekov[11] also considered the generalization problem of Hahn-Banach theorem. However, by his result in [11], he can only obtain the following weak result.

Proposition 2.1[11] Let $(B, \|\cdot\|)$ be a real separable reflexive Banach space. Then for any nonzero element $p \in L(\Omega, B)$ there exists a nonzero strongly bounded random linear functional f on $L(\Omega, B)$ such that $f(p) \neq \theta$ (the null of $L(\Omega, R)$) and $\|f\| = 1$.

Review 2.1 Let $\Omega = [0, 1]$, $\mathscr{A}$ = the σ-algebra of all Legesgue measurable subsets of $[0, 1]$ and μ = the Lebesgue measure on $\mathscr{A}$. Then $(\Omega, \mathscr{A}, \mu)$ is a complete probability space, and $L(\Omega, R)$ is exactly the linear space of all equivalence classes of the real Lebesgue measurable functions on $[0, 1]$, and thus $(L(\Omega, R), \mathscr{F}^+)$ is a complete Menger-PN space under W, it is well known that not even a nonzero continuous linear functional on $(L(\Omega, R), \mathscr{F}^+)$ exists. Proposition 2.1 shows there exists sufficiently many strongly bounded random linear functionals on $L(\Omega, R)$, this brings a new hope to look for a new theory of conjugate spaces instead of classical conjugate spaces in the study of Such Menger-PN spaces as $(L(\Omega, B), \mathscr{F})$. Therefore Sultanbekov's work[11] is of great importance.

Review 2.2 Unfortunately, not only because his result of Hahn-Banach theorem of strongly bounded random linear functionals is too limited, but also because, as Sultanbekov said in [11], $L(\Omega, B)'$ doesn't necessarily form a linear space, the notion of a strongly bounded random linear functional can not, eventually, lead to a satisfying theory of random conjugate spaces for $L(\Omega, B)$.

3 Some Reviews on Zhu Lin-hu's Work on Random Conjugate Spaces under the Framework of E-norm Spaces

Let $(B, \|\cdot\|)$ be a normed space over K. According to [22, 23], a mapping f: $(\Omega, \mathscr{A}, \mu) \times B \to K$ is called a random functional if $f(\cdot, b)$: $\Omega \to K$ is $\mathscr{A}$-measurable for each $b \in B$; a random functional $f: \Omega\times B\to K$ is called sample-linear (accordingly, sample-continuous) if $f(\omega,\cdot): B\to K$ is linear (correspondingly, continuous) for each ω in Ω; a random functional $f: \Omega\times B\to K$ is called linear if $P(\{\omega\in\Omega \mid f(\omega,\alpha b_1+\beta b_2)=\alpha f(\omega,b_1)+\beta f(\omega,b_2)\})=1$ for all $\alpha,\beta\in K$ and all $b_1, b_2\in B$.

Let $L^0(\Omega,K)$ and $L(\Omega,K)$ be the same as in Section 2 of this paper. Then a random functional $f: \Omega\times B\to K$ can be regarded as the mapping $\hat{f}: B\to L^0(\Omega,K)$ defined by $\hat{f}(b)=f(\cdot,b)$ $\forall b\in B$. Then it is clear that f is sample-linear iff $\hat{f}$ is linear and that f is linear iff the lifting $\tilde{f}$ of $\hat{f}$, namely $\tilde{f}: B\to L(\Omega,K)$ defined by $\tilde{f}(b)=$ the equivalence class of $\hat{f}(b)$ $(\forall b\in B)$, is linear. Obviously, the "sample-linearity" and the "linearity" of random functionals are essentially different from each other.

In [9], Wang posed the following query: let $(B,\|\cdot\|)$ be a real normed space, M a linear subspace of B and $f: \Omega\times M\to R$ a sample-linear and sample-continuous random functional, then can f be extended to a sample-linear and sample-continuous random functional on $\Omega\times B$? Hanš[24] gave an affirmative answer when B is separable, it remains to solve the nonseparable case.

Zhu[12] attempted to attack this problem by using the framework of E-norm spaces. Before we give his results, let us recall the following two propositions.

Proposition 3.1[25] Let $L^0(\Omega,R)$ be the set of all real-valued random variables on $(\Omega,\mathcal{A},P)$ and A be a subset of $L^0(\Omega,R)$. $\xi\in L^0(\Omega,R)$ is called an essential upper (lower) bound of A if $\xi(\omega)\geq a(\omega)$ $(\xi(\omega)\leq a(\omega))$ P-a.s. for each $a\in A$. Similarly, one can have a notion of an essential supremum (or infimum) of A. Then every subset A having an essential upper (lower) bound must have an essential supremum (infimum), and is unque in the sense of almost sure equality, denoted by $\vee A$ $(\wedge A)$, furthermore there exists a sequence $\{a_n\}$ in A such that $\vee_{n\geq 1}a_n=\vee A$ $(\wedge_{n\geq 1}a_n=\wedge A)$.

Proposition 3.2 below is merely an equivalent variant of Proposition 3.1, but it is more natural from the traditional lattice theory.

Proposition 3.2[20] Let $L(\Omega,R)$ be the set of all equivalence classes of the elements in $L^0(\Omega,R)$. Then $L(\Omega,R)$ is a complete lattice by the ordering $\leq$: $\xi\leq\eta$ iff $\xi^0(\omega)\leq\eta^0(\omega)$ P-a.s. for arbitrarily chosen representatives ξ^0 and η^0 of ξ and η respectively. Suppose A is a subset of $L(\Omega,R)$, if $\vee A$ $(\wedge A)$ exists then there exists a sequence $\{a_n\}$ in A such that $\vee_{n\geq 1}a_n=\vee A$ (resp., $\wedge_{n\geq 1}a_n=\wedge A$).

Denote the set $\{\xi\in L^0(\Omega,R) \mid \xi(\omega)\geq 0\ P\text{-a.s.}\}$ by $L_0^+(\Omega)$, and the set of all equivalence classes of the elements in $L_0^+(\Omega)$ by $L^+(\Omega)$.

Independent of Sultanbekov[11], Zhu introduced the following notion in [12]. Throughout his work in [12] Zhu always assumed any two elements that are equal almost surely in an E-norm space, $L_0^+(\Omega)$ and $L^0(\Omega,R)$ respectively, are identified. Zhu introduced Definition 3.1 below under the former assumption.

Definition 3.1[12] Let $(B,\|\cdot\|)$ be a real normed space and $(E,\mathcal{F})$ be a real E-norm space with base $(\Omega,\mathcal{A},P)$ and target $(B,\|\cdot\|)$. A linear operator f from E to $L^0(\Omega,R)$ is called a random linear functional, furthermore f is called almost surely bounded (briefly, a.s.

bounded) if there exists some $\xi \in L_0^+(\Omega)$ such that $|f(p)(\omega)| \le \xi(\omega)\cdot \|p(\omega)\|$ P-a.s. $\forall p \in E$, namely $P(\{\omega\in\Omega \mid |f(p)(\omega)| \le \xi(\omega)\cdot\|p(\omega)\|\}) = 1\ \forall p\in E$. Denote by E^* the linear space of all a.s. bounded random linear functionals on E, define $\mathscr{F}: S^* \to D^+$ by $F_f^*(t) = P(\{\omega\in\Omega \mid X_f^*(\omega) < t\})$ for all $t\in R$ and all $f\in E^*$, where $X_f^* = \wedge\{\xi\in L_0^+(\Omega) \mid |f(p)(\omega)| \le \xi(\omega)\cdot\|p(\omega)\|$ P-a.s. $\forall p\in E\}$, then Zhu asserted in [12] that $(E^*, \mathscr{F})$ is isomorphically isometric to a semi-norm generated space and hence it can be regarded as an E-norm space by Proposition 1.5, this E-norm space (still denoted by $(E^*, \mathscr{F})$) is called (by Zhu) the random conjugate space of $(E, \mathscr{F})$.

Definition 3.1 given by Zhu in [12] is full of serious vagueness on the linearity of random functionals, and his assertion that $(E^*, \mathscr{F})$ is an E-norm space is a vital mistake. Following are two reviews on his Definition 3.1.

Review 3.1 Since E-norm spaces are not necessarily Hausdorff spaces, it is in order to guarantee the two E-norm spaces $(E, \mathscr{F})$ and $L^0(\Omega, R)$ in Definition 3.1 to have the Hausdorff separation property that Zhu assumed in [12] any two elements that are equal almost surely both in $(E, \mathscr{F})$ and in $L^0(\Omega, R)$ are identified. However, once the hypothesis is made, then $(E, \mathscr{F})$ and $L^0(\Omega, R)$ are, strictly speaking, their quotient spaces $(\tilde{E}, \tilde{\mathscr{F}})$ and $(L(\Omega, R), \mathscr{F})$, as such a linear operator f from E to $L^0(\Omega, R)$ is, in principle, a linear operator from $\tilde{E}$ to $L(\Omega, R)$. If the hypothesis had not been made, according to the oringinal linear structure of E and $L^0(\Omega, R)$, f is a linear operator from E to $L^0(\Omega, R)$ iff $f(\alpha p_1 + \beta p_2)(\omega) = \alpha f(p_1)(\omega) + \beta f(p_2)(\omega)$ for all ω in Ω. Clearly, f being a linear operator from $\tilde{E}$ to $L(\Omega, R)$ is essentially different from f being a linear operator from E to $L^0(\Omega, R)$, and this two kinds of linearity can not coexist, in general, in a single definition. However, Zhu, in [12], sometimes consider f to be a linear operator from E to $L^0(\Omega, R)$, and sometimes consider f to be a linear operator from $\tilde{E}$ to $L(\Omega, R)$ only by a single definition, which leads to many confusions in his and sequent work on the topics of random conjugate spaces.

Review 3.2 If we remove the hypothesis that any two elements equal almost surely in $(E, \mathscr{F})$, $L_0^+(\Omega)$ and $L^0(\Omega, R)$ are identified, it is not very difficult for one to realize that Zhu′s assertion that $(E^*, \mathscr{F})$ is an E-norm space is false: since by Proposition 3.1 for any given $\alpha\in R$ and any two a.s. bounded random linear functionals f and g on E one, in general, can only obtain the following information:

1) $X_{\alpha f}^*(\omega) = |\alpha|\cdot X_f^*(\omega)$ P-a.s.;

2) $X_{f+g}^*(\omega) \le X_f^*(\omega) + X_g^*(\omega)$ P-a.s..

And one can not get the further information: there exists a single P-null set $\mathscr{N}$ such that for each ω in $\Omega\setminus\mathscr{N}$, $\|\cdot\|_\omega: E^* \to [0, +\infty)$ defined by $\|f\|_\omega = X_f^*(\omega)$ for all $f\in E^*$, is a semi-norm on E^*. Thus there is no argument for Zhu′s asertion that $(E^*, \mathscr{F})$ is a semi-norm-generated space and hence allows him to define $(E^*, \mathscr{F})$ to be an E-norm space, and it turned out to be a misuse of Proposition 1.5 that Zhu said $(E^*, \mathscr{F})$ to be an E-norm space in [12]. His misuse of Proposition 1.5 had let Zhu to make a vital mistake since this mistake makes him in [12] not to realize the importance of random normed spaces in the theory of random conjugate spaces!

With the above attendant shortcomings of his Definition 3.1 Zhu gave an analogue of the Hahn-Banach extension theorem for real linear spaces and some other vague conclusions as follows:

Proposition 3.3[12] Let $(E, \mathscr{F})$ be the same as in Definition 3.1, $M \subset E$ a real subspace and $f: M \to L^0(\Omega, R)$ an a.s. bounded random linear functional. Then there exists an a.s. bounded random linear functional $\tilde{f}$ on E such that 1) $\tilde{f}(p) = f(p)\ \forall p \in M$ and 2) $F^*_{\tilde{f}} = F^*_f$.

Review 3.3 Replacing the supremum and infimum principle for the system of real numbers by Proposition 3.1 or Proposition 3.2, Zhu made use of the similar techniques for classical Hahn-Banach theorem to prove Proposition 3.3. Simultaneously, Zhu said in [12] it is also obvious that the complex formulation of Proposition 3.3 holds. However Zhu′s so-called obvious reason for the complex formulation is that he inexplicitly employed the following fact: let $(E, \mathscr{F})$ be a complex E-norm space, $M \subset E$ a subspace and $f: M \to L^0(\Omega, C)$ an a.s. bounded random linear functional. Suppose $f_1: M \to L^0(\Omega, R)$ is the real part of f with an extension $\tilde{f}_1: E \to L^0(\Omega, R)$, then there exists a single $\mathcal{N}$-null set $\mathcal{N}$ such that for each ω in $\Omega \setminus \mathcal{N}$ the following two conditions are satisfied: 1) $\tilde{f}_1(\alpha p_1 + \beta p_2)(\omega) = \alpha \tilde{f}_1(p_1)(\omega) + \beta \tilde{f}_1(p_2)(\omega)$ for all reals α, $\beta \in R$ and all $p_1, p_2 \in E$, and 2) $|\tilde{f}_1(p)(\omega)| \leq X^*_f(\omega) \cdot \|p(\omega)\|$ for all $p \in E$. Since his desired single $\mathcal{N}$-null set $\mathcal{N}$ in general, seldom exists, in particular the property 2) can not be guaranteed at all by Proposition 3.1 or by any kind of linearity as described in our Review 3.2, and then Zhu′s conclusions on Hahn-Banach theorem for a.s. bounded random linear functionals on E-norm spaces hide great vagueness because of Definition 3.1 as well as false assertions.

Indeed, it is not very easy to give a proper review on Zhu′s above work as a part of the whole work in [12]. Just as stated in our Reviews 3.1, 3.2 and 3.3, this part contains lots of vagueness and vital mistakes; on the other hand, this part marks the beginning of the study of a.s. bounded random linear functionals. There is no doubt that this part of [12] is extremely motivating in the formative course of the theory of random conjugate spaces.

4 Some Reviews on Guo Tie-xin′s Work on the Theory of Random Conjugate Spaces under the Framework of Random Normed Spaces

Before Guo′s work[13] appeared, random metric theory is not only not systematic but also relatively surfacial, it occupies in [8] only one chapter "Random Metric Spaces" where the theory of E-spaces is still the subject of [8]; in particular, the notion of a random normed space was merely mentioned in a inexplicit way in Chapter 15 of [8]. However, a series of recent developments of random metric theory and its applications to functional analysis and random functional analysis have shown random metric theory, in particular, the theory of random normed spaces is both an outgrowth of several closely related branches in Mathematics and an extremely fruitful part of the theory of probabilistic metric spaces.

For the subject of this paper, let us first recall a random normed space in the sense of [8].

Definition 4.1[8] An ordered pair $(S, \mathscr{X})$ is called a random normed space (briefly an RN space) over K with base $(\Omega, \mathscr{A}, \mu)$ if S is a linear space over K and $\mathscr{X}$ is a mapping from S to $L^+_0(\Omega)$ such that the following hold:

(RN-1) $X_{\alpha p}(\omega) = |\alpha| \cdot X_p(\omega)$ a.s. for any $\alpha \in K$ and any p in S;

(RN-2) $X_p(\omega) = 0$ a.s. implies $p = \theta$ (the null in S);

(RN-3) $X_{p+q}(\omega) \leq X_p(\omega) + X_q(\omega)$ a.s. for any p, q in S.

Here $X_p = \mathscr{X}(p)$ is called the random norm of the vector p in S. If $\mathscr{X}$ only satisfies the above (RN-1) and (RN-3) then $\mathscr{X}$ is called a random seminorm, at this time $(S, \mathscr{X})$ is called a random semi-normed space; if, in addition, there exists a single μ-null set Γ of Ω such that for each $\omega \in \Omega \setminus \Gamma$, $\|\cdot\|_\omega: S \to [0, +\infty)$ defined by $\|p\|_\omega = X_p(\omega)\ \forall p \in S$, is an ordinary seminorm on S, then $(S, \mathscr{X})$ is said to be a uniform random semi-normed space.

Let $(E, \mathscr{F})$ be an E-norm space with base $(\Omega, \mathscr{A}, \mu)$ and target $(B, \|\cdot\|)$. Define $\mathscr{X}: E \to L^+_0(\Omega)$ by $X_p(\omega) = \|p(\omega)\|$ for all p in E and all ω in Ω, then $(E, \mathscr{X})$ is a uniform random semi-normed space [8]. In spite of this fairly well-known fact it is rather strange that the authors of the paper [26] still did not know the essential difference between an E-norm space and an random normed space.

Let $(S, \mathscr{X})$ be a random semi-normed space with base $(\Omega, \mathscr{A}, \mu)$. Define $\mathscr{F}: S \to D^+$ by $F_p(t) = \mu(\{\omega \in \Omega | X_p(\omega) < t\})$ for all p in S and all $t \in R$, then $(S, \mathscr{F}, W)$ is a Menger-PPN space, its (ε, λ)-linear topology is also called the (ε, λ)-linear topology of $(S, \mathscr{X})$.

Let $(B, \|\cdot\|)$ be a normed space. Then as in Section 2 of this paper the set $L^0(\Omega, B)$ of all B-valued $\mathscr{A}$-random variables on $(\Omega, \mathscr{A}, \mu)$ forms an E-norm space. Let $R(\mathscr{A}, B)$ be the set of all B-valued $\mathscr{A}$-random elements on $(\Omega, \mathscr{A}, \mu)$, then it is well known from [9] that $R(\mathscr{A}, B)$ does not form a linear space for many nonseparable spaces $(B, \|\cdot\|)$, and from Remark 1.5 it is impossible that the linear space generated by $R(\mathscr{A}, B)$ can always be made into an E-norm space, the following construct shows $R(\mathscr{A}, B)$ can be embedded into a random semi-normed space in a natural and useful way.

Proposition 4.1[13] Let $(B, \|\cdot\|)$ be a normed space, and $S = \{p: (\Omega, \mathscr{A}, \mu) \to B |$ there exists some nonnegative random variable ξ such that $\|p(\omega)\| \leq \xi(\omega)$ μ-a.s.$\}$. Define $\mathscr{X}: S \to L^+_0(\Omega)$ by $X_p = \wedge\{\xi \in L^+_0(\Omega) | \|p(\omega)\| \leq \xi(\omega)\ \mu\text{-a.s.}\}\ \forall p \in S$, then $(S, \mathscr{X})$ is a random semi-normed space with base $(\Omega, \mathscr{A}, \mu)$. Set $L^0(\mathscr{A}, B) =$ the linear topological closure of the linear space generated by $R(\mathscr{A}, B)$ in $(S, \mathscr{X})$, we still denote the limitation of $\mathscr{X}$ to $L^0(\mathscr{A}, B)$ by $\mathscr{X}$, then $(L^0(\mathscr{A}, B), \mathscr{X})$ is a random semi-normed space with base $(\Omega, \mathscr{A}, \mu)$, called all B-valued $\mathscr{A}$-random element generated random semi-normed space, and it is also complete if B is a Banach space. When B separable, $L^0(\mathscr{A}, B)$ is exactly $L^0(\Omega, B)$.

Proposition 4.1 shows random semi-normed spaces are of fundamental importance in random functional analysis. The following definition further shows random semi-normed spaces are also of fundamental importance in the theory of random conjugate spaces.

Definition 4.2[13] Let $(S, \mathscr{X})$ be a random semi-normed space over K with base $(\Omega, \mathscr{A}, \mu)$. A mapping $f: S \to L^0(\Omega, K)$ is called a random linear functional if $f(\alpha p + \beta q)(\omega) = \alpha f(p)(\omega) + \beta f(q)(\omega)$ a.s. for all p, q in S and all α, β in K, if, in addition, there exists a single μ-null set Γ of Ω such that $f(\alpha p + \beta q)(\omega) = \alpha \cdot f(p)(\omega) + \beta \cdot f(q)(\omega)$ for all ω in $\Omega \setminus \Gamma$, all p, q in S, and all α, β in K, then f is called a sample-linear random functional. A random linear functional f on S is called a.s. bounded if there exists some $\xi \in L^+_0(\Omega)$ such that

$|f(p)(\omega)| \leq \xi(\omega) \cdot X_p(\omega)$ a.s. for all $p \in S$. Denote the linear space of all a.s. bounded random linear functionals on $(S, \mathscr{X})$ under the pointwise addition and scalar multiplication operations by S^*, define $\mathscr{X}^*: S^* \to L_0^+(\Omega)$ by $X_f^* = \wedge \{\xi \in L_0^+(\Omega) \mid |f(p)(\omega)| \leq \xi(\omega) \cdot X_p(\omega)$ a.s. for all $p \in S\}$ for all $f \in S^*$, then it is easy to check from Proposition 3.1 that $(S^*, \mathscr{X}^*)$ is still a random semi-normed space with base $(\Omega, \mathcal{A}, P)$, called the random conjugate space of $(S, \mathscr{X})$.

Review 4.1 From the paragraph following Definition 4.1, an E-norm space can be naturally regarded as a uniform random semi-normed space, hence Definition 4.2 is, of course, suitable for an E-norm space. Review 3.2 shows us that one can only guarantee the random conjugate space of an E-norm space to be a random semi-normed space (generally speaking, this random semi-normed space is seldom an E-norm space). Thus the framework of an E-norm space is somewhat not self-sufficient for the theory of random conjugate spaces.

Review 4.2 Let $(S^*, \mathscr{X}^*)$ be the random conjugate space of a random semi-normed space $(S, \mathscr{X})$. Although $(S^*, \mathscr{X}^*)$, being a random semi-normed space, determines a Menger-PPN space $F_f^*(t) = P(\{\omega \in \Omega \mid X_f^*(\omega) < t\})$ for all $t \in R$ and all $f \in S^*$, the triangle inequality: $X_{f+g}^*(\omega) \leq X_f^*(\omega) + X_g^*(\omega)$ a.s. for all $f, g \in S^*$, is much stronger than the triangle inequality: $F_{f+g}^*(t) \geq W(F_f^*(r), F_g^*(s))$ for all $f, g \in S$ and all the $r, s \in R$ such that $t = r + s$, and the former triangle inequality is key to the deep development of random conjugate spaces, it is because of this that we have not called $(S^*, \mathscr{F}, W)$ the random conjugate space of $(S, \mathscr{X})$.

Review 4.3 For any two elements p and q in an E-norm space, $L_0^+(\Omega)$ and $L^0(\Omega, K)$ respectively, we say they are equal iff $p(\omega) = q(\omega)$ for all $\omega \in \Omega$, and thus in Definition 4.2 we have to select the framework of a random semi-normed space. It should also be pointed out that all possible vagueness occurring in Definition 3.1 have been removed by Definition 4.2, in particular the linearity and the sample-linearity for a random functional are also made clear in an explicit way. Based on Definition 4.2, the Hahn-Banach extension theorem of random linear functionals can be given in a concise way, as follows.

Proposition 4.2[13] Let $(S, \mathscr{X})$ be a random semi-normd space over K with base $(\Omega, \mathcal{A}, P)$, $M \subset S$ a linear subspace and $f: M \to L^0(P, K)$ an a.s. bounded random linear functional. Then there exists an a.s. bounded random linear functional $\tilde{f}: S \to L^0(\Omega, K)$ such that the following two properties hold:

1) $\tilde{f}(p)(\omega) = f(p)(\omega)$ for all $p \in M$ and all ω in Ω;

2) $X_{\tilde{f}}^*(\omega) = X_f^*(\omega)$ for all ω in Ω.

Further, if f is sample-linear, then $\tilde{f}$ can also be asked to be sample-linear.

Remark 4.1 For a rigorous proof of Proposition 4.2, see [Guo, 2]. In 2) of Proposition 4.2, since $X_f^* = \wedge \{\xi \in L_0^+(\Omega) \mid |f(p)(\omega)| \leq \xi(\omega) \cdot X_p(\omega)$ a.s. $\forall p \in M\}$ and $X_{\tilde{f}}^* = \wedge \{\xi \in L_0^+(\Omega) \mid |\tilde{f}(p)(\omega)| \leq \xi(\omega) \cdot X_p(\omega)$ a.s. $\forall p$ in $S\}$, and it is easy to see by Proposition 3.1 and by the process of the proof of Proposition 4.2 that $X_{\tilde{f}}^*(\omega) = X_f^*(\omega)$ a.s., thus one can see X_f^* is also an essential infimum of the set $\{\xi \in L_0^+(\Omega) \mid |\tilde{f}(p)(\omega)| \leq \xi(\omega) \cdot X_p(\omega)$ a.s. $\forall p \in S\}$, and hence we can take $X_{\tilde{f}}^*$ to be X_f^*.

Remark 4 2 When $K = R$, in the proof of Proposition 4 2 we adopted an idea of the proof of Proposition 3 3 given in [12]. When $K =$ the complex field C, there were many other scholars interested in the proof of Proposition 4 2 but their proofs are all wrong, the first right proof of it was given in [Guo, 13] who make full use of the separability of the complex number field C. Let K be the real number field R, $(S, \mathscr{X})$, M, f and $\mathscr{F}$ be the same as in Proposition 4 2, even if f is sample-linear and there exists a single _ -null set Γ_1 of K such that $|f(p)(k)| \leq X^*_f(k) \cdot X_p(k)$ for all $p \in M$ and all k in $K \setminus \Gamma_1$, one can not guarantee that there exists a single _ -null set Γ of K such that $|\mathscr{F}(p)(k)| \leq X^*_f(k) \cdot X_p(k)$ for all p in S and all $k \in K \setminus \Gamma$, either. Thus the proof of Proposition 4 2 for the case $K = C$ can not be completed by a completely mimicking way that is used in the proof of classical Hahn-Banach theorem.

Remark 4 3 In [13], Guo also showed the set of all $\mathscr{A}$-random elements from a probability space (K, $\mathscr{A}$, _) to a metric space (M, d) always forms a random pseudometric space (see also [2, Theorem 3 1]) with base (K, $\mathscr{A}$, _). Combining this, Proposition 4 2 and Definition 4 2, Guo first recognized the fundamental importance of random metric theory in random functional analysis and first put forward in [13] a new approach to random functional analysis. This new approach amounts to regarding random functional analysis as analysis founded on random metric theory, which was further enriched and made perfect in [14, 27].

5 Some Reviews on Guo's Work on a New Form of the Theory of Random Conjugate Spaces

Just as was shown in [2], both a random semi-normed space and a random linear functional defined on it can be regarded as a stochastic process, it is to enable us to obtain as much information about samples of these stochastic processes as possible that we are forced to develop the theory of random conjugate spaces under the framework of a random semi-normed space in Section 4 of this paper. However, when we deeply developed random metric theory and its applications to traditional functional analysis (see [4]) we often need a direct connection between random metric theory and functional analysis, this leads Guo in [1] to a new form of the definition of a random normed space together with its random conjugate space, namely the quotient-space form, since the normed spaces that are closely related to random normed spaces (e.g., Lebesgue-Bochner function spaces[20]) take the corresponding quotient- space form. In particular, based on this new form we smoothly presented the notion of a random normed module that has been the key bridge connecting random metric theory and functional analysis (see [2, 3]).

In this section, let (K, $\mathscr{A}$, _) be a e-finite measure space, and $(B, \|\cdot\|)$ be a given normed space over K. $L^0(\_, B)$ is the linear space of all B-valued _ -measurable functions on (K, $\mathscr{A}$, _), and $L(\_, B)$ is the linear space of all _ -equivalence classes of the elements in $L^0(\_, B)$, in particular when $B = K$, $L(\_, K)$ is an algebra over K under the ordinary addition, multiplication and scalar multiplication operations on _ -equivalence classes[20], we simply write 0 and 1 for the null element and the identity element respectively. Specially, for any element p in $L(\_, B)$, let $p^0 \in L^0(\_, B)$ be an arbitrarily chosen representative of p, we denote the μ-equivalence class of $\|p^0\|$ by $\|p\|$, where $\|p^0\|:(\Omega, \mathcal{A}, \mu)\to[0,+\infty)$ is defined by $\|p^0\|(\omega)=\|p^0(\omega)\|$ for all ω in Ω.

It is well known from [20] that $L(\mu, R)$ is a complete lattice by the ordering $\leq$: $\xi\leq\eta$ iff $\xi^0(\omega)\leq\eta^0(\omega)$ μ-a.e. for arbitrarily chosen representatives ξ^0 and η^0 of ξ and η respectively. We denote the set $\{\xi\in L(\mu, R)\mid\xi\geq 0\}$ by $L^+(\mu)$.

Definition 5.1[1] An ordered pair $(S, \mathcal{X})$ is called a random normed space (briefly, an RN space) over K with base $(\Omega, \mathcal{A}, \mu)$ if S is a linear space over K and if $\mathcal{X}$ is a mapping from S to $L^+(\mu)$ such that, writing X_p for $\mathcal{X}(p)$ for all p in S, the following hold:

1) $X_{\alpha p}=|\alpha|X_p$ for all $\alpha\in K$ and all $p\in S$;

2) $X_p=0$ implies $p=\theta$ (the null in S);

3) $X_{p+q}\leq X_p+X_q$ for all p and q in S.

X_p is called the random norm of the vector p in S. If $\mathcal{X}$ only satisfies 1) and 3) as above, then $\mathcal{X}$ is called a random seminorm on S, and $(S, \mathcal{X})$ is called a random semi-normed space.

If $(S, \mathcal{X})$ is an RN space over K with base $(\Omega, \mathcal{A}, \mu)$, and if, in addition, there exists another mapping $*: L(\mu, K)\times S\to S$ such that the following hold:

4) $(S, *)$ is a left module over the algebra $L(\mu, K)$;

5) $X_{\xi * p}=|\xi|\cdot X_p$ for all ξ in $L(\mu, K)$ and all p in S.

Then the triple $(S, \mathcal{X}, *)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{A}, \mu)$.

From now on, that we say that $(S, \mathcal{X})$ is an RN space always means $(S, \mathcal{X})$ is one in the sense of Definition 5.1 rather than Definition 4.1 unless otherwise stated.

Remark 5.1 As [1] showed, if $(S, \mathcal{X}, *)$ is an RN module over K with base $(\Omega, \mathcal{A}, \mu)$ then, according to 4) of Def 5.1, the module multiplication $*: L(\mu, K)\times S\to S$ can be regarded a natural extension of the scalar multiplication $\cdot: K\times S\to S$ when K and $\{\alpha\cdot 1\mid\alpha\in K\}$ are identified, where 1, as at the beginning of this section, denotes the identity element in $L(\mu, K)$, so 1) and 5) of Def 5.1 are, obviously, compatible. Thus once $*$ is understood we can simply write $(S, \mathcal{X})$ and $\xi\cdot p$ for $(S, \mathcal{X}, *)$ and $\xi * p$ respectively for any RN module $(S, \mathcal{X}, *)$, all p in S and all ξ in $L(\mu, K)$.

Remark 5.2 As was shown in [4], in Def 5.1 we employed μ-measurable functions instead of $\mathcal{A}$-measurable functions because $L^0(\Omega, K)\subset L^0(\mu, K)$ and each element of $L^0(\mu, K)$ is exactly $\tilde{\mathcal{A}}$-measurable, and particularly because $\tilde{\mathcal{A}}$ is the Lebesgue completion of $\mathcal{A}$ with respect to μ so that we can make full use of the lifting property established in [28].

Proposition 5.1[1] Let $(S, \mathcal{X})$ be an RN space over K with base $(\Omega, \mathcal{A}, \mu)$, and $\mathcal{F}_+(\mathcal{A})=\{A\in\mathcal{A}: 0<\mu(A)<+\infty\}$. For each $A\in\mathcal{F}_+(\mathcal{A})$, $\varepsilon>0$, and $0<\lambda<\mu(A)$, set $U_\theta(A, \varepsilon, \lambda)=\{p\in S\mid\mu(\{\omega\in A\mid X_p^0(\omega)<\varepsilon\})>\mu(A)-\lambda\}$ where X_p^0 is an arbitrarily chosen μ-measurable representative of X_p (since $\mu(\{\omega\in A\mid X_p^0(\omega)<\varepsilon\})$ is independent of a particular choice of X_p^0, we often write $\mu(\{\omega\in A\mid X_p(\omega)<\varepsilon\})$ for $\mu(\{\omega\in A\mid X_p^0(\omega)<\varepsilon\})$). Denote $\mathcal{U}(A)=\{U_\theta(A, \varepsilon, \lambda)\mid\varepsilon>0, 0<\lambda<\mu(A)\}$ for each $A\in\mathcal{F}_+(\mathcal{A})$, and $\mathcal{U}=\bigcup_{A\in\mathcal{F}_+(\mathcal{A})}\mathcal{U}(A)$, then $\mathcal{U}$ is a base of the neighborhood system at θ of some Hausdorff linear topology for S, called the (ε, λ)-linear topology of $(S, \mathcal{X})$, and the linear topology is induced by the quasinorm $|||\cdot|||: S\to[0,+\infty)$ defined by $|||p|||=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_{A_n}\frac{X_p}{1+X_p}d\mu\ \forall p\in S$, where $\{A_n\}$ is an arbitrarily chosen countable partition of Ω to $\mathscr{A}$. Clearly a sequence $\{p_n\}$ in $(S,\mathscr{X})$ converges in the (ε,λ)-linear topology to a point p in S iff $\{X_{p_n-p}\}$ converges in measure μ to 0 on each $A\in\mathscr{F}(A)$, hence we often called the (ε,λ)-linear topology the topology of convergence locally in measure $L(\mu,K)$, as an RN space (see Example 5.1 below), becomes a topological algebra over K when endowed with its (ε,λ)-linear topology, namely the algebra multiplication $\cdot: L(\mu,K)\times L(\mu,K)\to L(\mu,K)$ is jointly continuous with respect to the natural product topology. In particular, when $(S,\mathscr{X})$ is an RN module S becomes a topological module over the topological algebra $L(\mu,K)$ under the (ε,λ)-linear topologies of $(S,\mathscr{X})$ and $L(\mu,K)$ respectively, namely the module multiplication $\cdot: L(\mu,K)\times S\to S$ is jointly continuous.

Definition 5.2[1] Let $(S,\mathscr{X})$ be an RN space over K with base $(\Omega,\mathscr{A},\mu)$. A linear operator f from S to $L(\mu,K)$ is called a random linear functional on S, further f is called a.s. bounded if there exists some $\xi\in L^+(\mu)$ such that $|f(p)|\le\xi\cdot X_p$ for all $p\in S$. Denote the linear space of all a.s. bounded random linear functionals under the ordinary operations on linear operators by S^*, define $\mathscr{X}^*: S^*\to L^+(\mu)$ by $X^*_f=\wedge\{\xi\in L^+(\mu)\mid |f(p)|\le\xi\cdot X_p\ \forall p\in S\}\ \forall f\in S^*$, then $(S^*,\mathscr{X}^*)$ is an RN space over K with base $(\Omega,\mathscr{A},\mu)$. Define $\otimes: L(\mu,K)\times S^*\to S^*$ by $(\xi\otimes f)(p)=\xi\cdot(f(p))$ for all ξ in $L(\mu,K)$, all p in S and all $f\in S^*$, then $(S^*,\mathscr{X}^*,\otimes)$ is an RN module over K with base $(\Omega,\mathscr{A},\mu)$, still denoted by $(S^*,\mathscr{X}^*)$, called the random conjugate space of $(S,\mathscr{X})$.

Remark 5.3 In Definition 5.2, the set $\{\xi\in L^+(\mu)\mid |f(p)|\le\xi\cdot X_p\ \forall p\in S\}$ has a lower bound 0 in the complete lattice $L(\mu,R)$, and it is also dually directed, and thus X^*_f exists and $|f(p)|\le X^*_f\cdot X_p$ for all $f\in S^*$ and all $p\in S$.

Example 5.1 Define $\mathscr{X}: L(\mu,B)\to L^+(\mu)$ by $X_p=\|p\|\ \forall p\in L(\mu,B)$; define $*: L(\mu,K)\times L(\mu,B)\to L(\mu,K)$ by $\xi * p=\xi\cdot p$, where $\xi\cdot p$ is the μ-equivalence class of the μ-measurable function $\xi^0\cdot p^0$ defined by $(\xi^0\cdot p^0)(\omega)=\xi^0(\omega)\cdot p^0(\omega)\ \forall\omega\in\Omega$, here ξ^0 and p^0 are arbitrarily chosen representatives of ξ and p, respectively. Then $(L(\mu,B),\mathscr{X},*)$ is an RN module over K with base $(\Omega,\mathscr{A},\mu)$, so is $L(\mu,K)$.

Example 5.2 Let B' be the classical conjugate space of $(B,\|\cdot\|)$. Denote by $L^0(\mu,B',\omega^*)$ the linear space of all B'-valued ω^*-μ-measurable functions on $(\Omega,\mathscr{A},\mu)$ under the ordinary operations, and by $L(\mu,B',\omega^*)$ the linear space of all ω^*-μ-equivalence classes of the elements in $L^0(\mu,B',\omega^*)$. Define $\mathscr{X}: L(\mu,B',\omega^*)\to L^+(\mu)$ by $X_q=\vee\{|\langle b,q\rangle|\mid b\in B'$ and $\|b\|\le 1\}\ \forall q\in L(\mu,B',\omega^*)$, and $*: L(\mu,K)\times L(\mu,B',\omega^*)\to L(\mu,B',\omega^*)$ by $\xi * q=\xi\cdot q$ for all $q\in L(\mu,B',\omega^*)$ and all $\xi\in L(\mu,K)$, where, for an arbitrarily chosen representative q^0 of q, and for an arbitrarily chosen representative ξ^0 of ξ, $\xi\cdot q$ is the ω^*-μ-equivalence class of $\xi^0\cdot q^0$ defined by $(\xi^0\cdot q^0)(\omega)=\xi^0(\omega)\cdot q^0(\omega)\ \forall\omega\in\Omega$, and $\langle b,q\rangle$ is the μ-equivalence class of $\langle b,q^0\rangle$ defined by $\langle b,q^0\rangle(\omega)=\langle b,q^0(\omega)\rangle=(q^0(\omega))(b)$ for all $b\in B$, and all $\omega\in\Omega$. Then $(L(\mu,B',\omega^*),\mathscr{X},*)$ is an RN module over K with base $(\Omega,\mathscr{A},\mu)$.

Example 5.3 Let $(S^0,\mathscr{X}^0)$ be a random semi-normed space over K with base $(\Omega,\mathscr{A},\mu)$ in the sense of Definition 4.1, and $f^0: S^0\to L^0(\Omega,K)$ be an a.s. bounded random linear functional in the sense of Definition 4.2. For any $p^0 \in S^0$, set $p = \{q^0 \in S^0 | X^0_{p^0 - q^0}(\kappa) = 0$ a.s.$\}$ and $S = \{p | p^0 \in S^0\}$. Define $\mathscr{X}: S \to L^+(\mu)$ (note $L^+(\mu) = L^+(\mathrm{K})$) by $X_p =$ the μ-equivalence class of $X^0_{p^0}$ $\forall p \in S$, and $f: S \to L(\mu, K)$ by $f(p) =$ the μ-equivalence class of $f^0(p^0)$ $\forall p \in S$. Then it is easy to check that $(S, \mathscr{X})$ is an RN space over K with base $(\mathrm{K}, \mathscr{A}, \mu)$ in the sense of Definition 5.1 and that f is an a.s. bounded random linear functional on $(S, \mathscr{X})$ in the sense of Definition 5.2 with the property: $X^*_f =$ the μ-equivalence class of the random norm of f^0. Thus Definition 5.1 and Definition 5.2 provide the quotient-space forms of Definition 4.1 and Definition 4.2, respectively. With the aid of the Choice axiom, for every RN space $(S, \mathscr{X})$ and every a.s. bounded random linear functional f in the sense of Definition 5.1 and Definition 5.2 respectively there exist an RN space $(S^0, \mathscr{X}^0)$ and an a.s. bounded random linear functional f^0 on $(S^0, \mathscr{X}^0)$ in the sense of Definitions 4.1 and 4.2 respectively such that the former correspondence relations hold. This leads directly to the following quotient-space form of Proposition 4.2 (due to Guo).

Proposition 5.2[1] Let $(S, \mathscr{X})$ be an RN space over K with base $(\mathrm{K}, \mathscr{A}, \mu)$, $M \subset S$ a linear subspace and f an a.s. bounded random linear functional on M. Then there exists an $\tilde{f} \in S^*$ such that $\tilde{f}$ is an extension of f and $X^*_{\tilde{f}} = X^*_f$.

Corollary 5.1[1] Let $(S, \mathscr{X})$ be an RN space over K with base $(\mathrm{K}, \mathscr{A}, \mu)$, and $p \in S$ be a nonzero element. The there exists $f \in S^*$ such that $f(p) = X_p$ and $X^*_f = I_A$, where I_A denotes the μ-equivalence class of I_{A^0} and $A^0 = \{\kappa \in \mathrm{K} | \xi^0(\kappa) \neq 0\}$ for an arbitrarily chosen representative ξ^0 of X_p.

Remark 5.4 It is obvious that Corollary 5.1 includes Sultanbekov's Proposition 2.1 as an extremely special case.

For an RN module $(S, \mathscr{X})$, an a.s. bounded random linear functional on S has many nice properties such that the theory of random conjugate spaces has obtained a deep development for the past six years. Following are some convincing conclusions.

Proposition 5.3 [Guo 1] Let $(S, \mathscr{X})$ be an RN module over K with base $(\mathrm{K}, \mathscr{A}, \mu)$, and $f: S \to L(\mu, K)$ be a linear operator. Then we have:

1) f is a.s. bounded iff f is a continuous module homomorphism;

2) if f is a.s. bounded then $X^*_f = \bigvee\{|f(p)| \mid p \in S$ and $X_p \leq 1\}$ and there exists a sequence $\{p_n\}$ in $\{p \in S | X_p \leq 1\}$ such that $\{|f(p_n)|\}$ (in fact we can ask $\{f(p_n)\}$) converges to X^*_f in a nondecreasing way.

Proposition 5.4 [Guo, 4] The canonical mapping $T: L(\mu, B', \mathrm{K}^*) \to (L(\mu, B))^*$ defined by $T_q(p) = \langle p, q\rangle$ for all $p \in L(\mu, B)$ and all $q \in L(\mu, B', \mathrm{K}^*)$ (where T_q denotes $T(q)$) is a random-norm-preserving module isomorphism. If, replacing $L(\mu, B', \mathrm{K}^*)$ by $L(\mu, B')$, then T is a random-norm-preserving module isomorphism iff B' has the Radon-Nikodým property with respect to $(\mathrm{K}, \mathscr{A}, \mu)$.

Proposition 5.4 solves all representation problems about the random conjugate space of the random normed module $L(\mu, B)$. In a completely similar way to Definition 5.1 Guo introduced the notion of a random inner product module (briefly an RIP module) and proved the following Riesz representation theorem in [1].

Proposition 5.5[1] Let $(S, \mathcal{X})$ be a complete RIP module over K with base $(\Omega, \mathcal{A}, \mu)$. Then for any a.s. bounded random linear functional f on S there exists uniquely an element $q(f)$ in S such that $f(p) = X_{p, q(f)}$ $\forall p \in S$ and $\tilde{X}^*_f = \tilde{X}_{q(f)}$, where $\tilde{\mathcal{X}}: S \to L^+(\mu)$ is given by $\tilde{X}_q = \overline{\sqrt{X_{q,q}}}$ $\forall q \in S$ (note $(S, \tilde{\mathcal{X}})$ is an RN module).

Proposition 5.6 below connects the theory of random conjugate space to that of classical conjugate spaces. Let $1 \leq p \leq +\infty$ and $(S, \mathcal{X})$ be an RN module over K with base $(\Omega, \mathcal{A}, \mu)$. Define $\|\cdot\|_p: S \to [0, +\infty]$ by $\|g\|_p =$ the ordinary p-norm of X_g $\forall g \in S$, namely $\|g\|_p = \left(\int_\Omega (X_g)^p \mathrm{d}\mu\right)^{1/p}$ when $1 \leq p < +\infty$, $\|g\|_\infty =$ the μ-essential supremum of X_g. Then $(L^p(S), \|\cdot\|_p)$ is a normed space over K for every p, $1 \leq p \leq +\infty$, where $L^p(S) = \{g \in S \mid \|g\|_p < +\infty\}$, and every $L^p(S)$ is dense in S in the (ε, λ)-linear topology for $(S, \mathcal{X})$.

Proposition 5.6[6] Let $1 \leq p < +\infty$. Then the canonical mapping $T: (L^q(S^*), \|\cdot\|_q) \to (L^p(S), \|\cdot\|_p)'$ (the conjugate space of $(L^p(S), \|\cdot\|_p)$) defined by $T_f(g) = \int_\Omega f(g)\mathrm{d}\mu$ for all $f \in L^q(S^*)$ and all $g \in L^p(S)$, is an isometric isomorphism, where T_f denotes $T(f)$ for all $f \in L^q(S^*)$, $L^q(S^*) = \{f \in S^* \mid \|f\|_q \equiv$ the ordinary q-norm of $X^*_f < +\infty\}$ and p and q are a pair of conjugate numbers, namely $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 5.5 When $S = L(\mu, B)$ in Proposition 5.6, $L^p(S)$ is exactly the ordinary Lebesgue-Bochner function space $L^p(\mu, B)$ (see [19, 20]), and $L^q(S^*)$ is exactly $L^q(\mu, B', \mathrm{k}^*)$, and thus Proposition 5.6 unifies all representation theorems of the conjugate space of $L^p(\mu, B)$ (see [5] for details). Proposition 5.6 is of vital importance in characterizing random reflexive spaces. We say a complete random normed module $(S, \mathcal{X})$ is random reflexive if the canonical embedding $J: (S, \mathcal{X}) \to (S^{**}, \mathcal{X}^{**})$ defined by $(J(p))(f) = f(p)$ for all $p \in S$ and all $f \in S^*$, is surjective, where $(S^{**}, \mathcal{X}^{**})$ denotes the random conjugate space of $(S^*, \mathcal{X}^*)$. Then we have the following.

Proposition 5.7[6] Let $1 < p < +\infty$ be a given positive number. Then a complete RN module $(S, \mathcal{X})$ is random reflexive iff $(L^p(S), \|\cdot\|_p)$ is reflexive.

In [4], By Proposition 5.4 Guo proved $L(\mu, B)$ is random reflexive iff B is reflexive. In particular, Guo has recently proved the famous James theorem still holds for complete RN modules in [3].

Proposition 5.8[3] A complete RN module $(S, \mathcal{X})$ is random reflexive iff for every $f \in S^*$ there exists some $p \in S$ such that $X_p \leq 1$ and $f(p) = X^*_f$.

Review 5.1 From the viewpoint of traditional functional analysis, Definitions 5.1 and 5.2 are more natural than Definitions 4.1 and 4.2 respectively, in particular these propositions presented in this section are enough to convince anyone that Definitions 5.1 and 5.2 are most fruitful and that RN modules have played an essential role in the course of the deep development of random metric theory and its applications.

6 The Relations among Strongly Bounded, Topologically Bounded and a.s. Bounded Random Linear Functionals

In this section, let $(\Omega, \mathscr{A}, P)$ be a probability space. Suppose $(S, \mathscr{X})$ is an RN space over K with base $(\Omega, \mathscr{A}, P)$, then $(S, \mathscr{X})$ determines a Menger-PN space $(S, \mathscr{F}, W)$ over K as follows: $F_p(t) = P(\{\omega \in \Omega \mid X_p(\omega) < t\})$ for all $p \in S$ and all $t \in R$. Clearly, $L^\infty(S) = \{p \in S \mid$ there exists some $t > 0$ such that $F_p(t) = 1\}$ and for any $p \in L^\infty(S)$ it is easy to check that $\| p \|_\infty = \inf\{t > 0 \mid F_p(t) = 1\}$.

Throughout this section, let $(S^1, \mathscr{X}^1)$ and $(S^2, \mathscr{X}^2)$ be any two given RN modules over the identical scalar field K with the identical base $(\Omega, \mathscr{A}, P)$ unless otherwise stated, they determine the two Menger-PN spaces $(S^1, \mathscr{F}^1, W)$ and $(S^2, \mathscr{F}^2, W)$ respectively. A linear operator $T: S^1 \to S^2$ is called a.s. bounded if there exists some $\xi \in L^+(P)$ such that $X^2_{Tp} \le \xi \cdot X^1_p$ for all $p \in S^1$, $X_T = \wedge \{\xi \in L^+(P) \mid X^2_{Tp} \le \xi \cdot X^1_p$ for all $p \in S^1\}$ called the random norm of T.

Since the t-norm W (see Section 1) satisfies the condition $\sup_{0<a<1} W(a, a) = \sup_{0<a<1} \mathrm{Max}(2a - 1, 0) = 1$. According to Definition 1.2 and the fact that $(S^1, \mathscr{X}^1)$ and $(S^2, \mathscr{X}^2)$ are metrizable linear topological spaces one can easily see that a strongly bounded linear operator from S^1 to S^2 is topologically bounded linear operator and that an a.s. bounded linear operator from S^1 to S^2 is topologically bounded, namely continuous. The following Example 6.1 shows a strongly bounded (hence also topologically bounded) random linear functional defined on an RN module is not a.s. bounded.

Example 6.1[29, 2] Take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\mathscr{A}$ = the σ-algebra of all Lebesgue-measurable subsets of Ω and P = the Legesgue measure on $\mathscr{A}$, then $(\Omega, \mathscr{A}, P)$ is a probability space. Consider $S = L(P, R)$ and define $f: S \to S$ as follows:

$f(p) = \check{p}$ for all $p \in S$, where for each $p \in S$, let p_0 be an arbitrarily chosen representative of p, then $\check{p}$ stands for the P-equivalence class of $\check{p}_0$ defined by $\check{p}_0(\omega) = p_0(-\omega)$ for all $\omega \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Then it is obvious that $S = L(P, R)$ is a complete RN module over R with base $(\Omega, \mathscr{A}, P)$ (see Example 5.1), and f is a strongly bounded random linear functional on S: in fact, $F_{f(p)}(t) = F_p(t)$ for all $p \in S$ and all $t \in R$, namely f is also probabilistic-norm-preserving.

But f is not a.s. bounded. If there exists some $\xi \in L^+(P)$ such that $|f(p)| \le \xi \cdot X_p$ for all $p \in S$, namely $|\check{p}| \le \xi \cdot |p|$ $\forall p \in S$, however, this is impossible since, define p_0: $\left[-\frac{1}{2}, \frac{1}{2}\right] \to R$ by $p_0(\omega) = 0$ if $\omega \in \left[-\frac{1}{2}, 0\right)$, and by $p_0(\omega) = 1$ if $\omega \in \left[0, \frac{1}{2}\right]$, then p = the P-equivalence class of p_0 clearly does not satisfy $|\check{p}| \le \xi \cdot |p|$.

We knew from [1] that a linear operator T from S^1 to S^2 is a.s. bounded iff it is a continuous module homomorphism, and in this case $X_T = \wedge \{X^2_{Tp} \mid p \in S^1$ and $X^1_p \le 1\}$, in particular there exists a sequence $\{p_n\}$ in $\{p \in S^1 \mid X^1_p \le 1\}$ such that $\{X^2_{Tp_n}\}$ converges to X_T in a nondecreasing way. Thus it is also obvious that a strongly bounded module homomorphism from S^1 to S^2 must be a.s. bounded, conversely, we have the following:

Proposition 6.1 An a.s. bounded linear operator T from $(S^1, \mathscr{X}^1)$ to $(S^2, \mathscr{X}^2)$ is strongly bounded iff X_T is P-essentially bounded.

Proof (Sufficiency) If X_T is P-essentially bounded, and denote by M the P-essential supremum of X_T, then $0 \le M < +\infty$. Obviously $X^2_{Tp} \le X_T \cdot X^1_p \le (M+1) \cdot X^1_p$ for all $p \in S^1$, and hence also $F^2_{Tp}(t) \ge F^1_p\left(\frac{t}{M+1}\right)$ for all $p \in S^1$ and all $t \in R$, namely T is strongly bounded.

(Necessity) If T is strongly bounded, namely there exists a positive number M such that $F^2_{Tp}(t) \ge F^1_p\left(\frac{t}{M}\right)$ for all $p \in S^1$ and all $t \in R$. Therefore, if $p \in L^\infty(S^1)$, by the beginning of this section, namely there exists $t_0 > 0$ such that $F^1_p(t_0) = 1$, then $F^2_{Tp}(Mt_0) \ge F^1_p\left(\frac{Mt_0}{M}\right) = F^1_p(t_0) = 1$, namely $F^2_{Tp}(Mt_0) = 1$, this implies $Tp \in L^\infty(S^2)$. It is easy to check that for each $p \in L^\infty(S^1)$ $\|Tp\|_\infty = \inf\{t > 0 \mid F^2_{Tp}(t) = 1\} \le \inf\left\{t > 0 \mid F^1_p\left(\frac{t}{M}\right) = 1\right\} = M \cdot (\inf\{t > 0 \mid F^1_p(t) = 1\}) = M \cdot \|p\|_\infty$, namely the limitation of T to $L^\infty(S^1)$ is a bounded linear operator from $(L^\infty(S^1), \|\cdot\|_\infty)$ to $(L^\infty(S^2), \|\cdot\|_\infty)$.

Noting $\{p \in S^1 \mid X^1_p \le 1\}$ is exactly $\{p \in S^1 \mid \|p\|_\infty \le 1\}$, one can easily see T maps $\{p \in S^1 \mid X^1_p \le 1\}$ into $\{q \in S^2 \mid \|q\|_\infty \le M\}$. Since there exists a sequence $\{p_n\}$ in $\{p \in S^1 \mid X^1_p \le 1\}$ such that $\{X^2_{Tp_n}\}$ converges to X_T in a nondecreasing way, and since for each n $\|Tp_n\|_\infty \le M$, namely $X^2_{Tp_n}(\omega) \le M$ μ-a.s., this means also $X_T(\omega) \le M$ μ-a.s. So X_T is μ-essentially bounded.

This completes the proof.

Corollary 6.1 Let $(S, \mathscr{X})$ be an RN module over K with base $(\Omega, \mathscr{A}, \mu)$. Then an a.s. bounded random linear functional f on $(S, \mathscr{X})$ is strongly bounded iff X^*_f is μ-essentially bounded.

Proof Taking $S^1 = S$ and $S^2 = L(\mu, K)$, then our desired conclusion follows immediately from Proposition 6.1.

Remark 6.1 By Corollary 6.1 one can easily find an a.s. bounded (of course, continuous) random linear functional defined on an RN module such that it is not strongly bounded, this fact and Example 6.1 show the three notions of a topologically bounded (namely, continuous), strongly bounded and a.s. bounded random linear functional are essentially different: the first is properly weaker than the latter two, and neither of the latter two implies another; but if a topologically bounded or strongly bounded random linear functional defined on an RN module is a module homomorphism then it must be a.s. bounded.

Considering Proposition 1.4, Proposition 6.2 below is of new interest.

Proposition 6.2 Let $(S^1, \mathscr{X}^1)$ be an RN space over K with base $(\Omega, \mathscr{A}, \mu)$ and $(S^2, \mathscr{X}^2)$ be an RN module over K with base $(\Omega, \mathscr{A}, \mu)$. If there exist p_0 in S^1 and q_0 in S^2 such that $X^1_{p_0} \cdot X^2_{q_0} \ne 0$ then there exists both a nonzero a.s. bounded linear operator and a nonzero strongly bounded linear operator from S^1 to S^2, of course, there exists a nonzero topologically bounded (equivalently, a nonzero continuous) linear operator from S^1 to S^2.

Proof Since $(S^2, \mathscr{X}^2)$ is an RN module, we can, without loss of generality, suppose $X^2_{q_0} \le 1$ (if not, we can replace q_0 by $Q(X^2_{q_0}) \cdot q_0$, where $Q(X^2_{q_0})$ denotes the generalized inverse of $X^2_{q_0}$, then $Q(X^2_{q_0}) \cdot q_0$ satisfies our desire, see [1, Definition 1.1] for the definition of the generalized inverse).

By Corollary 5.1 there exists an a.s. bounded random linear functional f on $(S^1, \mathscr{X}^1)$

such that $X_f^* = I_A$ (where $A = [X_{p_0}^1 \neq 0]$) and $f(p_0) = X_{p_0}^1$. Define $T: S^1 \to S^2$ by $T(p) = (f(p)) \cdot q_0$ for all $p \in S^1$, then T is a nonzero linear operator and T is also a.s. bounded since $T(p_0) = f(p_0) \cdot q_0 = (X_{p_0}^1) \cdot q_0 \neq \theta$ (the null in S^2) and $X_{T(p)}^2 \leq |f(p)| \cdot X_{q_0}^2 \leq |f(p)| \leq X_f^* \cdot X_p^1 \leq X_p^1$ for all $p \in S^1$.

It is also obvious that $F_{T(p)}^2(t) \geq F_p^1(t)$ for all $t \in R$ and all $p \in S^1$, this implies T is also both strongly bounded and continuous.

This completes the proof.

Let $(S^1, \mathscr{X}^1)$ and $(S^2, \mathscr{X}^2)$ be any two RN spaces over K with base $(\Omega, \mathscr{A}, \mu)$ and $(S^2, \mathscr{X}^2)$ be an RN module. Denote by $SBL(S^1, S^2)$ the set of all strongly bounded linear operators from S^1 to S^2; by $CL(S^1, S^2)$ the set of all continuous (equivalently, all topologically bounded) linear operators from S^1 to S^2; and by $BL(S^1, S^2)$ the set of all a.s. bounded linear operators from S^1 to S^2. Here we do not intend to give any reviews on $SBL(S^1, S^2)$ since it is not necessarily a linear space, we have the following two reviews concerning the other two since they are both linear spaces with the addition and scalar multiplication as usual.

Review 6.1 First, the linear space $CL(S^1, S^2)$ becomes a left module over $L(\mu, K)$ under the module multiplication $*: L(\mu, K) \times CL(S^1, S^2) \to CL(S^1, S^2)$ given by $(a * T)(p) = a \cdot (T(p))$ for all $a \in L(\mu, K)$, all $T \in CL(S^1, S^2)$ and all $p \in S^1$. Second, we will introduce a linear topology for $CL(S^1, S^2)$ such that this linear topology is exactly the one of convergence of operators in $CL(S^1, S^2)$ uniformly on each bounded suset of $(S^1, \mathscr{X}^1)$ (where "bounded" means "linear topologically bounded", which is equivalent to "probabilistically bounded" in $(S^1, \mathscr{F}^1, W)$, namely the Menger-PN space determined by $(S^1, \mathscr{X}^1)$) as follows: denote by $\mathscr{B}$ the family of all bounded subsets of $(S^1, \mathscr{X}^1)$, then for each $E \in \mathscr{B}$ define $\mathscr{F}^E: CL(S^1, S^2) \to D^+$ by $F_T^E(t) = \sup_{x<t} \inf_{p \in E} F_{T(p)}^2(x)$ for all $t \in R$ and all $T \in CL(S^1, S^2)$, where $(S^2, \mathscr{F}^2, W)$ is the Menger-PN space determined by $(S^2, \mathscr{X}^2)$, it is not very difficult to prove $\mathscr{F}^E$ satisfies $F_{T_1+T_2}^E(t_1 + t_2) \geq W(F_{T_1}^E(t_1), F_{T_2}^E(t_2))$ for all T_1, T_2 in $CL(S^1, S^2)$, $(CL(S^1, S^2), \{\mathscr{F}^E\}_{E \in \mathscr{B}}, W)$ forms a so-called probabilistic locally convex space in terms of [30], the (ε, λ)-linear topology of $CL(S^1, S^2)$ determined by the family $\{\mathscr{F}^E\}_{E \in \mathscr{B}}$ is exactly our desired linear topology, this topology may be rather complicated since $\mathscr{B}$ is too complicated, up to now we have not even known whether it is metrizable (although it is always Hausdorff), we only know it is metrizable in the rather simple case when $(S^1, \mathscr{X}^1)$ admits a bounded neighborhood $N_\theta(\varepsilon_0, \lambda_0)$ at θ (the null in S^1), where $N_\theta(\varepsilon_0, \lambda_0) = \{p \in S^1 | F_p^1(\varepsilon_0) > 1 - \lambda_0\}$, $\varepsilon_0 > 0$ and $0 < \lambda_0 < 1$. Define $\mathscr{F}: CL(S^1, S^2) \to D^+$ by $F_T(t) = \sup_{x<t} \inf_{p \in N_\theta(\varepsilon_0, \lambda_0)} F_{T(p)}^2(x)$ for all $T \in CL(S^1, S^2)$ and all $t \in R$, then it is easy to see that $(CL(S^1, S^2), \mathscr{F}, W)$ is a Menger-PN space and that the (ε, λ)-linear topology determined by the single $\mathscr{F}$ is equivalent to the one determined by the family $\{\mathscr{F}^E\}_{E \in \mathscr{B}}$. But even for quite simple RN modules like $L(\mu, R)$ in Example 6.1 they do not admit any bounded neighborhood. Hence, generally speaking, $CL(S^1, S^2)$ does not possess so nice and simple structures as an RN module so that we can further develop it deeply. Clearly, for an RN space $(S, \mathscr{X})$ over K with base $(\Omega, \mathscr{A}, \mu)$, $CL(S, L(\mu, K))$ is exactly the linear space of all continuous random linear functionals on $(S, \mathscr{X})$, hence one can easily see why we have not defined $(CL(S, L(\mu, K)), \{\mathscr{F}^E\}_{E \in \mathscr{B}}, W)$ to be the random conjugate space of $(S,$

$\mathscr{X}$).

Review 6.2 Clearly, $BL(S^1, S^2)$ is an $L(\_, K)$-submodule of $CL(S^1, S^2)$, define $\mathscr{X}$: $BL(S^1, S^2) \to L^+(\_)$ by $X_T = \wedge \{a \in L^+(\_) \mid X^2_{T(p)} \le a \cdot X^1_p$ for all p in $S^1\}$ for all T in $BL(S^1, S^2)$, then it is easy to check that $(BL(S^1, S^2), \mathscr{X})$ is an RN module over K with base $(K, \mathscr{A}, \_)$. When $(S^1, \mathscr{X}^1)$ is also an RN module the (ε, λ)-linear topology of $(BL(S^1, S^2), \mathscr{X})$ is equivalent to the linear topology of convergence of operators in $BL(S^1, S^2)$ uniformly on each a.s. bounded subset of $(S^1, \mathscr{X}^1)$ (a set $E \subset S^1$ is called a.s. bounded if there exists $a \in L^+(\_)$ such that $X^1_p \le a \; \forall p \in E$). Generally speaking, an a.s. boundet set is always bounded, but the converse is false, and hence the (ε, λ)-linear topology of $(BL(S^1, S^2), \mathscr{X})$ is strictly weaker than the limitation of the (ε, λ)-linear topology of $(CL(S^1, S^2), \{\mathscr{F}\}_{E \in \mathscr{B}}, W)$ to $BL(S^1, S^2)$. But the fact that $BL(S^1, S^2), \mathscr{X})$ is an RN module is very important since we can make full use of the theory of RN modules to develop it deeply, in particular when $(S^1, \mathscr{X}^1)$ is an RN module every T in $BL(S^1, S^2)$ behaves very well, for example, T is a continuous module hommorphism and $X_T = \vee \{X^2_{T(p)} \mid p \in S^1$ and $X^1_p \le 1\}$, and further there exists a sequence $\{p_n\}$ in $\{p \in S^1 \mid X^1_p \le 1\}$ such that $\{X^2_{T(p_n)}\}$ converges to X_T in a nondecreasing manner. Even if $(S^1, \mathscr{X}^1)$ is merely an RN space we can also convert the study of $BL(S^1, S^2)$ to the case when $(S^1, \mathscr{X}^1)$ is an RN module, in [31] we succeeded in proving $(BL(S^1, S^2), \mathscr{X})$ is complete if $(S^2, \mathscr{X}^2)$ is complete by means of this converting way, in particular we proved in [31] that the random conjugate space of an RN space is always complete.

Returning to the classical case when $(S^1, \mathscr{X}^1)$ and $(S^2, \mathscr{X}^2)$ are both ordinary normed spaces, the case amounts to taking the base space $(K, \mathscr{A}, \_)$ to be trivial, namely $\mathscr{A} = \{K, H\}$, then $SBL(S^1, S^2)$, $CL(S^1, S^2)$ and $BL(S^1, S^2)$ all automatically reduce to the linear space of all bounded linear operators from S^1 to S^2, which is a normed space under the ordinary operator norm, denoted by $(B(S^1, S^2), \|\cdot\|)$. In the normed space $(S^1, \mathscr{X}^1)$, the topological boundedness and norm-boundedness for a set coincide, but when $(K, \mathscr{A}, \_)$ is not trivial, namely in the random normed space $(S^1, \mathscr{X}^1)$, the topological boundedness and the a.s. boundedness no longer concide (see [1, Definition 3.4 and Lemma 3.1]), and the distinctions between $SBL(S^1, S^2)$, $CL(S^1, S^2)$ and $BL(S^1, S^2)$ are obvious. Our above investigations show $(BL(S^1, S^2), \mathscr{X})$ is the best random generalization of the normed space of all bounded linear operators from a normed space to another. Therefore our Definition 5.2 is the best random generalization of the traditional conjugate space of a normed space.

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关于随机共轭空间的各种定义及随机线性泛函各种有界性的某些评论

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摘要: 中心目的是详细廉政论在随机共轭空间理论形成过程中所经历的三个阶段的工作, 尤其指出了这三个阶段工作之间的联系及本质差别; 给出了强有界、拓扑有界及几乎处处有界随机线性泛函之间的关系; 亦指出了在概率赋范空间上线性算子理论研究中目前存在的不足.

关键词: 概率赋范空间; E-范空间; 随机赋范空间; 强有界随机线性泛函; 拓扑有界的随机线性泛函; 几乎处处有界的随机线性泛函; 随机共轭空间