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受约束 Vacco 系统的几个 Noether 定理

顾书龙[1] 计国君[2] 张宏彬[3] 刘东生[1]

([1] 巢湖学院物理系, 巢湖 238000)
([2] 厦门大学管理学院, 厦门 361005)
([3] 上海大学应用数学和力学研究所, 上海 200072)

摘要: 研究受约束 Vacco 系统的 Noether 对称性. 基于受约束 Vacco 系统在 r-参数有限变换群 G_r 的无限小广义准对称变换下的不变性, 给出了受单面约束的 Vacco 系统的 Noether 定理及其逆定理. 受双面约束的 Vacco 系统的 Neother 定理为该定理的推论. 最后给出一个算例说明了结果的应用.

关键词: 分析力学; 单面约束 Vacco 系统; 守恒量; 对称性; Noether 定理
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Several Noether' s theorem of Vacco systems with constraints

Gu Shulong[1] Ji Guojun[2] Zhang Hongbin[3] Liu Dongsheng[1]

([1]Department of Physics, Chaohu College, Chaohu 238000, China)
([2]Management School, Xiamen University, Xiamen 361005, China)
([3]Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China)

Abstract: The Neother symmetry for Vacco systems with constraints is studied. Noether' s theorem and its inverse theorem of Vacco systems with unilateral constraints are given, which are based upon the invariant properties by introducing the generalized quasi-symmetry of the infinitesimal transformation for the transformation group G_r of r-parameter. And Noether' s theorem dealing with the Vacco system of bilateral constraints is its corollary. An example to illustrate the application of the result is given.

Key words: analytical mechanics; Vacco systems with unilateral constraints; conserved quantity; symmetry; Noether' s theorem

近年来, 约束力学系统的对称性与不变量的研究引起了力学、物理学和数学工作者的高度重视. 对称性不仅能帮助人们求得所研究问题的解, 也可以帮助人们去寻求新的运动规律. 我国学者张毅、梅凤翔[1]率先将广义的 Neother 定理从双面约束系统推广到单面约束系统, 建立了单面约束系统的 Neother 理论. 1990 年郭仲衡[2]在证明了对于一阶非线性非完整约束 d-δ运算是可以交换的之后, 对虚位移不引入 Appell-Цetaeb 定义, 得出一种新型方程的形式与 Vacco 动力学方程相同, 引起了分析力学研究者的很大兴趣, 并相继取得了一系列的研究成果[3~7]. 本文通过引进 r-参数有限变换群 G_r 的无限小变换的广义准对称概念, 研究并给出受单面约束的 Vacco 系统的 Noether 定理和 Noether 逆定理. 由于受双面约束的 Vacco 系统的 Noether 定理可作为本文定理的推论, 因此本文的研究更有普遍意义.

1 受约束 Vacco 系统的 Noether 定理

研究由 N 个质点组成的力学系统, 其位形由 n 个广义坐标 q_s ($s = 1, 2, \cdots, n$) 确定, 系统的运动受如下单面 Vacco 约束:

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作者简介: 顾书龙(1958—), 男, 副教授, gsl2142@sohu.com.

$$\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \geqslant 0 \qquad \beta = 1, 2, \cdots, g \tag{1}$$

则系统的运动方程可写成(本文约定重复脚码表示求和)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s - \lambda_\beta\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \phi_\beta}{\partial \dot{q}_s} - \frac{\partial \phi_\beta}{\partial q_s}\right) - \dot{\lambda}_\beta \frac{\partial \phi_\beta}{\partial \dot{q}_s},\ \lambda_\beta \geqslant 0,\ \lambda_\beta\phi_\beta = 0$$
$$s = 1, 2, \cdots, n;\ \beta = 1, 2, \cdots, g \tag{2}$$

式中, L 为系统的 Lagrange 函数, Q''_s 为非势广义力, λ_β 为约束乘子. 如果系统受约束, 则 $\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$, 如果系统脱离约束, 则 $\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) > 0$, 此时, 式(2)的第 1 组方程右端的后 2 项为零. 引进一般形式的 r-参数有限变换群 G_r:

$$\left.\begin{aligned} t^* &= g_0(t, \boldsymbol{q}_k, \dot{\boldsymbol{q}}_k, b_\alpha) \\ q_s^* &= g_s(t, \boldsymbol{q}_k, \dot{\boldsymbol{q}}_k, b_\alpha) \end{aligned}\right\} \qquad s, k = 1, 2, \cdots, n;\ \alpha = 1, 2, \cdots, r \tag{3}$$

式中, $b_\alpha(\alpha = 1, 2, \cdots, r)$ 为独立参数, 对应的无限小变换为

$$t^* = t + \Delta t,\ q_s^*(t^*) = q_s(t) + \Delta q_s \qquad s = 1, 2, \cdots, n \tag{4}$$

或其展开式为

$$t^* = t + \varepsilon_\alpha \xi_0^\alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\ q_s^*(t^*) = q_s(t) + \varepsilon_\alpha \xi_0^\alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \qquad s = 1, 2, \cdots, n;\ \alpha = 1, 2, \cdots, r \tag{5}$$

式中, ε_α 为无限小参数, 具有一阶小量.

首先, 若系统受约束, 即 $\phi_\beta = 0$, 构造积分

$$A = \int_{t_1}^{t_2} [L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_\beta(t)\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})]\,\mathrm{d}t \tag{6}$$

式(6)在变换前后的差为

$$A(\gamma^*) - A(\gamma) = \int_{t_1^*}^{t_2^*} [L(t^*, \boldsymbol{q}^*, \dot{\boldsymbol{q}}^*) + \lambda_\beta(t^*)\phi_\beta(t^*, \boldsymbol{q}^*, \dot{\boldsymbol{q}}^*)]\,\mathrm{d}t -$$
$$\int_{t_1}^{t_2} [L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_\beta(t)\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})]\,\mathrm{d}t \tag{7}$$

式中, γ, γ^* 分别为给定曲线与邻近曲线. $[t_1^*, t_2^*]$ 为与原积分区间 $[t_1, t_2]$ 相对应的积分区间, 将其对 ε 的主线性部分精确到一阶小量部分, 记为 ΔA, 因为

$$\mathrm{d}t^* = \mathrm{d}(t + \Delta t) = \mathrm{d}t[1 + (\Delta t)^{\cdot}] \tag{8}$$

则得

$$\Delta A = \int_1^2 \Big[\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_s}\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}\Delta \dot{q}_s + \lambda_\beta\phi_\beta\Delta t + L(\Delta t)^{\cdot} + \lambda_\beta\phi_\beta(\Delta t)^{\cdot} +$$
$$\lambda_\beta\left(\frac{\partial \phi_\beta}{\partial t}\Delta t + \frac{\partial \phi_\beta}{\partial q_s}\Delta q_s + \frac{\partial \phi_\beta}{\partial \dot{q}_s}\Delta \dot{q}_s\right)\Big]\,\mathrm{d}t \tag{9}$$

考虑等时变分与非等时变分的关系, 并考虑式(5), 得单面约束条件下 $\bar{\delta} q_s$ 满足

$$\bar{\delta} q_s = \varepsilon_\alpha \xi_s^\alpha - \varepsilon_\alpha \dot{q}_s \xi_0^\alpha \qquad s = 1, 2, \cdots, n \tag{10}$$

因此, 式(9)可写为

$$\Delta A = \int_1^2 \Big[\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t + \lambda_\beta\phi_\beta\Delta t) + \frac{\partial L}{\partial q_s}\bar{\delta} q_s + \frac{\partial L}{\partial \dot{q}_s}\bar{\delta} \dot{q}_s + \lambda_\beta\left(\frac{\partial \phi_\beta}{\partial q_s}\bar{\delta} q_s + \frac{\partial \phi_\beta}{\partial \dot{q}_s}\bar{\delta} \dot{q}_s\right)\Big]\,\mathrm{d}t \tag{11}$$

根据 Hölder 定义, 无论完整或非完整系统, 全部等时变分都采用下面的交换关系:

$$\bar{\delta} \dot{q}_s = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta} q_s \qquad s = 1, 2, \cdots, n \tag{12}$$

将式(12)代入式(11), 并根据式(5)得

$$\Delta A = \int_1^2 \Big\{\varepsilon_\alpha\Big[\frac{\mathrm{d}}{\mathrm{d}t}\Big(L\xi_0^\alpha + \lambda_\beta\phi_\beta\xi_0^\alpha + \frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_s^\alpha + \lambda_\beta\frac{\partial \phi_\beta}{\partial \dot{q}_s}\bar{\xi}_s^\alpha\Big)\Big] -$$
$$\Big[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \lambda_\beta\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \phi_\beta}{\partial \dot{q}_s} - \frac{\partial \phi_\beta}{\partial q_s}\right) + \dot{\lambda}_\beta\frac{\partial \phi_\beta}{\partial \dot{q}_s}\Big]\bar{\xi}_s^\alpha\Big\}\,\mathrm{d}t \tag{13}$$

式中, $\bar{\xi}_s^\alpha = \xi_s^\alpha - \dot{q}_s\xi_0^\alpha$, $s = 1, 2, \cdots, n$.

其次, 若系统脱离约束, 即 $\phi_\beta > 0$, 同前面进行类似的讨论, 可得

$$\Delta A=\int_{t_1}^{t_2}\varepsilon_\alpha\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(L\xi_0^\alpha+\frac{\partial L}{\partial\dot{q}_s}\xi_s^\alpha\right)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}\right)\xi_s^\alpha\right]\mathrm{d}t \tag{14}$$

于是,可得到下述定理.

定理 1 对受单面约束 $\phi_\beta=0$ 的 Vacco 系统,如果有限变换群 G_r 的无限小变换式(5)是广义准对称变换,则系统存在 r 个函数独立的第一积分,即

$$(L+\lambda_\beta\phi_\beta)\xi_0^\alpha+\left(\frac{\partial L}{\partial\dot{q}_s}+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right)\xi_s^\alpha+G^\alpha(t,\boldsymbol{q},\dot{\boldsymbol{q}})=C^\alpha\qquad\alpha=1,2,\cdots,r \tag{15}$$

证明 因无限小变换式(5)是广义准对称变换,故有

$$\Delta A+\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)+Q''_s\,\delta q_s\right]\mathrm{d}t=0 \tag{16}$$

式中,$\Delta G=\varepsilon_\alpha G^\alpha$.将式(13)代入式(16),注意到式(2)的第1组方程、ε_α 的独立性及积分区间$[t_1,t_2]$的任意性,得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(L+\lambda_\beta\phi_\beta)\xi_0^\alpha+\left(\frac{\partial L}{\partial\dot{q}_s}+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right)\xi_s^\alpha+G^\alpha\right]=0\qquad\alpha=1,2,\cdots,n \tag{17}$$

或者

$$(L+\lambda_\beta\phi_\beta)\xi_0^\alpha+\left(\frac{\partial L}{\partial\dot{q}_s}+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right)\xi_s^\alpha+G^\alpha=C^\alpha\qquad\alpha=1,2,\cdots,n \tag{18}$$

同理可证对受单面约束 $\phi_\beta>0$ 的 Vacco 系统,如果有限变换群 G_r 的无限小变换式(5)是广义准对称变换,则系统存在 r 个函数独立的第一积分,即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L\xi_0^\alpha+\frac{\partial L}{\partial\dot{q}}\xi_s^\alpha\right)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q''_s\right)\xi_s^\alpha=-\frac{\mathrm{d}}{\mathrm{d}t}G^\alpha\qquad\alpha=1,2,\cdots,n \tag{19}$$

定理 1 可称为受单面约束的 Vacco 系统的 Noether 定理,利用该定理可以由已知对称性求出系统的守恒量.需要指出的是,由于式(1)为不等式,则无限小变换的生成函数需同时满足 2 组方程(17)和(19).

因此,与受双面约束的 Vacco 系统相比较,受单面约束的 Vacco 系统无限小变换生成函数的选择范围变小,守恒量数目也相应减少.

$r=1$,定理 1 退化为如下推论.

推论 1 对于受单面约束的 Vacco 系统(1)、(2),如果有限变换群 G_r 的无限小变换的生成元 ξ_s, ξ_0,及规范函数 G 满足

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[(L+\lambda_\beta\phi_\beta)\xi_0+\left(\frac{\partial L}{\partial\dot{q}_s}+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right)\xi_s\right]-\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q''_s+\right.\\&\qquad\left.\lambda_\beta\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\phi_\beta}{\partial\dot{q}_s}-\frac{\partial\phi_\beta}{\partial q_s}\right)+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right]\xi_s=-\frac{\mathrm{d}}{\mathrm{d}t}G\\&\frac{\mathrm{d}}{\mathrm{d}t}\left(L\xi_0+\frac{\partial L}{\partial\dot{q}_s}\xi_s\right)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q''_s\right)\xi_s=-\frac{\mathrm{d}}{\mathrm{d}t}G\end{aligned}\right\}\qquad\alpha=1,2,\cdots,n \tag{20}$$

则系统存在守恒量

$$(L+\lambda_\beta\phi_\beta)\xi_0+\left(\frac{\partial L}{\partial\dot{q}_s}+\lambda_\beta\frac{\partial\phi_\beta}{\partial\dot{q}_s}\right)\xi_s+G=\mathrm{const} \tag{21}$$

若系统受如下双面 Vacco 约束:

$$\phi_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0\qquad\beta=1,2,\cdots,g \tag{22}$$

则定理 1 退化为如下推论.

推论 2 对于受双面约束的 Vacco 系统,如果有限变换群 G_r 的无限小变换式(5)是广义准对称变换,则系统存在 r 个形如式(15)的线性独立的第一积分.

推论 2 在文献[11]中也曾给出.

如果系统不受约束,则由定理 1 可得如下推论.

推论 3 对于双面理想完整非保守动力学系统,如果有限变换群 G_r 的无限小变换式(5)是广义准对称变换,则系统存在 r 个形如式(15)的线性独立的第一积分.

2 受约束 Vacco 系统的 Noether 逆定理

现在研究根据已知第一积分来寻求相应的无限小广义准对称变换问题.

假设受单面约束的 Vacco 系统(1) 和(2) 有 r 个线性独立的第一积分, 即

$$I^{\alpha}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = C^{\alpha} \qquad \alpha = 1, 2, \cdots, r \tag{23}$$

因此有

$$\frac{\mathrm{d}I^{\alpha}}{\mathrm{d}t} = \frac{\partial I^{\alpha}}{\partial t} + \frac{\partial I^{\alpha}}{\partial q_s}\dot{q}_s + \frac{\partial I^{\alpha}}{\partial \dot{q}_s}\ddot{q}_s \qquad \alpha = 1, 2, \cdots, r \tag{24}$$

将式(2) 的第 1 组方程两端同时乘以 ξ_s^{α} 并对 s 求和, 再将结果与式(24) 相加, 得

$$\frac{\partial I^{\alpha}}{\partial t} + \frac{\partial I^{\alpha}}{\partial q_s}\dot{q}_s + \frac{\partial I^{\alpha}}{\partial \dot{q}_s}\ddot{q}_s + \left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s + \lambda_{\beta}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s} - \frac{\partial \phi_{\beta}}{\partial q_s}\right) + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right]\xi_s^{\alpha} = 0$$
$$\alpha = 1, 2, \cdots, r \tag{25}$$

将式(25) 展开可得

$$\frac{\partial I^{\alpha}}{\partial t} + \frac{\partial I^{\alpha}}{\partial q_s}\dot{q}_s + \frac{\partial I^{\alpha}}{\partial \dot{q}_s}\ddot{q}_s + \left[\frac{\partial^2 L}{\partial \dot{q}_s \partial t} + \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\ddot{q}_k - \frac{\partial L}{\partial q_s} - Q''_s + \right.$$
$$\left.\lambda_{\beta}\left(\frac{\partial^2 \phi_{\beta}}{\partial \dot{q}_s \partial t} + \frac{\partial^2 \phi_{\beta}}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 \phi_{\beta}}{\partial \dot{q}_s \partial \dot{q}_k}\ddot{q}_k - \frac{\partial \phi_{\beta}}{\partial q_s}\right) + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right]\xi_s^{\alpha} = 0 \tag{26}$$

由于式(26) 沿任意轨道都成立, 所以 $\ddot{q}_k$ 的系数应为 0, 即

$$\frac{\partial I^{\alpha}}{\partial \dot{q}_k} - \left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k} + \lambda_{\beta}\frac{\partial^2 \phi_{\beta}}{\partial \dot{q}_s \partial \dot{q}_k}\right)\xi_s^{\alpha} = 0 \qquad s, k = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r \tag{27}$$

假设 $\det(h_{sk}) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k} + \frac{\partial^2 \phi_{\beta}}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0$, 则由式(27) 可得

$$\xi_s^{\alpha} = \bar{h}_{sk}\frac{\partial I^{\alpha}}{\partial \dot{q}_k} \qquad s, k = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r \tag{28}$$

式中, $\bar{h}_{sk}\bar{h}_{kl} = \delta_{sl}$, 若令

$$I^{\alpha} = (L + \lambda_{\beta}\phi_{\beta})\xi_0^{\alpha} + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right)\xi_s^{\alpha} + G^{\alpha} \qquad \alpha = 1, 2, \cdots, r \tag{29}$$

则

$$\xi_0^{\alpha} = (L + \lambda_{\beta}\phi_{\beta})^{-1}\left[I^{\alpha} - \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right)\xi_s^{\alpha} - G^{\alpha}\right] \qquad \alpha = 1, 2, \cdots, r \tag{30}$$

这样在已知系统第一积分 I^{α} 时, 由式(27)、(29) 可确定有限变换群 G_r 的无限小变换式(5).

若把式(28) 代入式(24), 可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(L + \lambda_{\beta}\phi_{\beta})\xi_0^{\alpha} + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right)\xi_s^{\alpha}\right] - \left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s + \right.$$
$$\left.\lambda_{\beta}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s} - \frac{\partial \phi_{\beta}}{\partial q_s}\right) + \lambda_{\beta}\frac{\partial \phi_{\beta}}{\partial \dot{q}_s}\right]\xi_s^{\alpha} = -\frac{\mathrm{d}}{\mathrm{d}t}G^{\alpha} \tag{31}$$

可见, 上面求出的有限变换群 G_r 的无限小变换是系统的广义准对称变换.

3　算　例

设单位质量的质点 Lagrange 函数为

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \tag{32}$$

它的运动受如下单面 Vacco 约束:

$$\phi = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - C \geqslant 0 \tag{33}$$

式中, C 为负常量. 试研究该系统的 Noether 对称性与守恒量.

在运动方程积分之前, 先求出约束乘子 $\lambda = \lambda_0$(常量), 若取生成函数和规范函数为

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_1, \ \xi_2 = \dot{q}_2, \ \xi_3 = \dot{q}_3, \ G = -\frac{1 + \lambda_0}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \tag{34}$$

则有

$$t^* = t, \ q_1^* = q_1 + \varepsilon\dot{q}_1, \ q_2^* = q_2 + \varepsilon\dot{q}_2, \ q_3^* = q_3 + \varepsilon\dot{q}_3 \tag{35}$$

容易验证式(35) 和规范函数 G 满足式(19), 是系统的广义准对称变换. 再由式(15) 得

$$I=\frac{1+\lambda_0}{2}(\dot q_1^2+\dot q_2^2+\dot q_3^2)=\mathrm{const} \tag{36}$$

式(36)正是系统的能量积分.

反之,由式(36)可求出对应的有限变换群 G_r 的无限小变换,由于

$$\frac{\partial I}{\partial \dot q_1}=(1+\lambda_0)\dot q_1,\quad \frac{\partial I}{\partial \dot q_2}=(1+\lambda_0)\dot q_2,\quad \frac{\partial I}{\partial \dot q_3}=(1+\lambda_0)\dot q_3 \tag{37}$$

又

$$\frac{\partial^2 L}{\partial \dot q_s\partial \dot q_k}+\lambda_0\frac{\partial^2 \phi}{\partial \dot q_s\partial \dot q_k}=(1+\lambda_0)\delta_{sk}\qquad s,k=1,2,3 \tag{38}$$

当 $\lambda_0\neq -1$ 时

$$h_{sk}=\frac{1}{1+\lambda_0}\delta_{sk} \tag{39}$$

故由式(27)得

$$\xi_1=\dot q_1,\ \xi_2=\dot q_2,\ \xi_3=\dot q_3 \tag{40}$$

将式(39)代入式(29),有

$$\xi_0=(L+\lambda_0\phi)^{-1}\left[-\frac{1+\lambda_0}{2}(\dot q_1^2+\dot q_2^2+\dot q_3^2)-G\right] \tag{41}$$

取

$$G=-\frac{1+\lambda_0}{2}(\dot q_1^2+\dot q_2^2+\dot q_3^2) \tag{42}$$

则

$$\xi_0=0 \tag{43}$$

将式(43)代入式(39),并考虑到 $\bar\xi_s=\xi_s-\dot q_s\xi_0$,得

$$\bar\xi_1=\dot q_1,\ \bar\xi_2=\dot q_2,\ \bar\xi_3=\dot q_3 \tag{44}$$

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