RAPID analysis of variable stiffness beams and plates: Legendre polynomial triple-product formulation

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Numerical integration techniques are commonly employed to formulate the system matrices encountered in the analysis of variable stiffness beams and plates using a Ritz based approach. Computing these integrals accurately is often computationally costly. Herein, a novel alternative is presented, the Recursive Analytical Polynomial Integral Definition (RAPID) formulation. The RAPID formulation offers a significant improvement in the speed of analysis, achieved by reducing the number of numerical integrations that are performed by an order of magnitude. A common Legendre Polynomial (LP) basis is employed for both trial functions and stiffness/load variations leading to a common form for the integrals encountered. The LP basis possesses algebraic recursion relations that allow these integrals to be reformulated as triple-products with known analytical solutions, defined compactly using the Wigner $(3j)$ coefficient. The satisfaction of boundary conditions, calculation of derivatives, and transformation to other bases is achieved through combinations of matrix multiplication, with each matrix representing a unique boundary condition or physical effect, therefore permitting application of the RAPID approach to a variety of problems. Indicative performance studies demonstrate the advantage of the RAPID formulation when compared to direct analysis using MATLAB's "integral" and "integral2".

1 Introduction

The competitive nature of the aerospace and other industries places increasing importance on the ability of designers to maximise performance from structural components. To achieve these goals the use of composite materials has steadily increased to meet these demands. Design in composite materials often requires the balancing of a multitude of design parameters and the resulting performance characteristics cannot always be predicted from physical insight alone. Numerical optimisation techniques are increasingly exploited to meet design goals as the parameter spaces become more complex. For example, in the design of lightweight aerospace structures it is common practice to analyse many potential load cases, various structural configurations, and the impact of layup tailoring and anisotropy. Furthermore with advances in the manufacture of variable stiffness components such as variable angle tow laminates (VAT), continuously varying stiffness profiles can now readily be achieved, e.g. [1, 2, 3], one such example of the design space's continuing expansion.

Increasing the number of design variables in an optimisation is not without cost. For efficient optimisation, the number of design variables that can be considered is intrinsically linked to the time taken for each analysis. In most instances there is no analytical relationship between design variables and the objective function thereby requiring computational analysis. This computational cost often prohibits the use of finite element analysis (FEA) and analytical, or semi-analytical, approximation is sought instead. Often it is sufficient to utilise an approximate method to identify optimised design candidates that can be subject to more detailed analysis. Indeed, it is commonplace when analysing beam and plate-like structures with varying stiffness profiles to exploit a Ritz based formulation.

In particular cases closed form solutions exist for a Ritz analysis, however, there is typically a requirement to compute many of the integrals associated with the problem numerically. It is, in the authors' experience, that the computation of such integrals dominate the analysis time. A comparison is made herein between the typical adaptive quadratures as employed by MATLAB and the triple product formulation using Legendre polynomials (LP). While alternative techniques exist, such as trapezoid and cuboid methods, Gaussian quadrature, and Monte

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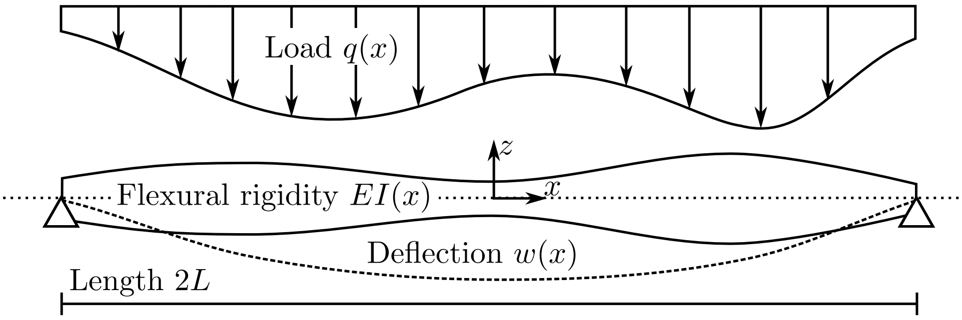


Figure 1: Diagram of a variable stiffness beam symmetric about its mid-plane, subject to a variable out of plane load.

Carlo techniques that offer potential runtime reduction, there is often an associated loss in accuracy, limiting the number of basis functions that can be reliably used [4]. Such a restricted basis expansion limits the ability to capture localised phenomena effectively. This paper outlines an alternative method for the computation of the integrals encountered directly using LP triple-products and the Wigner $(3j)$ coefficient. The use of this technique for the analysis of variable stiffness beams and plate-like structures is presented here here as the Recursive Analytical Polynomial Integral Definition (RAPID) formulation.

The results presented herein extend the technique highlighted in O'Donnell and Weaver [5]. The effectiveness of the RAPID formulation for plate analysis is demonstrated and additional insight into the formulation provided. The authors of this paper do not claim that the relationships they present for LPs are novel in their own right, indeed many of these results are well known, particularly beyond the engineering community, for example [6, 7, 8]. However, to the authors' best knowledge there is no existing application of these results to the analysis of variable stiffness plates that expands stiffness matrices and trial functions using a common LP basis in order to exploit the triple-product integral form to minimise numerical integration costs. In utilising the RAPID formulation an order of magnitude reduction in the number of numerical quadratures required, as compared to a typical Ritz analysis, is observed from the formulations definition. This efficiency increase is observed in practice via a comparative performance study contained herein.

The familiar Ritz formulation for variable stiffness beam structures and identify the integrals that must be evaluated is now outlined. The RAPID formulation is detailed and implemented for these integrals. A comparison with numerical quadrature is then presented. The approach is then extended to variable stiffness plate structures and a second comparative study presented.

2 Variable Stiffness Beams

Consider a Bernoulli beam aligned with the x axis having its origin located at the centre of the midspan, fig 1. Suppose the beam possesses a variable flexural rigidity along its length, for example caused by varying depth and/or cross sectional area, and the rate-of-change of these variations are sufficiently small that a one-dimensional analysis suffices. For brevity it is assumed that the variation in flexural rigidity is symmetric about the neutral axis, given by $(EI)(x)$, and the effects of a variable transverse loading, $q(x)$ are consider. An energetic formulation to determine the vertical deflection, w, is now defined. To aid analysis the following non-dimensionalisation is made,[1]

$$\tilde{x} = \frac{x}{L}, \qquad \tilde{q}(x) = \frac{q(x)}{q_0}, \qquad \widetilde{EI}(x) = \frac{(EI)(x)}{(EI)_0}, \qquad \tilde{w}(x) = \frac{w}{L}, \tag{1}$$

the energy of the system can then be written as,

$$\Pi = U + T = \frac{(EI)_0}{2L} \int_{-1}^{1} \widetilde{EI}(\tilde{x}) \left(\frac{d^2\tilde{w}(\tilde{x})}{d\tilde{x}^2} \right)^2 d\tilde{x} - q_0 L^2 \int_{-1}^{1} \tilde{q}(\tilde{x})\tilde{w}(\tilde{x})\, d\tilde{x}, \tag{2}$$

or alternatively,

$$\tilde{\Pi} = \tilde{\alpha} \int_{-1}^{1} \widetilde{EI}(\tilde{x}) \left(\frac{d^2\tilde{w}(\tilde{x})}{d\tilde{x}^2} \right)^2 d\tilde{x} - \int_{-1}^{1} \tilde{q}(\tilde{x})\tilde{w}(\tilde{x})\, d\tilde{x} \quad \text{with} \quad \tilde{\alpha} = \frac{(EI)_0}{2q_0 L^3}. \tag{3}$$

[1] Tilde is used to represent a non-dimensional quantity and a zero subscript the nominal normalising value throughout.

Letting the unknown displacements be approximated by a series of kinematically admissible trial functions, $\mathbf{S}(\tilde{x})$, with terms $S_i(\tilde{x})$,

$$\tilde{w}(\tilde{x}) \approx \sum_{i=0}^{N} \omega_i S_i(\tilde{x}), \tag{4}$$

the unknown coefficients w_i can be determined using the principle of minimum potential energy,

$$\frac{\partial \tilde{\Pi}}{\partial \omega_i} = 0, \tag{5}$$

defining a linear system of equations. Solving this system requires obtaining $\mathcal{O}(N^2)$ integrals of the form,

$$\int_{-1}^{1} \frac{d^2 S_i(\tilde{x})}{d\tilde{x}^2} \widetilde{EI}(\tilde{x}) \frac{d^2 S_j(\tilde{x})}{d\tilde{x}^2}\, d\tilde{x}. \tag{6}$$

The choice of trial function, $\mathbf{S}(\tilde{x})$, affects how efficiently these integrals can be computed and how many terms are required for a good approximation to the solution. It is often observed that the computational bottleneck in the analysis of systems such as these is computing integrals of the form given in eq (6). While this may not be prohibitive for an individual analysis it is significant for optimisation studies where many designs must be considered. A method outlining how recursive relationships for LPs can be exploited to decrease the number of quadratures is presented. It is noted that the LPs have been exploited successfully in structural analysis problems and are amenable to capturing local variations without an excessive number of terms, e.g. [9].

3 Utilising Legendre Polynomials

The LP matrix, $\mathbf{P}$, is defined for a finite expansion of $(N+1)$ terms,

$$\mathbf{P}^{\intercal} = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_i & \cdots & P_N \end{bmatrix}, \tag{7}$$

where P_i is the i$^{\text{th}}$ LP. This form allows many of the following results to be expressed in terms of matrix multiplication. The LPs form an orthogonal basis with respect to a weighting function of 1,

$$\int_{-1}^{1} P_m(x) P_n(x) = \frac{2}{2n+1} \delta_{mn}, \tag{8}$$

where δ_{mn} is the Kroneker delta. Each successive term may be generated by convenient recursive relations,

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x). \tag{9}$$

Utilising the recursion relations first explored by Adams [6], the product of two Legendre polynomials may be written as,

$$P_m(x) P_n(x) = \sum_{l=|m-n|}^{m+n} \overbrace{\begin{pmatrix} l & m & n \\ 0 & 0 & 0 \end{pmatrix}^2}^{\text{Wigner }(3j)} (2l+1) P_l(x) \tag{10}$$

where the Wigner $(3j)$ coefficient function is utilised for compactness. Olver *et. al.* [7] gives the general definition of the Wigner $(3j)$ coefficient. For consistency with existing definitions the notation common in the physics and chemistry communities where rows m and j relate to quantum angular momentum of a system is adopted,

$$\text{Wigner }(3j) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{11}$$

Note that the Wigner $(3j)$ coefficient is not a matrix, the function returns a single scalar value and is differentiated here by the use of curved parenthesis.

For this investigation the special case of interest, where $m_i = 0$ and $j_i \in \mathbb{N}$, the Wigner $(3j)$ coefficient is given by

$$\begin{pmatrix} k & l & m \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{cases} \frac{(2s-2k)!(2s-2l)!(2s-2m)!}{(2s+1)!} \left[\frac{s!}{(s-k)!(s-l)!(s-m)!}\right]^2 & \begin{array}{l} 2s \text{ is even and } l,\ m, \text{ and } n \\ \text{satisfy triangle inequality} \end{array}, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Table 1: Examples of different boundary conditions that are satisfied by the modified basis $\mathbf{P}_{\mathrm{BC}}$ are presented. The basis modifications polynomials $\zeta(\tilde{x})$ and their equivalent form in Legendre polynomials are given. The multiplication matrix $\mathbf{C}_{\mathrm{BC}}$ can be obtained by use of eq (10).

Modified Basis Requirements	Polynomial modifier, $\zeta(\tilde{x})$	Equivalent Legendre polynomial
$\mathbf{P}_{\mathrm{BC}}(\pm 1) = 0$	$\tilde{x}^2 - 1$	$P_2 - P_0$
$\mathbf{P}_{\mathrm{BC}}(\pm 1) = \mathbf{P}''_{\mathrm{BC}}(\pm 1) = 0$	$\tilde{x}^4 - 6\tilde{x}^2 + 5$	$P_4 + 15P_2 + 14P_0$
$\mathbf{P}_{\mathrm{BC}}(\pm 1) = \mathbf{P}'_{\mathrm{BC}}(\pm 1) = 0$	$\tilde{x}^4 - 2\tilde{x}^2 + 1$	$3P_4 - 10P_2 + 7P_0$
$\mathbf{P}_{\mathrm{BC}}(\pm 1) = \mathbf{P}'_{\mathrm{BC}}(-1) = \mathbf{P}''_{\mathrm{BC}}(1) = 0$	$\tilde{x}^4 - \tilde{x}^3 - 3\tilde{x}^2 + \tilde{x} + 2$	$4P_4 - 7P_3 - 25P_2 + 7P_1 + 21P_0$
$\mathbf{P}_{\mathrm{BC}}(-1) = \mathbf{P}'_{\mathrm{BC}}(-1) = \mathbf{P}''_{\mathrm{BC}}(1) = \mathbf{P}'''_{\mathrm{BC}}(1) = 0$	$\tilde{x}^4 - 4\tilde{x}^3 + 6\tilde{x}^2 + 28\tilde{x} + 17$	$P_4 - 7P_3 - 20P_2 + 112P_1 + 84P_0$

with $2s = k + l + m$. The triangle inequality must hold on l, m, n for the factorials in the second term of equation (12) to be defined.[2]

In order to satisfy the kinematic requirements a modified basis, $\mathbf{P}_{\mathrm{BC}}$, can be obtained by pre-multiplying the Legendre basis by a polynomial, $\zeta(\tilde{x})$, that satisfies the required boundary conditions,

$$\mathbf{P}_{\mathrm{BC}} = \zeta(\tilde{x})\mathbf{P}, \tag{13}$$

where $\zeta(\tilde{x})$ can be determined in a similar manner to Jaunky *et. al.* [10]. For convenience this polynomial can be written in terms of Legendre Polynomials. For example consider a modified basis $\mathbf{Z}$ that is zero at $\tilde{x} = \pm 1$. This boundary constraint is satisfied by $\zeta = (\tilde{x}^2 - 1)$ which is similarly achieved using a Legendre polynomial, $P_2(\tilde{x}) - P_0(\tilde{x})$. Thus the modified basis is,

$$Z_n(\tilde{x}) = (P_2(\tilde{x}) - P_0(\tilde{x}))P_n(\tilde{x}). \tag{14}$$

Using equation (10) this modification can be written as a weighted sum,

$$Z_n(\tilde{x}) = \left[\sum_{l=|n-2|}^{n+2} \begin{pmatrix} l & 2 & n \\ 0 & 0 & 0 \end{pmatrix}^2 (2l+1)P_l(\tilde{x})\right] - P_n(\tilde{x}), \tag{15}$$

which can be represented by matrix multiplication,

$$\mathbf{Z} = \mathbf{C}_Z\mathbf{P}, \tag{16}$$

where $\mathbf{C}_{\mathbf{z}}$ is a $(N+1) \times (N+3)$ non-square matrix, indicating that the $(N+2)^{\mathrm{th}}$ order LP term is required to represent the N^{th} term of $\mathbf{Z}$. The components are

$$C_{Z\,ij} = \begin{cases} \begin{pmatrix} 2 & i-1 & j-1 \\ 0 & 0 & 0 \end{pmatrix}^2 (2j-1) - 1 & \text{if } i = j, \\ \begin{pmatrix} 2 & i-1 & j-1 \\ 0 & 0 & 0 \end{pmatrix}^2 (2j-1) & \text{if } |i-3| \leq j-1 \leq i+1 \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}. \tag{17}$$

Other modified bases can be obtained in a similar manner and recast into the form,

$$\mathbf{P}_{\mathrm{BC}} = \mathbf{C}_{\mathrm{BC}}\mathbf{P}, \tag{18}$$

where $\mathbf{C}_{\mathrm{BC}}$ is typically a non-square coefficient matrix reflecting the desired kinematic boundary conditions, table 1.

A similar approach can be utilised to perform a basis transformation in order to utilise other polynomial bases. For example, the Chebyshev polynomials are often utilised in the analysis of plates for their beneficial convergence properties, e.g [11, 12]. Efficient transformations between Chebyshev and Legendre polynomials exist and may be used to determine the appropriate coefficient matrix for transformation [8]. Using a similar matrix multiplication approach the derivatives of the LP basis may be obtained,

$$\frac{d}{dx}P_{n+1}(x) = \begin{cases} \sum_{k=0,2,4,\ldots}^{n} (2k+1)P_k(x) & \text{if } n \text{ is even} \\ \sum_{k=1,3,5,\ldots}^{n} (2k+1)P_k(x) & \text{if } n \text{ is odd} \end{cases} \tag{19}$$

[2]Triangle inequality: $|a-b| < c < a+b$ for three sides of a triangle a, b, and c.

The coefficients of the $(N+1)\times N$ differentiation matrix, $\mathbf{C}_\delta$, are

$$C_{\delta\, ij} = \begin{cases} 2j-1 & \text{if } i>j \text{ and } i+j \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

giving

$$\frac{d}{d\tilde{x}}\mathbf{P} = \mathbf{C}_{\boldsymbol{\delta}}\mathbf{P} \tag{21}$$

The second derivative, $\mathbf{C}_{2\delta}$, is a $(N+1)\times(N-1)$ matrix and follows as the product of two differentiation matrices of compatible sizes,

$$\frac{d^2}{d\tilde{x}^2}\mathbf{P} = \mathbf{C}_\delta\mathbf{C}_\delta^*\mathbf{P} = \mathbf{C}_{2\delta}\mathbf{P}, \tag{22}$$

where $\mathbf{C}_\delta^*$ is the differentiation matrix as before but of reduced size $N\times(N-1)$. Higher derivatives follow by further multiplication of appropriately sized matrices. By combining both the modified basis and differentiation matrices the derivatives of $\mathbf{Z}$ can be obtained in terms of the LP basis,

$$\frac{d^2}{d\tilde{x}^2}\mathbf{Z} = \mathbf{C}_Z\mathbf{C}_{2\delta}^*\mathbf{P}. \tag{23}$$

To ensure the desired $N+1$ term output, the size of $\mathbf{C}_{2\delta}^*$ must be $(N+3)\times(N+1)$ and follows the same definition as $\mathbf{C}_{2\delta}$. The dimension change required is dependent on the modified basis matrix and thus varies in accordance with the boundary conditions imposed. For the remainder of this paper it is assumed that all required coefficient matrices are dimensionally compatible and for clarity there is no distinction indicated between matrices of the same type but having different size. In general it may be stated,

$$\frac{d^k}{dx^k}\mathbf{P}_{\mathrm{BC}} = \mathbf{C}_{\mathrm{BC}}\mathbf{C}_{k\delta}\mathbf{P}. \tag{24}$$

Suppose that a matrix, $\mathbf{F}$, defined in integral form,

$$\mathbf{F} = \int_{-1}^{1} \frac{d^k}{d\tilde{x}^k}\mathbf{P}_{\mathrm{BC}} f(\tilde{x}) \frac{d^k}{d\tilde{x}^k}\mathbf{P}_{\mathrm{BC}}^\intercal \, d\tilde{x}, \tag{25}$$

is comparable to the commonly encountered integrals previously discussed, eq (6). Letting $f(\tilde{x})$ be approximated by a suitably accurate LP expansion,

$$f(\tilde{x}) \approx \sum_{i=0}^{M} f_i P_i, \tag{26}$$

allows, $\mathbf{F}$ to be written in the form,

$$\begin{aligned} \mathbf{F} &\approx \int_{-1}^{1}\sum_{i=0}^{M} \left(\mathbf{C}_{\mathrm{BC}}\mathbf{C}_{k\delta}\mathbf{P}\right) f_i P_i \left(\mathbf{C}_{\mathrm{BC}}\mathbf{C}_{k\delta}\mathbf{P}\right)^\intercal \, d\tilde{x} \\ &\approx \mathbf{C}_{\mathrm{BC}}\mathbf{C}_{k\delta}\left[\int_{-1}^{1}\sum_{i=0}^{M}\mathbf{P}(f_i P_i)\mathbf{P}^\intercal\right]\mathbf{C}_{k\delta}^\intercal\mathbf{C}_{\mathrm{BC}}^\intercal, \\ &= \mathbf{C}_{\mathrm{BC}}\mathbf{C}_{k\delta}\sum_{i=0}^{M} f_i\left[\int_{-1}^{1}\mathbf{P}\mathbf{P}^\intercal P_i\, d\tilde{x}\right]\mathbf{C}_{k\delta}^\intercal\mathbf{C}_{\mathrm{BC}}^\intercal. \end{aligned} \tag{27}$$

this result follows directly from the linearity of integration. This manipulation may appear counter productive, having replaced a single integral associated with a component of $\mathbf{F}$ with a sum of M integrals. However, as all integrals are now expressible in terms of triple-products of a common basis function these M integrals can be computed directly via the relationship,

$$\int_{-1}^{1} P_i P_j P_k \, d\tilde{x} = 2\begin{pmatrix} i & j & k \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{28}$$

Thus there is no need to perform the M integrations numerically and the complete solution to $\mathbf{F}$ is the computation of a summation of M, $(N+1) \times (N+1)$ matrices. This feature allows significant computational savings to be realised as the computational expense has been reduced from calculating $\mathcal{O}(N^2)$ integrals directly solving eq (25) to compute $\mathbf{F}$, to $\mathcal{O}(M)$, integrals to compute the weighting coefficients of f_k, and an algebraic summation of known matrices to obtain an approximation of $\mathbf{F}$, eq (27). Herein referring to the adoption of such integral/summation interchange to improve computational cost for the analysis of variable stiffness beams and plates as the Recursive Analytical Polynomial Integral Definition (RAPID) formulation. It is noted that where the variation function $f(\tilde{x})$, can be represented exactly by a LP series expansion the approximation becomes an equality and the two methods, direct integration and RAPID formulation, give identical results. An application of the RAPID formulation for variable stiffness beams and quantification of the performance benefit via an indicative study is now presented.

4 RAPID Formulation - Beams

Consider the non-dimensional formulation of the variable stiffness beam presented in eq (3). The variable stiffness and load can be represented using LP series expansions,

$$\widetilde{EI}(\tilde{x}) \approx \sum_{j=0}^{M_1} \zeta_k P_j, \qquad \tilde{q}(\tilde{x}) \approx \sum_{k=0}^{M_2} \eta_k P_k. \tag{29}$$

The unknown displacement function is represented by an appropriately modified basis,

$$w(x) \approx \sum_{i=0}^{N} \omega_i P_{\mathrm{BC}\,i} = \boldsymbol{\omega}^{\intercal} \mathbf{P}_{\mathrm{BC}} = \boldsymbol{\omega}^{\intercal} \mathbf{C}_{\mathrm{BC}} \mathbf{P}, \tag{30}$$

having made use of eq (18). Following from the general definition, eq (24), the resulting system, eq (5), can then be written as,

$$\frac{d\tilde{\Pi}}{d\omega} = \mathbf{0} = \left[\alpha \int_{-1}^{1} \frac{d\mathbf{P}_{\mathrm{BC}}}{d\tilde{x}^2} \frac{d\mathbf{P}_{\mathrm{BC}}^{\intercal}}{d\tilde{x}^2} \widetilde{EI}\, d\tilde{x} \right] \boldsymbol{\omega} - \left[\int_{-1}^{1} \mathbf{P}_{\mathrm{BC}} \tilde{q}\, d\tilde{x} \right], \tag{31}$$

or in RAPID form,

$$\frac{d\tilde{\Pi}}{d\omega} = \mathbf{0} \approx \left[\alpha \mathbf{C}_{\mathrm{BC}} \mathbf{C}_{2\delta} \sum_{j=0}^{M_1} \zeta_k \left[\int_{-1}^{1} \mathbf{P}\mathbf{P}^{\intercal} P_j\, d\tilde{x} \right] \mathbf{C}_{2\delta}^{\intercal} \mathbf{C}_{\mathrm{BC}}^{\intercal} \right] \boldsymbol{\omega} - \left[\mathbf{C}_{\mathrm{BC}} \sum_{k=0}^{M_2} \eta_k \left[\int_{-1}^{1} \mathbf{P} P_k\, d\tilde{x} \right] \right]. \tag{32}$$

The efficiency gains of the RAPID formulation is now presented.

4.1 Computational Efficiency

To compare the computational efficiency and accuracy of the two approaches four beam systems with variable stiffness and loading are considered. The four cases are detailed in table 2. In order to compare the accuracy of the RAPID formulation (TP) with a typical analysis the results obtained via direct numerical quadrature (DI) are computed. It is assumed that stiffness and load expansions contain an equal number of terms, $M = M_1 = M_2$. The error from the RAPID approach is quantified via the norm of the difference between the two methods. This is calculated for the resulting system matrices associated with energy components U and T, together with the solution vector $\mathbf{w}$.[3] The expansion for the unknown displacement function is computed for $N = 15$ terms. The results presented in fig 2 demonstrate typical convergence behaviour as would be expected for any series approximation.

To provide a representative comparison a computational analysis is undertaken using a 64-bit CPU E8400 @ 3.00GHz with 3.8 GiB RAM using MATLAB 2013b [13]. It is expected that some further performance gains may be obtained using more refined algorithms and alternative languages. However, the objective of this study is to identify overall trends in behaviour not detailed benchmarking of the implementation. The change in computational run-time through the use of the RAPID approach is observed, fig 3. The runtime is calculated as the median of five identical analyses. The RAPID computation time is compared to the runtime associated with direct numerical integration from MATLAB's "integral" function [14] - calculated with the default parameters. As

[3] Numerical integration does not give the exact solution so the error presented is relative to another approximation. In this context error between the approaches serves sufficiently well to provide a comparative measure of accuracy, but not error relative to the true solution.

Table 2: Representative loading and stiffness functions investigated for computational benchmarking.

Case	$\widetilde{EI}(\tilde{x})$	$\tilde{q}(\tilde{x})$
Case 1	$-0.5x^8 + 0.1x^3 - 0.3x + 1.2$	$-0.3x^4 - 0.5x$
Case 2	$0.75 + 0.5\cos(\pi x)$	$\sin(6\pi x)\cos(4\pi x)$
Case 3	$0.5 + \exp(-2x^2)$	$\exp(-10(x-0.5)^2) - \exp(-10(x+0.3)^2)$
Case 4	$0.5 + \exp(-2x^2)$	$\exp(-100(x-0.5)^2) - \exp(-100(x+0.3)^2)$

the unknown expansion terms are increased beyond $N \gtrsim 7$ RAPID analysis is computationally more efficient when compared to direct quadrature. As is expected the computational analysis time for the triple-product approach increases with the number of terms, M, due to calculating the increasing number of expansion coefficients, fig 3a. From inspection of fig 2 it is observed that an expansion of $M = 20$ terms provides a reasonable approximation of the stiffness and load variations. The computational efficiency gains, when increasing the number of terms in the displacement trial functions expansion, N, is shown in fig 3b. At low-resolution there is no advantage in using the RAPID formulation, however as the number of terms increases, to levels typically used [9, 15], significant computational savings are realised. This behaviour is consistent with the prediction of an order of magnitude reduction in the number of quadratures required.

It is noted that in these comparisons the computation time for determining the required coefficients of the triple-products matrices are included in the run-time. If multiple configurations were investigated, for example in an optimisation study, further savings could be realised. The triple-product and coefficients matrices would not need to be calculated multiple times. In fact, as the exact evaluation of the Wigner $(3j)$ coefficient is possible [16], closed form solutions can be obtained when stiffness and load variation is expressible as a finite LP series offering potential improvements over numerical quadrature. Indeed, if the required stiffness matrices are expressed as LP expansions *a priori* no numerical quadrature is required. An extension of the RAPID formulation to the analysis of variable stiffness plates is now presented.

5 Variable Stiffness Plates

Consider a plate aligned with the xy plane having its origin located at the central mid-point of the plate, fig 4. The plate has variable stiffness properties, for example due to thickness variations. As before, it is assumed that the rates-of-change of these variations are sufficiently small that two-dimensional plate analysis suffices. An energetic formulation based on classical laminate theory (CLT) with $\mathbf{d}$ representing bending stiffness given by its usual definition is utilised [17]. For purposes of brevity it is assumed that there is no-coupling between in-and out-of-plane behaviour and consider the out-of-plane response only. This simplification does not reduce the applicability of the RAPID approach which can be adapted for more general problems.

Proceeding with a series of non-dimensionalisations,

$$\tilde{x} = \frac{x}{L_x}, \qquad \tilde{y} = \frac{y}{L_y}, \qquad \tilde{\mathbf{d}}(x,y) = \frac{\mathbf{d}(x,y)}{d_0}, \qquad \tilde{q} = \frac{q(x,y)}{q_0} \qquad \tilde{w}(x,y) = \frac{2w}{L_x + L_y} \tag{33}$$

The curvatures, $\boldsymbol{\kappa}$, can be written in terms of deflection,

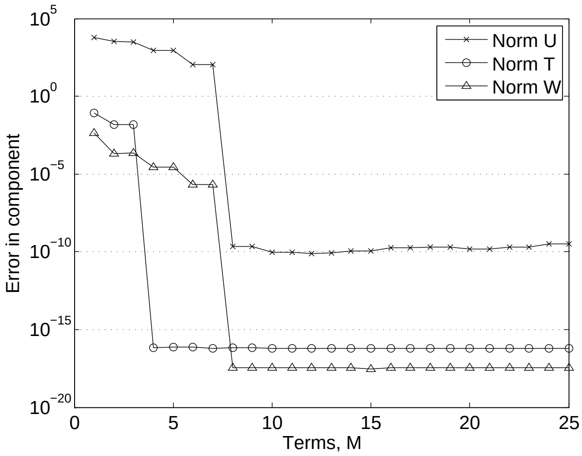
$$\boldsymbol{\kappa}^{\mathsf{T}} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} & -\frac{\partial^2 w}{\partial y^2} & -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{34}$$

allowing the non-dimensional energy to be written as,

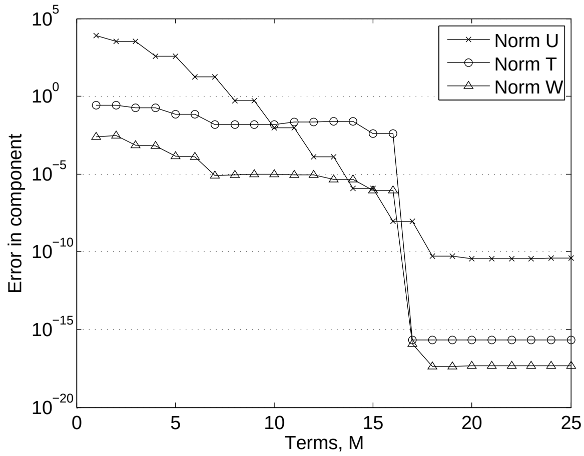
$$\begin{aligned}
\tilde{\Pi} = {} & \alpha_{11} \iint\limits_{\tilde{A}} \tilde{d}_{11} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\right)^2 d\tilde{A} + \alpha_{12} \iint\limits_{\tilde{A}} \tilde{d}_{12} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\right)\left(\frac{\partial^2 \tilde{w}}{\partial \tilde{y}^2}\right) d\tilde{A} + \alpha_{16} \iint\limits_{\tilde{A}} \tilde{d}_{16} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\right)\left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}\partial \tilde{y}}\right) d\tilde{A} \\
& + \alpha_{22} \iint\limits_{\tilde{A}} \tilde{d}_{22} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{y}^2}\right)^2 d\tilde{A} + \alpha_{26} \iint\limits_{\tilde{A}} \tilde{d}_{26} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{y}^2}\right)\left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}\partial \tilde{y}}\right) d\tilde{A} + \alpha_{66} \iint\limits_{\tilde{A}} \tilde{d}_{66} \left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}\partial \tilde{y}}\right)^2 d\tilde{A} \\
& - \iint\limits_{\tilde{A}} \tilde{q}\tilde{w}\, d\tilde{A}
\end{aligned} \tag{35}$$

where $\tilde{A}$ is the unit square, with

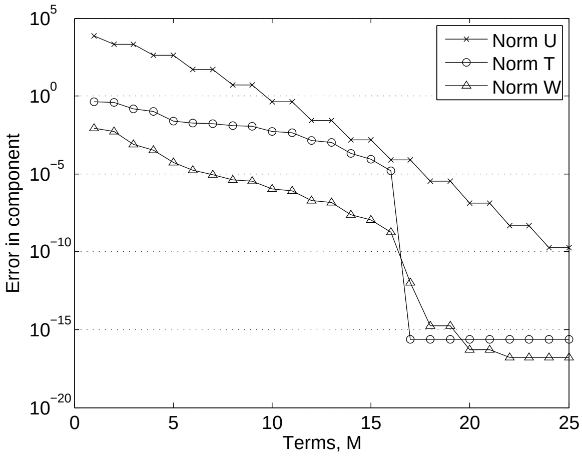
$$\begin{aligned}
\alpha_{11} &= \frac{d_0}{2q_0}\frac{L_x + L_y}{4L_x^4}, & \alpha_{12} &= \frac{d_0}{q_0}\frac{L_x + L_y}{4L_x^2 L_y^2}, & \alpha_{16} &= 2\frac{d_0}{q_0}\frac{L_x + L_y}{4L_x^3 L_y}, \\
\alpha_{22} &= \frac{d_0}{2q_0}\frac{L_x + L_y}{4L_y^4}, & \alpha_{26} &= 2\frac{d_0}{q_0}\frac{L_x + L_y}{4L_x L_y^3}, & \alpha_{66} &= 2\frac{d_0}{q_0}\frac{L_x + L_y}{4L_x^2 L_y^2}.
\end{aligned} \tag{36}$$



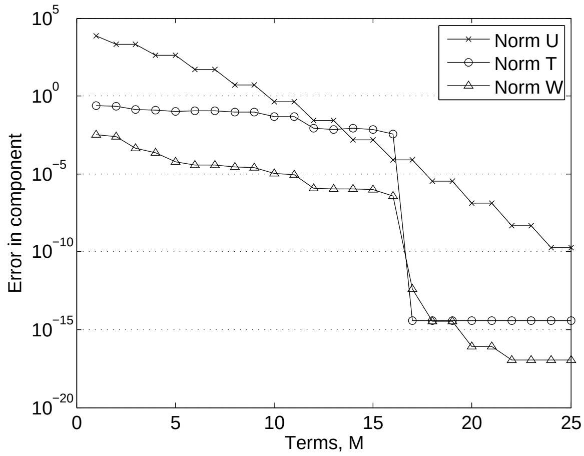
(a) Case 1.



(b) Case 2.

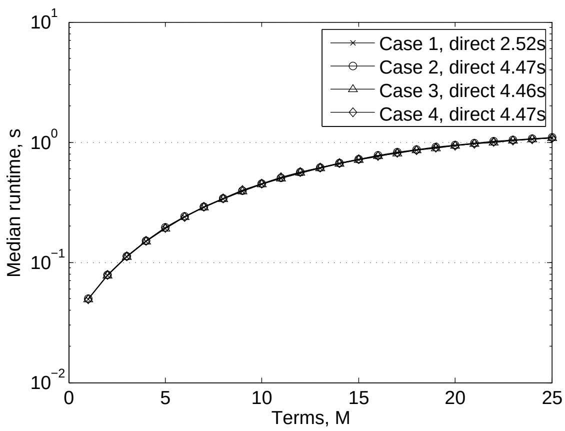


(c) Case 3.

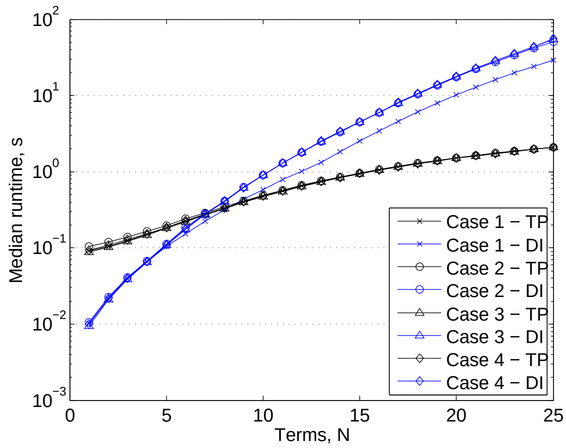


(d) Case 4.

Figure 2: Convergence of RAPID formulation to results obtained via direct numerical integration for increasingly refined stiffness/load approximation with M the number of terms for the four cases in table 2.



(a) Increased runtime with increasing M and fixed $N = 15$, using TP. DI time is provided for comparison as it does not vary with M.



(b) Change in runtime with increasing N for fixed $M = 20$ for TP and DI. For $N \gtrsim 7$, TP shows runtime improvements.

Figure 3: Runtime for RAPID triple-product integration (TP) and direct numerical integration (DI) for each of the four cases in table 2. Several trend lines are co-located indicating that the integration technique, rather than the variation of stiffness/load functions, determines the computational cost.

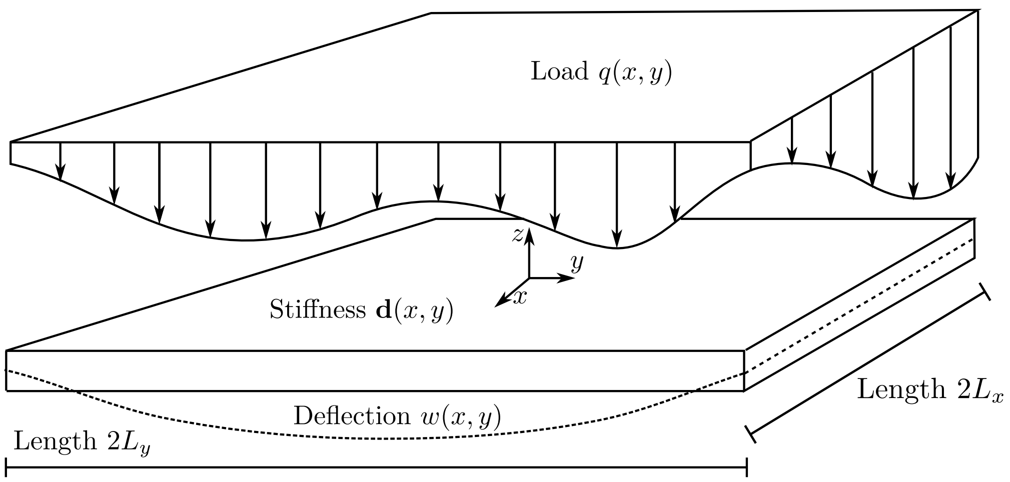


Figure 4: Diagram of a variable stiffness plate symmetric about its mid-plane, subject to a variable out of plane load.

Letting the unknown displacement function be approximated by a kinematically admissible two-dimensional trial function $\mathbf{S}(\tilde{x}, \tilde{y})$ with terms $S_i(\tilde{x}, \tilde{y})$ that can be composed of basis functions in the $\tilde{x}$ and $\tilde{y}$ directions,

$$\tilde{w}(\tilde{x}) \approx \boldsymbol{\omega}\mathbf{S} = \sum_{i=0}^{N} \omega_i S_i(\tilde{x}, \tilde{y}) = \sum_{j=0}^{N_x} \sum_{k=0}^{N_y} \omega_{jk} S_j(\tilde{x}) S_k(\tilde{y}) \tag{37}$$

defines a system of equations defined by

$$\frac{\partial \tilde{\Pi}}{\omega_i} = 0. \tag{38}$$

Solving this system requires evaluating $\mathcal{O}(N_x N_y)^2$ two-dimensional integrals, for example,

$$\iint_{\tilde{A}} \frac{\partial^2 S_i}{\partial \tilde{x}^2} \tilde{d}_{11} \frac{\partial^2 S_i}{\partial \tilde{x}^2} \, d\tilde{A}. \tag{39}$$

Similarities between eq (39) and the one-dimensional case, eq (6) are observed. It is possible to exploit the RAPID formulation in a similar fashion for two-dimensional problems using Kronecker multiplication in order to improve computational efficiency.

6 Two-Dimensional Legendre Polynomials

The techniques outlined for one-dimensional analysis can be readily extended to the two-dimensional integrals. To account for behaviour in each direction the number of trial functions required is often the square of that required for a one-dimensional analysis. Evidently, the potential for improvements in computational efficiency are therefore expected to be more pronounced for the two-dimensional case. The two-dimensional LP basis using Kronecker multiplication may be defined,

$$\mathbf{P}(\tilde{x}, \tilde{y}) = \mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y}) = \begin{bmatrix} P_0(\tilde{x})\mathbf{P}(\tilde{y}) \\ P_1(\tilde{x})\mathbf{P}(\tilde{y}) \\ \vdots \\ P_N(\tilde{x})\mathbf{P}(\tilde{y}) \end{bmatrix}. \tag{40}$$

As in the one-dimensional case a modified basis, satisfying kinematic boundary conditions, can be obtained by application of appropriate coefficient matrices in the x and y directions,

$$\begin{aligned} \mathbf{P}_{\mathrm{BC}}(\tilde{x}, \tilde{y}) &= \mathbf{P}_{\mathrm{BC}x}(\tilde{x}) \otimes \mathbf{P}_{\mathrm{BC}y}(\tilde{y}), \\ &= [\mathbf{C}_{\mathrm{BC}x}\mathbf{P}(\tilde{x})] \otimes [\mathbf{C}_{\mathrm{BC}y}\mathbf{P}(\tilde{y})], \\ &= [\mathbf{C}_{\mathrm{BC}x} \otimes \mathbf{C}_{\mathrm{BC}y}][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})], \end{aligned} \tag{41}$$

where $\mathbf{P}_{\mathrm{BC}x,y}$ is the modified basis and $\mathbf{C}_{\mathrm{BC}x,y}$ are the required coefficient matrices. Partial differentiation can be similarly achieved using matrix multiplication by $\mathbf{C}_{\delta x,y}$. Using the derivative matrix defined previously, eq (19), together with the identity matrix $I_{x,y}$ in the stationary direction,

$$\frac{\partial}{\partial \tilde{x}}\mathbf{P}(\tilde{x},\tilde{y}) = [\mathbf{C}_\delta \otimes \mathbf{I}_y][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})], \qquad \frac{\partial}{\partial \tilde{y}}\mathbf{P}(\tilde{x},\tilde{y}) = [\mathbf{I}_x \otimes \mathbf{C}_\delta][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})]. \tag{42}$$

Combining these results gives,

$$\frac{\partial^{i+j}}{\partial \tilde{x}^i \partial \tilde{y}^j}\mathbf{P}_{\mathrm{BC}}(\tilde{x},\tilde{y}) = [\mathbf{C}_{\mathrm{BC}x}\mathbf{C}_{i\delta} \otimes \mathbf{C}_{\mathrm{BC}y}\mathbf{C}_{j\delta}][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})]. \tag{43}$$

A typical system integral is,

$$\mathbf{F} = \iint\limits_{\tilde{A}} \frac{\partial^{i+j}}{\partial \tilde{x}^i \partial \tilde{y}^j}\mathbf{P}_{\mathrm{BC}}(\tilde{x},\tilde{y}) f(\tilde{x},\tilde{y}) \frac{\partial^{i+j}}{\partial \tilde{x}^i \partial \tilde{y}^j}\mathbf{P}^{\intercal}_{\mathrm{BC}}(\tilde{x},\tilde{y})\, d\tilde{A} \tag{44}$$

Letting $f(\tilde{x},\tilde{y})$ be represented by a series expansion,

$$f \approx= \sum_{i=0}^{M} f_i P_i(\tilde{x},\tilde{y}) = \sum_{k=0}^{M_x}\sum_{k=0}^{M_y} f_{jk} P_j(\tilde{x}) P_k(\tilde{y}), \tag{45}$$

the integrals $\mathbf{F}$ can be written in terms of LP basis functions,

$$\begin{aligned}
\mathbf{F} &\approx \iint\limits_{\tilde{A}} \sum_{k=0}^{M_x}\sum_{k=0}^{M_y} [\mathbf{C}_{\mathrm{BC}x}\mathbf{C}_{i\delta} \otimes \mathbf{C}_{\mathrm{BC}y}\mathbf{C}_{j\delta}][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})] f_{jk} P_j(\tilde{x}) P_k(\tilde{y}) \, [[\mathbf{C}_{\mathrm{BC}x}\mathbf{C}_{i\delta} \otimes \mathbf{C}_{\mathrm{BC}y}\mathbf{C}_{j\delta}][\mathbf{P}(\tilde{x}) \otimes \mathbf{P}(\tilde{y})]]^{\intercal} \, d\tilde{A} \\
&\approx [\mathbf{C}_{\mathrm{BC}x}\mathbf{C}_{i\delta} \otimes \mathbf{C}_{\mathrm{BC}y}\mathbf{C}_{j\delta}] \sum_{k=0}^{M_x}\sum_{k=0}^{M_y} f_{jk} \left[\iint\limits_{\tilde{A}} \mathbf{P}(\tilde{x})\mathbf{P}^{\intercal}(\tilde{x})P_j(\tilde{x}) \otimes \mathbf{P}(\tilde{y})\mathbf{P}^{\intercal}(\tilde{y})P_k(\tilde{y})\, d\tilde{A}\right] [\mathbf{C}^{\intercal}_{i\delta}\mathbf{C}^{\intercal}_{\mathrm{BC}x} \otimes \mathbf{C}^{\intercal}_{j\delta}\mathbf{C}^{\intercal}_{\mathrm{BC}y}] \\
&\approx \mathbf{C}\left[\sum_{k=0}^{M_x}\sum_{k=0}^{M_y} f_{jk} \left[\int_{-1}^{1} \mathbf{P}(\tilde{x})\mathbf{P}^{\intercal}(\tilde{x})P_j(\tilde{x})\, d\tilde{x}\right] \otimes \left[\int_{-1}^{1} \mathbf{P}(\tilde{y})\mathbf{P}^{\intercal}(\tilde{y})P_k(\tilde{y})\, d\tilde{y}\right]\right]\mathbf{C}^{\intercal}.
\end{aligned} \tag{46}$$

where $\mathbf{C}$ is a coefficient matrix that captures all the required partial differentiation and kinematic constraints. It is noted that if there is an equally refined mesh in each direction [4],

$$\left[\int_{-1}^{1} \mathbf{P}(\tilde{x})\mathbf{P}^{\intercal}(\tilde{x})P_i(\tilde{x})\, d\tilde{x}\right] = \left[\int_{-1}^{1} \mathbf{P}(\tilde{y})\mathbf{P}^{\intercal}(\tilde{y})P_i(\tilde{y})\, d\tilde{y}\right], \tag{47}$$

since the $\tilde{x}$ and $\tilde{y}$ triple products integrals differ only by the variable of integration. The analytical triple-products need only be calculated in one direction offering further efficiency gains. In the example presented $\mathcal{O}(N_x^2 N_y^2)$ numerical quadratures have been replaced by the $\mathcal{O}(M_x M_y)$ required to calculate the weighting coefficients f_{jk} and a summation of known matrices. The efficiency savings are the square of the savings made for for the one-dimensional case, highlighting the potential for the triple-product approach when applied to plates. Variable stiffness plates are now considered. proceed to consider a variable stiffness plate.

7 RAPID Formulation - Plates

The variation in load over the plate can be described via a series expansion using the two-dimensional LP series,

$$\tilde{q}(\tilde{x},\tilde{y}) \approx \sum_{j=0}^{M_x}\sum_{k=0}^{M_y} q_{jk} P_j(\tilde{x}) P_k(\tilde{y}). \tag{48}$$

The variation of each of the stiffness matrices components can be similarly described. Solving eq (38), results in a system of the form,

$$\frac{\partial \tilde{\Pi}}{\partial \boldsymbol{\omega}} = \Lambda \boldsymbol{\omega} - \boldsymbol{\Gamma}, \tag{49}$$

[4] For a non-equal refinement the a subset of the larger can be used to define the smaller.

Table 3: Representative loading and thickness variation investigated for two-dimensional computational benchmarking.

Case	$H(\tilde{x},\tilde{y})$	$\tilde{q}(\tilde{x},\tilde{y})$
Wide	$\frac{1}{2}+\exp(-2(x^2+y^2))$	$\sin(2\pi y)+2\cos(\pi y)$
Narrow	$\frac{1}{2}+\exp(-10(x^2+y^2))$	$\sin(8\pi y)+2\cos(4\pi y)$

where

$$\begin{aligned}\Lambda = {} & \alpha_{11}\mathbf{C}_{\mathrm{BC},xx}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{11\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},xx}^{\intercal} + \alpha_{12}\mathbf{C}_{\mathrm{BC},xy}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{12\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},yy}^{\intercal} \\ & + \alpha_{16}\mathbf{C}_{\mathrm{BC},xx}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{16\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},xy}^{\intercal} + \alpha_{22}\mathbf{C}_{\mathrm{BC},yy}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{22\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},yy}^{\intercal} \\ & + \alpha_{26}\mathbf{C}_{\mathrm{BC},yy}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{26\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},xy}^{\intercal} + \alpha_{66}\mathbf{C}_{\mathrm{BC},xy}\left[\sum_{j=0}^{M_x}\sum_{k=0}^{M_y} d_{66\,jk}\boldsymbol{\Omega}_{jk}\right]\mathbf{C}_{\mathrm{BC},xy}^{\intercal} \qquad (50)\end{aligned}$$

with

$$\boldsymbol{\Omega}_{jk} = \left[\int_{-1}^{1}\mathbf{P}(\tilde{x})\mathbf{P}^{\intercal}(\tilde{x})P_j(\tilde{x})\,d\tilde{x}\right]\otimes\left[\int_{-1}^{1}\mathbf{P}(\tilde{y})\mathbf{P}^{\intercal}(\tilde{y})P_k(\tilde{y})\,d\tilde{y}\right], \qquad (51)$$

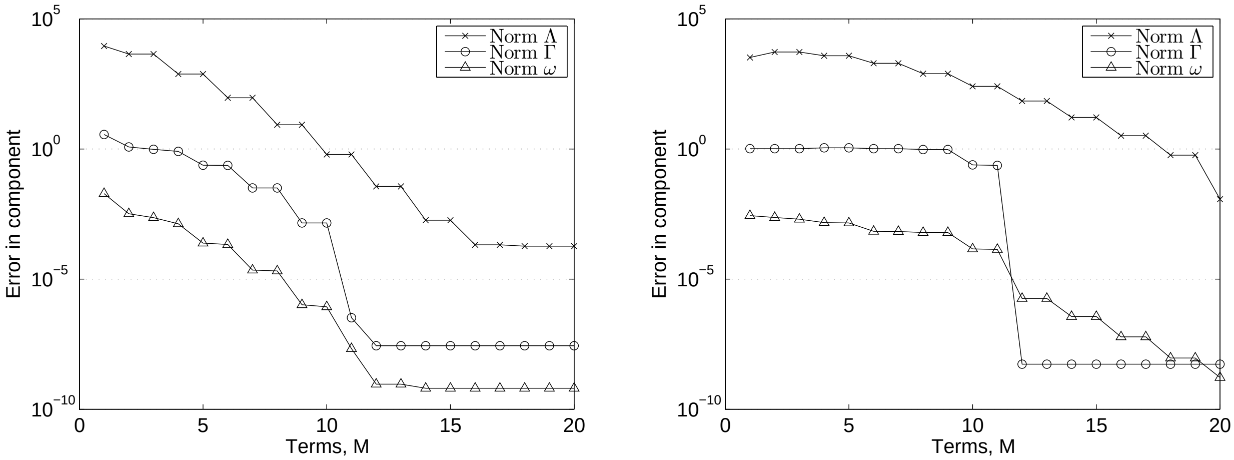
and with appropriate coefficient matrices enforcing the boundary conditions and derivatives and

$$\boldsymbol{\Gamma} = \sum_{j=0}^{M_x}\sum_{k=0}^{M_y} q_{jk}\left[\left[\int_{-1}^{1}\mathbf{P}(\tilde{x})P_j(\tilde{x})\,d\tilde{x}\right]\otimes\left[\int_{-1}^{1}\mathbf{P}(\tilde{y})P_k(\tilde{y})\,d\tilde{y}\right]\right], \qquad (52)$$

completely defining the system in terms of common LP integrals. The computational time required to complete the analysis can now be compared.

7.1 Computational Efficiency

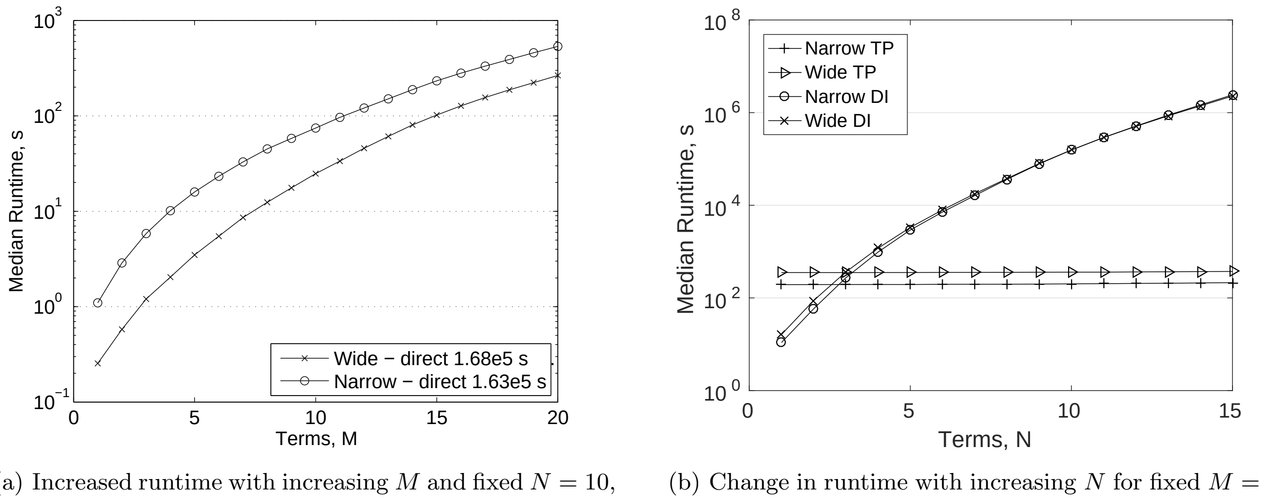
So as to simplify the comparison between the RAPID and direct integration formulations a plate, composed of an isotropic material with a thickness variation, $H(\tilde{x},\tilde{y})$, that causes a variation in stiffness is considered. The cases investigated are listed in table 3. For direct integration MATLAB's "integral2" function is utilised [13, 18]. As for the one-dimensional case the norm of the difference between the two approaches is used to demonstrate convergence for $\boldsymbol{\Lambda}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\omega}$, the matrices associated with bending energy, out-of-plane work done and the unknown solutions coefficients. The convergence results are presented in fig 5, where $M_x = M_y = M$, represents the accuracy of the series approximation. The results are calculated for a trial function expansion with $N_x = N_y = 10$ terms. The cost, as shown in fig 6a demonstrates increased runtime with larger M similar to those observed for the one-dimensional case. Fig 6b compares the computational cost for increasing N, for fixed $M = 20$. As was observed in the one-dimensional case at very low refinement there is no advantage to the RAPID formulation, however as the number of terms increases the efficiency gains become significant. For the case of two-dimensional analysis it is observed that as the total number of numerical quadratures is the square of N, owing to expansion in two directions, thus the efficiency savings are more pronounced than for the one-dimensional analysis.



(a) Wide.

(b) Narrow.

Figure 5: Convergence of RAPID formulation to results obtained via direct numerical integration for increasingly refined stiffness/load approximation with M the number of terms for the two cases in table 3.



(a) Increased runtime with increasing M and fixed $N = 10$, using RAPID formulation. Direct integration (DI) time is provided for comparison.

(b) Change in runtime with increasing N for fixed $M = 20$ for TP and DI. For $N \gtrsim 4$, TP shows runtime improvements.TP runtime increases with N however it appears constant when when compared to DI on a log-scale.

Figure 6: Runtime for RAPID triple-product integration (TP) and direct numerical integration (DI) for each of the two cases in table 3. The reduction in runtime is more significant for plate analysis requiring trial function expansion in both the x and y directions thus the total number of trial functions is N^2.

8 Conclusion

An alternative method for computing integrals encountered in a typical Ritz based analysis for variable stiffness beams and plates has been presented. This method is computationally efficient compared to direct numerical quadrature that is typically employed for analysis of these problems. Herein, the Recursive Analytical Polynomial Integral Definition (RAPID) formulation presented utilises Legendre polynomial (LP) triple products to replace the $\mathcal{O}(N^2)$ integrals encountered with a summation of known analytical results. To implement this approach at most $\mathcal{O}(M)$ numerical integrations are required to calculate the expansion coefficients for the stiffness variation. Typically, $M \approx N$, therefore the number of integrations required has been reduced by an order of magnitude when compared to a direct numerical integration approach. The calculation of these integrals is often the computational bottleneck in problems of this type therefore the RAPID formulation provides significant runtime reductions.

The order of magnitude decrease predicted in the computational runtime has been observed when comparing the RAPID formulation to MATLAB's adaptive quadrature functions, "integral" and "integral2" [13, 14, 18]. Furthermore, convergence of the RAPID formulation, in-line with a series approximation, is observed and the variance between the RAPID approach and direct integration discussed. These results demonstrate that our new approach is both viable and effective.

The imposition of various boundary conditions and derivatives of the trial function can be achieved via matrix multiplication, reducing most integrals to a common form. Such a simplification aids the investigation into various designs. The computational advantage of this approach is significant, particularly when utilised for component design and optimisation. In fact, as the exact evaluation of the Wigner $(3j)$ coefficients is possible, closed form solutions can be obtained if the stiffness and load variation is expressible exactly as a finite Legendre polynomial series. By utilising the RAPID approach an increase in the number of terms in the trial function is computationally favourable when compared to direct integration. Thus the capture of localised features may be achieved more efficiently than state-of-the-art approaches currently employed.

Data access statement

The data necessary to support the conclusions are included in the paper.

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References

[1] S. T. Ijsselmuiden, M. M. Abdalla, and Z. Gürdal, "Optimization of variable-stiffness panels for maximum buckling load using lamination parameters," *AIAA Journal*, vol. 48, pp. 134–143, Jan 2010.

[2] B. Kim, P. Weaver, and K. Potter, "Computer aided modelling of variable angle tow composites manufactured by continuous tow shearing," *Composite Structures*, vol. 129, pp. 256 – 267, 2015.

[3] G. Zucco, R. Groh, A. Madeo, and P. Weaver, "Mixed shell element for static and buckling analysis of variable angle tow composite plates," *Composite Structures*, vol. 152, pp. 324 – 338, 2016.

[4] P. Davis and P. Rabinowitz, *Methods of Numerical Integration*. Dover Books on Mathematics Series, Dover Publications, 2007.

[5] M. O'Donnell and P. Weaver, "Rapid analysis of variable stiffness plates: Legendre polynomial triple product formulation," in *The Royal Aeronautical Society's 4th Aircraft Structural Design Conference*, 10 2014.

[6] J. C. Adams, "On the expression of the product of any two Legendre's coefficients by means of a series of Legendre's coefficients," *Proceedings of the Royal Society of London*, vol. 27, pp. 63–71, 1878.

[7] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*. New York, NY, USA: Cambridge University Press, 1st ed., 2010.

[8] N. Hale and A. Townsend, "A fast, simple, and stable Chebyshev-Legendre transform using an asymptotic formula," *SIAM Journal on Scientific Computing*, vol. 36, no. 1, pp. A148–A167, 2014.

[9] Z. Wu, G. Raju, and P. M. Weaver, "Buckling analysis of vat plate using energy method," in *53rd AIAA/ASME Structures, Structural Dynamics and Materials Conference*, (Hawaii), pp. 1–12, AIAA, 2012.

[10] N. Jaunky, N. F. Knight, and D. R. Ambur, "Buckling analysis of general triangular anisotropic plates using polynomials," *AIAA Journal*, vol. 33, pp. 2414–2417, Dec 1995.

[11] D. Zhou, F. Au, Y. Cheung, and S. Lo, "Three-dimensional vibration analysis of circular and annular plates via the Chebyshev-Ritz method," *International Journal of Solids and Structures*, vol. 40, no. 12, pp. 3089 – 3105, 2003.

[12] J. Boyd and R. Petschek, "The relationships between Chebyshev, Legendre and Jacobi polynomials: The generic superiority of Chebyshev polynomials and three important exceptions," *Journal of Scientific Computing*, vol. 59, pp. 1–27, 4 2014.

[13] MATLAB, *version 8.2.0.701 (R2013b)*. Natick, Massachusetts: The MathWorks Inc., 2013.

[14] L. Shampine, "Vectorized adaptive quadrature in {MATLAB}," *Journal of Computational and Applied Mathematics*, vol. 211, no. 2, pp. 131 – 140, 2008.

[15] E. Cosentino and P. M. Weaver, "Prebuckling and buckling of unsymmetrically laminated composite panels with stringer run-outs," *AIAA Journal*, vol. 47, pp. 2284–2297, October 2009. AIAA/ASME/ASCE/AHS/ASC 50th Structures, Structural Dynamics, and Materials Conference, Palm Springs, CA, MAY 02-07, 2009.

[16] A. J. Stone and C. P. Wood, "Root-rational-fraction package for exact calculation of vector-coupling coefficients," *Computer Physics Communications*, vol. 21, no. 2, pp. 195 – 205, 1980.

[17] E. H. Mansfield, *The Bending and Stretching of Plates*. The Bending and Stretching of Plates, Cambridge University Press, 2005.

[18] L. Shampine, "Matlab program for quadrature in 2d," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 266 – 274, 2008.