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On k-Strong Distance in Strong Oriented Graphs

Miao Huifang *Guo Xiaofeng*
(School of Mathematical Sciences, Xiamen University, Xiamen Fujian 361005)

Abstract For a nonempty vertex set S in a strong digraph D, the strong distance $d(S)$ of S is the minimum size (the number of edges) of a strong subdigraph of D containing the vertices of S. If S contains k vertices, then $d(S)$ is referred to as the k-strong distance of S. For an integer $k\geq 2$ and a vertex v of a strong digraph D, the k-strong eccentricity $se_k(v)$ of v is the maximum k-strong distance $d(S)$ among all sets S of k vertices in D containing v. The minimum k-strong eccentricity among the vertices of D is the k-strong radius of D $srad_k(D)$ and the maximum k-strong eccentricity is the k-strong diameter of D $sdiam_k(D)$. In this paper, we will show that for any integers r, d with $k+1\leq r, d\leq n$, there exist strong tournaments T' and T'' of order n such that $srad_k(T')=r$ and $sdiam_k(T'')=d$. And we also give an upper bound on the k-strong diameter of strong oriented graphs.

Key words Digraph; Strong distance; Strong radius; Strong diameter
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1 Introduction

The familiar distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest (u,v)-path in G. Equivalently, this distance is the minimum size of a connected subgraph of G containing u and v. This concept was extended by Chartrand G et al. in [1] to strong digraphs, in particular to strong oriented graphs. We refer to [2] for graph theory notation and terminology not described here.

A digraph D is strongly connected if for every pair u, v of distinct vertices of D, there is both a directed (u,v)-path and a directed (v,u)-path in D. A digraph D is an oriented graph if there is no cycle of length two. In this paper, we will be interested in strong oriented graphs. The underlying graph of a strong oriented graph is necessarily 2-edge connected.

Let D be a strong oriented graph. The order and size of D are denoted by $n(D)$ and $m(D)$. For two vertices u and v of D, the strong distance $sd(u,v)$ between u and v is the minimum size of a strong subdigraph of D containing u and v. A (u,v)-geodesic is a strong subdigraph of D of size $sd(u,v)$ containing u and v. If $u\neq v$, then $3\leq sd(u,v)\leq m(D)$. A strong oriented graph D

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is a strong (u,v)-path, if there is no proper strong subdigraph of D containing u and v. An oriented graph D is simply called a strong path, if D is a strong (u,v)-path for some pair u,v of vertices of D. We note that a (u,v)-geodesic is a strong (u,v)-path, while a strong (u,v)-path need not be a (u,v)-geodesic. Chartrand G et al. have made some elementary observation concerning strong paths and gave the following properties in [1].

Theorem 1. 1 [1] If D is a strong (u,v)-path, then D contains a unique directed (u,v)-path and a unique directed (v,u)-path.

Theorem 1. 2 [1] If D is a strong path of order $n\geq 3$ and size m, then

$$\frac{m}{n-1}\leq\frac{5}{3}.$$

For a nonempty vertex set S in a connected graph G, the Steiner distance $d(S)$ of S is the minimum size of a connected subgraph of G containing S. This concept was extended to strong digraph by Zhang P et al. in [3]. For a nonempty vertex set S in a strong digraph D, the strong Steiner distance $d(S)$ was defined by Zhang P et al. as the minimum size of a strong subdigraph of D containing S. They also referred to such a subdigraph as a Steiner subdigraph with respect to S, or, simply, S-subdigraph. Since D itself is strong, $d(S)$ is defined for every nonempty vertex set S of D. If $|S|=k$, then $d(S)$ is referred to as the k-strong Steiner distance (or simply k-strong distance) of S. Thus, $3\leq d(S)\leq m(D)$, for each vertex set S in a strong digraph D with $|S|\geq 2$. When $k=2$, the 2-strong distance is the strong distance studied in [1, 4]. For example, in a strong oriented graph D of Fig. 1, let $S_1=\{s,v,x\}$, $S_2=\{v,x,z\}$, $S_3=\{s,x,y\}$. Then the 3-strong distance of S_1, S_2 and S_3 are $d(S_1)=3$, $d(S_2)=4$ and $d(S_3)=5$, respectively.

It was showed in [3] that k-strong distance satisfies an extension of triangle inequality.

Theorem 1. 3 [3] For an integer $k\geq 2$, let S_1, S_2 and S_3 be vertex sets in a strong oriented graph with $|S_i|=k$ for $1\leq i\leq 3$. If $S_1\subseteq S_2\cup S_3$ and $S_2\cap S_3\neq\varnothing$, then

$$d(S_1)\leq d(S_2)+d(S_3).$$

Let v be a vertex of a strong oriented graph D of order $n\geq 3$ and let k be an integer with $2\leq k\leq n$. The k-strong eccentricity $se_k(v)$ is defined by

$$se_k(v)=\max\{d(S)\mid v\in S\subseteq V(D),|S|=k\}.$$

The k-strong diameter $sdiam_k(D)$ is

$$sdiam_k(D)=\max\{se_k(v)\mid v\in V(D)\};$$

while the k-strong radius $srad_k(D)$ is defined by

$$srad_k(D)=\min\{se_k(v)\mid v\in V(D)\};$$

The 3-strong eccentricity of each vertex of a strong oriented graph D is showed in Fig. 2. Thus, $srad_3(D)=8$ and $sdiam_3(D)=12$.

For any integer $k\geq 2$, the k-strong radius and k-strong diameter of a strong oriented graph satisfy the following familiar inequality.

Theorem 1. 4 [3] Let $k\geq 2$ be an integer. For every strong oriented graph D,

$$srad_k(D)\leq sdiam_k(D)\leq 2srad_k(D).$$

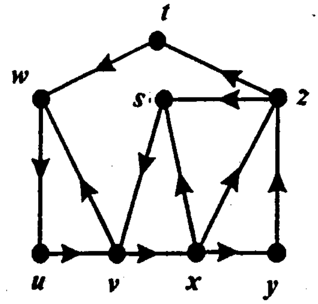

Fig. 1 A strong oriented graph D.

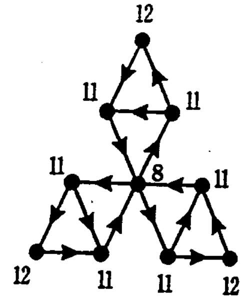

Fig. 2 A strong oriented graph D with $srad_3(D)=8$ and $sdiam_3(D)=12$.

In [4], Chartrand G et al. showed that, for any integers r, d with $3\leq r\leq k+1$ and $3\leq d\leq 2k+1$, there exist strong tournaments T' and T'' of order $2k+1$ such that $srad_2(T')=r$ and $sdiam_2(T'')=d$. Dankelmann P et al. presented an upper bound on the strong radius of a strong oriented graph D of order n as $srad_2(D)\leq n$ in [5]. In [1], Chartrand G el al. gave an upper bound on the strong diameter of a strong oriented graph D as following.

Theorem 1.5 [1] If D is a strong oriented graph of order $n\geq 3$, then

$$sdiam_2(D)\leq [5(n-1)/3].$$

In this paper, we shall make some observation concerning the k-strong distance of strong tournament T of order n. First we will show that for any integers r, d with $k+1\leq r, d\leq n$, there exist strong tournaments T' and T'', such that $srad_k(T')=r$ and $sdiam_k(T'')=d$. And we also give an upper bound on the k-strong diameter $sdiam_k(D)$ for $3\leq k\leq n$.

2 The k-Strong Distance of Strong Tournaments

In this section, we consider strong tournaments of order $n\geq 4$. Since every strong tournament T is hamiltonian, clearly, $k\leq srad_k(T)\leq sdiam_k(T)\leq n$ for $2\leq k\leq n$. When $k=2$, it has been studied in [4]. While $k=n$, it is clear that $srad_n(T)=sdiam_n(T)=n$. In the following, first we shall consider the k-strong distance for $3\leq k\leq n-2$.

Lemma 2.1 For any integer k with $3\leq k\leq n-2$, there exist strong tournaments T' and T'' of order $n\geq 5$ such that

$$srad_k(T')=r_k \text{ and } sdiam_k(T'')=d_k,$$

where r_k, d_k are integers with $k+1\leq r_k\leq n-1$ and $k+2\leq d_k\leq n$.

Proof For a given $k\geq 3$, we now construct a strong tournament T of order n such that $srad_k(T)=n-p+k-1$ and $sdiam_k(T)=n-p+k$ for all p with $k\leq p\leq n-2$.

Let $V(T)=\{v_1, v_2, \cdots, v_n\}$. We partition the vertex set of T into two subsets $V_1=\{v_1, v_2, \cdots, v_p\}$ and $V_2=\{v_{p+1}, \cdots, v_n\}$, where $k\leq p\leq n-2$. Furthermore, let

$$A(T)=\{(v_i, v_j)\mid 1\leq i<j\leq p\}$$

$$\cup \{(v_i, v_{i+1}) \mid p+1 \leq i \leq n-1\}$$
$$\cup \{(v_j, v_i) \mid p+1 \leq i < j \leq n, i \neq j-1\}$$
$$\cup \{(v_i, v_j) \mid v_i \in V_1, v_j \in V_2\} \cup \{(v_n, v_1)\} - \{(v_1, v_n)\} \quad \text{(see Fig. 3)}.$$

Let $S=\{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If $i_k \leq p$ and $v_1 = v_{i_1} \in S$, then the directed $(k+1)$-cycle $v_1 v_{i_2} \cdots v_{i_k} v_n v_1$ is a S-subdigraph. So $d(S)=k+1$. If $i_k \leq p$ and $v_1 \notin S$, then the directed $(k+2)$-cycle $v_1 v_{i_1} v_{i_2} \cdots v_{i_k} v_n v_1$ is a S-subdigraph. So $d(S)=k+2$. Otherwise, let i_j be the smallest subscript satisfying $p < i_j \leq n$. If $v_1 = v_{i_1} \in S$, then the directed $(n-i_j+j)$-cycle $v_1 v_{i_2} \cdots v_{i_{j-1}} v_{i_j} v_{i_j+1} \cdots v_n v_1$ is a S-subdigraph. So $d(S)=n-i_j+j$. If $v_1 \notin S$, then the directed $(n-i_j+j+1)$-cycle $v_1 v_{i_1} v_{i_2} \cdots v_{i_{j-1}} v_{i_j} v_{i_j+1} \cdots v_n v_1$ is a S-subdigraph. So $d(S)=n-i_j+j+1$. So we have $se_k(v_1)=\max\{k+1, n-i_j+j \mid p<i_j \leq n, 1 \leq j \leq k\}=n-p+k-1$; $se_k(v_i)=\max\{k+1, k+2, n-i_j+j, n-i_j+j+1 \mid p<i_j \leq n, 1 \leq j \leq k\}=n-p+k$ for $2 \leq i \leq p$; $se_k(v_{p+1})=\max\{n-i_j+j, n-i_j+j+1 \mid p<i_j \leq n, 1 \leq j \leq k\}=n-p+k$; $se_k(v_j)=\max\{n-i_j+j, n-i_j+j+1 \mid p<i_j \leq n, 1 \leq j \leq k\}=n-p+k-1$ for $p+2 \leq j \leq n$.

Therefore, $srad_k(T)=n-p+k-1$ and $sdiam_k(T)=n-p+k$ for any p with $k \leq p \leq n-2$, implying that, for any integers r_k, d_k with $k+1 \leq r_k \leq n-1$ and $k+2 \leq d_k \leq n$, there exist strong tournaments T' and T'' of order $n \geq 5$ such that $srad_k(T')=r_k$ and $sdiam_k(T'')=d_k$. □

Now we will show that, for any integer k with $3 \leq k \leq n-1$, the tournament T of order n satisfying $srad_k(T)=n$ exists.

Lemma 2.2　For any integer k with $3 \leq k \leq n-1$, there exists a strong tournament T of order $n \geq 4$ such that

$$srad_k(T)=sdiam_k(T)=n.$$

Proof　Let $V(T)=\{v_1, v_2, \cdots, v_n\}$. And

$$A(T)=\{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$$
$$\cup \{(v_j, v_i) \mid 1 \leq i < j \leq n, i \neq j-1\}$$

(see Fig. 4).

For any vertex v_i, let $S_i=\{v_i\} \cup \{v_1, v_n\}, i=1,2,\cdots,n$. Since $sd(v_1, v_n)=n$, we have $d(S_i) \geq sd(v_1, v_n)=n$. On the other hand, T has a Hamilton cycle. So $d(S_i)=n$, which implies that $se_3(v_i)=n, i=1,2,\cdots,n$. Hence, $srad_3(T)=sdiam_3(T)=n$.

By the definition of k-strong radius and k-strong diameter, for any k with $4 \leq k \leq n-1$, $srad_k(T) \geq srad_3(T)=n$, $sdiam_k(T) \geq sdiam_3(T)=n$, And T has a Hamilton cycle. So $srad_k(T)=sdiam_k(T)=n$, for any k with $3 \leq k \leq n-1$. □

By Lemmas 2.1 and 2.2, we have the following result.

Theorem 2.3　For any integer k with $3 \leq k \leq n-1$, there exists a strong tournament T of order $n \geq 4$ such that

$$srad_k(T)=r,$$

for every r with $k+1 \leq r \leq n$. □

Lemma 2.1 shows that for any k with $3 \leq k \leq n-2$, there exists strong tournament T such that $sdiam_k(T)=d$ for any d with $k+2 \leq d \leq n$. In the following we show that the strong

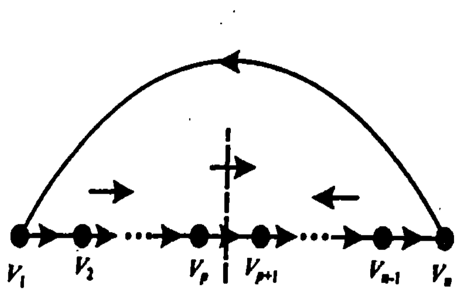

Fig. 3　The strong tournament T.

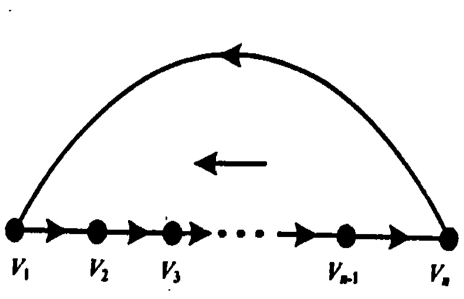

Fig. 4　The strong tournament T.

tournament T with $sdiam_k(T)=k+1$ can be found.

Lemma 2.4　For any integer k with $3\leq k\leq n-1$, there exists a strong tournament T_n of order $n\geq 4$ such that

$$sdiam_k(T_n)=k+1.$$

Proof　Let $V(T_n)=\{v_1,v_2,\cdots,v_n\}$.

Let T_k, T_{k+1} be the strong tournaments constructed in Lemma 2.2 with $sdiam_{k-1}(T_k)=k$ and $sdiam_k(T_{k+1})=k+1$.

Now, we construct T_{n+2} from T_n by replacing exactly one vertex of a directed 3-cycle C_3 with a copy of T_n, and for any vertex $v\in V(T_n)$, (v_{n+1},v), $(v,v_{n+2})\in A(T_{n+2})$ (see Fig. 5) for $n\geq k$.

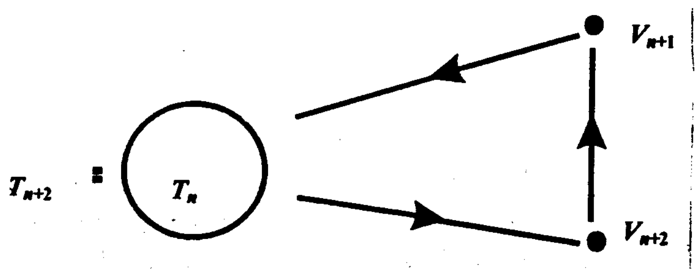

Fig. 5　Strong tournament T_{n+2}.

Now we will show that for any i, n with $0\leq i\leq k-2$, $n>k$, and any $k-i$ vertices of T_n, there is a directed path just containing all these $k-i$ vertices in T_n; and $sdiam_k(T_n)=k+1$.

We show it by induction on n. When $n=k+1$, by the construction of T_{k+1}, for any $k-i$ vertices, there is a directed path just containing all these $k-i$ vertices in T_{k+1}; and $sdiam_k(T_{k+1})=k+1$.

When $n=k+2$. By the construction of T_k, for any $k-i$ vertices, there is a directed path just containing all these $k-i$ vertices in T_k. Let $S=\{v_{t_1},v_{t_2},\cdots,v_{t_{k-i}}\}\subseteq V(T_{k+2})$. If $S\subseteq V(T_k)$, then it is clear that there is a directed path just containing all these $k-i$ vertices in T_{k+2}; and d

$(S) \leq k$. If S contains only one vertex of $\{v_{k+1}, v_{k+2}\}$, we may assume that $v_{t_{k-i}} = v_{k+1} \in S$, and T_k exists a directed path just containing the vertices $v_{t_1}, v_{t_2}, \cdots, v_{t_{k-i-1}}$. Then $v_{k+1}P$ is a directed path in T_{k+2} and the directed $(k-i+1)$-cycle $v_{k+1}Pv_{k+2}v_{k+1}$ is a S-subdigraph in T_{k+2}. So $d(S) = k-i+1$. If $\{v_{k+1}, v_{k+2}\} \subseteq S$, we may assume that $v_{t_{k-i-1}} = v_{k+1}, v_{t_{k-i}} = v_{k+2}$, and T_k contains a directed path Q just containing the vertices $v_{t_1}, v_{t_2}, \cdots, v_{t_{k-i-2}}$. Then $Qv_{k+2}v_{k+1}$ is a directed path in T_{k+2} and the directed $(k-i)$-cycle $v_{k+1}Qv_{k+2}v_{k+1}$ ia a S-subdigraph in T_{k+2}. So $d(S) = k-i$. Therefore, $se_k(v_j) = k+1$ for $1 \leq j \leq k+2$, which implies $sdiam_k(T_{k+2}) = k+1$.

Assume that when $n = m > k$, for any $k-i$ vertices of T_m, there is a directed path just containing all these $k-i$ vertices; and $sdiam_k(T_m) = k+1$. Consider the strong tournament T_{m+2} constructed from T_m. Let $S = \{v_{t_1}, v_{t_2}, \cdots, v_{t_{k-i}}\} \subseteq V(T_{m+2})$. If $S \subseteq V(T_m)$, then by the induction hypothesis, there is a directed path just containing all these $k-i$ vertices in T_{m+2}; and $d(S) \leq k+1$. If S contains only one vertex of $\{v_{m+1}, v_{m+2}\}$, we may assume $v_{t_{k-i}} = v_{m+1}$. By the induction hypothesis, T_m exists a directed path P just containing the vertices $v_{t_1}, v_{t_2}, \cdots, v_{t_{k-i-1}}$. Similar to the case $n = k+2$, $v_{m+1}P$ is a directed path just containing the vertex of S in T_{m+2}; and $d(S) = k-i+1$. If $\{v_{m+1}, v_{m+2}\} \subseteq S$, we may assume that $v_{t_{k-i-1}} = v_{m+1}, v_{t_{k-i}} = v_{m+2}$, and T_m contains a directed path Q just containing the vertices $v_{t_1}, v_{t_2}, \cdots, v_{t_{k-i-2}}$. Similar to the case $n = k+2$, there is a directed path $Qv_{m+2}v_{m+1}$ just containing the vertex of S in T_{m+2}; and $d(S) = k-i$. Hence, $se_k(v_j) = k+1$ for $1 \leq j \leq m+2$, which implies $sdiam_k(T_{m+2}) = k+1$. □

Lemmas 2.1, 2.2 and 2.4 give the following result.

Theorem 2.5 For any integer k with $3 \leq k \leq n-1$, there exists a strong tournament T of order $n \geq 4$ such that

$$sdiam_k(T) = d,$$

for every d with $k+1 \leq d \leq n$. □

3 A Bound on k-Strong Diameter

The k-strong diameter of a strong oriented graph is at least k. In this section, we present an upper bound on the k-strong diameter of a strong oriented graph of order $n \geq 3$.

An ear decomposition of a digraph D is a sequence $X = \{P_0, P_1, \cdots, P_r\}$, where P_0 is a directed cycle and each P_i is a directed path or a directed cycle with the following properties:

(a) P_i and P_j are arc-disjoint when $i \neq j$.

(b) For each $i = 1, 2, \cdots, r$. Let D_i be the digraph induced by $\bigcup_{j=0}^{i} A(P_j)$. If P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of P_i are distinct vertices of D_{i-1} and no other vertex of P_i belongs to $V(D_{i-1})$.

(c) $\bigcup_{j=0}^{r} A(P_j) = A(D)$.

An ear P_i is trivial if $|A(P_i)| = 1$. In [2], it has been showed that every strong digraph has an ear decomposition $X = \{P_0, P_1, \cdots, P_r\}$. We now show that for every S-subdigraph D' of a strong oriented graph, $m(D') \leq 2n(D') - 3$.

Lemma 3.1 Let D' be a S-subdigraph with respect to a vertex subset S of a strong oriented graph D, where $|S| = k \geq 3$. Then $m(D') \leq 2n(D') - 3$.

Proof Let $X = \{P_0, P_1, \cdots, P_r\}$ be an ear decomposition of D'. By the minimality of $m(D')$, X does not contain any trivial ear, and P_0 contains at least 3 arcs. Hence, each P_i, $i = 1, 2, \cdots, r$ contains at least one internal vertex, and $r \leq n(D') - |V(P_0)| \leq n(D') - 3$. Let $m_i = |A(P_i)|$, we can make the following estimate:

$$m(D') = \sum_{i=0}^{r} m_i = |V(P_0)| + \sum_{i=1}^{r} (m_i - 1) + r = n(D') + r$$
$$\leq n(D') + n(D') - 3 \leq 2n(D') - 3,$$

where equality only holds if $|V(P_0)| = 3$ and each P_i, $i = 1, 2, \cdots, r$, has length 2. □

Theorem 1.5 gives an upper bound on 2-strong diameter. In the following, we will give an upper bound on k-strong diameter for $3 \leq k \leq n$.

Theorem 3.2 If D is an oriented graph of order $n \geq 3$, then

$$sdiam_k(D) \leq 2n - 3,$$

for every k with $3 \leq k \leq n$.

Proof Let $S \subseteq V(D)$ such that $d(S) = sdiam_k(D)$, where $|S| = k$. Let D' be a S-subdigraph in D with respect to S. By Lemma 3.1, then

$$sdiam_k(D) = m(D') \leq 2n(D') - 3 \leq 2n - 3.$$ □

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强定向图的 k-强距离

缪惠芳　　郭晓峰

(厦门大学数学科学学院, 福建 厦门 361005)

摘　要　对强连通有向图 D的一个非空顶点子集 S, D中包含 S的具有最少弧数的强连通有向子图称为 S的 Steiner子图, S的强 Steiner距离 $d(S)$等于 S的 Steiner子图的弧数. 如果 $|S| = k$, 那么 $d(S)$称为 S的 k-强距离. 对整数 $k \geq 2$和强有向图 D的顶点 v, v的 k-强离心率 $se_k(v)$为 D中所有包含 v的 k个顶点的子集的 k-强距离的最大值. D中顶点的最小 k-强离心率称为 D的 k-强半径, 记为 $srad_k(D)$, 最大 k-强离心率称为 D的 k-强直径, 记为 $sdiam_k(D)$. 本文证明了, 对于满足 $k+1 \leq r, d \leq n$的任意整数 r, d, 存在顶点数为 n的强竞赛图 T'和 T'', 使得 $srad_k(T') = r$和 $sdiam_k(T'') = d$; 进而给出了强定向图的 k-强直径的一个上界.

关键词　有向图; 强距离; 强半径; 强直径