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Graph-Theoretic Simplicial Complexes, Hajos-type Constructions, and k-Matchings

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DISSERTATION

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fulfillment of the requirements for
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ABSTRACT OF DISSERTATION

Graph-Theoretic Simplicial Complexes, Hajós-type Constructions, and k-Matchings

A graph property is monotone if it is closed under the removal of edges and vertices. Given a graph G and a monotone graph property P, one can associate to the pair (G, P) a simplicial complex, which serves as a way to encode graph properties within faces of a topological space. We study these graph-theoretic simplicial complexes using combinatorial and topological approaches as a way to inform our understanding of the graphs and their properties.

In this dissertation, we study two families of simplicial complexes: (1) neighborhood complexes and (2) k-matching complexes. A neighborhood complex is a simplicial complex of a graph with vertex set the vertices of the graph and facets given by neighborhoods of each vertex of the graph. In 1978, Lovász used neighborhood complexes as a tool for studying lower bounds for the chromatic number of graphs. In Chapter 2, we will prove results about the connectivity of neighborhood complexes in relation to Hajós-type constructions and analyze randomly generated graphs arising from two Hajós-type stochastic algorithms using SageMath. Chapter 3 will focus on k-matching complexes. A k-matching complex of a graph is a simplicial complex with vertex set given by edges of the graph and faces given sets of edges in the graph such that each vertex of the induced graph has degree at most k. We pursue the study of k-matching complexes and investigate 2-matching complexes of wheel graphs and caterpillar graphs.

KEYWORDS: Neighborhood complex, graphs, homotopy, matchings, connectivity

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Dedicated to every beautiful soul.
May you grow to find your passions and live them out loud.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1 Overview

The main goal of topological combinatorics is to study the structure of combinatorial objects using topological tools. In this dissertation, we will focus on the study of graphs and their properties through an associated simplicial complex. Ideally, we can use topological tools to obtain the homotopy type of the simplicial complex, which would lead to a full understanding of the topological properties. When it is infeasible to reach a homotopy type due the complexity of the graph we can deduce information about the space through topological invariants, such as connectivity, homology, and Euler characteristic.

Recall that a finite, simple graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is a pair of elements in $V(G)$. We say two vertices are *adjacent* if they are joined by an edge in the graph and two edges or an edge and a vertex are *incident* if they share a common vertex. Let G and H be two finite, simple graphs. The *neighborhood* $N_G(v)$ of a vertex $v \in G$ is is the subset of vertices $K \subset V(G)$ that are adjacent to v and an element $w \in K$ is called a *neighbor* of G. For a vertex $v \in V(G)$, the *degree of* v, $\deg(v)$ is the number of edges incident to v.

If $V(G) \cap V(H) = \{x\}$, the *wedge sum* $G \underset{x}{\vee} H$ of G with H over x is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let $\{u, v\} \in E(G)$ and w a new vertex not in $V(G)$. The *subdivision of* $\{u, v\} \in G$ is obtained by deleting $\{u, v\}$ and adding w to $V(G)$ and $\{u, w\}, \{w, v\}$ to $E(G)$. A vertex $v \in V(G)$ is a leaf if its neighborhood contains exactly one vertex. The *chromatic number* of a graph G, denoted $\chi(G)$ is the minimum number of colors needed to color the vertices of the graph G such that no two adjacent vertices share the same color.

A *partial order* is a binary relation $\leq$ on a set P that is reflexive ($a \leq a$ for all $a \in P$), antisymmetric (if $a \leq b$ and $b \leq a$ then $a = b$), and transitive (if $a \leq b$ and $b \leq c$ then $a \leq c$). A *partially ordered set (poset)* is any set P equipped with a partial order $\leq$. For elements $a, b \in P$ such that $a \prec b$ we say a is *covered* by b or b covers a if there does not exist an element c such that $a \leq c \leq b$. A *Hasse diagram* $\mathcal{P}$ is a directed graphical representation of a poset P such that the vertex set $V(\mathcal{P}) = V(P)$ and edges represent pairs of elements (a, b) such that $a \prec b$ in P.

1.1 Simplicial Complexes

Definition 1.1.1. An *(abstract) simplicial complex* Δ on a set X is a collection of subsets of X such that

(i) $\emptyset \in \Delta$

(ii) If $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of a simplicial complex are called *faces* and an $n-$ *simplex* is the collection of all subsets of $[n + 1] = \{1, 2, \ldots, n + 1\}$. A subcomplex Γ of a complex Δ is a subcollection of Δ which satisfies (i) and (ii). We will make no distinction between an abstract simplicial complex Δ and an arbitrary geometric realization $|\Delta|$ of Δ as a topological space. The *link* of a vertex v in a simplicial complex X is $\text{link}_X(v) := \{\sigma \in X : v \notin \sigma, \{v\} \cup \sigma \in X\}$. Further, the *star* of a vertex v in X is $\text{star}_X(v) := \{\sigma \in X : v \in \sigma\}$.

For simplicial complexes Δ and Δ', the topological *join* is $\Delta * \Delta' = \{\sigma \cup \sigma' : \sigma \in \Delta, \sigma' \in \Delta'\}$. A simplicial complex Δ is said to be a *cone* with *cone point* $\{x\} \in \Delta$ if for every face $\sigma \in \Delta$ we have $\sigma \cup \{x\} \in \Delta$, that is the simplicial complex $\Delta' * x$ for some Δ'. Note that every cone is contractible. The suspension of a space Δ is denoted $\Sigma(\Delta)$ and is the join of Δ with two discrete points.

A topological space X is called *k-connected* if for every $0 \leq \ell \leq k$, every continuous map from the boundary of $B^{\ell+1}$, the unit ball in $(\ell+1)$-dimensional Euclidean space, into X can be extended to a continuous map from all of $B^{\ell+1}$ to X. Equivalently, the higher homotopy groups $\pi_\ell(X)$ vanish for all dimensions $\ell \leq k$.

Neighborhood Complexes

An influential example of topological combinatorics is Lovász's 1978 work [22] studying lower bounds for chromatic numbers through the construction of neighborhood complexes, answering a long-standing conjecture by Kneser. Over 20 years ealier, Kneser [16] conjectured

Conjecture 1.1.2 (Kneser, 1955)**.** *If we split the n-subsets of a $(2n+k)$-element set into $k+1$ classes, one of the classes will contain two disjoint n-subsets.*

The upperbound of this conjecture can be quickly verified by showing that we can split $(2n+k)$-element set into $k+2$ classes such that any two n-subsets in the same class intersect. Let K_i denote the class of subsets whose first element is i. Then, the sets $K_1, K_2, ..., K_{k+1}$, and $K_{k+2} \cup \cdots \cup K_{k+n+1}$ is a partition of the n-subsets into $k+2$ subsets such that any two n-subsets in the same class intersect.

In 1978, Lovász [22] proved the lower bound also holds. His approach involved constructing the Kneser graph $KG_{n,k}$ to have vertices which are n-subsets of $\{1, 2, \ldots, 2n+k\}$ where an edge exists between two vertices if and only if the n-subsets are disjoint and reinterpreting Conjecture1.1.2 into showing the chromatic number of the Kneser graph $KG_{n,k}$ is $k+2$.

Proving the chromatic number of a graph is a challenge within itself leading Lovász to defining the neighborhood complex of a graph G [22]. For any graph G, the *neighborhood complex of G*, denoted $\mathcal{N}(G)$, is the simplicial complex with vertex set $V(G)$ and facets given by $N_G(v)$ for all $v \in V(G)$, where $N_G(v)$ denotes the neighbors of v in G (not including v).

Example 1.1.3 (Neighborhood complex)**.** Given the graph on the left of Figure 1.1 there are three neighborhoods $\{1,3\}$, $\{2,3,4\}$, and $\{1,2,4\}$ which can be seen as maximal faces in the neighborhood complex, depicted to the right.

Lovász used the neighborhood complex $\mathcal{N}(G)$ along with Borsuk-Ulam's Theorem to provide a general lower bound for the chromatic number of graphs and a sharp lower bound for the chromatic number of the Kneser graphs.

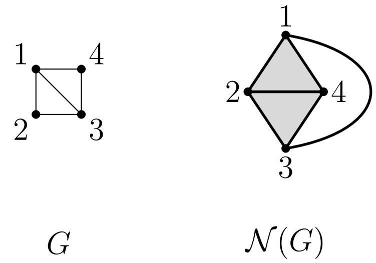

Figure 1.1

Theorem 1.1.4 (Lovász [22])**.** *If $\mathcal{N}(G)$ is k-connected, then $\chi(G) \geq k+3$.*

There are several famous families of graphs, e.g. Kneser and stable Kneser graphs, for which these topological lower bounds (or equivalent techniques) yield the only known proofs of their chromatic numbers. We note here that $\mathcal{N}(G)$ is 0-connected if and only if it is path-connected, and being 0-connected implies having chromatic number greater than 2. For connected bipartite graphs, having a disconnected $\mathcal{N}(G)$ characterizes this family. The following proposition justifies our assumptions throughout this work that when G is connected with $\chi(G) \geq 3$, $\mathcal{N}(G)$ is path-connected.

Proposition 1.1.5. *The complex $\mathcal{N}(G)$ is path-connected if and only if G is connected and not bipartite.*

Proof. If G is not connected, then it is immediate that $\mathcal{N}(G)$ is not connected. Let G be a connected bipartite graph with bipartition $V(G) = A \uplus B$, where $\uplus$ denotes disjoint union. For all $a \in A, N(a) \subseteq B$ and for all $b \in B, N(b) \subseteq A$. Therefore, the neighborhood complex induced by B is disjoint from the neighborhood complex induced by A and $\mathcal{N}(G) = \mathcal{N}(A) \uplus \mathcal{N}(B)$, hence $\mathcal{N}(G)$ is not connected.

Suppose now that G is connected and not bipartite, so there exists an odd cycle $C = c_0, c_1, ..., c_n$ in G. We prove that between any two vertices $x, y \in V(G)$ there is a walk of even length, from which it follows that x and y are connected by a path in $\mathcal{N}(G)$. Since G is connected, there exists a walk W_x from x to some $c_x \in C$, and a walk W_y from y to some $c_y \in C$. Since C has odd length, there exists a walk C_{odd} in C of odd length from c_x to c_y, and there also exists a walk C_{even} in C of even length from c_x to c_y. For any given even/odd parities of W_x and W_y, one can connect x and y by a path of even length that starts with W_x, continues through either C_{odd} or C_{even}, and concludes with W_y. Thus, x and y are path-connected in $\mathcal{N}(G)$. $\square$

In Chapter 2 we will consider the relationship between connectivity and neighborhood complexes. In general, when $\mathcal{N}(G)$ is k-connected it is possible for there to be arbitrarily large gaps between $\chi(G)$ the chromatic number of G and $k+3$. Matoušek and Ziegler [Remark (H1) in [24]] make a remark that implies that if G does not contain a 4-cycle, then $\mathcal{N}(G)$ is at most 0-connected. Thus, the neighborhood complex of a graph where the girth, i.e. the length of the shortest cycle, is greater than 4 is not 1-connected.

Matching Complexes

Since the influential work of Lovász, interest in graph-theoretic simplicial complexes continued to rise. Another such complex of interest is a matching complex. A *matching complex* of a graph G, denoted $M_1(G)$, is a simplicial complex with vertices given by edges of G and faces given by matchings of G, where a matching is a subset of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most 1. Some matching complexes that have been studied in detail are the full matching complex $M_1(K_n)$, where K_n is the complete graph on n vertices, and the chessboard complex $M_1(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph with block size m and n. The original motivation for the full Matching complex was established through the work of Brown and Quillen to study the structure of the order complex of the non-trivial abelian subgroups of G generated by transpositions [3, 4, 25].

Results about $M_1(K_n)$ and $M_1(K_{m,n})$ include connectivity bounds and rational homology. For a general survey on matching complexes see [33]. The homotopy type of matching complexes is a bit more mysterious. The homotopy type of matching complexes for paths and cycles [18], for forests [23], and for the $\left(\lfloor \frac{n+m+1}{3} - 1 \rfloor\right)$ skeleton of $M_1(K_{m,n})$ for all m, n and $M_1(K_{m,n})$ when $2m - 1 \leq n$ [34] is known to be either a point, sphere, or wedge of spheres, but beyond these classes the homotopy type of matching complexes is unclear. In further analysis of the full matching complex and, analogously, the chessboard complex it was discovered that 3-torsion appeared in higher homology groups of both complexes [28]. Leading one to question, what graph structures give rise to torsion in matching complexes?

In [14], Jonsson defines the bounded degree complex $BD_n^\lambda(G)$ with a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the complex of subgraphs of a graph G with n vertices such that the degree of vertex x_i is at most λ_i, which is a natural generalization of matching complexes. When $\lambda = (d, \ldots, d)$ we write $BD_n^d(G) := BD_n^{(d,d,..,d)}(G)$. The bounded degree complex $BD_n^1(K_n)$ is the matching complex on complete graphs, that is $M_1(K_n)$. Jonsson primarily focuses on the connectivity of $BD_n^\lambda(K_n)$ considering the outcome for graphs with and without loops. For $d \geq 2$, $BD_n^d(G)$ is the *d-matching complex* on G with vertices given by edges in G and faces by d-matchings in G, where a d-matching is a subset of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most d. Bounded degree complexes are generalizations of matching complexes that involve relaxing the incidence conditions on the vertices. Using this more general family of complexes we can learn about matching complexes. For example, in Section 3.5 we use bounded degree complexes to inductively study k-matching complexes. For a further survey of bounded degree complexes see [33].

In Section 3.2, we connect our results to the connectivity results in [14]. The focus of Chapter 3 will be the topology of $M_2(G) := BD_n^2(G)$, the 2-matching complex of G. Since a matching of G is also a 2-matching of G, the matching complex of G is a subcomplex of the 2-matching complex of G, with $M_1(G) \subset M_2(G)$.

Definition 1.1.6. A *2-matching complex* of a graph G, denoted $M_2(G)$ is a simplicial complex with vertices given by edges of G and faces subsets of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most 2.

Example 1.1.7. See Figure 1.2 consisting of the graph G, its matching complex $M_1(G)$, and its 2-matching complex $M_2(G)$. The 2-matching complex of G consists of 5 maximal faces. Namely, (1) **{a,c,d}**, (2) **{a,c,e}** (3) **{b,c,d}**, (4) **{b,c,e}**, (5) **{a,b,d,e}**. Note that $M_1(G)$ is homotopy equivalent to $S^0 \vee S^1$, $M_2(G)$ is homotopy equivalent to S^2 a 2-sphere, and that $M_1(G) \subseteq M_2(G)$.

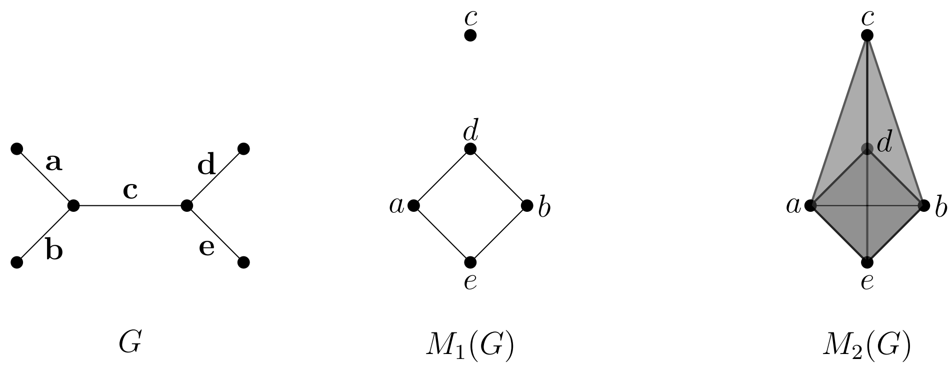

Figure 1.2: A graph G with its 1-matching and 2-matching complex.

The rest of the overview provides the relevant background on the tools that we will use with a primary focus on discrete Morse theory and the Matching Tree Algorithm.

1.2 Discrete Morse Theory

Discrete Morse theory was first developed by R. Forman [6] and has since become a powerful tool for topological combinatorialists. The main idea of the theory is to pair faces within a simplicial complex in such a way that we obtain a sequence of collapses yielding a homotopy equivalent cell complex.

Definition 1.2.1. A *partial matching* in a poset P is a partial matching in the underlying graph of the Hasse diagram of P, i.e., it is a subset $M \subseteq P \times P$ such that

- $(a, b) \in M$ implies $b \succ a$; i.e. $a < b$ and no c satisifies $a < c < b$.
- each $a \in P$ belongs to at most one element in M.

When $(a, b) \in M$, we write $a = d(b)$ and $b = u(a)$.

A partial matching on P is called *acyclic* if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \prec u(a_m) \succ a_1$$

with $m \geq 2$ and all $a_i \in P$ being distinct.

Given an acyclic partial matching M on a poset P, an element c is *critical* if it is unmatched. If every element is matched by M, M is called *perfect*. We are now able to state the main theorem of discrete Morse theory as given in [18, Theorem 11.13].

Theorem 1.2.2 (Kozlov [18])**.** *Let Δ be a polyhedral cell complex and let M be an acyclic matching on the face poset of Δ. Let c_i denote the number of critical i-dimensional cells of Δ. The space Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single 0-dimensional cell in the case where the empty set is paired in the matching.*

The following theorem shows there is an intimate relationship between linear extensions and acyclic matchings [18].

Theorem 1.2.3 (Kozlov, Theorem 11.2)**.** *A partial matching on a poset P is acyclic if and only if there exists a linear extension of $\mathcal{L}$ of P such that x and $u(x)$ follow consecutively.*

Since x and $u(x)$ follow consecutively in the linear extension, when we refer to these elements in the linear extension we will use the notation $(x, u(x))$ and consider them as a pair of consecutive elements in the poset.

It is often useful to create acyclic partial matchings on different sections of the face poset of a simplicial complex and then combine them to form a larger acyclic partial matching on the entire poset. This process is detailed in the following theorem known as the *Cluster Lemma* in [14] and the *Patchwork Theorem* in [18].

Theorem 1.2.4 ([14], [18])**.** *Assume that $\varphi : P \to Q$ is an order-preserving map. For any collection of acyclic matchings on the subposets $\varphi^{-1}(q)$ for $q \in Q$, the union of these matchings is itself an acyclic matching on P.*

A common way to obtain an acyclic matching is to *toggle* on an element x in the vertex set of a face poset P.

Definition 1.2.5. Let P be the face poset of a simplicial complex Δ and $Q \subseteq P$ a subposet. *Toggling* on an element x in the vertex set of Q is a partial matching that pairs subsets $a \in Q$, $x \notin a$ with $a \cup \{x\}$, whenever possible.

Lemma 1.2.6. *Toggling provides an acyclic partial matching.*

Proof. To see this suppose we toggle on the element x. Start with an element $a_1 \in P$ such that $x \notin a_1$, $x \in u(a_1)$. Any element $a_2 \prec u(a_1)$ with $a_2 \neq a_1$ contains x since $(a_1, u(a_1)) \in M$. Hence, there is no element $u(a_2)$, and a cycle cannot be created. □

Additionally, using the patchwork theorem, we see that performing repeated toggling yields an acyclic matching.

Lemma 1.2.7. *Let P be a poset with vertex set $V(P)$ and $\{x_1, x_2, ..., x_n\}$ with $x_i \in V(P)$ for all i be a sequence of toggling elements of P. Repeatedly toggling on x_1, then x_2 and so on, yields an acyclic matching on P.*

Proof. Let Q be a poset with elements $\mathcal{R}$ and Y_i for $i \in [n]$ with relations given by $Y_i \prec Y_{i+1}$ for all $i \in [n-1]$ and $Y_n \prec \mathcal{R}$. Recursively define $D_i := \{\alpha \in P | \{x_i\} \in \alpha \text{ or } \alpha \cup \{x_i\} \in P \text{ and } \alpha \notin D_j \text{ for } j \leq i-1\}$. Define $\varphi : P \to Q$ by $\varphi^{-1}(Y_1) := D_1 \cup \{x_1\}$, $\varphi^{-1}(Y_i) := D_i$, for $2 \leq i \leq n$, and the remaining elements to $\mathcal{R}$. The map φ is well-defined and order-preserving. On $\varphi^{-1}(Y_i)$ toggle on x_i, which is an acyclic matching by Lemma 1.2.6. The union of which forms an acyclic matching on P by Theorem 1.2.4. □

1.3 Matching Tree Algorithm (MTA)

In [1], the authors detail the Matching Tree Algorithm which provides an acyclic discrete Morse matching on the face poset of an independence complex of a graph G. An *independence complex* $Ind(G)$ of a graph G is a simplicial complex in which the vertices are given by vertices of G and faces are given by independent sets of vertices. The matching complex of a graph G is equal to the independence complex of the *line graph* of G where the vertices of the line graph are the edges of the graph and two vertices are adjacent if the corresponding edges are incident in the graph. In Section 3.4, we will use the Matching Tree Algorithm to find the homotopy type of the 1-matching complex of a wheel graph, by looking at the independence complex of the line graph.

Let G be a simple graph with vertex set $V = V(G)$. Bousquet-Mélou, Linusson, and Nevo motivate the MTA with the following algorithm. Let Σ denote the independence complex of G. Take a vertex $p \in V$ and let $N(p)$ denote the set of its neighbors. Define $\Delta = \{I \in \Sigma : I \cap N(p) = \emptyset\}$. For $I \in \Delta$ and $p \notin I$, the set of pairs $(I, I \cup \{p\})$ form a perfect matching of Δ and hence a matching of Σ. The vertex p is called a *pivot*.

Notice that the unmatched elements of Σ are those containing at least one element of $N(p)$. Choose an unmatched vertex and continue the process as many times as possible. This algorithm will give rise to a rooted tree, called a *matching tree of* Σ, whose nodes represent sets of unmatched elements. Some of the nodes are reduced to the empty set, and all others are of the form

$$\Sigma(A, B) = \{I \in \Sigma : A \subseteq I \text{ and } B \cap I = \emptyset\},$$

where

$$A \cap B = \emptyset \text{ and } N(A) := \bigcup_{a \in A} N(a) \subseteq B.$$

The root of the tree is $\Sigma(\emptyset, \emptyset)$, which is equal to the set of all the independent sets of G. As we traverse the tree the sets $\Sigma(A, B)$ will become smaller and the leaves of the tree will have cardinality 0 or 1.

The following presentation of the Matching Tree Algorithm follows [10]. Begin with the root node $\Sigma(\emptyset, \emptyset)$ and at each node $\Sigma(A, B)$ where $A \cup B \neq V$ apply the following procedure:

(1) If there is a vertex $v \in V \setminus (A \cup B)$ such that $N(v) \setminus (A \cup B) = \emptyset$, then v is called a *free vertex*. Give $\Sigma(A, B)$ a single child labeled $\emptyset$.

(2) Otherwise, if there is a vertex $v \in V \smallsetminus (A \cup B)$ such that $N(v) \smallsetminus (A \cup B)$ is a single vertex w, then v is called a *pivot* and w a *matching vertex*. Give $\Sigma(A, B)$ a single child labeled $\Sigma(A \cup \{w\}, B \cup N(w))$.

(3) When there is no vertex that satisfies (1) or (2) and $A \cup B \neq V$, choose a *tentative pivot* in $V' = V \smallsetminus (A \cup B)$ and give $\Sigma(A, B)$ two children $\Sigma(A \cup \{v\}, B \cup N(v))$, which we call the *right child*, and $\Sigma(A, B \cup \{v\})$, which we call the *left child*.

Remark 1.3.1. Step (3) is motivated by the observation that if v has at least two neighbors, say w and w' then some of the unmatched sets I contain w, and some others don't, but if they do not contain w than they must contain w'.

The following theorem is the main theorem for the Matching Tree Algorithm, which is due to Bousquet-Mélou, Linusson, and Nevo [1], but is stated as it appears in Braun and Hough [2].

Theorem 1.3.2. *A matching tree for G yields an acyclic partial matching on the face poset of $Ind(G)$ whose critical cells are given by the non-empty sets $\Sigma(A, B)$ labeling non-root leaves of the matching tree. In particular, for each set $\Sigma(A, B)$, the set A yields a critical cell in Ind(G).*

Thus far, we have provided combinatorial tools for determining the homotopy type of simplicial complexes. It is also possible to use more topological methods to approach homotopy type. As we will see in Section 3.5 this approach requires inductively determining the homotopy type of complexes of interest and appropriately "gluing" these spaces over a common subspace. For a more detailed discussion see [8, Section 4.G] and [18, Section 15.2]. The following lemma follows from [8, Proposition 4G.1], where $X \vee Y$ is considered as a homotopy colimit.

Lemma 1.3.3. *Let X and Y be two spaces such that $X \simeq_f X'$ and $Y \simeq_g Y'$, then $X \vee Y \simeq X' \vee Y'$.*

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Chapter 2 Hajós-type Constructions and Neighborhood Complexes

2.1 Introduction

Given a finite graph G, recall that the neighborhood complex $\mathcal{N}(G)$ is the simplicial complex with vertex set $V(G)$ and facets given by neighborhoods of vertices $v \in V(G)$. In this chapter, we investigate the interaction of the topology of neighborhood complexes and k-constructibility. The constructibility we are interested in results from the operations of *Hajós merge*, $G_1 \Delta_H G_2$, applied to two disjoint graphs G_1 and G_2, and of *vertex identification*, $\operatorname{vid}(G, L)$, applied to a graph G and a list of pairs of nonadjacent vertices in G.

Definition 2.1.1. A graph is called *Hajós k-constructible* if it is a complete graph K_k or if it can be constructed from K_k by successive applications of the following two operations:

- (Hajós Merge) If G_1 and G_2 are already-obtained disjoint graphs, then to the disjoint union $G_1 \uplus G_2$ remove an edge (x_1, y_1) from G_1 and an edge (x_2, y_2) from G_2, identify x_1 with x_2, and add the edge (y_1, y_2). We abuse notation and denote the resulting graph $G_1 \Delta_H G_2$.

- (Vertex Identification) Identify two nonadjacent vertices in an already-obtained graph H, where we ignore the presence of multiple edges. If L is a list of pairs of nonadjacent vertices in G, then $\operatorname{vid}(G, L)$ is the graph obtained by identifying all those pairs of vertices.

Any construction of a graph using this process will be called a *Hajós construction*. An example of a Hajós merge is given in Figure 2.1. One can verify that $\chi(G_1 \Delta_H G_2) \geq \min(\chi(G_1), \chi(G_2))$, and that $\chi(\operatorname{vid}(G, L)) \geq \chi(G)$ showing if G is Hajós k-constructible, then $\chi(G) \geq k$.

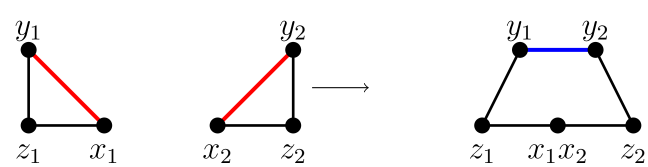

Figure 2.1: An example of a Hajós merge, $K_3 \Delta_H K_3$.

Our two main results show that applying one of a broad family of Hajós merges or one of a large class of vertex identifications results in at least one copy of S^1 as a wedge summand of the neighborhood complex of the resulting graph. Specifically:

Theorem 2.3.3. For two connected graphs G_1 and G_2 with edges (x_1, y_1) and (x_2, y_2) respectively such that either

1. $\chi(G_1), \chi(G_2) \geq 3$ and neither (x_1, y_1) nor (x_2, y_2) is a bridge or

2. $\chi(G_1) \geq 3$, $\chi(G_2) \geq 4$, and (x_2, y_2) is not a bridge,

$\mathcal{N}(G_1 \Delta_H G_2)$ is homotopy equivalent to a wedge of at least one copy of S^1 with another space.

Theorem 2.3.5. Let G be a connected graph that is not bipartite with $v, w \in V(G)$ such that $\mathrm{d}(v, w) \geq 5$ and $G' := \mathrm{vid}(G, [v, w])$ with the resulting identified vertex denoted vw. Then $\mathcal{N}(G') \simeq \mathcal{N}(G) \bigvee S^1 \bigvee S^1$.

If $\mathcal{N}(G)$ includes an S^1-wedge summand, then the rank of the first homology group of $\mathcal{N}(G)$, called the first Betti number of $\mathcal{N}(G)$, is positive. In contrast to these results, we prove Theorem 2.3.2 and Theorem 2.3.3, in which we show that for a restricted class of vertex identifications using vertices at distance less than five, the first Betti number of the neighborhood complex does not increase. In Theorem 2.3.5, we prove that arbitrarily large decreases in the first Betti number can result from a single vertex identification.

By our above results, if G is constructible where the final step is one of these Hajós merges or a vertex identification using vertices with distance at least five, then $\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G))) \geq 1$. It is interesting to compare this fact with the following theorem of Kahle regarding neighborhood complexes of random graphs. Recall that the random graph $G(n, p)$ denotes the probability space of all graphs on a labeled vertex set of size n with each edge inserted independently with probability p.

Theorem 2.1.2 (Kahle [15])**.** *If $p = 1/2$ and $\epsilon > 0$ then almost always $\widetilde{H}_\ell(\mathcal{N}(G(n, p))) = 0$ for $\ell \leq (1 - \epsilon) \log_2(n)$.*

From this perspective, we expect neighborhood complexes of graphs to have trivial first Betti numbers. Thus, we infer that a "typical" Hajós construction of a graph will end in a vertex identification using two vertices that are at distance less than or equal to four.

We conclude this chapter with a discussion of the results of computational experiments involving different approaches to randomly generating Hajós-type constructions. In particular, we define two stochastic algorithms for generating graphs of chromatic number at least k and use these algorithms to analyze the number of vertices and edges in the resulting graphs and their first Betti numbers.

Motivation

Proper graph colorings are of great interest in combinatorics and the study of chromatic numbers for graphs is a frequent focus. While coarse bounds on the chromatic number can be found easily, determining the chromatic number is NP-complete. In 1978, Lovász advanced the study of chromatic numbers through the construction of the neighborhood complex $\mathcal{N}(G)$ of a graph G. Intuitively, $\mathcal{N}(G)$ is capturing the relations of the vertices with their neighbors and the topological connectivity of $\mathcal{N}(G)$ is measuring the complexity of continuous deformations of the neighborhoods in the graph. Lovász [22] proved that the topological connectivity of $\mathcal{N}(G)$ gives a general lower bound for the chromatic number of G, and then showed that this provides a sharp lower bound for Kneser graphs. Since this original result, there has been steady development regarding our understanding of neighborhood complexes of graphs and various topological lower bounds for chromatic numbers [18].

Another area of interest related to graph colorings is the characterization of k-chromatic graphs, i.e. graphs with chromatic number k. In 1961, Hajós [7] characterized graphs with chromatic number at least k through the concept of a k-constructible subgraph, showing that if if $\chi(G) \geq k$, then G contains a Hajós k-constructible subgraph. Urquhart later strengthened Hajós' result.

Theorem 2.1.3 (Urquhart [31])**.** *For every $3 \leq t \leq k$, every k-chromatic graph is t-constructible.*

In the proof, Urquhart describes a Hajós-type construction which involves an "Ore merge," an operation that involves the Hajós merge followed by a restricted series of vertex identifications.

This theorem has been the subject of continued investigation in recent years, including connections to computation complexity [9, 21], explicit Hajós constructions [12, 13, 31], and extensions [11, 20]. Also, the end behavior of Hajós constructions was considered for k-critical graphs, where a k-chromatic graph is *k-critical* if for every proper subgraph H there exists some $j < k$ such that H is j-chromatic. Theorem 2.1.3 implies that k-critical graphs are k-constructible. Jensen and Royle [12] proved the existence of k-critical graphs that do not have a Hajós sequence consisting of exclusively k-critical graphs.

Theorem 2.1.4 (Jensen and Royle [12])**.** *For every $k \geq 4$ there exists a k-critical graph that allows no Hajós k-construction where all intermediary graphs are k-critical.*

The proof involves finding graphs that satisfy the three specifications in the proposition below for $k \geq 4, k \neq 8$, which forces some structure on the end behavior of certain Hajós constructions.

Proposition 2.1.5 (Jensen and Royle [12])**.** *If G*

1. *is 3-connected,*
2. *has chromatic number at least k, and*
3. *for every $v \in V(G)$ and every pair $u_1, u_2 \in N_G(v)$ there exists a $(k-1)$-coloring φ of $G - v$ such that $\varphi(w) \neq \varphi(u_i)$ for all $w \in N_G(v) \setminus \{u_i\}, i = 1, 2,$*

then G is a k-critical graph such that the last step of any possible Hajós k-construction of G consists of a vertex identification on a graph that is not critical.

The proof of this result involved exploring the operations necessary for the final steps of the Hajós construction sequence, showing that the key to constructing k-critical graphs is to end in a vertex identification.

Building on these narratives, we are led to ask how the connectivity of the neighborhood complex of a given graph is affected by the Hajós merge and vertex identification operations. Further, we ask what end behavior of a Hajós construction sequence is necessary to obtain a graph with a highly (topologically) connected neighborhood complex.

We explore the topological effects of Hajós merges and vertex identifications in Section 2.2, providing insight into restrictions on Hajós-type constructions for graphs with highly connected neighborhood complexes. In Section 2.3 we investigate the impact of "short-distance" identifications on the first Betti number of $\mathcal{N}(G)$. In Section 2.4, we briefly investigate the topological effects of DHGO compositions, a generalization of the Hajós merge. Finally, in Section 2.5, we introduce two graph construction algorithms based on different Hajós-type constructions, discuss the outcomes of computational experiments using these, and conclude with open problems.

2.2 Topological effects of Hajós-type operations

In this section, we investigate the effects of Hajós merges and vertex identifications on $\mathcal{N}(G)$. Our main results are Theorem 2.2.3, Theorem 2.2.5, and Corollary 2.2.6, in which we prove that a broad family of Hajós merges and vertex identifications with distance at least five result in S^1-wedge summands in the neighborhood complex. We assume throughout this section that G_1 and G_2 are connected graphs with chromatic number at least 3. Recall that a *bridge* in a graph G is an edge whose deletion increases the number of connected components of G.

Lemma 2.2.1. *$G_1 \Delta_H G_2$ has a bridge if at least one of the edges used in the Hajós merge is a bridge.*

Proof. Suppose that G_1 and G_2 are two connected graphs with edges (x_1, y_1) and (x_2, y_2) respectively. Suppose that under the Hajós construction y_1 and y_2 get identified as y_1y_2, while (x_1, y_1) and (x_2, y_2) get deleted and (x_1, x_2) is added to create $G_1 \Delta_H G_2$. Let $G_i' := G_i \setminus (x_i, y_i)$. Suppose that (x_1, y_1) is a bridge in G_1 such that $G_1' := A \uplus B$, then the Hajos merge produces a graph with (x_1, x_2) a bridge between A and $\text{vid}(G_2' \uplus B, [y_1, y_2])$. □

Lemma 2.2.2. *Let G be a connected non-bipartite graph such that $G' := G \setminus (x, y)$ is connected bipartite. Then, $\mathcal{N}(G') = A \uplus B$ with A and B path-connected, where $N_{G'}(x), N_{G'}(y) \subseteq A$ and $x, y \in B$.*

Proof. Since the deletion of the edge (x, y) results in a bipartite graph, (x, y) must be part of all the odd cycles and no even cycles in G. Hence, by the connectivity of G', there must be an even path p from x to y and no such odd path. Therefore, there is no even path from y to any neighbor $v \in N_{G'}(x)$ or from x to any neighbor $w \in N_{G'}(y)$, which implies y and $N_{G'}(x)$ are not in the same connected component.

Similarly for x and $N_{G'}(y)$. In addition, notice that x and $N_{G'}(x)$ cannot be in the same component because they would need to be connected by an even path in G' creating an odd cycle, a contradiction to G' being bipartite. Similarly for y and $N_{G'}(y)$. Since a connected bipartite graph gives rise to a neighborhood complex with two connected components, the result follows. $\square$

Theorem 2.2.3. *For two connected graphs G_1 and G_2 with edges (x_1, y_1) and (x_2, y_2) respectively such that either*

1. $\chi(G_1) \geq 3$*,* $\chi(G_2) \geq 4$*, and (x_2, y_2) is not a bridge, or*
2. $\chi(G_1), \chi(G_2) \geq 3$ *and neither (x_1, y_1) nor (x_2, y_2) is a bridge.*

Then $\mathcal{N}(G_1 \Delta_H G_2)$ is homotopy equivalent to a wedge of at least one copy of S^1 with another space.

Proof. Suppose under the Hajós construction y_1 and y_2 get identified as y_1y_2, while (x_1, y_1) and (x_2, y_2) get deleted and (x_1, x_2) is added to create $G_1 \Delta_H G_2$. Let $G_i' := G_i \setminus (x_i, y_i)$.

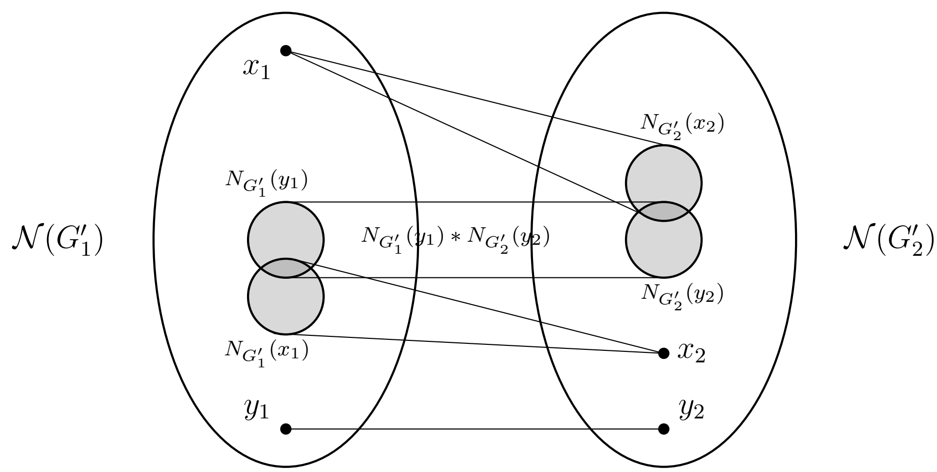

Figure 2.2: Building $\mathcal{N}(G_1 \Delta_H G_2)$ as in the proof of Theorem 2.2.3.

First, consider the effect of identifying y_1 and y_2. Since $y_1 \in \mathcal{N}(G_1')$ and $y_2 \in \mathcal{N}(G_2')$ are elements in separate connected components, merging the two vertices in the disjoint union $\mathcal{N}(G_1') \uplus \mathcal{N}(G_2')$ is homotopy equivalent to attaching a 1-cell between y_1 and y_2, as shown in Figure 2.2. In addition, merging y_1 and y_2 has the effect of joining the faces $N_{G_1'}(y_1)$ and $N_{G_2'}(y_2)$ in $\mathcal{N}(G_1') \uplus \mathcal{N}(G_2')$, which adds $N_{G_1'}(y_1) * N_{G_2'}(y_2)$ to $\mathcal{N}(G_1') \uplus \mathcal{N}(G_2')$. Since $N_{G_1'}(y_1) \cap N_{G_2'}(y_2) = \emptyset$, this join connects a face of $\mathcal{N}(G_1')$ to a face of $\mathcal{N}(G_2')$ through a contractible join which is homotopy equivalent to attaching a 1-cell between the contractions of $N_{G_1'}(y_1)$ and $N_{G_2'}(y_2)$, respectively.

Now, we consider the effect of adding edge (x_1, x_2). Notice $N_{G_1'}(x_1) \in \mathcal{N}(G_1')$ and $x_2 \in \mathcal{N}(G_2')$, so adding the edge (x_1, x_2) has the effect of coning $N_{G_1'}(x_1)$ over

x_2. Since $N_{G_1'}(x_1)$ is contractible, this is homotopy equivalent to attaching a 1-cell between x_2 and the contraction of $N_{G_1'}(x_1)$. The analysis is similar for $N_{G_2'}(x_2)$ and x_1, as shown in Figure 2.2.

It remains to show that there is at least one S^1 wedge summand in our resulting neighborhood complex. We consider two cases.

Case 1: Assume $\chi(G_1) \geq 3$ and $\chi(G_2) \geq 4$, with (x_2, y_2) not a bridge. Thus, G_2' is connected and $\chi(G_2') \geq 3$. If $\chi(G_1') \geq 3$ and G_1' is connected, then there exists a path in the resulting complex from y_1 to y_2 that lies outside of the 1-simplex connecting y_1 and y_2, and thus we can deform the attaching map for the 1-cell between y_1 and y_2 along this path to create our S^1.

If $\chi(G_1') \geq 3$ and G_1' is not connected, then G_1' is a disjoint union of two connected graphs, call them X and Y, where $\{x_1\}$ and $N_{G_1'}(x_1)$ are both in X and $\{y_1\}$ and $N_{G_1'}(y_1)$ are both in Y. Since $\chi(G_1') \geq 3$ and X and Y are not connected by a bridge in G_1', it is impossible to have $\chi(X) = 2 = \chi(Y)$. Thus, without loss of generality $\chi(X) \geq 3$ and hence $\mathcal{N}(X)$ is path-connected. Thus, there exists a continuous path in $\mathcal{N}(X)$ from x_1 to $N_{G_1'}(x_1)$, and deforming the cone from x_1 to $N_{G_2'}(x_2)$ along this path results in an S^1 summand.

The only remaining case is that $\chi(G_1') = 2$ and G_1' is connected. Note that G_1' cannot be disconnected, as otherwise adding the edge (x_1, y_1) to G_1' would result in $\chi(G_1) = 2$, a contradiction. By Lemma 2.2.2, at least one of the connected components of $\mathcal{N}(G_1')$ must contain at least two elements from the set $\{\{x_1\}, \{y_1\}, N_{G_1'}(x_1), N_{G_1'}(y_1)\}$. By using a path homotopy argument similar to that used above, this implies the existence of an S^1 summand in the neighborhood complex.

Case 2: Assume $\chi(G_1) \geq 3$ and $\chi(G_2) \geq 3$, with (x_1, y_1) and (x_2, y_2) not bridges. Our assumption implies that both G_1' and G_2' are connected. Thus, if both $\chi(G_1') \geq 3$ and $\chi(G_2') \geq 3$, then both $\mathcal{N}(G_1')$ and $\mathcal{N}(G_2')$ are path-connected, and a similar homotopy argument to the ones in the previous case implies the existence of an S^1-wedge summand.

Next, suppose without loss of generality that $\chi(G_2') \geq 3$ and $\chi(G_1') = 2$. By Lemma 2.2.2, at least one of the connected components of $\mathcal{N}(G_1')$ must contain at least two elements from the set $\{\{x_1\}, \{y_1\}, N_{G_1'}(x_1), N_{G_1'}(y_1)\}$. By using a path homotopy argument similar to that used above, this implies the existence of an S^1 summand in the neighborhood complex

Finally, suppose that $\chi(G_1') = \chi(G_2') = 2$, and thus Lemma 2.2.2 implies that $\mathcal{N}(G_1')$ and $\mathcal{N}(G_2')$ each have two path-connected components. Let $\mathcal{N}(G_1') = A \uplus B$ and $\mathcal{N}(G_2') = C \uplus D$. From Lemma 2.2.2 we can assume $x_1, y_1 \in A$; $N_{G_1'}(x_1), N_{G_1'}(y_1) \in B$; $x_2, y_2 \in C$; and $N_{G_2'}(x_2), N_{G_2'}(y_2) \in D$. It follows that, up to homotopy equivalence, the Hajós construction attaches 1-simplices between C and A, A and D, D and B, and B and C. Thus, we have found that $\mathcal{N}(G_1 \Delta_H G_2)$ is homotopy equivalent to the space obtained by starting with $\mathcal{N}(G_1') \uplus \mathcal{N}(G_2')$, contracting each of the faces formed by $N_{G_1'}(x_1)$, $N_{G_2'}(x_2)$, $N_{G_1'}(y_1)$, and $N_{G_2'}(y_2)$ to a point and then attaching 1-cells as indicated above. Note that if any of these faces intersect, then contracting their union leads to the same structure. Attaching the 1-cells between $\mathcal{N}(G_1') \uplus \mathcal{N}(G_2')$ as indicated creates a wedge summand of at least one copy of S^1. $\square$

Suppose v and w are vertices in a graph G and recall that the *distance* $\mathrm{d}_G(v,w)$ denotes the minimum number of edges in a path from v to w, where $\mathrm{d}_G(v,w)=\infty$ if v and w are not in the same component of G. When G is clear, we will write $\mathrm{d}(v,w)$ for $\mathrm{d}_G(v,w)$.

Lemma 2.2.4. *Let G be a graph with vertices v and w such that $\mathrm{d}(v,w)\geq 5$ and $G' := \mathrm{vid}(G,[v,w])$ with the resulting identified vertex denoted vw. If $\sigma \subseteq N_{G'}(vw)$ with $\sigma \cap N_G(v) \neq \emptyset$ and $\sigma \cap N_G(w) \neq \emptyset$, then $\sigma \notin \mathcal{N}(G)$.*

Proof. Suppose that $\sigma \in \mathcal{N}(G)$. Define $N_G(v)\cap\sigma$ to be $\{v_1,\ldots,v_m\}$ and $N_G(w)\cap\sigma$ to be $\{w_1,\ldots,w_\ell\}$. Since $N_G(v)\cap N_G(w)=\emptyset$, there exists a vertex $x\in G$ such that $\{v_1,\ldots,v_m,w_1,\ldots,w_\ell\}\cup\sigma \subseteq N_G(x)$. In particular, there exists $v_i\in\{v_1,\ldots,v_m\}$ such that $\mathrm{d}(v_i,w_j)=2$ for some $w_j\in\{w_1,\ldots,w_\ell\}$. This implies $\mathrm{d}(v,w)\leq \mathrm{d}(v,v_i)+\mathrm{d}(v_i,w_j)+\mathrm{d}(w_j,w)=2+1+1=4$, a contradiction to $\mathrm{d}(v,w)\geq 5$. □

Theorem 2.2.5. *Let G be a connected graph that is not bipartite with $v,w\in V(G)$ such that $\mathrm{d}(v,w)\geq 5$ and $G' := \mathrm{vid}(G,[v,w])$ with the resulting identified vertex denoted vw. Then $\mathcal{N}(G')\simeq\mathcal{N}(G)\bigvee S^1\bigvee S^1$.*

Proof. By Proposition 1.1.5, $\mathcal{N}(G)$ consists of one path-connected component, so the simplices $N_G(v), N_G(w), \{v\}$, and $\{w\}$ are all in the same path-connected component. Further, since $\mathrm{d}(v,w)\geq 5$, we have that the sets $N_G(v), N_G(w), \{v\},\text{and}\{w\}$ are pairwise disjoint. The operation of identifying v and w in G adds to $\mathcal{N}(G)$ the join of $N_G(v) * N_G(w)$ and identifies v and w in $\mathcal{N}(G)$. By Lemma 2.2.4, no faces of $N_G(v) * N_G(w)$ other than $N_G(v)$ and $N_G(w)$ are present in $\mathcal{N}(G)$. Therefore, the join is homotopy equivalent to attaching a 1-cell between the contractions of the facets given by $N_G(v)$ and $N_G(w)$, respectively. In addition, the identification of the simplices $\{v\}$ and $\{w\}$ is homotopy equivalent to attaching a 1-cell between $\{v\}$ and $\{w\}$, as shown in Figure 2.3. Hence, $\mathcal{N}(G')\simeq\mathcal{N}(G)\bigvee S^1\bigvee S^1$.

□

Corollary 2.2.6. *Let G be a bridgeless j-chromatic graph with $j\geq 4$ such that $\mathcal{N}(G)$ is i-connected with $i>0$. For any Hajós construction of G, there exists an H such that the final step of the construction is a vertex identification $\mathrm{vid}(H,[v,w])$ where v and w satisfy $\mathrm{d}_H(v,w)\leq 4$.*

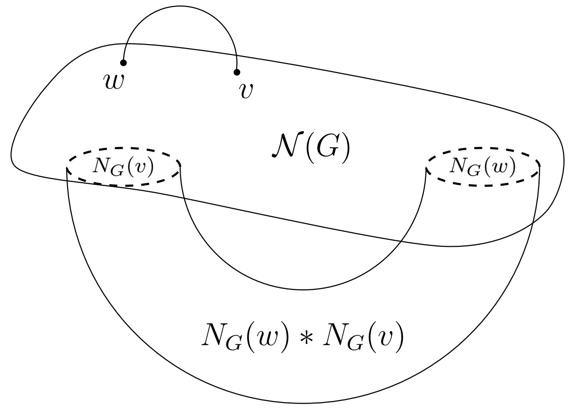

Figure 2.3: Through an identification of v and w in G where $\mathrm{d}(v, w) \geq 5$, we add a 1-cell $\{v, w\}$ and the join $N(w) * N(v)$ to $\mathcal{N}(G)$ to obtain a space homotopy equivalent to $\mathcal{N}(\mathrm{vid}(G, [v, w]))$.

2.3 Identifications of Vertices at Short Distances

Corollary 2.2.6 demonstrates the importance in Hajós-type constructions of identifications of pairs of vertices at distance strictly less than five. Since the typical Hajós merge or vertex identification at distance strictly greater than four results in a wedge summand of S^1 for $\mathcal{N}(G)$, in order to produce a Hajós-type construction of G where $\mathcal{N}(G)$ is i-connected for $i > 0$, it must be the case that vertex identifications at short distances eliminate these wedge summands, and thus reduce the first homology group of $\mathcal{N}(G)$. Therefore, we are motivated in this section to consider the effect of such vertex identifications on these wedge summands and on the first Betti number of $\mathcal{N}(G)$, i.e. the rank of the first homology group of $\mathcal{N}(G)$. Our main results are Theorems 2.3.2 and 2.3.3, in which we show that for a restricted class of vertex identifications using vertices at distance less than five, the first Betti number of the neighborhood complex does not increase. We also prove in Theorem 2.3.5 that the number of wedge summands eliminated by a single vertex identification can be arbitrarily large.

Lemma 2.3.1. *If G is a graph with two vertices v and w such that p is a path from v to w of length four, then $N_G(v)$ and $N_G(w)$ are connected via an edge in $\mathcal{N}(G)$ between two of their respective vertices.*

Proof. Let $p = (v, x_1, x_2, x_3, w)$. Then, $x_1 \in N_G(v)$ and $x_3 \in N_G(w)$. In addition, $x_1, x_3 \in N_G(x_2)$. Therefore, $N_G(v)$ and $N_G(w)$ are connected through the edge (x_1, x_3) in $\mathcal{N}(G)$. $\square$

Theorem 2.3.2. *Let G be a graph with vertices v and w such that* $\mathrm{d}(v, w) \geq 3$ *and there exists a path of length four from v to w. Let* $G' := \mathrm{vid}(G, [v, w])$*. Then*

$$\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G'))) \leq \mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G)))\,.$$

Proof. Recall that the *link* of a vertex v in a simplicial complex X is $\mathrm{link}_X(v) := \{\sigma \in X : v \notin \sigma, \{v\} \cup \sigma \in X\}$. Further, the *star* of a vertex v in X is $\mathrm{star}_X(v) := \{\sigma \in X : v \in \sigma\}$. By Lemma 2.3.1, $N_G(v)$ and $N_G(w)$ are connected via an edge in $\mathcal{N}(G)$, though they are disjoint faces in $\mathcal{N}(G)$. Since $\mathrm{d}(v,w) \geq 3$, both v and w are disjoint from these neighborhoods.

Through the identification of v and w, there is a two-step alteration of the neighborhood complex $\mathcal{N}(G)$ that produces $\mathcal{N}(G')$:

1. $N_G(v)$ and $N_G(w)$ are joined to become the face τ with vertex set $N_G(v) \cup N_G(w)$;
2. any faces containing v or w are deleted, while a new vertex vw is created with
$$\mathrm{link}_{\mathcal{N}(G')}(vw) := \mathrm{link}_{\mathcal{N}(G)}(v) \cup \mathrm{link}_{\mathcal{N}(G)}(w)\,.$$

We first address step (1): Consider the complex $\mathcal{N}(G) \cup \tau$. Since τ is contractible and thus has zero homology, by the Mayer-Vietoris sequence we obtain:

$$\cdots \to \widetilde{H_1}(\mathcal{N}(G) \cap \tau) \to \widetilde{H}_1(\mathcal{N}(G)) \to \widetilde{H_1}(\mathcal{N}(G) \cup \tau) \to \widetilde{H_0}(\mathcal{N}(G) \cap \tau) \to \cdots$$

Combining the fact that $\mathcal{N}(G) \cap \tau$ is a subcomplex of $N_G(v) * N_G(w)$ with Lemma 2.3.1, it follows that $\mathcal{N}(G) \cap \tau$ is connected. Hence, $\widetilde{H_0}(\mathcal{N}(G) \cap \tau) = 0$. So, $\widetilde{H_1}(\mathcal{N}(G))$ surjects onto $\widetilde{H_1}(\mathcal{N}(G) \cup \tau)$, implying that

$$\mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G))) \geq \mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G) \cup \tau))\,.$$

We next address step (2), in which we go from $\mathcal{N}(G) \cup \tau$ to $\mathcal{N}(G')$: Since $\mathrm{star}_{\mathcal{N}(G')}(vw)$ is a cone, it is contractible, and thus

$$\begin{aligned}\mathcal{N}(G') = \mathcal{N}(\mathrm{vid}(G,[v,w])) &\simeq \mathcal{N}(\mathrm{vid}(G,[v,w]))/\mathrm{star}_{\mathcal{N}(G')}(vw) \\ &\cong (\mathcal{N}(G) \cup \tau)/(\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w))\end{aligned}$$

where the homeomorphism of the latter two quotients follows from the fact that $\mathrm{link}_{\mathcal{N}(G')}(vw) := \mathrm{link}_{\mathcal{N}(G)}(v) \cup \mathrm{link}_{\mathcal{N}(G)}(w)$ and the links of v in both $\mathcal{N}(G)$ and $\mathcal{N}(G) \cup \tau$ are the same, and similarly for w. Hence, we have

$$\widetilde{H}_1(\mathcal{N}(G')) \cong \widetilde{H}_1((\mathcal{N}(G) \cup \tau)/(\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w)))\,.$$

It follows from the existence of a path of length four from v to w that $\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w)$ is connected, and combining this with the long exact sequence for the pair $(\mathcal{N}(G) \cup \tau, \mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w))$ we obtain

$$\begin{aligned}\cdots &\to \widetilde{H_1}(\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w)) \to \widetilde{H}_1(\mathcal{N}(G) \cup \tau) \\ &\to \widetilde{H_1}((\mathcal{N}(G) \cup \tau)/(\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w))) \\ &\to \widetilde{H_0}(\mathrm{star}_{\mathcal{N}(G)\cup\tau}(v) \cup \mathrm{star}_{\mathcal{N}(G)\cup\tau}(w)) = 0\,.\end{aligned}$$

So, $\widetilde{H_1}(\mathcal{N}(G) \cup \tau)$ surjects onto $\widetilde{H_1}(\mathcal{N}(G'))$, implying

$$\mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G))) \geq \mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G) \cup \tau)) \geq \mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G')))\,.$$

$\square$

Theorem 2.3.3. *Let G be a graph with vertices v and w. Define U to be the set of common neighbors of v and w, $A := N_G(v) \setminus U$, and $B := N_G(w) \setminus U$. Suppose that*

1. $\mathrm{d}(v, w) = 2$,
2. *there is no path of length* 3 *from v to w in G, and*
3. $\left(\bigcup_{a \in A} N_G(a)\right) \cap \left(\bigcup_{b \in B} N_G(b)\right) = \emptyset$.

If $G' := \mathrm{vid}(G, [v, w])$, then

$$\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G'))) \leq \mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G)))\,.$$

Proof. Consider the disjoint union

$$V(G) = \{v, w\} \uplus U \uplus A \uplus B \uplus X$$

where U is the set of common neighbors of v and w, $A := N_G(v) \setminus U$, $B := N_G(w) \setminus U$, and X contains the remaining vertices in G. Since G contains no path of length 3 from v to w, the following must be true:

- there is no edge between any two elements of U;
- v is adjacent to every element of U and A;
- w is adjacent to every element of U and B;
- there is no edge between U and A, and similarly for U and B;
- there are no restrictions on the structure of edges within A, B, or X or the edges connecting elements of X with elements of A, U, or B, respectively.

Note that since v and w share a common neighbor, we have $N_G(v) \cap N_G(w) = U \neq \emptyset$. Through the identification of v and w, there are two changes of the neighborhood complex $\mathcal{N}(G)$ that produce $\mathcal{N}(G')$. First, the face $\tau := N_G(v) \cup N_G(w)$ is added to the complex — note that since $N_G(v) \cap N_G(w) \neq \emptyset$ in this case, τ is not the join of these two simplices. Also note that $\tau = U \cup A \cup B$, so $\tau \cap X = \emptyset$. We can make an identical argument to that given in the first step of Theorem 2.3.2 to show that $\mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G))) \geq \mathrm{rank}(\widetilde{H_1}(\mathcal{N}(G) \cup \tau))$, except that we argue $\mathcal{N}(G) \cap \tau$ is connected as a result of the property that $N_G(v) \cap N_G(w) \neq \emptyset$.

For the second step, we will show that $\mathcal{N}(G) \cup \tau$ is homotopy equivalent to $\mathcal{N}(G')$, and thus

$$\widetilde{H}_1(\mathcal{N}(G')) \cong \widetilde{H}_1(\mathcal{N}(G) \cup \tau)\,,$$

from which our result will follow. Because we have assumed that $\left(\bigcup_{a \in A} N_G(a)\right) \cap \left(\bigcup_{b \in B} N_G(b)\right) = \emptyset$, in $\mathcal{N}(G) \cup \tau$ the edge (v, w) is contained in only $\bigcup_{u \in U} N_G(u)$, while also

$$v \in \left(\bigcup_{a \in A} N_G(a)\right) \cup \left(\bigcup_{u \in U} N_G(u)\right) \text{ and } w \in \left(\bigcup_{b \in B} N_G(b)\right) \cup \left(\bigcup_{u \in U} N_G(u)\right).$$

The subcomplex $\bigcup_{u\in U} N_G(u) \subseteq \mathcal{N}(G) \cup \tau$ therefore has the structure of the join of (v, w) with the subcomplex of $\bigcup_{u\in U} N_G(u)$ induced by X; within this subcomplex, we make a discrete Morse matching by pairing any cell σ containing v but not w with $\sigma \cup \{w\}$. By Theorem 1.2.2, the resulting space, which we denote $\mathcal{X}$, is homotopy equivalent to $\mathcal{N}(G) \cup \tau$. The topological deformation resulting from this matching has the effect of contracting the joined subcomplex along the edge (v, w), and thus $\mathcal{X}$ is a simplicial complex where $\bigcup_{b\in B} N_G(b)$ has been added to the link of v. This is precisely the description of the link of vw in $\mathcal{N}(G')$, which completes the proof. $\square$

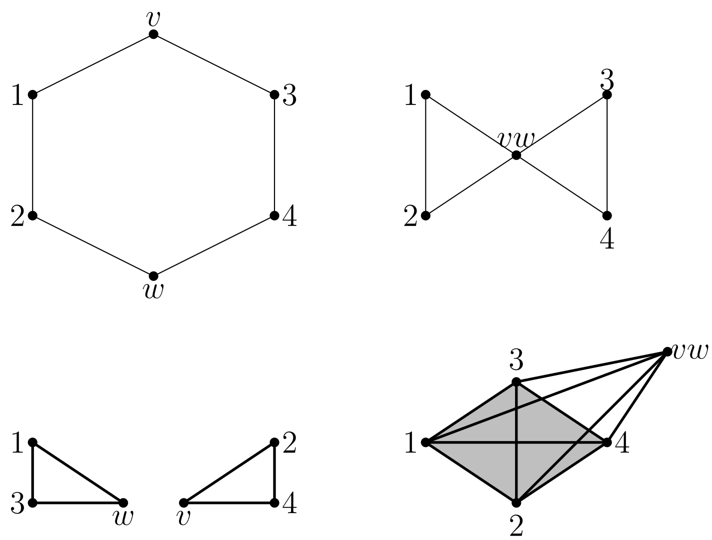

Figure 2.4: Top left: Graph G with $\mathrm{d}(v, w) = 3$; Top right: Graph $G' := \mathrm{vid}(G, (v, w))$; Bottom left: $\mathcal{N}(G)$; and Bottom right: $\mathcal{N}(G')$. Observe that $\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G))) = 2$ and $\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G'))) = 3$.

Remark 2.3.4. Without a path of length four, when $\mathrm{d}(v, w) = 3$ we cannot conclude in general that $\mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G'))) \leq \mathrm{rank}(\widetilde{H}_1(\mathcal{N}(G)))$, since our Mayer-Vietoris argument no longer holds. As an example, consider the graph G depicted in Figure 2.4 and perform a vertex identification on v and w. An analogous consideration when G is a cycle on 9 vertices yields similar results without increasing $\chi(G)$.

We next demonstrate that a single vertex identification can result in an arbitrarily large decrease in the size of the first homology of $\mathcal{N}(G)$. Further, this decrease can arise from the elimination of an arbitrarily large number of S^1 wedge summands.

Theorem 2.3.5. *For every $n \geq 5$, there exists a graph G_n and a single vertex identification in G_n resulting in a graph G'_n such that:*

- *$\mathcal{N}(G_n)$ is a wedge of $2n + 5$ circles, and*
- *$\mathcal{N}(G'_n)$ has trivial first homology.*

Proof. The theorem follows from Propositions 2.3.7 and 2.3.8. □

We define our desired G_n as follows.

Definition 2.3.6. Let G_n be the graph with vertex and edge sets defined as follows:

$$V(G_n) = \{X, Y, Z\} \cup \{iA, iB, iC : 1 \leq i \leq n\}$$

and

$$\begin{aligned} E(G_n) := \quad & \{(X, Y), (Y, Z), (1A, X), (nC, X)\} \\ & \cup \{(iB, X), (iA, iB), (iB, iC), (iC, iA) : 1 \leq i \leq n\} \\ & \cup \{(jC, Z), (jC, (j+1)A) : 1 \leq j \leq n-1\} \\ & \cup \{(kA, Z) : 2 \leq k \leq n\} \end{aligned}$$

Define $G'_n := \operatorname{vid}(G_n, [X, Z])$.

G_3 and G'_3 are shown in Figure 2.5. One can show that $\chi(G_n) = 4$ for all n, and that it is possible to provide an explicit Hajós construction of G_n starting with iterated Hajós merges using n copies of K_4, followed by a sequence of vertex identifications.

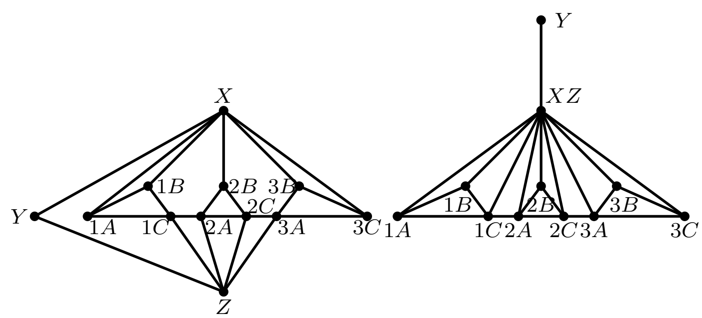

Figure 2.5: G_3 is displayed on the left and $G'_3 = \operatorname{vid}(G_3, [X, Z])$ on the right. G_n along with G'_n defines a family of graphs for which the decrease in the rank of the first homology group after one vertex identification can be arbitrarily large.

Proposition 2.3.7. *For $n \geq 5$, $\mathcal{N}(G_n) \simeq \bigvee_{2n+5} S^1$.*

Proof. Let P_n denote the face poset of $\mathcal{N}(G_n)$. Every vertex $v \in G_n$ generates a subposet isomorphic to a boolean algebra B_d containing the subsets of $N_{G_n}(v)$, where d is the cardinality of $N_{G_n}(v)$. The neighborhood of X generates a subposet isomorphic to B_{n+3} and the neighborhood of Z generates a subposet isomorphic to B_{2n-1}. The remaining vertices have degree 2, 3, or 4 giving rise to subposets isomorphic to B_2, B_3, and B_4, respectively. To provide notation for this, let $\mathrm{NP}(v) := \{p \in P_n : p \subseteq N_{G_n}(v)\}$ and $\mathrm{NP}(v)_{\geq 1} := \{p \in \mathrm{NP}(v) : |p| \geq 1\}$, where NP stands for *Neighborhood Poset*. The strategy for this proof is to make systematic acyclic

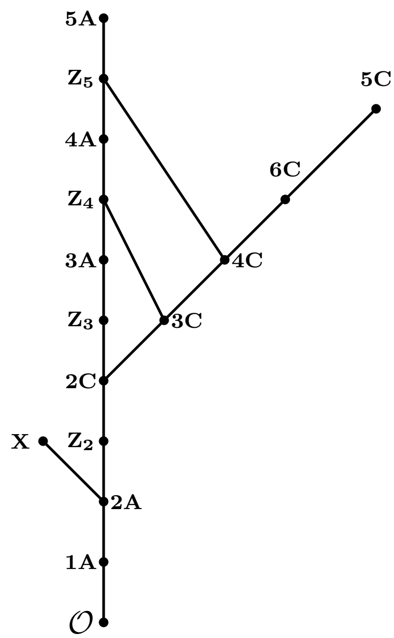

Figure 2.6: Q_6, poset example for proof of Proposition 2.3.7.

matchings on parts of the small Boolean algebras, then to quotient the resulting space by the subcomplex of $\mathcal{N}(G_n)$ given by $N_{G_n}(X) \cup N_{G_n}(Z)$, which is contractible.

In order to create our systematic acyclic matchings, we will apply Theorem 1.2.4. Define Q_n to be the poset on the elements

$$\{\mathcal{O}, \mathbf{X}, \mathbf{Z_2}, \dots, \mathbf{Z_{(n-1)}}, \mathbf{1A}, \mathbf{2A}, \dots, \mathbf{(n-1)A}, \mathbf{2C}, \mathbf{3C}, \dots, \mathbf{nC}\}$$

with cover relations given by:

- $\mathcal{O} \prec \mathbf{1A} \prec \mathbf{2A} \prec \mathbf{Z_2} \prec \mathbf{2C} \prec \mathbf{Z_3} \prec \mathbf{3A}$;
- $\mathbf{2A} \prec \mathbf{X}$;
- $\mathbf{2C} \prec \mathbf{3C} \prec \cdots \prec \mathbf{(n-2)C} \prec \mathbf{nC} \prec \mathbf{(n-1)C}$;
- $\mathbf{jA} \prec \mathbf{Z_{j+1}} \prec \mathbf{(j+1)A}$ for all $3 \leq j \leq n-2$; and
- $\mathbf{jC} \prec \mathbf{Z_{j+1}}$ for all $3 \leq j \leq (n-2)$.

Q_6 is illustrated in Figure 2.6. To distinguish target elements in Q_n from the vertices of G_n, we write the elements of Q_n in bold.

We next define a poset map $\Gamma_n : P_n \to Q_n$. The preimage of an element $\alpha \in Q_n$ is denoted as $\Gamma_n^{-1}(\alpha)$. We begin by setting $\Gamma_n^{-1}(\mathcal{O}) = \mathrm{NP}(X) \cup \mathrm{NP}(Z)$. No matching will take place on this subposet of P_n, and the corresponding subcomplex will remain contractible. This accounts for both of the large Boolean algebras in P_n. What remains unmapped are a handful of cells in each smaller Boolean algebra, which we must handle individually, leading to a long list of preimages. For each element in Q_n, we define Γ_n^{-1} in a manner that makes it straightforward (though tedious) to verify that the resulting map is order preserving. For the elements in Q_n that are less than or equal to **2C**, we make the following assignments. Observe that when we remove elements from our neighborhood posets in the mapping below, it is due to the fact that those elements were already mapped in a previous preimage assignment. The preimages are:

- $\Gamma_n^{-1}(\mathbf{1A}) = \{\{1A, 1B, 2A\}, \{1A, 2A\}, \{1B, 2A\}\}$;
- $\Gamma_n^{-1}(\mathbf{2A}) =$
$$(\mathrm{NP}(Y) \cup \mathrm{NP}(1C) \cup \mathrm{NP}(2C) \cup \mathrm{NP}(2B)) \\ \setminus \{\{1A, 1B, 2A\}, \{1A, 2A\}, \{1B, 2A\}, \{1A, 1B\}, \{2A, 3A\}, 1A, 1B, 2A, 2B, 2C, 3A\}$$
;
- $\Gamma_n^{-1}(\mathbf{X}) = (\mathrm{NP}(1B) \cup \mathrm{NP}(1A) \cup \mathrm{NP}(nB) \cup \mathrm{NP}(nC))_{\geq 1}$;
- $\Gamma_n^{-1}(\mathbf{Z_2}) = \{\{2B, 2C, Z\}, \{2B, 2C\}, \{2C, Z\}\}$; and
- $\Gamma_n^{-1}(\mathbf{2C}) =$
$$(\mathrm{NP}(2A) \cup \mathrm{NP}(3A))_{\geq 1} \\ \setminus \{\{2B, 2C, Z\}, \{2B, 2C\}, \{2B, Z\}, \{1C, 2C\}, \{2C, 3C\}, \{2C, Z\}\}$$
.

At this point, all of the elements of P_n contained in $\mathrm{NP}(X)$, $\mathrm{NP}(Z)$, $\mathrm{NP}(Y)$, $\mathrm{NP}(1C)$, $\mathrm{NP}(1A)$, $\mathrm{NP}(2A)$, $\mathrm{NP}(3A)$, $\mathrm{NP}(nC)$, $\mathrm{NP}(1B)$, $\mathrm{NP}(nB)$, $\mathrm{NP}(2B)$, and $\mathrm{NP}(2C)$ have been assigned an image under Γ_n. Next, we map the remaining elements of P_n to elements of Q_n that are above **2C**:

- $\Gamma_n^{-1}(\mathbf{jC}) = \mathrm{NP}((j+1)A)_{\geq 1} \setminus \{\{jC, Z\}, \{jC, (j+1)C\}\}$ for each $3 \leq j \leq (n-2)$;
- $\Gamma_n^{-1}(\mathbf{Z_k}) = \{\{kA, kB, Z\}, \{kA, kB\}\}$ for each $3 \leq k \leq (n-1)$;
- $\Gamma_n^{-1}(\mathbf{kA}) =$
$$(\mathrm{NP}(kC) \cup \mathrm{NP}(kB))_{\geq 1} \\ \setminus \{\{kA, kB, Z\}, \{kA, kB\}, \{kB, Z\}, \{kA, Z\}, \{kA, (k+1)A\}\}$$
for each $3 \leq k \leq (n-1)$;
- $\Gamma_n^{-1}(\mathbf{nC}) = \{\{(n-1)C, nB, nC\}, \{(n-1)C, nC\}, \{(n-1)C, nB\}\}$; and
- $\Gamma_n^{-1}((\mathbf{n-1})\mathbf{C}) =$
$$\mathrm{NP}(nA)_{\geq 1} \\ \setminus \{\{(n-1)C, nB, nC\}, \{(n-1)C, nB\}, \{(n-1)C, nC\}, \{(n-1)C, Z\}, \{nB, nC\}\}$$
.

To complete our proof, we define a matching on each $\Gamma_n^{-1}(\mathbf{t})$ for $\mathbf{t} \in Q_n \setminus \mathcal{O}$. For each $\sigma \in \Gamma_n^{-1}(\mathbf{t})$ that does not contain the vertex t (the bold symbol is an element of Q_n, the unbolded symbol is a vertex of G_n), we pair σ with $\sigma \cup \{t\}$ — if the element is $\mathbf{Z_i}$, we use the vertex Z as the unbolded version of $\mathbf{Z_i}$. Because for each of these matchings there is a unique element that is used for the pairing assignment, it is a quick exercise to confirm that these are all acyclic matchings. Thus, by Theorem 1.2.4, we have defined an acyclic matching on P_n. In addition to $\Gamma_n^{-1}(\mathcal{O})$, the corresponding critical (i.e. unmatched) cells in each preimage are:

- $\Gamma_n^{-1}(\mathbf{1A}) : \{1A, 2A\}$;
- $\Gamma_n^{-1}(\mathbf{2A}) : \{2A, 2B\}$, $\{X, Z\}$;
- $\Gamma_n^{-1}(\mathbf{X}) : \{1A, X\}$, $\{1C, X\}$, $\{1B, X\}$, $\{nA, X\}$, $\{nB, X\}$, $\{nC, X\}$;
- $\Gamma_n^{-1}(\mathbf{Z_2}) : \{2C, Z\}$;
- $\Gamma_n^{-1}(\mathbf{2C}) : \{2C, 3B\}$;
- $\Gamma_n^{-1}(\mathbf{jC}) : \{jC, (j+1)B\}$ for $3 \leq j \leq n-2$, which gives rise to $n-4$ critical 1-cells;
- $\Gamma_n^{-1}(\mathbf{kA}) : \{kA, X\}$ for $3 \leq k \leq n-1$, which gives rise to $n-3$ critical 1-cells;
- $\Gamma_n^{-1}(\mathbf{Z_k})$ has no critical cells for any $3 \leq k \leq (n-1)$;
- $\Gamma_n^{-1}(\mathbf{nC}) : \{(n-1)C, nC\}$; and
- $\Gamma_n^{-1}((\mathbf{n-1})\mathbf{C})$ has no critical cells.

This gives a total of $2n+5$ critical 1-cells. Since $N_{G_n}(X) \cup N_{G_n}(Z)$ forms a contractible subcomplex of the critical complex for this matching, we can contract this subcomplex yielding $\mathcal{N}(G_n) \simeq \bigvee_{2n+5} S^1$. $\square$

Proposition 2.3.8. *For $n \geq 5$, we have $\mathcal{N}(G'_n) \simeq \bigvee_{2n-1} S^2$.*

Proof. Let P'_n denote the face poset of $\mathcal{N}(G'_n)$. Define Q'_n to be the poset on the elements

$$\{\mathcal{O}', \mathcal{T}\} \cup \{\mathbf{jA}, \mathbf{jC} : 2 \leq j \leq (n-1)\}$$

with cover relations given by $\mathcal{O}' \prec \mathbf{2A}$, $\mathbf{iA} \prec \mathbf{iC} \prec (\mathbf{i+1})\mathbf{A}$ for $2 \leq i \leq (n-2)$, and maximal element $\mathcal{T}$. Thus, Q'_n is totally ordered. As in the previous proof, it is helpful to think of P'_n as a union of subsets of neighborhoods of the vertices of G'_n, adjusted to remove the subsets contained in the neighborhood of XZ. Similarly to our previous proof, our strategy will be to leave the subsets of the neighborhood of XZ unmatched, pair off subsets of the remaining vertices in $\mathcal{N}(G'_n)$, and then quotient out the neighborhood of XZ at the end.

We define a poset map $\Gamma'_n : P'_n \to Q'_n$. As before, let $\mathrm{NP}(v) := \{p \in P_n : p \subseteq N_{G_n}(v)\}$ and $\mathrm{NP}(v)_{\geq 1} := \{p \in \mathrm{NP}(v) : |p| \geq 1\}$. Begin by setting $(\Gamma'_n)^{-1}(\mathcal{O}') =$

NP(XZ); no matching will take place on this subposet of P'_n and the corresponding subcomplex will remain contractible. For each of the remaining elements in Q_n, we define $(\Gamma'_n)^{-1}$ in a manner that makes it straightforward to verify that the resulting map is order-preserving. We first define

$$(\Gamma'_n)^{-1}(\mathbf{2A}) = (\mathrm{NP}(1C) \cup \mathrm{NP}(2B) \cup \mathrm{NP}(2C))_{\geq 1} \setminus \mathrm{NP}(XZ)\,.$$

Next, to account for NP($1A$), NP($1B$), NP($2A$), NP($3A$), and NP($3B$), set

$$\begin{aligned}(\Gamma'_n)^{-1}(\mathbf{2C}) =& \{\{1C, 2B, 2C, XZ\}, \{2C, 3B, 3C, XZ\}, \{1A, 1C, XZ\}, \{1B, 1C, XZ\},\\ & \{1C, 2B, XZ\}, \{2C, 3C, XZ\}, \{2B, 2C, XZ\}, \{2C, 3B, XZ\},\\ & \{3B, 3C, XZ\}, \{3A, 3C, XZ\}, \{1C, 2C, XZ\}, \{1C, XZ\}, \{3B, XZ\},\\ & \{3C, XZ\}\}\,.\end{aligned}$$

We next make two types of iterated assignments, to account for the neighborhoods of the remaining vertices of G'_n, and a final assignment for maximal triangles in $\mathcal{N}(G'_n)$:

- $(\Gamma'_n)^{-1}(\mathbf{kA}) = \{\{kA, kB, (k+1)A, XZ\}, \{kB, (k+1)A, XZ\}, \{kA, (k+1)A, XZ\}, \{(k+1)A, XZ\}, \{kA, kB, XZ\}\}$ for $3 \leq k \leq n$, where we do not include the four sets in $(\Gamma'_n)^{-1}(\mathbf{nA})$ containing an element labeled $(n+1)A$, thus $|(\Gamma'_n)^{-1}(\mathbf{nA})| = 1$;

- $(\Gamma'_n)^{-1}(\mathbf{kC}) = \{\{kC, (k+1)B, (k+1)C, XZ\}, \{(k+1)B, (k+1)C, XZ\}, \{kC, (k+1)C, XZ\}$,
$\{kC, (k+1)B, XZ\}, \{(k+1)B, XZ\}, \{(k+1)C, XZ\}\}$ for $3 \leq k \leq n$; and

- $(\Gamma'_n)^{-1}(\mathcal{T}) = \{\{kA, kC, XZ\} : 4 \leq k \leq n\}$.

All cells in $(\Gamma'_n)^{-1}(\mathcal{T})$ are unmatched. As in the previous proof, for each $\sigma \in \Gamma_n^{-1}(\mathbf{t})$ that does not contain the vertex t (the bold symbol is an element of Q_n, the unbolded symbol is a vertex of G_n), we pair σ with $\sigma \cup \{t\}$. For each matching there is a unique element used for pairing and therefore all matchings are acyclic. Thus by Theorem 1.2.4, we have defined a matching on P'_n. Other than the elements in $(\Gamma'_n)^{-1}(\mathcal{O})$, our critical cells are:

- $(\Gamma'_n)^{-1}(\mathbf{2A})$ has no critical cells;

- $(\Gamma'_n)^{-1}(\mathbf{2C}) : \{1A, 1C, XZ\}$, $\{1B, 1C, XZ\}, \{2B, 2C, XZ\}$, $\{3A, 3C, XZ\}$;

- $(\Gamma'_n)^{-1}(\mathbf{kA}) : \{kA, kB, XZ\}$ for $3 \leq k \leq n$, which gives rise to $n-2$ critical 2-cells;

- $(\Gamma'_n)^{-1}(\mathbf{kC})$ has no critical cells for $3 \leq k \leq n$; and

- $(\Gamma'_n)^{-1}(\mathcal{T}) : \{kA, kC, XZ\}$ for $4 \leq k \leq n$, which gives rise to $n-3$ critical 2-cells.

This gives a total of $2n-5+4 = 2n-1$ critical 2-cells. Since $(\Gamma'_n)^{-1}(\mathcal{O})$ corresponds to a contractible subcomplex of the critical complex for this matching, we can contract this subcomplex and obtain $\mathcal{N}(G'_n) \simeq \bigvee_{2n-1} S^2$.

□

2.4 Topological Effects of DHGO Compositions

Recall that a graph G is *k-extremal* if G is k-chromatic and k-critical, i.e. any subgraph of G has chromatic number lower than k, and G has the minimum number of edges possible among such graphs on n vertices. According to Kostochka and Yancey [17], Ore suggested that the best possible construction of sparse critical graphs can be obtained by starting with an extremal graph on at most $2k$ vertices and repeatedly using K_k as G_2 in DHGO-compositions, a generalization of Hajós merges. We study these compositions in this section.

Definition 2.4.1. For a graph G and $u \in V(G)$, a *split* of u, denoted $\mathrm{sp}(G,u)$, is a new graph G'' with vertex set $V(G'') = V(G) \setminus \{u\} + \{u', u''\}$, where $G \setminus \{u\} \cong G'' \setminus \{u', u''\}$, $N(u') \cup N(u'') = N(u)$, and $N(u') \cap N(u'') = \emptyset$.

Definition 2.4.2. Let G_1 and G_2 be graphs such that $x_1 \in V(G_1)$ and $(x_2, y_2) \in E(G_2)$. A *DHGO-composition*, denoted $D(G_1, G_2)$, is a graph that is created by deleting (x_2, y_2), splitting x_1 into x'_1 and x''_1 with positive degrees, and identifying x'_1 with x_2 and x''_1 with y_2.

A Hajós merge is a special case of a DHGO-composition, which motivates us to consider the topological effect on $\mathcal{N}(G)$ of these more general operations.

Lemma 2.4.3. *Let G be a connected non-bipartite graph such that $G'' := sp(G,u)$ is connected bipartite where $N(u') \cup N(u'') = N(u)$ and $N(u') \cap N(u'') = \emptyset$. Then, $\mathcal{N}(G'') = A \uplus B$, where $N_{G''}(u'), u'' \in A$ and $N_{G''}(u''), u' \in B$.*

Proof. Since $\chi(G'') = 2$, u must be part of all odd cycles in G. This implies that there is an odd path from u' to u'', so $u'', N(u') \in A$ and $N_{G''}(u''), u' \in B$. If u' and u'' are in the same connected component, there is an even path between u' and u'', implying there is only one connected component, which is a contradiction to being connected bipartite. □

Theorem 2.4.4. *Consider two connected graphs G_1 and G_2 with $x_1 \in V(G_1)$ and $(x_2, y_2) \in E(G_2)$ such that:*

1. *$\chi(G_1), \chi(G_2) \geq 3$ and (x_2, y_2) is not a bridge, and*
2. *$D(G_1, G_2)$ is built using a connected* $\mathrm{sp}(G_1, x_1)$.

Then, $\mathcal{N}(D(G_1, G_2))$ is a wedge of at least one copy of S^1 with another space.

Proof. The result follows analogously to Theorem 2.2.3 with an adaptation arising in the steps of the operation. Let $G_1' := \mathrm{sp}(G_1, x_1)$ and $G_2' := G_2 \setminus (x_2, y_2)$ with the DGHO-composition identifying x_1', x_2 and x_1'', y_2. It follows as in Theorem 2.2.3, the effect of identifying x_1' with x_2 is homotopy equivalent to attaching a 1-simplex, (x_1', x_2), and joining $N_{G_1'}(x_1') * N_{G_2'}(x_2)$. Similarly, the identification of x_1'' with y_2 gives rise to the attachment of (x_1'', y_2) and joining $N_{G_1'}(x_1'') * N_{G_2'}(y_2)$.

It remains to show that there is at least one S^1 wedge summand in our resulting neighborhood complex. Assume $\chi(G_1) \geq 3$ and $\chi(G_2) \geq 3$, with (x_2, y_2) not a bridge. Thus, G_2' is connected and $\chi(G_2') \geq 2$. Since $G_1' = \mathrm{sp}(G_1, x_1)$ is connected, analogously to the proof of Theorem 2.2.3 it suffices to show that one of the connected components of $\mathcal{N}(G_1')$ contains at least two elements from $\{\{x_1'\}, \{x_1''\}, N_{G_1'}(x_1'), N_{G_1'}(x_1'')\}$. This follows from Lemma 2.4.3 when $\chi(G_1') \geq 2$ and $\chi(G_2') \geq 3$.

Suppose that $\chi(G_1') = \chi(G_2') = 2$. Then, $\mathcal{N}(G_1')$ and $\mathcal{N}(G_2')$ each have two connected components. That is, $\mathcal{N}(G_1') = A \uplus B$ and $\mathcal{N}(G_2') = C \uplus D$ have two connected components, respectively. From Lemma 2.4.3, we can assume $N_{G_1'}(x_1'), x_1'' \in A$ and $N_{G_1'}(x_1''), x_1' \in B$. From Lemma 2.2.2, we can assume $x_2, y_2 \in C$ and $N_{G_2'}(x_2), N_{G_2'}(y_2) \in D$. Hence, up to homotopy equivalence the DGHO-composition attaches 1-cells between A and C, C and B, B and D, and D and A. The result follows. □

2.5 Graph Construction Algorithms, Experimental Results, and Open Problems

In this section, we report on results regarding computational experiments using SageMath [30] via CoCalc.com [26]. We describe two stochastic algorithms that produce graphs using Hajós constructions and Urquhart constructions (defined below). Using these algorithms, we generate sets of graphs and analyze the resulting distributions of their sizes, orders, and the ranks of the first homology groups of their neighborhood complexes. We conclude the section with several open problems.

Two Hajós-type construction algorithms

We define in Appendix A two algorithms that we call the Constructible Random Algorithm (CRA) and the Urquhart Random Algorithm (URA). The CRA is a stochastic algorithm implementing the recursive definition of k-constructible graphs given in Defintion 2.1.1. We use a probability p to determine the likelihood of selecting a Hajós merge or vertex identification at each step of the algorithm; when p is small, vertex identifications are favored. To define the URA, we require the notion of Ore constructibility.

Definition 2.5.1. A graph is *Ore k-constructible* if it is a complete graph K_k or if it can be constructed from K_k by successive applications of the following operation:

- (Ore Merge) Suppose G_1 and G_2 are already-obtained disjoint graphs with respective edges (x_1, y_1) and (x_2, y_2). Let μ be a bijection from a subset of $V(G_1)$ to $V(G_2)$ so that $x_1 \notin \mathrm{domain}(\mu)$ and $x_2 \notin \mathrm{range}(\mu)$, and $\mu(y_1) \neq y_2$. Form the

Hajós merge on G_1 and G_2 using the two edges, then identify the vertex pairs $[x, \mu(x)]$. We abuse notation and denote the resulting graph $\mathrm{Ore}(G_1, G_2)$.

Any construction of a graph using this process will be called an *Ore construction*.

Note that an Ore merge arises from a single Hajós merge followed by a sequence of restricted vertex identifications. Urquhart [31] proved that the families of Hajós constructible and Ore constructible graphs are equivalent.

Theorem 2.5.2 (Urquhart [31])**.** *For a graph G and $k \geq 2$, the following conditions are equivalent:*

- *G is Hajós-k-constructible;*
- *G is Ore-k-constructible.*

In the proof of Theorem 2.5.2, Urquhart showed that for $k \geq 3$ every k-chromatic graph G can be obtained by the following construction.

Definition 2.5.3. Suppose that a graph G is obtained by applying Ore merges to a sequence of graphs $G_1, G_2, \ldots, G_\ell$ each containing K_k. We will call a construction of G using this approach an *Urquhart construction*, and write (again abusing notation) $G = \mathrm{Urq}(G_1, \ldots, G_\ell)$.

Inspired by this chain of ideas, the URA repeatedly generates graphs using a slightly-restricted version of the Urquhart construction. These minor restrictions ensure that the input graphs for the Urquhart construction are connected. We use random choices for various parameters and vertex/edge selections in the URA, leading to the stochasticity of the algorithm.

Experimental Results

Using an implementation in SageMath, we ran the Constructible Random Algorithm for $p \in \{0.02, 0.1, 0.5\}$ and $k \in \{3, 4, 5, 6\}$, generating various numbers of graphs. For each of the three values of p when $k = 5$ and for $p = 0.02$ when $k = 6$, we include in Appendix A a plot of the number of vertices versus number of edges for the graphs generated, and a histogram of the first Betti numbers of the resulting neighborhood complexes. We also provide in Table 2.1 the percent of graphs for which the first Betti number was trivial. Similarly, we ran the Urquhart Random Algorithm for $k \in \{3, 4, 5, 6\}$ and $m = n = 12$; we provide in Appendix A and Table 2.2 the same data report as for our CRA generated data.

It is interesting to note that the URA-generated graphs frequently have zero first Betti number as shown in Figures A10, A12, and A14 and Table 2.2, and also have a reasonable distribution of number of vertices versus number of edges, as shown in Figures A9, A11, and A13. On the one hand, this matches our expectation from Theorem 2.1.2 that many graphs will have a trivial first Betti number; on the other hand, it is somewhat surprising that we so frequently have an Urquhart construction with final step being a vertex identification of two vertices at short distance from

Table 2.1: CRA results.

k	p	# graphs generated	fraction of first Betti numbers equal to zero
3	0.02	10,000	0.25
3	0.10	10,000	0.065
3	0.50	10,000	0.0018
4	0.02	10,000	0.61
4	0.10	10,000	0.32
4	0.50	10,000	0.01
5	0.02	10,000	0.88
5	0.10	10,000	0.54
5	0.50	2,939	0.03
6	0.02	10,000	0.88
6	0.10	10,000	0.44
6	0.50	2,894	0.08

Table 2.2: URA results, with $m = n = 12$ for all cases.

k	# graphs generated	fraction of first Betti numbers equal to zero
3	2000	0.67
4	2000	0.74
5	2000	0.81
6	2000	0.83

each other. We obtain these distributions with only 2000 graphs sampled. It is worth noting that by the nature of Urquhart constructions beginning from supergraphs of K_k, the URA produces graphs that would potentially require a large number of recursive steps in order to be produced by the standard Hajós constructions.

The CRA data is somewhat more complicated, in that the probability p that is selected as input for the algorithm plays a significant role in the outcomes. Further, in order to obtain interesting data, it is necessary to generate a larger number of graphs; while it is possible for all of the cases we consider to generate 10000 graphs with CRA, it was not always reasonable using SageMath via CoCalc to compute the first Betti numbers for all these graphs. When Hajós merges are prioritized, on average, the number of vertices in the resulting graphs is much larger than when vertex identifications are prioritized. With a larger number of vertices, it is computationally more expensive to compute the Betti number and as a result only the first approximately 2900 graphs generated by the algorithm were able to be computed, as shown in Table 2.1 for $k = 5$ and $k = 6$ with $p = 0.50$. When $p = 0.50$, Hajós merges are equally likely as vertex identifications in CRA, so it is not surprising that Table 2.1 and Figure A2 show that most of the graphs produced have a positive first Betti number. Also, because a Hajós merge of G_1 and G_2 results in a graph with $|V(G_1)| + |V(G_2)| - 1$ vertices and $|E(G_1)| + |E(G_2)| - 1$ edges, the plot in Figure A1 is not particularly surprising. While Table 2.1 and Figures A3 and A4 show that the situation improves for $p = 0.10$, it is when $p = 0.02$ that we see behavior similar to the URA output.

In summary, based on these initial investigations, the URA appears to be quite effective at producing sample sets of graphs that have a broad distribution of number of vertices against number of edges and also have a large percentage of graphs with zero first Betti number. The CRA is also capable of generating reasonable sample sets, but it is less clear how long it would take the CRA with $p \approx 0.02$ to produce a sample set of graphs with a large average number of vertices. To accomplish this task, the URA appears to be better suited.

Further questions

The original motivation inspiring the definition of $\mathcal{N}(G)$ was to provide lower bounds for the chromatic numbers of Kneser graphs [22]. These topological approaches have also been used subsequently to find sharp lower bounds for chromatic numbers of other graphs, e.g. the stable Kneser graphs [27]. In this work we have shown that Hajós-type constructions of these and other graphs with highly-connected neighborhood complexes are constrained in significant ways, leading to the following:

Problem 2.5.4. *Find Hajós, Ore, and/or Urquhart constructions for Kneser and stable Kneser graphs, or for other graphs with highly-connected neighborhood complexes.*

The operations of Hajós merge and vertex identification leading to Hajós constructibility are foundational in all this work. However, both Urquhart's results and our experimental data has demonstrated that the specific form of Urquhart constructions given in Definition 2.5.3 serve as both a powerful theoretical tool and a useful ingredient in algorithms for sampling k-constructible graphs. This motivates their further study, and thus we define the following families of graphs.

Definition 2.5.5. Let $k \geq 3$, $m \geq k$, and $n \geq 1$. Define $\mathrm{Urq}(k, m, n)$ to be the set of graphs of the form $\mathrm{Urq}(G_1, \dots, G_j)$, where $1 \leq j \leq n$ and for each $i = 1, \dots, j$ the graph G_i is connected, G_i is a supergraph of K_k, and $|V(G_i)| \leq m$.

Thus, $\mathrm{Urq}(k, m, n)$ is the set of graphs that are Urquhart-k-constructible using at most n component graphs (all connected), each having no more than m vertices. Given a k-constructible graph G with $k \geq 3$, Urquhart's proof of Theorem 2.1.3 implies that there exist m, n such that $G \in \mathrm{Urq}(k, m, n)$. The following problems are inspired by classical questions in k-constructibility and topological properties of neighborhood complexes.

Problem 2.5.6. *For a fixed $k \geq 3$, identify infinite families of graphs G for which good upper bounds can be given on the values of m and n where $G \in \mathrm{Urq}(k, m, n)$.*

Problem 2.5.7. *What is the distribution of the chromatic numbers of the graphs in $\mathrm{Urq}(k, m, n)$?*

Problem 2.5.8. *For each fixed i, what is the distribution of the ranks of the i-th homology groups for $\mathcal{N}(G)$ over $G \in \mathrm{Urq}(k, m, n)$?*

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Chapter 3 Two-Matching Complexes

3.1 Introduction

In this chapter, we will use discrete Morse Theory and the Matching Tree Algorithm to prove homotopical results about 2-matching complexes. In Section 3.2, we consider a class of graphs for which the homotopy type of the 2-matching complex is contractible.Then, in Section 3.3 we look at graphs whose homotopy type of the 2-matching complex changes from a sphere to a point with the addition of leaves. We end this section with a constructible algorithm to maximize the number of additional leaves that can be added to a certain family of graphs without changing the homotopy type of $M_2(G)$. In Section 3.4, we define k-matching sequences and look at wheel graphs as a first example. We conclude with k-matchings of perfect caterpillar graphs and future directions.

3.2 Contractibility in 2-matching complexes

We begin this section by exploring properties of 2-matching complexes. Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. Recall a vertex $v \in V(G)$ is a leaf if its neighborhood contains exactly one vertex. For a graph G with $v \in V(G)$, *attaching a leaf to v in G* refers to the process of adding a new vertex w to $V(G)$ and $\{v, w\}$ to $E(G)$. Given a graph G with two leaves u, v and edges $\{v_1, u\}$ and $\{v_2, v\}$, define $G_{(u,v)}$ to be the graph obtained by identifying u and v, labeled uv. That is $E(G_{(u,v)}) = E(G) \setminus \{\{v_1, u\}, \{v_2, v\}\} \cup \{\{v_1, uv\}, \{v_2, uv\}\}$ and $V(G_{(u,v)}) = V(G) \setminus \{u, v\} \cup \{uv\}$.

Definition 3.2.1. For a graph $G = (V(G), E(G))$ with max degree three, the *clawed graph of G*, denoted $C(G)$ or CG is the graph obtained by subdividing every $e \in E(G)$ and attaching a (possibly empty) set of leaves to every $v \in V(G)$ so that $\deg(v) = 3$ for all v. The graph G is called the *core* of $C(G)$. See Example 3.2.2.

If $|E(G)|$ and $|V_{\leq 2}(G)|$ denote the number of edges and the number of vertices with degree less than or equal to 2 in a graph G, respectively, and L is the number of leaves of G, the process of clawing G introduces $|E(G)| + |V_{\leq 2}(G)| + L$ new vertices and $|V_{\leq 2}(G)| + L$ new edges.

Example 3.2.2. Clawing a graph as shown in Figure 3.1: (A) Begin with a graph G, (B) Subdivide each edge (depicted with open circles), (C) attach a set of leaves to each vertex of G so that $\deg(v) = 3$ for all $v \in V(G)$. We say graph G is the core graph of $C(G)$ or $C(G)$ is the clawed graph with respect to G. $|E(G)| = 4 = |V_{\leq 2}(G)|$ and $L = 3$ so the total number of vertices added is 11 and the total number of new (leaf) edges is 7.

Definition 3.2.3. For an edge set $H \subseteq E(G)$, let $V(H)$ denote the set of vertices supported by H. That is, $V(H) := \bigcup_{e \in H} V(e)$. An *induced claw unit* of a graph is a

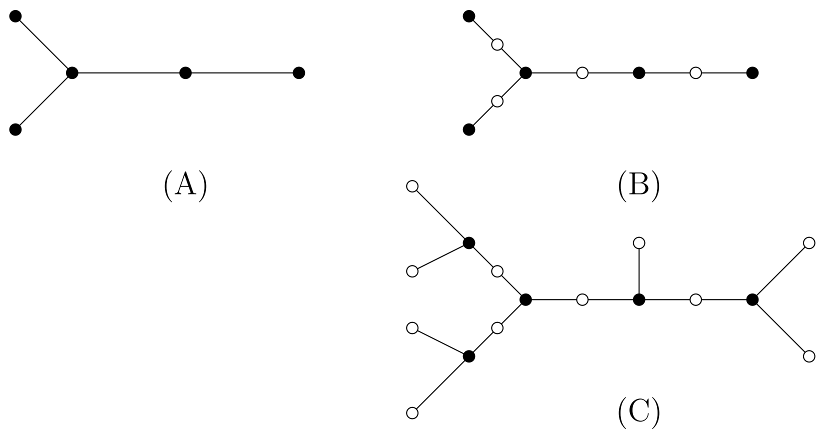

Figure 3.1: Clawing a graph.

$K_{1,3}$ subgraph with 1 vertex of degree 3 in G and 3 vertices of degree less than or equal to 2 in G (Figure 3.2.)

We will be interested in deleting an induced claw unit in a graph. To do so, we consider an induced claw unit c to be defined by the unique degree 3 vertex, call it v. We abuse notation and use $G \setminus c$ to denote the vertex deleted subgraph of $G \setminus \{v\}$, the graph obtained by deleting v and all the edges incident to it.

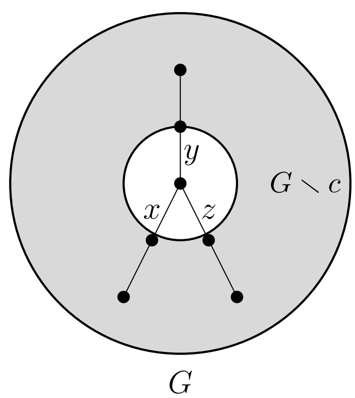

Figure 3.2: The edge set $\{x, y, z\}$ defines an induced claw unit, call it c, of graph G. The shaded region is the graph $G \setminus c$.

We will use discrete Morse theory to determine the homotopy type of clawed graphs. We observe now that induced claw units in graphs behave nicely with 2-matching complexes.

Proposition 3.2.4. *Let $c \in G$ be an induced claw unit with edge set $E(c) = \{x, y, z\}$. The following sets are in bijection with each other:*

(i) The set of 2-matchings of $G \setminus c$,

(ii) The set of 2*-matchings containing* $\{y, z\}$*, and*

(iii) The set of 2*-matchings containing* x *and not* y *or* z*.*

Proof. For any 2-matching m in $G \setminus c$, both $m \cup \{x\}$ (not containing y or z) and $m \cup \{y, z\}$ are 2-matchings in G. Notice that x and $\{y, z\}$ cannot be in a 2-matching together since they all meet at a degree three vertex. $\square$

Example 3.2.5. Consider the graph in Figure 3.3. There is exactly 1 induced claw unit, call it c, given by the edge set $\{x, y, z\}$. The set $\{e\}$ is the only 2-matching of $G \setminus c$. Notice that the 2-matchings contianing $\{y, z\}$ consists of exactly $\{e, y, z\}$ and 2-matchings containing x and not y or z consists of exaclty $\{e, x\}$.

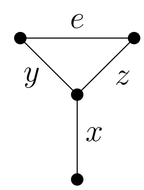

Figure 3.3

We turn our attention to a general connectivity result of $M_2(G)$ for any graph G. Since $\mathcal{F}(M_2(G))$ the face poset of a 2-matching complex of G has vertex set consisting of faces of $M_2(G)$ with an order relation of containment, for $a, b \in \mathcal{F}(M_2(G))$, $a \prec b$ if $b = a \cup e$ for some $e \in E(G)$. In relation to Figure 3.3, suppose we define a partial matching of $\mathcal{F}(M_2(G))$ by toggling on x, where $x \in E(G)$. Then, the matchings remaining after toggling are exactly those that contain $\{y, z\}$ and therefore are in bijection with 2-matchings of $G \setminus c$ by Proposition 3.2.4. Hence, if you have two edge-disjoint induced claw units c_1 and c_2 in G, the choice of toggle edge in c_1 and c_2 and the order in which one toggles is irrelevant.

Lemma 3.2.6. *Let* G *be a simple, finite graph and* $\mathcal{C} = \{c_1, ..., c_n\}$ *be a collection of induced claw units in* G *with* $E(c_i) := \{x_i, y_i, z_i\}$ *for each* c_i*. Then the connectivity of* $M_2(G)$ *is at least* $2|\mathcal{C}| - 2$*. Further, if we fix the toggle edge in each* c_i*, say* x_i*, then every critical cell remaining after toggling on all of the* x_i*'s will consist of* $\{y_i, z_i\}$ *for all* i*, regardless of order.*

Proof. Let $P := \mathcal{F}(M_2(G))$ be the face poset of the 2-matching complex of G. We define a partial (discrete Morse) matching on P by (arbitrarily) fixing x_i as the toggle edge for each c_i. Our claim is that for any permutation $\pi \in \mathfrak{S}_n$, the unmatched subposet that remains after toggling on $x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \ldots, x_{\pi(n)}$ is the upper-order ideal $P_{\geq\{y_{\pi(1)}, z_{\pi(1)}, y_{\pi(2)}, z_{\pi(2)}, \ldots, y_{\pi(n)}, z_{\pi(n)}\}}$. Since permutations can be generated by a sequence of transpositions, it suffices to consider the unmatched subposet obtained from toggling x_1, then x_2 and the unmatched subposet obtained from toggling x_2, then x_1.

Suppose first that we toggle on x_1. The edge $x_1 \in E(G)$ forms a 2-matching with all 2-matchings of G that do not contain both y_1 and z_1 so the unmatched cells of P

are precisely the elements containing both y_1 and z_1. That is, the unmatched subposet that remains is $P_{\geq\{y_1,z_1\}}$. Now, toggling on x_2 matches all of the 2-matchings of G that contain y_1, z_1, but do not contain y_2, z_2. All elements $a \in P_{\geq\{y_1,z_1\}}$ with $\{x_2\} \in a$ will be paired with $b := a \smallsetminus \{x_2\}$ through toggling on x_2 and all elements b are in $P_{\geq\{y_1,z_1\}}$ since $\{y_1, z_1\} \in b$. Notice that all matchings in $P_{\geq\{y_1,z_1\}}$ are in bijection with 2-matchings in $G \smallsetminus c_1$ by Proposition 3.2.4 and $c_2 \in G \smallsetminus c_1$.

Hence, the unmatched subposet that remains after toggling on x_1 then x_2 is precisely $P_{\geq\{y_1,z_1,y_2,z_2\}}$. An analogous argument shows that the same upper order ideal remains after toggling first on x_2 and then x_1. By induction, we get that the unmatched subposet that remains after toggling on $x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \dots x_{\pi(n)}$ is the upper-order ideal $P_{\geq\{y_{\pi(1)},z_{\pi(1)},y_{\pi(2)},z_{\pi(2)},\dots,y_{\pi(n)},z_{\pi(n)}\}}$. Two elements from each induced claw unit contribute to $\{y_{\pi(1)}, z_{\pi(1)}, y_{\pi(2)}, z_{\pi(2)}, \dots, y_{\pi(n)}, z_{\pi(n)}\}$ and an acyclic matching has been produced such that all unmatched sets are of size at least $2|\mathcal{C}|$. It follows that the connectivity of $M_2(G)$ is at least $2|\mathcal{C}| - 2$. □

Definition 3.2.7. Let G be any graph. A *claw-induced partial matching* is an acyclic partial matching on $\mathcal{F}(M_2(G))$ obtained by toggling on elements in the vertex set of $\mathcal{F}(M_2(G))$ corresponding to edges in induced claw units of G, whenever possible.

Observation 3.2.8. If G and H are two graphs with leaf vertices $v_1 \in V(G)$ and $v_2 \in V(H)$, it is immediate for $G \underset{v_1\sim v_2}{\vee} H$ we have

$$M_2(G \underset{v_1\sim v_2}{\vee} H) = M_2(G) * M_2(H),$$

where $*$ denotes the topological join.

Proposition 3.2.9. *If $v_1, v_2 \in V(G)$ are two leaf vertices of a graph G, then $M_2(G) = M_2(G_{(v_1,v_2)})$.*

Proof. Let G be a graph with x, y two leaf nodes and $m \in M_2(G)$ be a 2-matching. A 2-matching $H \subseteq E(G)$ of a graph G consists of vertices $v \in V(H)$ with degree at most 2. Identifying two leaf vertices of G does not affect the 2-matching since the identified vertex has degree 2. So m is also a 2-matching for $G_{(x,y)}$, the graph which identifies x and y. □

Theorem 3.2.10. *Let G be a graph with $e = \{x, y\} \in E(G)$ such that $deg(x) \leq 2$ and $deg(y) \leq 2$. Then $M_2(G)$ is contractible.*

Proof. Since both endpoints of e have degree at most 2, e may be included in any 2-matching of $G \smallsetminus e$ and $M_2(G) \simeq e * M_2(G \smallsetminus e)$. Hence, $M_2(G)$ is a cone and therefore contractible. □

We observed in Theorem 3.2.10 that graphs that contain an edge with endpoints of degree less than or equal to 2 form a large class of graphs that have contractible 2-matching complexes. We will now explore 2-matching complexes that are close to but not contractible. We begin by considering clawed paths of even length. In the following proposition, we use the well-known fact that for two spheres S^m and S^n, $S^m * S^n \simeq S^{m+n+1}$.

Proposition 3.2.11. *For $n \geq 0$, let CP_n be a clawed path with respect to a path of length n. Then, $M_2(CP_n) \simeq S^{2n+1}$.*

Proof. Since P_0 consists of one vertex and no edges, we have $CP_0 = K_{3,1}$. See Figure 3.4. It follows that $M_2(CP_0) \simeq S^1$. Consider now a clawed path of length 1, CP_1 consists of two copies of $K_{3,1}$ intersecting at one vertex. By Observation 3.2.8 we have $M_2(CP_1) = M_2(CP_0 \vee CP_0) = M_2(CP_0) * M_2(CP_0) = S^1 * S^1 \simeq S^{1+1+1} = S^3$. Continuing inductively, we have $M_2(CP_n) = M_2(CP_{n-1} \vee CP_0) \simeq S^{2(n-1)+1} * S^1 \simeq S^{2n-2+3} = S^{2n+1}$. $\square$

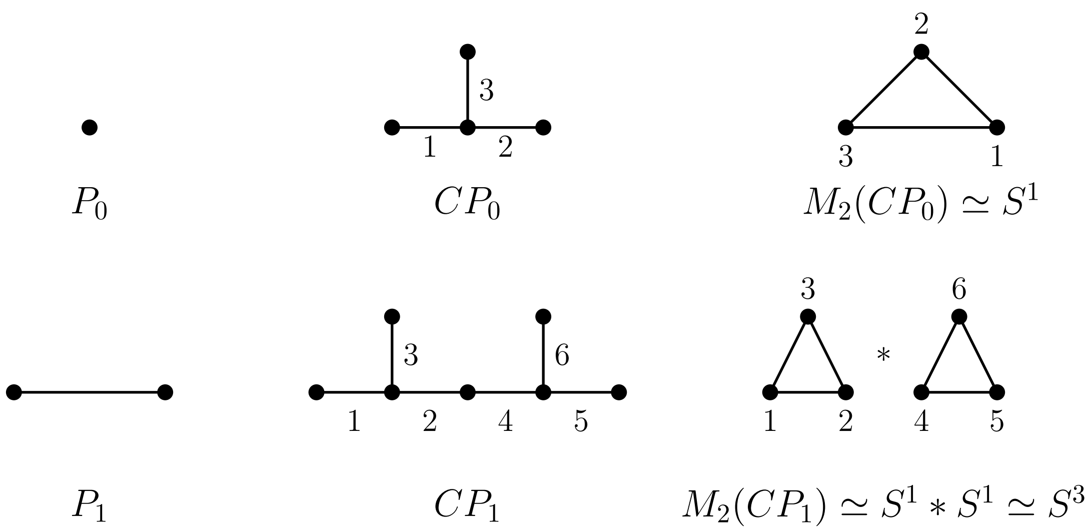

Figure 3.4: $M_2(CP_0)$ and $M_2(CP_1)$ as in the proof of Proposition 3.2.11.

Corollary 3.2.12. $M_2(CP_{n-1}) \simeq M_2(CC_n) \simeq S^{2n-1}$.

Proof. The result follows from Proposition 3.2.9, see Figure 3.5. $\square$

In the next proposition, we see that, even further, the 2-matching complex for a clawed cycle shares its homotopy type with the 2-matching complex of a fully whiskered cycle.

Definition 3.2.13. A *fully whiskered graph* $W(G)$ is a graph in which a leaf is attached to every vertex of the graph G.

Proposition 3.2.14. *Let WC_m denote a fully whiskered $2m$-cycle graph for $m \geq 3$. $M_2(WC_m) \simeq S^{2m-1}$.*

Proof. Label the edges of the cycle by $1, 2, ..., 2m$ and each leaf edge by $x_{i,i+1}$ for $i \in [2m-1]$, and $x_{1,2m}$, where the index corresponds to the incident edges in the cycle as in Figure 3.6. Let the edge set $c_i := \{x_{i,i+1}, i, i+1\}$ for each $i \in \{1, 3, 5, \ldots, 2m-1\}$ denote an induced claw of WC_m. Then the collection $\mathcal{C} = \{c_1, c_3, \ldots, c_{2m-1}\}$ defines

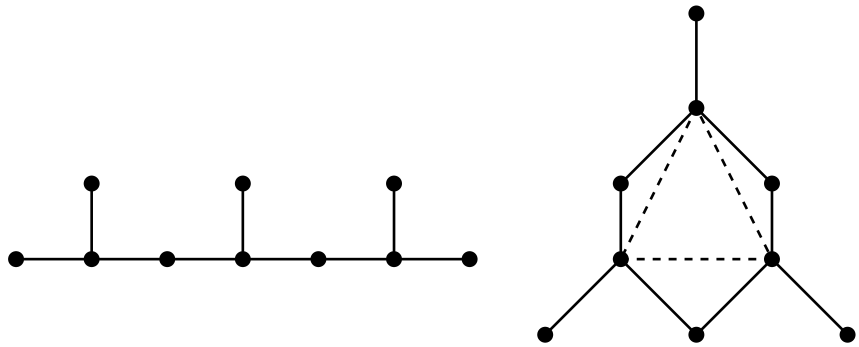

Figure 3.5: On the left graph CP_2, the clawed path of length 2 and on the right CC_3, the clawed 3-cycle obtained by identifying the endpoints of CP_2. The core 3-cycle is shown with dashed lines.

a family of m induced claw units that are edge disjoint. If this were not the case, then one edge $j \in E(WC_m)$ would be an edge in two claws, but by the labeling system this would mean that $j = j + 1$ which is a contradiction to the edge labels on the cycle. Following the proof of Lemma 3.2.6, for each $i \in \{1, 3, 5, \ldots, 2m-1\}$ let $x_{i,i+1}$ be the toggle edge in the discrete Morse matching on the face poset of $M_2(WC_m)$. By Lemma 3.2.6, we know the connectivity of $M_2(WC_m)$ is at least $2(m) - 2$. Further, every unmatched cell contains $\{1, 2, \ldots, 2m\}$, that is all of the edges in the even cycle. All of the edges in the cycle forms a maximal two matching of WC_m and hence $\{1, 2, \ldots, 2m\}$ is the only critical cell of the discrete Morse matching and $M_2(WC_m) \simeq S^{2m-1}$. □

Corollary 3.2.15. *$M_2(WC_n) \simeq M_2(CC_n)$ for $n \geq 3$.*

In Proposition 3.2.14, we considered fully whiskered $2m$-cycle graphs because we are interested in aligning this result with clawed path graphs, but there is no reason why we could not apply the same reasoning for fully whiskered odd-cycle graphs.

Theorem 3.2.16. *Let WC_n^d denote a fully whiskered n-cycle graph for odd n. Then, $M_2(WC_n^d) \simeq S^{n-1}$.*

Proof. Using the same claw-induced partial matching as in the proof of Proposition 3.2.14 for all $i \in \{1, 3, \ldots, n-2\}$, the remaining unmatched cells must contain $\{1, 2, \ldots, n-1\}$. These cells form an upper order ideal in the partially matched face poset of $M_2(WC_n^d)$ and include precisely $\{x_{n,1}, 1, 2, ..., n-1, x_{n-1,n}\}$, $\{1, 2, ..., n\}$, $\{x_{n,1}, 1, 2, ..., n-1\}$, $\{1, 2, ..., n-1, x_{n-1,n}\}$, and $\{1, 2, ..., n-1\}$. Performing a final toggle on the edge $x_{n-1,n}$, we obtain one critical cell, $\{1, 2, ..., n\}$ and hence $M_2(WC_n^d) \simeq S^{n-1}$. □

We saw in Corollary 3.2.15 that $M_2(CC_n) \simeq M_2(WC_n) \simeq S^{2n-1}$ and it is no coincidence that CC_n is a subgraph of WC_n. The next lemma shows that there are certain degree two vertices such that attaching a leaf does not affect the homotopy type of the 2-matching complex. We call such vertices *attaching sites*.

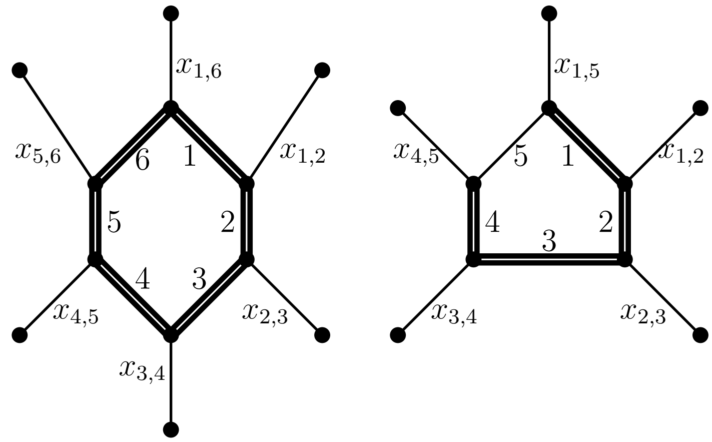

Figure 3.6: On the left a complete matching on WC_3 as in Proposition 3.2.14 highlighted with a double line and on the left the partial matching on WC_5^d as in Remark 3.2.16 highlighted with a double line.

Lemma 3.2.17. *Let $CG := C(G)$ be a clawed graph with vertex set $V(CG)$, edge set $E(CG)$, and $v \in V(CG)$ a degree two vertex with $e_1, e_2 \in E(CG)$ the two incident edges to v. Define a complete claw-induced partial matching on P, the face poset of $M_2(CG)$. Then both edges e_1 and e_2 are in a critical cell if and only if attaching a leaf to v does not change the homotopy type. Further, if at least one edge, $\{e_1, e_2\}$ is not in any critical cell obtained from the complete claw-induced partial matching of P, the 2-matching complex of CG with a leaf attached to v is contractible.*

Proof. Since CG is a clawed graph and $\deg(v) = 2$, v is the intersection of two claws c_1 and c_2. For each claw, one of the edges is a toggle edge and two are in critical cells. If e_1 and e_2 are in some critical cell; then they are in all critical cells since this would mean that one of the other edges in c_1 and c_2 are toggled on. Attaching a leaf w to v does not give rise to any additional cells since this would imply that e_1, e_2, and the edge $\{v, w\}$ are all in a 2-matching together, but this is not possible because they are all incident to a common vertex.

Suppose now that no critical cell contains both e_1 and e_2 (but perhaps contains one). Then attaching a leaf w to v gives rise to several new critical cells, under the same matching $\mathcal{M}$. For each critical cell X in the claw-induced partial matching on P, $X \cup \{w, v\}$ is a critical cell in the claw-induced partial matching on $\mathcal{F}(M_2(CG \cup \{w, v\}))$. Therefore, every critical cell can be further matched by toggling on $\{w, v\}$ and $M_2(CG \cup \{w, v\})$ is contractible. $\square$

Theorem 3.2.18. *For a clawed graph CG, $M_2(CG) \simeq S^{\frac{2}{3}n-1}$ where $n = |E(CG)|$.*

Proof. The clawed graph CG consists of a collection of claws that have pairwise intersection of at most 1 vertex, that is a collection of $\frac{1}{3}n$ induced claw units. Each claw in this claw decomposition of G gives rise to one toggle edge and two edges in the critical cell. By Lemma 3.2.6, the connectivity of $M_2(CG)$ is at least $\frac{2}{3}n - 2$. Further the complete claw-induced partial matching will consist of one critical cell consisting of two edges of each claw which defines a maximal matching on the clawed graph CG. Since a graph has $\frac{1}{3}n$ claws and two of every one belongs in the critical cell, the critical cell has size $\frac{2}{3}n$ and $M_2(CG) \simeq S^{\frac{2}{3}n-1}$. □

We can relate these findings back to [14, Theorem 12.5] which gives a general connectivity bound for these complexes. Recall that a clique of a graph is a induced subgraph that is isomorphic to a complete graph. For a real number ν, a family of sets Δ is $AM(\nu)$ if Δ admits an acyclic matching such that all unmatched sets are of dimension $\lceil \nu \rceil$. For $\lambda = (\lambda_1, ..., \lambda_n)$ define $|\lambda| = \sum_{i=1}^{n} \lambda_i$. For a sequence $\mu = (\mu_1, ..., \mu_n), n \geq 1$, define

$$\alpha(n, \mu) = \min\{\alpha : BD_n^{\lambda} \text{ is } AM(\frac{|\lambda| - \alpha}{2} - 1)\}.$$

Theorem 3.2.19. *(Thm 12.5, [14]) Let G be a graph on the vertex set V. Let $\{U_1, .., U_t\}$ be a clique partition of G and let $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mu = (\mu_1, ..., \mu_n)$ be sequences of nonnegative integers such that $\lambda_i \leq \mu_i$ for all i. Then $BD_n^{\lambda}(G)$ is $\lceil \nu \rceil - 1$ connected, where*

$$\nu = \frac{|\lambda|}{2} - \frac{1}{2} \sum_{j=1}^{t} (\alpha(|U_j|, \mu_{U_j}) - 1$$

Proposition 3.2.20. *Theorem 3.2.18 is an example where Theorem 3.2.19 is not sharp.*

Proof. To show this we need to choose a clique partition. By construction of the clawed graphs, the best we can do is choosing a partition of 2- and 1-cliques. Let $\lambda = (2, 2, ..., 2) = \mu$. By [14, Lemma 12.6], all values of α are 2 and any μ with $\lambda_i < \mu_i$ for $i = 1, 2$ would give rise to larger α values. So, the lower bound on connectivity is given by $\nu = \frac{|\lambda|}{2} - \frac{1}{2} \sum_{j=1}^{t} 2 - 1$. Let T denote the number of claws in G. Since $|\lambda| = 2|V(G)|$ and $t = T + (|v| - 2T) = \frac{|E|}{3} + (|V| - 2\frac{|E|}{3})$, ν simplifies to $|v| - (\frac{|E|}{3} + |v| - 2\frac{|E|}{3}) - 1 = \frac{|E|}{3} - 1$. From Theorem 3.2.18, the actual dimension of the 2-matching complex is $\frac{|E|}{3}$, greater than the lower bound obtained from Theorem 3.2.19. □

3.3 Clawed Non-separable Graphs

Suppose we have a graph with potential attaching sites, i.e. vertices of degree 2. It is natural to ask, which of these degree 2 vertices are actually attaching sites with respect to some matching. In addition, once we start attaching leaves, how many can we attach before the 2-matching complex becomes contractible? To analyze these questions, we will focus our attention on clawed non-separable graphs. Our overall goal of this section will be to maximize the number of attaching sites in a clawed graph by pairing toggle edges in the graph.

Definition 3.3.1. A *non-separable, i.e. 2-connected, graph* is a connected graph in which the removal of any one vertex results in a connected graph.

Non-separable graphs can be classified through the following construction [5, Proposition 3.1.1]:

1. Begin with a graph $G := n$-cycle
2. Choose two vertices of G, say v_1 and v_2.
3. Attach the two endpoints of a path to v_1 and v_2 respectively.
4. Set G to be this new graph and return to (2).

Using this construction we can define a clawed non-separable graph.

Definition 3.3.2. A *clawed non-separable graph* is a graph obtained through the following construction.

1. Begin with G a clawed n-cycle, that is $G := C(C_n)$.
2. Choose two leaves of G, say v_1 and v_2.
3. For each endpoint x in a path P, let one of the leaves attached to x be an endpoint of the clawed path, CP. Attach the two endpoints of a clawed path to v_1 and v_2 respectively.
4. Set G to be this new graph and return to (2).

We can use the construction of clawed non-separable graphs to get an understanding of the relationship between the number of claws in a clawed non-separable graph and the number of leaves. This will eventually lead us to finding an upper bound for the number of attaching sites in such a graph. Recall that an *attaching site* is a degree two vertex such that attaching a leaf does not affect the homotopy type of the resulting 2-matching complex.

Proposition 3.3.3. *Let T be the number of claws in a clawed non-separable graph and L the number of leaves. Then T and L have the same parity modulo 2.*

Proof. It is clear that for the clawed graph of a non-separable n-cycle the parity of T and L is the same. Then, by construction two leaves are chosen, changing the number of leaves but keeping the parity the same. For each additional claw we add another leaf and the parity remains the same. □

A consequence of this proposition is that there is an even number of possible toggle edges that are not in induced claw units that contain a leaf. Our strategy for obtaining an upper bound for the maximum number of attaching sites will be to *pair* the toggle edges such that two *paired* toggle edges are incident to each other.

Theorem 3.3.4. *Let $C(H)$ be the clawed graph of a non-separable graph H such that $C(H)$ has T claws. Then, the upper bound for the maximum number of leaves that can be added before changing the homotopy type of the 2-matching complex of $C(H)$ is T.*

Proof. The total number of possible attaching sites is given by $\frac{3T-L}{2}$ since each claw has three vertices with degree less than three, we need to remove the number of leaves since the degree is one, and then divide by two since all remaining vertices are the intersection of two claws. Now to find the maximum number of attaching sites we subtract away the minimum number of vertices that have at least one edge that is toggled on.

There is one toggled edge per claw and for any claw that has a leaf we can choose the leaf as the toggle edge, which will maximize the number of attaching sites since no additional leaf can be added to either endpoint of a leaf edge. The most ideal matching pairs the toggled edges, so minimally we have $\frac{T-L}{2}$ vertices that cannot be sites.

Hence, we have a maximum of $\frac{3T-L}{2} - \frac{T-L}{2} = \frac{2T}{2} = T$ attaching sites. □

The strategy in the proof of Theorem 3.3.4 was to *pair* toggle edges as a way to maximize the number of attaching sites. We provide two examples (Figures 3.7 and 3.8) in which the toggle edges are depicted with a solid line and the edges in the critical cell are depicted as double lines. In Figure 3.7, we have an example of a clawed non-separable graph together with a partial matching which attains the maximum number of attaching sites, namely 5.

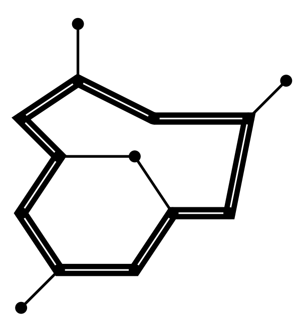

Figure 3.7: Clawed non-separable graph which attains the maximum attaching sites.

It is not always the case that we can achieve the upper bound for the number of attaching sites for clawed non-separable graphs. In Figure 3.8, we see that after toggling on the leaf edges and doing our best to pair the inner toggle edges we are still left with two independent induced claw units that are surrounded by edges that are already in the critical cell. No matter which edge we choose in either of these induced claw units as the toggle edge, we will decrease the total number of possible attaching sites and thereby the number of possible attaching sites is less than the maximum.

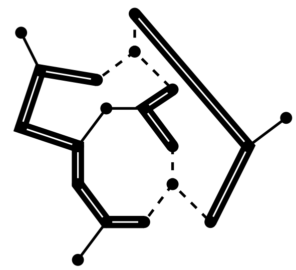

Figure 3.8: Clawed non-separable graph which does not achieve maximum attaching sites.

We end this section with a constructible algorithm to obtain a maximal number of attaching sites in a clawed non-separable graph.

This constructible algorithm to obtain a maximal number of attaching sites prioritizes using leaf edges as toggle edges followed by pairing non-leaf toggle edges. Using Lemma 3.2.6, we may arbitrarily choose one of the three edges in each of our claws without changing the homotopy type generated by the claw-induced partial matching. At each step we are bringing together as many of the toggle edges as possible to attain the maximal number of attaching sites. Figure 3.9 provides an example.

1. Begin with a clawed n-cycle and a claw decomposition $C = \{c_1, ..., c_n\}$. Choose all the leaves as toggle edges such that all edges in the cycle are in the critical cell.

2. Choose two claws, c_i and c_j to attach the next clawed path. Notice that c_i and c_j are induced claw units that contain a leaf, which we call *leaf-claws*. Change the matching on these two leaf-claws so that:

   a) For each of the chosen leaf claws c_i and c_j: If the leaf claw is incident to a previously chosen or currently chosen leaf claw change the matching to match the toggle edges of these two leaf-claws, prioritizing the leaf claws incident to only one previously chosen leaf-claws. In doing so the number of attaching sites will either remain the same or increase.

3. For the new clawed path, let all of the leaves be the toggle edges.

4. Return to (2).

This algorithm returns the maximum number of attaching sites. Consider taking a claw-induced matching on a clawed non-separable graph. If it was possible to increase the number of attaching sites by modifying this matching, one of two scenarios may be present:

(i) there exists a leaf-claw such that the toggle edge is not the leaf edge, or

(ii) there exists a pair of incident claws such that neither one has a toggle edge that is already incident to another toggle edge.

Through this algorithm, all leaves are toggle edges so (i) is not present. Notice that if (ii) appeared in this construction it would arise from step (2) of the algorithm when we add a new clawed path, but during that step we are re-orienting so that whenever possible toggle edges are incident to each other.

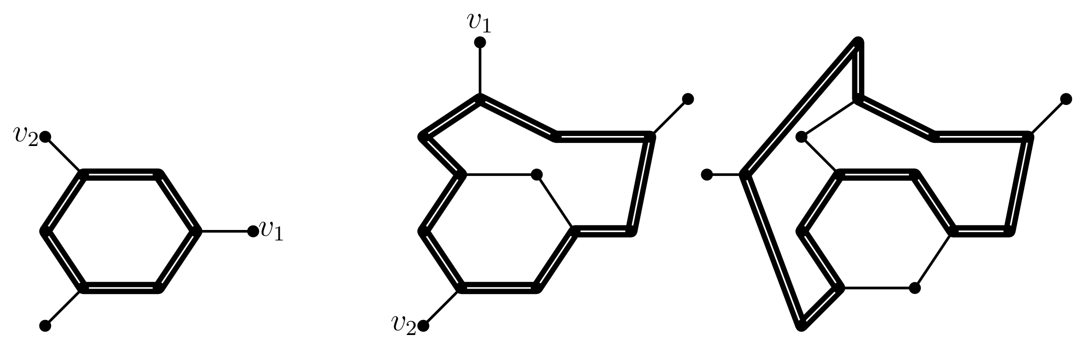

Figure 3.9: On the left most picture we start with a clawed cycle. Choosing two points, v_1 and v_2 we attach a clawed path of length 2. Since the chosen two leaf claws are incident, we pair the toggle edges of each. Then we choose two more vertices, v_1 and v_2 and continue. In this step there is one claw unit that is incident to two leaf claws and the other claw unit is incident to one.

3.4 k-matching sequences

We now turn our attention to the relationship between 1-matchings and 2-matchings. Define a *k-matching sequence* as the sequence $(M_1(G), M_2(G), M_3(G), \ldots, M_n(G))$, up to homotopy, for $1 \leq k \leq n$ and where $M_n(G)$ is a contractible space. The n-matching complex $M_n(G)$ is a cone, hence contractible precisely when there is an edge $e \in E(G)$ with both endpoints having max degree n. In this section we will look at the k-matching sequence for wheel graphs.

Let W_n be a wheel graph on n vertices, that is a graph formed by connecting every vertex of a $n-1$ cycle to a single universal vertex. Label the edges of the cycle with $c_0, ..., c_{n-2}$ and inner edges by $\ell_0, \ell_1, \ldots, \ell_{n-2}$, where ℓ is used to symbolize "leg" edges, such that c_i shares a vertex with ℓ_{i-1} and ℓ_i modulo $n-1$. See Figure 3.10.

We will determine the homotopy type of the 1-matching complex and 2-matching complex of wheel graphs. In the proof of Theorem 3.4.2, we will first focus on the "legs" or spokes of the wheel and then on the outer cycle. In [19], Kozlov proves the following proposition which will come in handy.

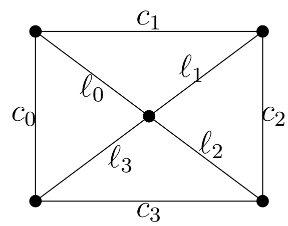

Figure 3.10: W_5 and the labeling used in Theorems 3.4.2 and 3.4.4.

Proposition 3.4.1 (Kozlov, [19] Proposition 5.2)**.** *For $n \geq 1$, let C_n denote the cycle of length n. The homotopy type of the independence complex of the cycle graph is*

$$Ind(C_n) \simeq \begin{cases} S^{\nu_n} \vee S^{\nu_n} & n \equiv 0 \ mod\ 3 \\ S^{\nu_n} & n \not\equiv 0 \ mod\ 3. \end{cases}$$

where $\nu_n = \lceil \frac{n-4}{3} \rceil$.

Theorem 3.4.2. *Let W_n be a wheel graph on n vertices. Then, for $k \in \mathbb{N}$, the homotopy type of $M_1(W_n)$ is given by:*

$$M_1(W_n) \simeq \begin{cases} S^{\nu_n} \vee S^{\nu_n} & n \equiv 1 \ mod\ 3 \\ \bigvee_{n-2} S^{\nu_n} & n \equiv 2 \ mod\ 3 \\ \bigvee_{n} S^{\nu_n} & n \equiv 0 \ mod\ 3 \end{cases}$$

where $\nu_n = \lceil \frac{n-4}{3} \rceil$.

Proof. The strategy of this proof will be to use the Matching Tree Algorithm on the line graph of W_n, see Figure 3.11. The line graph of W_n, denoted $L(W_n)$ is given by a complete graph on $n-1$ vertices, labeled $\ell_0, \ldots, \ell_{n-2}$ and an $(n-1)$-cycle graph $c_0, \ldots, c_{n-2}$ with the additional edges $\{c_j, \ell_{j-1}\}$ and $\{c_j, \ell_j\}$ where j is calculated modulo $n-1$. We derive the homotopy type of $M_1(W_n)$ by defining an acyclic (discrete Morse) matching on the face poset of the independence complex of $L(W_n)$ using the Matching Tree Algorithm.

Let P denote the face poset of $Ind(L(W_n))$. To begin we start with a tentative pivot ℓ_0 which gives rise to two children $\Sigma(\emptyset; \ell_0)$ and $\Sigma(\ell_0; \ell_1, \ldots, \ell_{n-2}, c_0, c_1)$. We first address the right child $\Sigma(\ell_0; \ell_1, \ldots, \ell_{n-2}, c_0, c_1)$. The elements of $V \setminus (A \cup B)$ are $c_2, c_3, \ldots c_{n-2}$. Since c_2 has exactly 1 neighbor in $V \setminus (A \cup B)$, use c_2 as a pivot leading to 1 child $\Sigma(\ell_0, c_3; \ell_1, \ldots, \ell_{n-2}, c_0, c_1, c_2, c_4)$ where c_3 is the matching vertex. Continue in this fashion consecutively choosing the pivot $c_{f(2)}, c_{f(3)}, \ldots, c_{f(k)}$ where $3k < n-1$ and $f(i) = j + 3(i) \text{ mod } n-1$, with j the index on the tentative pivot of this branch, namely the index of ℓ_j.

Notice $\frac{n-1}{3}$ is the number of groups of 3 that we can break the $(n-1)$ cycle into, where each group consists of 1 pivot and 2 neighbors of that pivot. Hence,

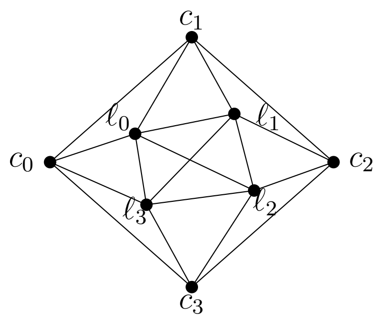

Figure 3.11: LW_5, the line graph of W_5.

for $n \equiv 0 \bmod 3$ and $n \equiv 2 \bmod 3$ $\frac{n-1}{3}$ is not a whole number meaning that all vertices in the outer cycle are either in A or B at the time we reach $c_{f(k)}$. Therefore $\ell_0, c_{f(1)}, c_{f(2)}, \ldots, c_{f(k)}$ is the single critical cell of this branch.

When $n \equiv 1 \bmod 3$, $\frac{n-1}{3}$ is a whole number and we have a group of 3 left over when we reach $c_{f(k)}$, 2 of which are already in A. Therefore, we have an isolated vertex and an empty leaf results, i.e. there are no critical cells of this branch.

Now, turning our attention to $\Sigma(\emptyset, \ell_0)$ we iterate this process using ℓ_1 as our tentative vertex. Due to the symmetry of $L(W_n)$, each branch beginning with $\Sigma(\ell_j, N(\ell_j))$ will either result in an empty leaf or a single critical cell as described above. The general structure of our matching tree can be seen in Figure 3.12.

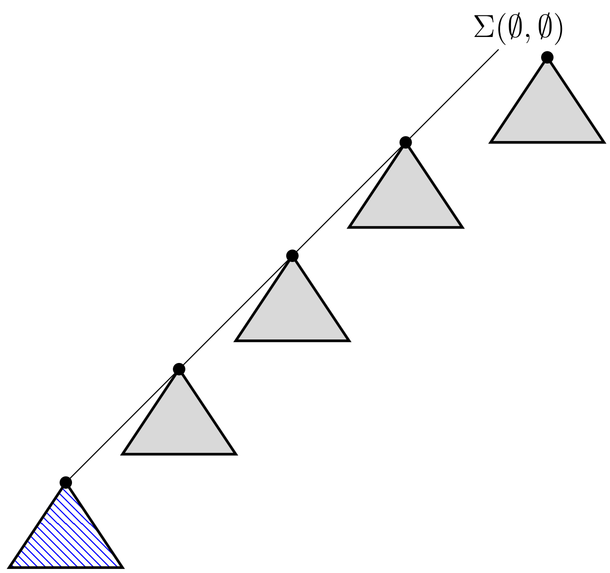

Figure 3.12: The shaded branches have identical structure with the first element of each branch starting with $A = \emptyset$. The last stripped branch is representative of the outer cycle.

Once all vertices of the complete graph have been chosen as tentative vertices, we are left with one child $\Sigma(\emptyset; \ell_0, \dots \ell_{n-2})$ and $V \setminus (A \cup B)$ consists of only vertices on the outside cycle. When $n \equiv 0 \bmod 3$, and $m = n-1 \equiv 2 \bmod 3$, Proposition 3.4.1 states there exists a matching tree with one critical cell of size $\nu_n + 1$. Additionally, from each of the other branches we have critical cells of size $\nu_n + 1$. By Theorem 1.3.2, the homotopy type is a wedge of spheres, $M_1(W_n) \simeq \bigvee_n S^{\nu_n}$ when $n \equiv 0 \bmod 3$.

When $n \equiv 1 \bmod 3$, and $m = n-1 \equiv 0 \bmod 3$, each of the branches resulting from vertices of the complete graph are empty. Hence, $M_1(W_n) \simeq M_1(C_m) = Ind(C_m) \simeq S^{\nu_n}$.

Finally, when $n \equiv 2 \bmod 3$, and $m = n-1 \equiv 1 \bmod 3$, a subtle shift occurs. Notice that $\nu_n = \nu_m + 1$ when $m = n-1$ so Proposition 3.4.1 says we have one critical cell of size ν_n and each of the $n-1$ branches gives rise to a critical cell of size $\nu_n + 1$. We now argue that we can further match the cells $\alpha := \{\ell_{n-2}, c_{f(1)}, \dots, c_{f(k)}\}$ and $\beta := \{c_{f(0)}, \dots, c_{f(k)}\}$. We do so by showing that there exists a linear extension which is a modification of the linear extension $\mathcal{L}$ that was generated by the Matching Tree Algorithm with $u(\beta) = \alpha$, which by Theorem 1.2.3 gives us that there is an acyclic matching with α and β paired, as desired.

First note that $\{\ell_{n-2}, c_{f(1)}, \dots, c_{f(k)}\}$ is a facet in the independence complex of $L(W_n)$ for $n \equiv 2 \bmod 3$ which means it is a maximal element of the face poset. Since $\beta \prec \alpha \in P$, β is a coatom.

We claim for any pair $(x, u(x))$ for which $\beta <_P x$ or $\beta <_P u(x)$ (i.e. $\beta <_{\mathcal{L}} (x, u(x))$, α is incomparable to x and to $u(x)$. If $\beta \prec_P x \prec_P u(x)$, then α is incomparable to x and incomparable to $u(x)$ since $\beta \prec \alpha$ and α is maximal. Suppose β is incomparable to x and $\beta <_P u(x)$. Since $\beta \prec \alpha$, α is incomparable to $u(x)$. Since β is incomparable to x, $\beta < u(x)$, and $x \prec u(x)$ it must be that $\beta \cup x \subseteq u(x)$. In addition, α and β differs by 1 element and if $x < \alpha$ this would mean $\alpha = \beta \cup x$ which is a contradiction to the incomparability of $u(x)$.

This means that any pair $(x, u(x))$ in $\mathcal{L}$ such that $\beta < (x, u(x))$ can be moved above α. The only concern is if there exists elements $(y, u(y))$ such that $(y, u(y)) <_{\mathcal{L}} \alpha$ and $(y, u(y)) >_{\mathcal{L}} (x, u(x))$ but this is not possible as this means $(y, u(y)) >_{\mathcal{L}} (x, u(x)) >_{\mathcal{L}} \beta$ and we have seen $(y, u(y))$ is incomparable to α.

Finally, we note that for any pair $(y, u(y))$ such that $(y, u(y)) <_{\mathcal{L}} \alpha$, we have seen $\beta \not<_{\mathcal{L}} (y, u(y))$ and therefore it is either the case that $(y, u(y))$ is incomparable to β or $(y, u(y)) <_{\mathcal{L}} \beta$.

Hence, we can rearrange $\mathcal{L}$ so that $u(\beta) = \alpha$ which implies pairing α and β forms an acyclic matching. It follows from Theorem 1.3.2 that this homotopy type for $M_1(W_n) \simeq \bigvee_{n-2} S^{\nu_n}$. □

The next theorem show that for $n \geq 6$, $M_2(W_n)$ is contractible.

Lemma 3.4.3 ([14, Lemma 4.3]). *Let Δ_0 and Δ_1 be disjoint families of subsets of a finite set such that $\tau \nsubseteq \sigma$ if $\sigma \in \Delta_0$ and $\tau \in \Delta_1$. If $\mathcal{M}_i$ is an acyclic matching on Δ_i for $i = 0, 1$ then $\mathcal{M}_0 \cup \mathcal{M}_1$ is an acyclic matching on $\Delta_0 \cup \Delta_1$.*

Theorem 3.4.4. *Let W_n be a wheel graph on n vertices. Then, for $k \in \mathbb{N}$, the homotopy type of $M_2(W_n)$ is given by:*

$$M_2(W_n) \simeq \begin{cases} S^2 \vee S^2 \vee S^2 & n = 4 \\ S^3 \vee S^3 & n = 5 \\ pt & n \geq 6. \end{cases}$$

Proof. Let P_n be the face poset of $M_2(W_n)$. See figure 3.10 for an example of the labeling of W_n. Our strategy will be to define acyclic matchings on subposets of P_n and then apply Theorem 1.2.4. Define Q_n to be a poset on the elements $\{\mathbf{c_0}, \mathbf{c_2}, \mathcal{R}\}$ given by the relations $\mathbf{c_0} \prec \mathbf{c_2} \prec \mathcal{R}$. The target elements in Q_n are in bold to differentiate them from vertices of W_n. Now, we define the poset map $\Gamma_n : P_n \to Q_n$ by defining the preimage $\Gamma_n^{-1}(\alpha)$ for each $\alpha \in Q_n$.

- For $n = 4$ let $\Gamma_n^{-1}(\mathcal{R}) := \{\{c_1, c_2, \ell_2\}, \{c_2, \ell_2\}, \{c_2, \ell_2, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_2, \ell, \ell_2\}\}$.
- For $n \geq 5$ let $\Gamma_n^{-1}(\mathcal{R}) := \{m \in M_2(W_n) | \{c_1, \ell_0, \ell_1\} \subseteq m$ or $\{c_1, \ell_0, c_3, \ell_2\} \subseteq m$ or $\{c_{n-2}, \ell_{n-2}, c_1, \ell_1\} \subseteq m$ or $\{c_{n-2}, \ell_{n-2}, c_3, \ell_2\} \subseteq m\}$.
- $\Gamma_n^{-1}(\mathbf{c_2}) := \{m \in M_2(W_n) | \{c_1, \ell_0\} \subseteq m$ or $\{c_{n-2}, \ell_{n-2}\} \subseteq m\} \setminus \Gamma_n^{-1}(\mathcal{R})$
- $\Gamma_n^{-1}(\mathbf{c_0}) = \{m \in M_2(W_n) | \{c_0\} \subseteq m$ or $m \cup \{c_0\} \in M_2(W_n)\}$.

Since every 2-matching of W_n either contains c_0, $\{c_1, \ell_0\}$, or $\{c_{n-2}, \ell_{n-2}\}$, elements of P_n have been assigned an image under Γ_n and, by definition, Γ_n is order-preserving poset map. For the preimages $\Gamma_n^{-1}(\mathbf{c_0})$ and $\Gamma_n^{-1}(\mathbf{c_2})$ perform a toggle on c_0 and c_2, respectively. That is, for each $\sigma \in \Gamma_n^{-1}(\alpha)$ that does not contain α, pair σ with $\sigma \cup \{\alpha\}$. By Lemma 1.2.6, these matchings are acyclic. In addition, both of these toggles result in a perfect (discrete Morse) matching. Notice that what remains are the elements of $\Gamma_n^{-1}(\mathcal{R})$ which is a set of disjoint subposets for $n \geq 5$ where each of the sets $\{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\}$, and $\{c_{n-2}, \ell_{n-2}, c_3, \ell_2\}$ are the minimal vertices of the respective subposets. Since the (poset) join between any two of these elements would contain more than two leg edges, which is not possible in a 2-matching, these posets are pairwise disjoint.

<u>Claim</u>: Each subposet either consists of 1 element or is associated to a contractible subcomplex for $n \geq 4$.

Recall that any subset of edges in a disjoint union of paths forms a 2-matching. Each of the sets $\{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\}$, and $\{c_{n-2}, \ell_{n-2}, c_3 \ell_2\}$ contains two leg edges and two cycle edges. Hence the possible edges that we union with any of these elements to form a 2-matching form a disjoint union of paths when $n \geq 6$. When $n \geq 7$, toggling on $\mathbf{c_4}$ will pair away all of the remaining cells since $\mathbf{c_4}$ can be in any 2-matching containing the sets $\{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\}$, and $\{c_{n-2}, \ell_{n-2}, c_3 \ell_2\}$. For $n = 6$, toggles can be made with $\mathbf{c_1}, \mathbf{c_3}$ and $\mathbf{c_4}$. Therefore, by Lemma 3.4.3, $M_2(W_n) \simeq$ pt when $n \geq 6$.

When $n = 5$, $\Gamma_n^{-1} = \{\{c_1, \ell_1, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_3, \ell_3, \ell_2, c_1\}, \{c_1, \ell_0, \ell_2, c_3\}, \{c_1, \ell_1, \ell_3, c_3\}\}$. Toggling on $\mathbf{c_1}$ and $\mathbf{c_3}$ leaves 2 critical 3-cells, namely $\{c_1, \ell_0, \ell_2, c_3\}, \{c_1, \ell_1, \ell_3,$

$c_3\}$. Hence, $M_2(W_5) \simeq S^3 \vee S^3$. When $n = 4$, $\Gamma_n^{-1}(\mathcal{R}) := \{\{c_1, c_2, \ell_2\}, \{c_2, \ell_2\}, \{c_2, \ell_2, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_2, \ell_1, \ell_2\}\}$ and toggling on $\mathbf{c_1}$ leaves 3 critical 2-cells $\{c_1, \ell_0, \ell_1\}, \{c_2, \ell_1, \ell_2\}$, and $\{c_2, \ell_2, \ell_0\}$. Hence, $M_2(W_4) = S^2 \vee S^2 \vee S^2$.

□

Since $M_3(W_n) \simeq \text{pt}$, we have the k-matching sequence of W_4 is $(S^{\nu_n} \vee S^{\nu_n}; S^2 \vee S^2 \vee S^2; \text{pt})$ and for W_5 is $(\bigvee_{n-2} S^{\nu_n}; S^3 \vee S^3;\ \text{pt})$ where $\nu = \lceil \frac{n-4}{3} \rceil$.

3.5 Two-matching complexes of caterpillar graphs

A *caterpillar graph* is a tree in which every vertex is on a central path or only one edge away from the path. A *perfect m-caterpillar of length n*, denoted G_n is a caterpillar graph with m legs at each vertex on the central path of n vertices (see Figure 3.13). We conclude the paper with a derivation of 2-matching complexes of perfect m-caterpillar graphs.

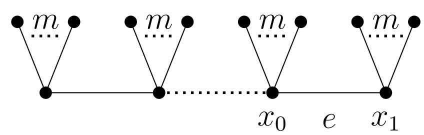

Figure 3.13: A perfect m-caterpillar of length n.

In [10], Jelić Milutinović et. al. calculate the homotopy type of $M_1(G_n)$ using topological techniques.

Theorem 3.5.1. *[10, Theorem 5.4] For $m \geq 2$, let G_n be a perfect m-caterpillar graph of length $n \geq 1$. Then the homotopy type of $M(G_n)$ is given by:*

$$M(G_n) \simeq \begin{cases} \bigvee_{t=0}^{k} \bigvee_{\alpha_t} S^{k-1+t} & \text{if } n = 2k \\ \bigvee_{t=0}^{k} \bigvee_{\beta_t} S^{k+t} & \text{if } n = 2k+1 \end{cases} \tag{3.1}$$

where $\alpha_t = \binom{k+t}{k-t}(m-1)^{2t}$ and $\beta_t = \binom{k+1+t}{k-t}(m-1)^{2t+1}$.

As we will now see the homotopy type of $M_2(G_n)$ is also a wedge of spheres.

Definition 3.5.2. Let G_n be a perfect m-caterpillar of length n with the right most edge along the central path $e = \{x_0, x_1\}$. Define $BD(G_n)$ as the bounded degree complex whose vertices are given by edges in G_n and faces are given by subgraphs H of G_n such that the $\deg(x_1) \leq 1$ and the degree of any other vertex is at most 2 in H.

In order to obtain the 2-matching complex of G_n, we will inductively use the bounded degree complex $BD(G_{n-1})$ to build up to $M_2(G_n)$. Namely, our progression will be:

$$M_2(G_{n-1}) \to BD(G_n) \to M_2(G_n) \to BD(G_{n+1}) \to M_2(G_{n+1}) \to \dots.$$

Notice that the only difference between $BD(G_n)$ and $M_2(G_n)$ is the possible degree of the last vertex on the central path. This will allow us to build an inductive argument on m-perfect caterpillar graphs. Let Σ_m denote the join with m distinct points.

Lemma 3.5.3. $BD(G_n) \cong \Sigma_m(M_2(G_{n-1})) \vee \Sigma(BD(G_{n-1}))$

Proof. Let m denote the number of legs off of each vertex along the central path as seen in Figure 3.13. For a bounded degree complex $BD(G_n)$, let $e = \{x_0, x_1\}$ be the right most edge along the central path and consider subgraphs H such that $\deg(x_1) \leq 1$ in H. We can decompose these bounded degree subgraphs into those that contain e and those that do not. Namely, if we exclude e, the bounded degree graphs are given by $M_2(G_{n-1}) * M_1(St_m)$ where St_m is a star graph on m edges, and if we include e the bounded degree subgraphs are given by $e * BD(G_{n-1})$. These two complexes share $BD(G_{n-1})$ as a common subcomplex and hence

$$BD(G_n) \cong M_2(G_{n-1}) * M_1(St_m) \bigcup_{BD(G_{n-1})} e * BD(G_{n-1}).$$

Since $e * BD(G_{n-1})$ is a contractible space we get

$$\begin{aligned} BD(G_n) &\cong M_2(G_{n-1}) * M_1(St_m) \bigcup_{BD(G_{n-1})} e * BD(G_{n-1})/e * BD(G_{n-1}) \\ &\cong \Sigma_m(M_2(G_{n-1}))/BD(G_{n-1}), \end{aligned}$$

where $\Sigma_m(X)$ is X join a set of m discrete points. Since $BD(G_{n-1}) \subseteq M_2(G_{n-1})$ we see that $BD(G_{n-1})$ is contractible in $\Sigma_m(M_2(G_{n-1}))$. Hence,

$$BD(G_n) \simeq \Sigma_m(M_2(G_{n-1})) \vee \Sigma(BD(G_{n-1})).$$

□

Lemma 3.5.4. $M_2(G_n) \cong M_2(G_{n-1}) * M_2(St_m) \vee \Sigma(\Sigma_m(BD(G_{n-1})))$

Proof. Let m be the number of legs off each vertex of the central path as seen in Figure 3.13. For the 2-matching complex $M_2(G_n)$, let $e = \{x_0, x_1\}$ be the right most edge along the central path and consider 2-matchings of G_n. Following the argument analogously to Lemma 3.5.3 we can decompose these 2-matchings into those that contain e and those that do not. Hence,

$$M_2(G_n) \cong M_2(G_{n-1}) * M_2(St_m) \bigcup_{BD(G_{n-1}) * M_1(St_m)} e * BD(G_{n-1}) * M_1(St_m).$$

Since $e * BD(G_{n-1}) * M_1(St_m)$ is contractible, we obtain

$$M_2(G_n) \cong M_2(G_{n-1}) * M_2(St_m)/BD(G_{n-1}) * M_1(St_m).$$

Further, since $BD(G_{n-1}) \subseteq M_2(G_{n-1})$ and $M_1(St_m) \subseteq M_2(St_m)$ we get that $BD(G_{n-1}) * M_1(St_m) \subseteq M_2(G_{n-1}) * M_2(St_m)$ is contractible and $M_2(G_n) \cong M_2(G_{n-1}) * M_2(St_m) \vee \Sigma(\Sigma_m(BD(G_{n-1})))$. □

Theorem 3.5.5. *Let G_n denote a perfect m-caterpillar graph of length n. Then,*

(i) the homotopy type of $BD(G_n)$ and $M_2(G_n)$ are wedges of spheres of varying dimensions for all $n \geq 1$,

(ii) the total number of spheres in $BD(G_{i+1})$ and $M_2(G_{i+1})$ is given by the coefficient of t^i in the series

$$\sum_{i \geq 0} \mathcal{A}_i t^i = \sum_{j \geq 0} \mathcal{B}_j t^j = \frac{x}{1-(1+y)t-(x^2-y)t^2}$$

where $x = (m-1)$ and $y = \binom{m-1}{2}$, and

(iii) $M_2(G_i) \simeq \bigvee_{j \geq 0} \bigvee_{\beta_{i,j}} S^{i+j}$ where $\beta_{i,j}$ the number of spheres of dimension $i+j$ is the coefficient of $r^i t^j$ in $B(r,t,x,y) = \sum_{i,j \geq 0} b_{i,j} r^i t^j = \frac{x}{1-rt-(x^2-y)r^2t^3-yrt^2}$ where $x = (m-1)$ and $y = \binom{m-1}{2}$.

Proof. (i) Since $BD(G_1) = M_1(St_m) \simeq \bigvee_{(m-1)} S^0$ and $M_2(G_1) = M_2(St_m) \simeq \bigvee_{\binom{m-1}{2}} S^1$, (i) follows from Lemmas 3.5.3, 3.5.4, and 1.3.3.

(ii) Let $\mathcal{A}_i$ denote the total number of spheres in the homotopy type of $BD(G_{i+1})$ and $\mathcal{B}_i$ be the total number of spheres in the homotopy type of $M_2(G_{i+1})$. From Lemmas 3.5.3, 3.5.4, and 1.3.3 we know $\mathcal{A}_0 = x := (m-1), \mathcal{B}_0 = y := \binom{m-1}{2}$, and $\mathcal{A}$, $\mathcal{B}$ follow the recursions:

$$\mathcal{A}_i = \mathcal{A}_{i-1} + x\mathcal{B}_{i-1} \tag{3.2}$$

$$\mathcal{B}_i = x\mathcal{A}_{i-1} + y\mathcal{B}_{i-1}. \tag{3.3}$$

Using equations 3.2 and 3.3, we see that $\mathcal{A}_i = (1+y)\mathcal{A}_{i-1} + (x^2-y)\mathcal{A}_{i-2}$. Let $A(t) = \sum_{i \geq 0} \mathcal{A}_i t^i$. Multiplying by $(1-(1+y)t-(x^2-y)t^2)$ and solving we obtain

$$A(t) = \frac{x}{1-(1+y)t-(x^2-y)t^2}.$$

The argument for $B(t) = \sum_{j \geq 0} \mathcal{B}_i t^i$ is analogous.

(iii) Let $\alpha_{i,j}$ be the total number of spheres of dimension j in $BD(G_{i+1})$ and $\beta_{i,j}$ the total number of spheres of dimension j in $M_2(G_{i+1})$. Using that $BD(G_1) \simeq \bigvee_{(m-1)} S^0$

and $M_2(G_1) \simeq \bigvee_{\binom{m-1}{2}} S^1$, and Lemmas 3.5.3, 3.5.4, and 1.3.3 we obtain the following initial conditions

$$\alpha_{0,0} = x := (m-1)$$
$$\beta_{0,1} = y := \binom{m-1}{2}$$
$$\alpha_{0,j} = 0 \text{ for } j \geq 1$$
$$\beta_{0,j} = 0 \text{ for } j \geq 2$$
$$\alpha_{i,0} = 0 \text{ for } i \geq 1$$
$$\beta_{i,0} = 0 \text{ for } i \geq 0$$

Additionally, $\alpha_{i,j}$ and $\beta_{i,j}$ follow the recursions

$$\alpha_{i,j} = \alpha_{i-1,j-1} + x(\beta_{i-1,j-1}) \tag{3.4}$$

$$\beta_{i,j} = x\alpha_{i-1,j-2} + y\beta_{i-1,j-2}. \tag{3.5}$$

Using equations 3.4 and 3.5, we can see that

$$\beta_{i,j} = \beta_{i-1,j-1} + (x^2 - y)(\beta_{i-2,j-3} + y(\beta_{i-2,j-3}).$$

Let $B(r,t,x,y) = \sum_{i,j \geq 0} b_{i,j} r^i t^j$ and multiply by $1 - rt - (x^2 - y)r^2t^3 - yrt^2$. When we solve and use the initial conditions we find that

$$B(r,t,x,y) = \frac{x}{1 - rt - (x^2 - y)r^2t^3 - yrt^2}$$

and the result follows from substituting $(m-1)$ for x and $\binom{m-1}{2}$ for y. $\square$

Remark 3.5.6. The table of the homotopy types for $BD(G_n)$ and $M_2(G_n)$, for small values of n, can be found in Appendix B. From Theorem 3.5.5 (iii), notice that the number of spheres in each dimension is given by a polynomial in x and y. If we set $x = y = 1$, we can see that the number of terms in the sum given by the coefficient of $r^i t^j$ is a binomial coefficient:

$$B(r,t,1,1) = \frac{1}{1 - rt(1+t)} = \sum_{k \geq 0} r^k t^k (1+t)^k$$

and the coefficient of $[r^i t^j] = \binom{i}{j-i}$.

3.6 Future directions.

The original motivation for this chapter was to study 1-matching complexes through the lens of k-matching complexes for $k \geq 2$. We end with a few open questions. Our exploration of 2-matching complexes led to observations about the flexibility of the homotopy type and how the homotopy type of clawed non-separable graphs changes (or doesn't change) as new leaves are added. One avenue to explore in this direction involves understanding the interaction between clawed non-separable graphs and additional leaves.

Question 3.6.1. Ranging over all clawed non-separable graphs, what is the average maximum number of leaves that can be added without affecting the homotopy type of the resulting 2-matching complex?

We have already seen that there are some graphs in which the maximum can be obtained and other graphs in which there is an obstruction to doing so. It would be interesting to know if clawed non-separable graphs tend to have structural properties that obstruct obtaining the maximum and can we expect the maximum number of leaves to be evenly distributed over all such graphs.

We can also ask about properties of graphs more generally.

Question 3.6.2. Given a graph, how can we determine when leaves can be attached without affecting the resulting homotopy type of the 2-matching complex?

In Sections 3.4 and 3.5 , we defined the k-matching complex of a graph and explored two examples, wheel graphs and perfect caterpillar graphs. Theorems 3.5.1 and 3.5.5 show that the homotopy type of $M_1(G_n)$ and $M_2(G_n)$ are both wedges of spheres with combinatorial structure. A future direction of this work would be to further understand the k-matching complex of families of graphs such as perfect caterpillar graphs and trees in general. In [32], I conjectured

Conjecture 3.6.3. *The k-matching complex of caterpillar graphs are homotopy equivalent to a wedge of spheres,*

which was recently proved in a paper by Singh. In [29], the author proves k-matching complexes of forests are contractible or wedges of spheres and determines the number of d-dimensional spheres in k-matching complexes of perfect caterpillar graphs. The next step is to begin to look at k-matching sequences of graphs which are known to have torsion in their 1-matching complexes such as the full and chessboard complexes.

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Appendices

Appendix A: Algorithms and Experimental Data

Input : A complete graph on k vertices, a probability $p \in (0, 1)$, and a positive integer t.

Output: A list `GraphList` of t k-constructible graphs.

Set $i = 0$;
Initialize `GraphList` as a list containing only K_k;
while $i < t$ **do**
  Generate a random value $0 < r < 1$;
  if $r > p$ and $i > 0$ **then**
    Randomly select a non-complete graph G from `GraphList`;
    Randomly choose a pair of non-adjacent vertices (v, w) in G;
    Set $G' := \mathrm{vid}(G, [v, w])$;
  else
    Randomly select two graphs (possibly equal) G_1 and G_2 from `GraphList`;
    Randomly select one edge from each of G_1 and G_2;
    Set $G' := G_1 \Delta_H G_2$, performing the Hajós merge with the two selected edges;
  end
  if G' is not isomorphic to any element of `GraphList` **then**
    Append G' to `GraphList`;
    Increase i by one;
  else
    Continue;
  end
end

Algorithm 1: Constructible Random Algorithm (CRA)

```
Input  : Positive integers $k \geq 3$ and $t, m, n \geq 1$.
Output: A list GraphList of $t$ Urquhart-$k$-constructible graphs.

Set $i = 0$;
Initialize GraphList as an empty list;
while $i < t$ do
    Generate a random integer $1 \leq r_c \leq n$;
    Initialize $L$ as an empty list;
    for $1 \leq s \leq r_c$ do
        Set $G_s = K_k$;
        Generate a random integer $1 \leq r \leq m$;
        Add $r$ new vertices named $v_1, \ldots, v_r$ to $G_s$;
        for $1 \leq j \leq r$ do
            Select a random subset $\emptyset \neq S \subseteq V(K_k)$;
            Add to $G_s$ the edges $\{(v_j, w) : w \in S\}$;
        end
        Select a random graph $H$ on vertex set $\{v_1, \ldots, v_r\}$;
        Add the edges of $H$ to $G_s$;
        Append $G_s$ to $L$;
    end
    while $L$ contains more than one element do
        Randomly select two graphs $G_1, G_2 \in L$;
        Delete $G_1$ and $G_2$ from $L$;
        Randomly select one edge from each of $G_1$ and $G_2$;
        Choose a random integer $1 \leq \ell \leq \min\{|V(G_1)|, |V(G_2)|\} - 1$;
        Select $\ell$ pairs of disjoint vertices from $G_1$ and $G_2$ satisfying the
          criteria to be used in an Ore merge;
        Append to $L$ the new graph $\mathrm{Ore}(G_1, G_2)$ formed using these
          selections;
    end
    Define $G'$ to be the unique remaining element of $L$;
    if $G'$ is not isomorphic to any element of GraphList then
        Append $G'$ to GraphList;
        Increase $i$ by one;
    else
        Continue;
    end
end
```

Algorithm 2: Urquhart Random Algorithm (URA)

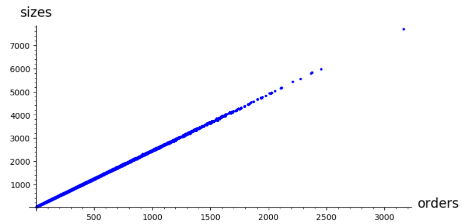

Figure A1: Orders versus sizes for 2939 CRA-generated graphs, $k = 5$, $p = 0.50$

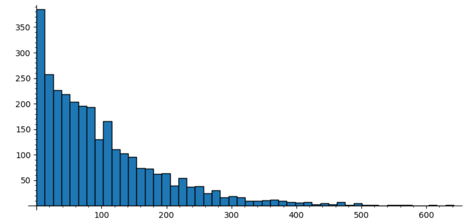

Figure A2: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 2939 CRA-generated graphs, $k = 5$, $p = 0.50$

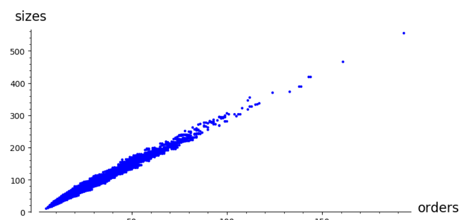

Figure A3: Orders versus sizes for 10000 CRA-generated graphs, $k = 5$, $p = 0.10$

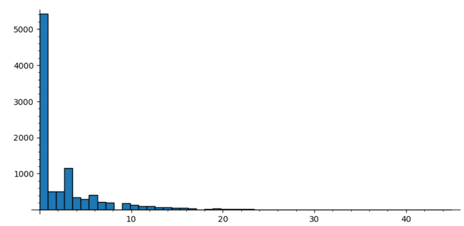

Figure A4: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 10000 CRA-generated graphs, $k = 5$, $p = 0.10$

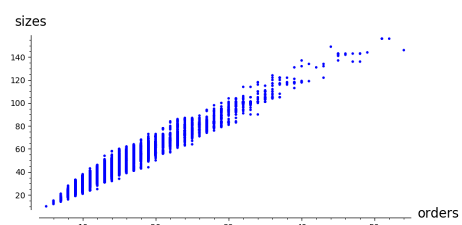

Figure A5: Orders versus sizes for 10000 CRA-generated graphs, $k = 5$, $p = 0.02$

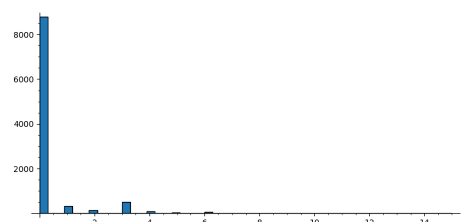

Figure A6: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 10000 CRA-generated graphs, $k = 5$, $p = 0.02$

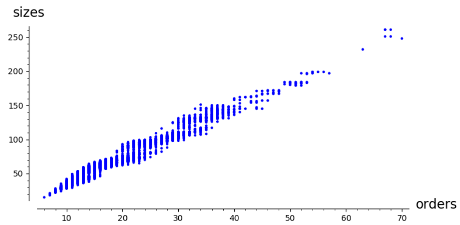

Figure A7: Orders versus sizes for 10000 CRA-generated graphs, $k = 6$, $p = 0.02$

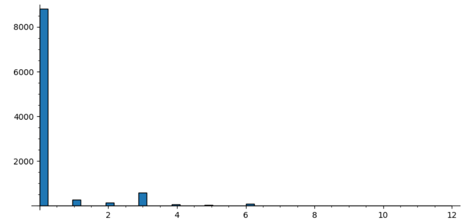

Figure A8: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 10000 CRA-generated graphs, $k = 6$, $p = 0.02$

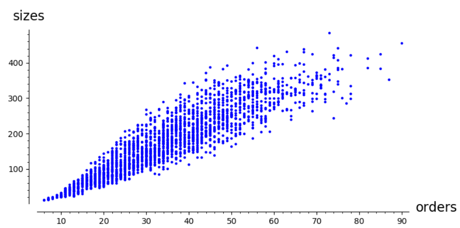

Figure A9: Orders versus sizes for 2000 URA-generated graphs, $k = 4$, $m = n = 12$

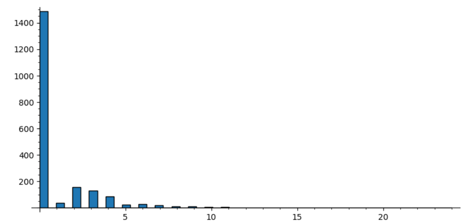

Figure A10: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 2000 URA-generated graphs, $k = 4$, $m = n = 12$

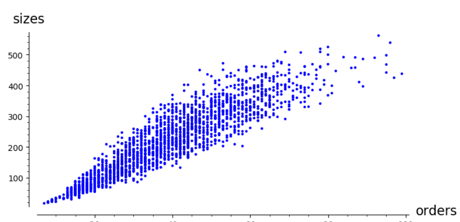

Figure A11: Orders versus sizes for 2000 URA-generated graphs, $k = 5$, $m = n = 12$

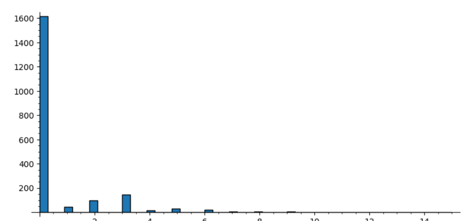

Figure A12: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 2000 URA-generated graphs, $k = 5$, $m = n = 12$

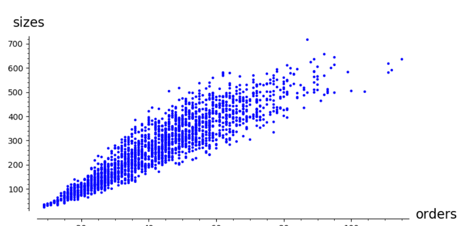

Figure A13: Orders versus sizes for 2000 URA-generated graphs, $k = 6$, $m = n = 12$

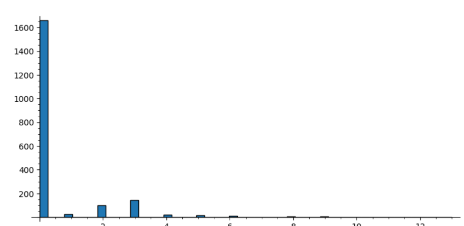

Figure A14: Histogram of first Betti numbers for $\mathcal{N}(G)$ for 2000 URA-generated graphs, $k = 6$, $m = n = 12$

Appendix B: Homotopy Type of $M_2(G_n)$ and $BD(G_n)$.

The homotopy type of $BD(G_n)$ and $M_2(G_n)$ are wedges of spheres in varying dimension. In Lemmas 3.5.3 and 3.5.4 , and Theorem 3.5.5 we saw that we can inductively define the homotopy type of these complexes. The following table depicts and example for the recursions 3.4 and 3.5, and the homotopy types of $BD(G_n)$ and $M_2(G_n)$ for small values of n.

Complex \ Dim	S^0	S^1	S^2	S^3	S^4	S^5	S^6
$BD(G_1)$	$(m-1)$	0	0	0	0	0	0
$M_2(G_1)$	0	$\binom{m-1}{2}$	0	0	0	0	0
$BD(G_2)$	0	$(m-1)$	$\binom{m-1}{2}(m-1)$	0	0	0	0
$M_2(G_2)$	0	0	$(m-1)^2$	$\binom{m-1}{2}^2$	0	0	0
$BD(G_3)$	0	0	$(m-1)$	$(m-1)^3+$ $(m-1)\binom{m-1}{2}$	$(m-1)\binom{m-1}{2}^2$	0	0
$M_2(G_3)$	0	0	0	$(m-1)^2$	$2(m-1)^2\binom{m-1}{2}$	$\binom{m-1}{2}^3$	0
$BD(G_4)$	0	0	0	$(m-1)$	$2(m-1)^3+$ $(m-1)\binom{m-1}{2}$	$2(m-1)^3\binom{m-1}{2}+$ $(m-1)\binom{m-1}{2}^2$	$(m-1)\binom{m-1}{2}^3$

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