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Multiple Periodic Solutions for Nonlinear Difference Equations

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Abstract Multiple solutions for a class of nonlinear difference equations are obtained by variational methods. Our results generalize a recent result of Cai, Yu and Guo [Comput. Math. Appl., 52 (2006), 1630–1647], and the argument here is considerably simpler.

Key words periodic solutions for difference equations; palais-Smale condition; three critical points theorem; Clark's theorem

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1 Introduction

In a recent paper [1], Cai, Yu and Guo considered the existence of multiple m-periodic solutions for the nonlinear difference equations of the form

$$\Delta(p_n(\Delta x_{n-1})^\delta) + f(n, x_n) = 0, \qquad n \in \mathsf{Z}; \tag{1}$$

where $m \geq 2$ is a fixed integer, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ is a real sequence such that $p_n > 0$, $p_{n+m} = p_n$ for all $n \in \mathsf{Z}$; f is a continuous function on $\mathsf{Z} \times \mathsf{R}$ such that

$$f(n+m, z) = f(n, z), \qquad \text{for all } (n, z) \in \mathsf{Z} \times \mathsf{R};$$

and $\delta > 0$. Throughout this paper, the convention $(-1)^\delta = -1$ is made.

Let $F(n, z) = \int_0^z f(n, s)\,\mathrm{d}s$. Assuming in addition that $f(n, z)$ satisfies the following conditions:

(H_1) for any $z \in \mathsf{R}$, $F(n, z) \geq 0$,

$$\lim_{z \to 0} \frac{f(n, z)}{z^\delta} = 0, \tag{2}$$

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(H_2) there exists $R_2 > 0$ and $\beta > \delta + 1$ such that for $n \in \mathsf{N}$ and $z \in \mathsf{R}$ with $|z| \geq R_2$,

$$zf(n, z) \geq \beta F(n, z) > 0.$$

It follows from (2) that $f(n, 0) = 0$, therefore $x_n = 0$ is a trivial m-periodic solution of (1). In [1] Theorem 3.1, under the assumptions (H_1) and (H_2), the authors obtained two nontrivial m-periodic solutions for the problem (1) by using variational methods and the linking theorem [2] Theorem 5.3.

In this paper we will generalize their result. We impose the following conditions on the nonlinearity $f(n, z)$:

(f_0) there exists $r > 0$ such that $F(n, z) \geq 0$ for all $|z| \leq r$, and $\lim\limits_{z\to 0} \dfrac{F(n,z)}{|z|^{1+\delta}} = 0$,

(f_∞) $\lim\limits_{|z|\to\infty} \dfrac{F(n,z)}{|z|^{1+\delta}} = +\infty$ uniformly in $n \in \mathsf{Z}$.

Then we have

Theorem 1 Assume that (f_0) and (f_∞) are satisfied, then the problem (1) has at least two nontrivial solutions.

Remark The limit in (f_0) implies that $f(n, 0) = 0$, so $x_n = 0$ is a trivial m-periodic solution of (1).

Note that the condition (H_1) of Cai, Yu and Guo [1] is a `global` assumption, while our (f_0) only requires $F(n, z) \geq 0$ for `small` $|z|$. By an easy computation, it is easy to see that (H_2) implies that there exists $a_1 > 0$, $a_2 > 0$ such that

$$F(n, z) \geq a_1 |z|^\beta - a_2.$$

Since $\beta > \delta + 1$, we see that (H_2) is stronger that our (f_∞). Therefore, our Theorem 1 generalizes [1] Theorem 3.1 considerably.

If $f(n, z)$ is odd in z, then we can obtain better result.

Theorem 2 Assume that (f_0) and (f_∞) are satisfied, if $f(n, z) = -f(n, -z)$ for all $(n, z) \in \mathsf{Z} \times \mathsf{R}$, then the problem (1) has at least $m - 1$ pairs of nontrivial solutions.

This symmetric case has not been considered in [1]. The proof of [1] Theorem 3.1 is based on the linking theorem [2] Theorem 5.3, which requires some tedious estimates. We shall prove Theorem 1 and Theorem 2 using the three critical points theorem [3], [4] and the Clark's theorem [5], [2]. It turns out that our approach is considerable simpler.

The three critical points theorem and the Clark's theorem have played a significant role in the study of differential equations. Recently, Liu [6] applied these theorems to difference equations and obtained some interesting results. This work is motivated by Liu [6].

2 Proofs of the theorems

As in [1], we define the linear operations on

$$E_m = \left\{x = \{x_n\}_{n\in \mathsf{Z}} : x_n \in \mathsf{R},\ \mathsf{x}_{n+m} = \mathsf{x}_n,\ n \in \mathsf{Z}\right\}$$

in an obvious way, then define the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ on E_m as follows:

$$\langle x, y\rangle = \sum_{n=1}^{m} x_n y_n, \qquad \|x\| = \left(\sum_{n=1}^{m} x_n^2\right)^{1/2}, \qquad x, y \in E_m.$$

Then, E_m is a m-dimensional Hilbert space and linear isomorphic to R^m.

We define a functional $I : E_m \to \mathsf{R}$,

$$I(x) = \frac{1}{\delta+1}\sum_{n=1}^{m} p_{n+1}(\Delta x_n)^{\delta+1} - \sum_{n=1}^{m} F(n, x_n), \qquad x = \{x_n\} \in E_m.$$

Since $f \in C(\mathsf{Z}\times\mathsf{R}, \mathsf{R})$, it follows that $I \in C^1(E_m, \mathsf{R})$. According to [1], the critical points of I are exactly the m-periodic solutions of (1). So we have to find critical points of I. For this purpose we need the following results.

Proposition 1[3,4] Let E be a Banach space, $\varphi \in C^1(E, \mathsf{R})$ satisfies the Palais-Smale (PS) condition and is bounded from below. Suppose φ has a *local linking* at the origin 0, namely, there are a decomposition $E = Y \oplus W$ and a positive real number $\rho > 0$ such that $k = \dim Y < \infty$,

$$\varphi(x) < \varphi(0) \text{ for } x \in Y,\ 0 < \|x\| \le \rho, \qquad \varphi(x) \ge \varphi(0) \text{ for } x \in W,\ \|x\| \le \rho; \tag{3}$$

then φ has at least three critical points.

Recall that φ satisfies the (PS) condition, if any sequence $\{x^{(i)}\}$ such that $\{\varphi(x^{(i)})\}$ is bounded and $\varphi'(x^{(i)}) \to 0$, has a convergent subsequence.

Proposition 2[2,5] Let E be a Banach space and $\varphi \in C^1(E, \mathsf{R})$ be an even functional satisfying the (PS) condition and $\varphi(0) = 0$. Assume that φ is bounded from below and there are $\rho > 0$ and a k-dimensional linear subspace Y of E such that

$$\sup_{x\in Y, \|x\|=\rho} \varphi(x) < 0,$$

then φ possesses at least k pairs of critical points.

Note that these critical points are nonzero, because the values of φ over these points are negative, see the proof of [2] Theorem 9.1 for the details.

Now we are ready to prove our theorems.

Lemma 3 If (f_∞) holds, then $I(x) \to -\infty$ as $\|x\| \to \infty$.

Proof The proof of this lemma is slightly difference from that of [1] Lemma 3.1. Let

$$v_2 = \max_{1\le n\le m} p_n > 0.$$

By (f_∞), there exists $C > 0$ such that

$$F(n,z) \ge \left(\frac{2^{\delta+2}v_2}{\delta+1} + 1\right)|z|^{\delta+1} - C. \tag{4}$$

Remember $x_{n+m} = x_n$ and $p_{n+m} = p_n$, we obtain

$$\begin{aligned} I(x) &\le \frac{1}{\delta+1}\sum_{n=1}^{m} p_{n+1}|x_{n+1} - x_n|^{\delta+1} - \sum_{n=1}^{m} F(n,x_n) \\ &\le \frac{v_2}{\delta+1}\sum_{n=1}^{m}\left[2^{\delta+1}(|x_{n+1}|^{\delta+1} + |x_n|^{\delta+1})\right] - \sum_{n=1}^{m} F(n,x_n) \\ &\le \frac{2^{\delta+2}v_2}{\delta+1}\sum_{n=1}^{m}|x_n|^{\delta+1} - \left(\frac{2^{\delta+2}v_2}{\delta+1} + 1\right)\sum_{n=1}^{m}|x_n|^{\delta+1} + Cm \\ &= -\sum_{n=1}^{m}|x_n|^{\delta+1} + Cm \to -\infty, \qquad \text{as } \|x\| = \left(\sum_{n=1}^{m} x_n^2\right)^{1/2} \to \infty, \end{aligned}$$

the desired result follows.

Proof of Theorem 1 Let

$$W = \left\{x = \{x_n\}_{n\in Z} : x_n = x \in \mathrm{R},\ \mathrm{n} \in \mathrm{Z}\right\},$$

and Y be the orthogonal complement of W in E_m. Then $E_m = W \oplus Y$, $\dim Y = m - 1$.

It has been proven in [1] Page 1643–1644 that the limit in (f_0) implies the existence of an $\eta > 0$ such that

$$I(x) > 0, \qquad \text{for } x \in Y \cap \partial B_\eta, \tag{5}$$

where $B_\eta = \{x \in E_m : \|x\| \le \eta\}$ and ∂B_η its boundary. Note that by the argument there, (5) is still valid if we decrease η. Therefore, there exists $\rho \in (0,r)$ such that

$$I(x) > 0, \qquad \text{for } x \in Y \cap B_\rho. \tag{6}$$

If $x \in W$, $\|x\| \le \rho$, then $|x_n| \le \rho < r$ and we have $\Delta x_n = 0$. Thus by (f_0) we obtain

$$I(x) = -\sum_{n=1}^{m} F(n,x_n) \le 0.$$

Therefore, $-I$ has a local linking at the origin 0. By Lemma and the fact that $\dim E_m < \infty$, it is easy to see that $-I$ is bounded from below and satisfies the (PS) condition. Applying Proposition 1, $-I$ has at least three critical points. Therefore, I has two nonzero critical points, which are nontrivial m-periodic solutions of the problem (1).

Remark 4 In [1], after obtaining (5), in order to apply the linking theorem, some tedious estimates are involved and the global condition $F(n,z) \geq 0$ for all $(n,z) \in \mathrm{Z} \times \mathrm{R}$ is needed. Our argument above does not need this global condition, and simplifies the proof considerably.

Proof of Theorem 2 If $f(n,z) = -f(n,-z)$ for all $(n,z) \in \mathrm{Z} \times \mathrm{R}$, then I is an even functional. We know that $I(0) = 0$, $-I$ is bounded from below and satisfies the (PS) condition. By (5), since $Y \cap B_\eta$ is compact, we have

$$\sup_{x \in Y, \|x\| = \eta} (-I)(x) < 0.$$

Now the desired result follows from Proposition 2.

Remark 5 By the proof of Lemma , we see that replacing (f_∞) with the following

$$\liminf_{|z| \to \infty} \frac{F(n,z)}{|z|^{1+\delta}} > \frac{2^{\delta+2} v_2}{\delta + 1},$$

the conclusion of Theorem 1 and Theorem 2 is still valid.

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非线性差分方程的多重周期解

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摘 要 用变分方法得到一类非线性差分方程多重周期解的存在性. 我们的结果推广了 Cai, Yu 和 Guo [Comput. Math. Appl., 52 (2006), 1630–1647] 的结果，并且这里给出的证明显著地简化了.

关键词 差分方程的周期解； Palais-Smale 条件；三临界点定理； Clark 定理