Unknown Dynamics Estimator Based Output-Feedback Control for Nonlinear Pure-Feedback Systems

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***Abstract*—Most existing adaptive control designs for nonlinear pure-feedback systems have been derived based on backstepping or dynamic surface control (DSC) methods, requiring full system states to be measurable. The neural networks (NNs) or fuzzy logic systems (FLSs) used to accommodate uncertainties also impose demanding computational cost and sluggish convergence. To address these issues, this paper proposes a new output feedback control for uncertain pure-feedback systems without using backstepping and function approximator. A coordinate transform is first used to represent the pure-feedback system in a canonical form to evade using the backstepping or DSC scheme. Then the Levant's differentiator is used to reconstruct the unknown states of the derived canonical system. Finally, a new unknown system dynamics estimator with only one tuning parameter is developed to compensate for the lumped unknown dynamics in the feedback control. This leads to an alternative, simple approximation-free control method for pure-feedback systems, where only the system output needs to be measured. The stability of the closed-loop control system including the unknown dynamics estimator and the feedback control is proved. Comparative simulations and experiments based on a PMSM test-rig are carried out to test and validate the effectiveness of the proposed method.**

***Index Terms*—Pure-feedback systems; output-feedback control; unknown dynamics estimator; Levant's differentiator.**

I. Introduction

During the past decades, many control design methodologies have been proposed for various nonlinear systems, e.g. Brunovsky systems [1], strict-feedback systems [2, 3], and pure- feedback systems [4, 5]. Among different system formulations, pure-feedback systems can cover the aforementioned nonlinear dynamics. However, the non-affine properties of the states and control input involved in the pure-feedback systems create certain difficulties in the control design for such systems compared with the other system formulations [6]. In viewing the control design for pure-feedback systems, e.g. [4-8] and references therein, it is found that the most commonly used method is to reformulate the pure-feedback systems into strict-feedback systems via the mean value theorem and then apply the backstepping scheme [9], which was originally derived for strict-feedback systems. Following this idea, a special affine-in-control pure-feedback system was studied in [4, 5], where the implicit function theorem is applied to assert the existence of the desired control actions. Generic non-affine systems were studied in [10], where the ISS approach and small gain theorem are used to relax the imposed assumptions. Similarly, by using the backstepping scheme, control designs for pure-feedback systems with time-delays [11], dead-zone input [8] and hysteresis [6] have been subsequently considered. However, all of above control designs were derived based on the lengthy and complicated backstepping procedure. One of the well-known drawbacks of backstepping is the 'explosion of complexity', which stems from the repeated calculation of the derivatives of virtual control actions. To tackle this problem, dynamic surface control (DSC) [12, 13] was developed using a low-pass filter in each step to approximate these derivatives. Although DSC has been tailored for nonlinear pure-feedback systems [7, 8, 14], it again follows a similar recursive synthesis as the backstepping, because the non-affine functions have to be transformed into strict-feedback forms.

In fact, most of existing control methods for pure-feedback systems [4, 5, 7, 8, 10, 11, 15] use the backstepping or DSC techniques, such that the control implementation and stability analysis are lengthy and complicated. Moreover, all of the above control designs assume that all the system states are available or directly measurable, which may not be true in practice. In this respect, only a few output-feedback control approaches have been investigated recently, e.g. [16-20] and references therein, where different adaptive observers with function approximators were used to reconstruct system states.

On the other hand, function approximation-based control has been proved as a powerful method to address the unknown uncertainties and nonlinearities, and thus attracted increasing attentions in the control community. In this method, neural networks (NNs) [4, 5, 7, 8, 10, 11, 15, 21-23] or fuzzy logic systems (FLSs) [18, 24-27] have been incorporated into adaptive control designs, to handle the unknown nonlinearities [28-30], where the unknown weights of NNs or FLSs can be online updated based on the gradient based adaptive laws to retain the closed-loop stability. This approximation based adaptive control was also extended for uncertain pure-feedback systems [4-8]. However, for such adaptive backstepping control designs, multiple function approximators have to be used to obtain each virtual control actions, which make them computational demanding.

Moreover, although function approximators have been used in the control designs for uncertain systems, there is merely unique guidelines to select the topology of NNs and FLSs, and analyze their approximation accuracy. Specifically, in order to obtain satisfactory performance, the number of parameters (e.g. NN weights) to be online updated is very large [31-33].

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Nevertheless, in most existing function approximation based controls, adaptive laws used to online update the unknown NN or FLS weights are driven by the control errors [34-38]. Although theoretical studies have shown that the closed-loop system is stable, the estimated weights may not converge to their ideal values [39], and they may suffer from the parameter drifting issue when a high gain adaptation is used. Thus, the parameter tuning of approximation-based adaptive control is generally difficult [40, 41]. The above mentioned issues partially result in the gap between the elegant theoretical studies and rare practical applications of these approximation-based adaptive control methods.

Following the above discussions, we find that developing alternative output-feedback control for pure-feedback systems without using backstepping and any function approximators [42] has not been fully solved, and deserves further investigation. Hence, this paper aims to present a new output-feedback control design for nonlinear unknown pure-feedback systems. We first represent the pure-feedback system in a Brunovsky form by defining new coordinate system states, which helps to evade the backstepping scheme. Then Levant's differentiator [43, 44] is used to reconstruct the immeasurable states of the derived canonical system with guaranteed finite-time convergence. Finally, instead of using NNs or FLSs, we will develop a new simple unknown dynamics estimator by tailoring the idea of unknown input observer [45, 46] to handle the lumped unknown nonlinearities. This unknown dynamics estimator uses first-order filter operations on the measured system dynamics, and has only one scalar (e.g. filter constant) to be set, whilst the exponential convergence is achieved. In this case, we do not need to select the topology of NNs or FLSs [31, 47, 48]. The sluggish online learning in the function approximators is also avoided. Finally, only the system output is required in the control implementation. Theoretical studies are all verified by using both simulations and experiments based on a practical servo system driven by a PMSM.

Compared with existing control methods of pure-feedback systems, the main contributions of this paper can be stated as:

1) A new output-feedback control is proposed for nonlinear pure-feedback systems without using the backstepping or DSC schemes. This is achieved by using a coordinate transform to reformulate a pure-feedback system into a Brunovsky form. Then, the suggested control is considerably simpler than the backstepping based methods [4, 5, 7, 8, 10, 11, 15].

2) A new unknown dynamics estimator inspired by [45, 46] is investigated to address the lumped unknown dynamics, such that function approximators and the corresponding online learning are avoided. This approximation-free control has not only faster convergence but also reduced computational burden than function approximation based methods.

3) The suggested control requires the system output only rather than the full system states. Hence, it is more attractive in terms of practical control implementation. Experiments are also carried out to validate its effectiveness.

The paper is structured as: Section II presents the problem formulation; Coordinate transform is described in Section III; Section IV gives the output-feedback control design with the differentiator and unknown dynamics estimator; Section V provides comparative simulations and experiments; Section VI gives some conclusions.

II. Problem formulation

In this paper, we consider the following nonlinear nonaffine pure-feedback systems

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \quad \vdots \\ \dot{x}_i = f_i(x_1, x_2, \cdots x_{i+1}), \qquad i = 2, \cdots, n-1, \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, x_2, \cdots x_n, u) \end{cases} \tag{1}$$

where $\bar{x}_i = [x_1, x_2 \cdots x_i]^T \in \mathbb{R}^i, i = 1, \cdots n$ is the system states; $y = x_1 \in \mathbb{R}$ denotes the system output and $u \in \mathbb{R}$ is the control input. $f_i(\cdot), i = 1, \cdots n$ are nonlinear unknown smooth functions.

The objective of control design is to find an appropriate control action u based on the measured output y only, such that the output y of system (1) follows a given trajectory y_d without using any function approximators (e.g. NNs, FLSz) and backstepping or DSC schemes.

To facilitate the following control design, we have

Assumption 2.1: The desired trajectory y_d and its derivatives $y_d^1, \cdots, y_d^n$ are bounded.

Assumption 2.2 [4, 5]: The functions $f_i(\bar{x}_{i+1})$ are continuous with respect to the state x_{i+1} and $x_{n+1} = u$, and the signs of $\partial f_i(x_1, \cdots, x_{i+1}) / \partial x_{i+1}$ are known. Without loss of generality, we assume that all of these signs are positive in this paper.

Remark 2.1: Assumption 2.2 is the well-known controllability condition for pure-feedback system (1), which is sufficient to guarantee that the system states $\bar{x}_i$ can be manipulated by x_{i+1} without encountering the control singularity problem. This condition has been widely used in the literature, e.g. [4, 5], and can be fulfilled in most of practical systems. The control design for the system with negative control gains $\partial f_i(x_1, \cdots, x_{i+1}) / \partial x_{i+1}$ can be carried out in a similar way as the case with positive gains to be presented in the paper. For systems with unknown signs of control coefficients, Nussbaum functions [6] can be further used in the control design.

Remark 2.2: Although adaptive control design for system (1) has been studied during the past decade [4, 5, 7, 8, 10, 11, 15], most existing methods were derived by using the backstepping scheme. Moreover, the unknown functions in these controls are addressed by using multiple function approxmators (NNs or FLZs) in each backstepping step. Consequently, these control designs and the stability analysis are complicated, and their implementations require significant computational costs.

The aim of this paper is to develop a new output-feedback control design for pure-feedback system (1) without using backstepping and any function approximation techniques. The first idea is to introduce a coordinate transform to reformulate the original system (1) into a canonical system. Then, the Levant's differentiator is used to reconstruct the unknown system states of the transformed system. Finally, a new unknown dynamics estimator will be investigated and used in the control design to compensate for the unknown lumped dynamics.

III. COORDINATE TRANSFORM

To avoid using the backstepping scheme, we first introduce

a coordinate transform to represent system (1). For this purpose, we define coordinate variables as [42]

$$\begin{cases} z_1 = x_1 \\ z_i = \dot{z}_{i-1}, \qquad i = 2,\cdots,n \end{cases}. \tag{2}$$

Then we know $z_2 = \dot{z}_1 = f_1(x_1, x_2)$, and then can calculate its derivative as

$$\dot{z}_2 = \frac{\partial f_1(x_1,x_2)}{\partial x_1} f_1(x_1,x_2) + \frac{\partial f_1(x_1,x_2)}{\partial x_2} f_2(x_1,x_2,x_3). \tag{3}$$

From the fact that $f_2(x_1,x_2,x_3)$ is a continuous function of x_3, one may apply the *Mean-Value Theorem* [49] such that

$$f_2(x_1,x_2,x_3) = f_2(x_1,x_2,0) + \frac{\partial f_2(x_1,x_2,x_3)}{\partial x_3}\Big|_{x_3=x_3^\theta} x_3, \tag{4}$$

where $x_3^\theta = \theta x_3$ for any constant $0 < \theta < 1$.

Substituting (4) into (3) yields

$$\dot{z}_2 = \phi_2(x_1,x_2) + \chi_2(x_1,x_2,x_3^\theta)x_3, \tag{5}$$

where $\phi_2(x_1,x_2) = \frac{\partial f_1(x_1,x_2)}{\partial x_1} f_1(x_1,x_2) + \frac{\partial f_1(x_1,x_2)}{\partial x_2} f_2(x_1,x_2,0)$ and $\chi_2(x_1,x_2,x_3^\theta) = \frac{\partial f_1(x_1,x_2)}{\partial x_2}\frac{\partial f_2(x_1,x_2,x_3)}{\partial x_3}\Big|_{x_3=x_3^\theta}$ are continuous functions of $\bar{x}_2 = [x_1,x_2]^T \in \mathbb{R}^2$.

By applying similar mathematical manipulations on (5), one can obtain for $i = 3$ that

$$\begin{aligned} \dot{z}_3 = \ddot{z}_2 &= \sum_{j=1}^{2} \frac{\partial \phi_2(\bar{x}_2)}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{2} \frac{\partial \chi_2(\bar{x}_3)}{\partial x_j} f_j(\bar{x}_{j+1}) x_3 \\ &+ \left(\frac{\partial \chi_2(\bar{x}_3)}{\partial x_3} x_3 + \chi_2(\bar{x}_3) \right) f_3(x_1,\cdots,x_4) \\ &= \phi_3(\bar{x}_3) + \chi_3(\bar{x}_3, x_4^\theta) x_4 \end{aligned}, \tag{6}$$

where $\phi_3(\bar{x}_3) = \sum_{j=1}^{2} \frac{\partial \phi_2(\bar{x}_2)}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{2} \frac{\partial \chi_2(\bar{x}_3)}{\partial x_j} f_j(\bar{x}_{j+1}) x_3 + \left(\frac{\partial \chi_2(\bar{x}_3)}{\partial x_3} x_3 + \chi_2(\bar{x}_3) \right) f_3(\bar{x}_3, 0)$, $\chi_3(\bar{x}_3, x_4^\theta) = \left(\frac{\partial \chi_2(\bar{x}_3)}{\partial x_3} x_3 + \chi_2(\bar{x}_3) \right) \frac{\partial f_3(\bar{x}_4)}{\partial x_4}\Big|_{x_4=x_4^\theta}$ are continuous functions of $\bar{x}_3 \in \mathbb{R}^3$.

Similarly, we can obtain for any $i = 4,\cdots,n-1$

$$\begin{aligned} \dot{z}_i = \ddot{z}_{i-1} &= \sum_{j=1}^{i-1} \frac{\partial \phi_{i-1}(\bar{x}_{i-1})}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{i-1} \frac{\partial \chi_{i-1}(\bar{x}_i)}{\partial x_j} f_j(\bar{x}_{j+1}) x_i \\ &+ \left(\frac{\partial \chi_{i-1}(\bar{x}_i)}{\partial x_i} x_i + \chi_{i-1}(\bar{x}_i) \right) f_i(x_1,\cdots,x_{i+1}) \\ &= \phi_i(\bar{x}_i) + \chi_i(\bar{x}_i, x_{i+1}^\theta) x_{i+1} \end{aligned}, \tag{7}$$

where $\phi_i(\bar{x}_i) = \sum_{j=1}^{i-1} \frac{\partial \phi_{i-1}(\bar{x}_{i-1})}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{i-1} \frac{\partial \chi_{i-1}(\bar{x}_i)}{\partial x_j} f_j(\bar{x}_{j+1}) x_i + \left(\frac{\partial \chi_{i-1}(\bar{x}_i)}{\partial x_i} x_i + \chi_{i-1}(\bar{x}_i) \right) f_i(\bar{x}_i, 0)$, $\chi_i(\bar{x}_{i+1}, x_{i+1}^\theta) = \left(\frac{\partial \chi_{i-1}(\bar{x}_i)}{\partial x_i} x_i + \chi_{i-1}(\bar{x}_i) \right) \frac{\partial f_i(\bar{x}_{i+1})}{\partial x_{i+1}}\Big|_{x_{i+1}=x_{i+1}^\theta}$ are continuous functions of $\bar{x}_i \in \mathbb{R}^i$.

For $i = n$, we can further calculate the derivative of z_n as

$$\begin{aligned} \dot{z}_n = \ddot{z}_{n-1} &= \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}(\bar{x}_{n-1})}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{n-1} \frac{\partial \chi_{n-1}(\bar{x}_n)}{\partial x_j} f_j(\bar{x}_{j+1}) x_n \\ &+ \left(\frac{\partial \chi_{n-1}(\bar{x}_n)}{\partial x_n} x_n + \chi_{n-1}(\bar{x}_n) \right) f_n(\bar{x}_n, u) \\ &= \phi_n(\bar{x}_n) + \chi_n(\bar{x}_n, u^\theta) u \end{aligned}, \tag{8}$$

where $\phi_n(\bar{x}_n) = \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}(\bar{x}_{n-1})}{\partial x_j} f_j(\bar{x}_{j+1}) + \sum_{j=1}^{n-1} \frac{\partial \chi_{n-1}(\bar{x}_n)}{\partial x_j} f_j(\bar{x}_{j+1}) x_n + \left(\frac{\partial \chi_{n-1}(\bar{x}_n)}{\partial x_n} x_n + \chi_{n-1}(\bar{x}_n) \right) f_n(\bar{x}_n, 0)$, $\chi_n(\bar{x}_n, u^\theta) = \left(\frac{\partial \chi_{n-1}(\bar{x}_n)}{\partial x_n} x_n + \chi_{n-1}(\bar{x}_n) \right) \frac{\partial f_n(\bar{x}_n, u)}{\partial u}\Big|_{u=u^\theta}$ are continuous functions of $\bar{x}_n \in \mathbb{R}^n$, respectively.

Based on the coordinate transform given in (2)-(8), the original pure-feedback system (1) is represented as the following canonical system

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_i = z_{i+1}, \qquad i = 1,\cdots,n-1, \\ \dot{z}_n = \phi_n(\bar{x}_n) + \chi_n(\bar{x}_n, u^\theta) u \end{cases} \tag{9}$$

where $y = x_1 = z_1$ is the output of system (1), $\chi_n(\bar{x}_n, u^\theta) \in \Omega$, $\Omega \in \mathbb{R}^n \times \mathbb{R}$ and the derived system (9). In this sense, the original control objective, i.e. make the output y of system (1) track a given trajectory y_d, can be achieved by controlling the Brunovsky system (9).

It should be noted that the above coordinate transform is used for analysis only, i.e. it is not used in the practical control implementation. The motivation for introducing this transform is to represent the pure-feedback system (1) as a Brunovsky form (9). Consequently, the following control design based on system (9) is simpler than the conventional backstepping method [4, 5, 7, 8, 10, 11, 15], while achieving better transient control response. In fact, a simple feedback linearization-like control can be designed for system (9), whilst the tedious backstepping procedure with multiple NNs or FLSs are avoided. Specifically, only the measured system output y is used in the following control design.

Lemma 3.1: The derived function $\chi_n(\bar{x}_n, u^\theta)$ in (9) is positive over a compact set Ω.

Proof: From Assumption 2.2, we know $\partial f_i(x_1,\cdots,x_{i+1}) / \partial x_{i+1}$ with x_{i+1} and $x_{n+1} = u$ is true, such that $\chi_i(\bar{x}_{i+1}, x_{i+1}^\theta)$ derived based on (3)-(7) are all positive. Then, from the definition of $\chi_n(\bar{x}_n, u^\theta)$ given in (8), we can verify its positiveness. ◇

Lemma 3.1 indicates that system (9) is controllable without encountering the control singularity problem.

IV. OUTPUT-FEEDBACK CONTROL DESIGN

In this section, we will present a new output-feedback control design for the derived canonical system (9) to achieve the output tracking of the original system (1). There are two difficulties in the control design for system (9): 1) the system states $z_i, i = 2,\cdots,n$ in (9) are not available although the output $y = x_1 = z_1$ can be measured; 2) the lumped nonlinear functions $\phi_n(\bar{x}_n)$, $\chi_n(\bar{x}_n, u^\theta)$ in (9) are unknown.

To tackle the first problem, we will first use a specific state observer to reconstruct $z_i, i = 2,\cdots,n$ by using the output x_1 only. Then based on the observed states, we will suggest a new

unknown dynamics estimator to address the unknown dynamics $\phi_n(\bar{x}_n)$ and $\chi_n(\bar{x}_n, u^\theta)$, which has only one tuning parameter and thus is easily to use the control implementation. Consequently, the widely used function approximators (e.g. NNs or FZs) and backstepping scheme are all avoided. The proposed control system structure can be illustrated in Fig.1.

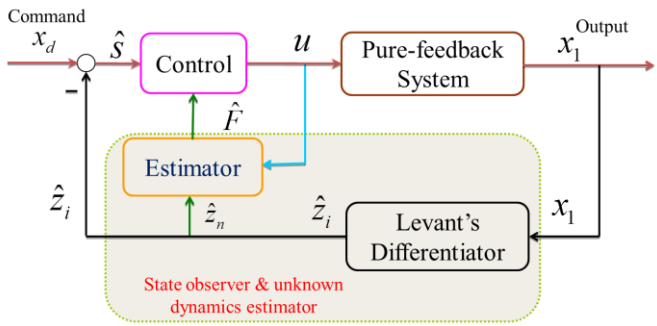

Fig.1 Schematic of the proposed control system.

A. *Levant's differentiator*

We first use an observer to reconstruct the unknown states $z_i, i = 2\cdots n$ of system (9). It is shown in (9) that $z_i, i = 2\cdots n$ are the high order derivatives of the measured system output $y = x_1$. Thus, the following Levant's differentiator [43, 44] can be used

$$\begin{cases} \dot{\hat{z}}_1 = \omega_1 \\ \dot{\hat{z}}_i = \omega_i \qquad i = 2,\cdots,n-1 \\ \dot{\hat{z}}_n = \omega_n \\ \dot{\hat{z}}_{n+1} = \omega_{n+1} \\ \omega_1 = -\lambda_1 \left|\hat{z}_1 - x_1\right|^{n/(n+1)} sign(\hat{z}_1 - x_1) + \hat{z}_2 \\ \quad \vdots \\ \omega_i = -\lambda_i \left|\hat{z}_i - \omega_{i-1}\right|^{(n-i+1)/(n-i+2)} sign(\hat{z}_i - \omega_{i-1}) + \hat{z}_{i+1} \\ \quad \vdots \\ \omega_n = -\lambda_n \left|\hat{z}_n - \omega_{n-1}\right|^{1/2} sign(\hat{z}_n - \omega_{n-1}) + \hat{z}_{n+1} \\ \omega_{n+1} = -\lambda_{n+1} sign(\hat{z}_{n+1} - \omega_n) \end{cases}, \tag{10}$$

where $\lambda_i, i = 1,\cdots,n+1$ are positive observer gains. In this observer, $\hat{z}_i, i = 1\cdots n$ can be taken as the reconstructed states of $z_i, i = 1\cdots n$ in (9), respectively.

The salient feature of this Levant's differentiator is that it has a finite-time error convergence property as proved in [43, 44]:

Lemma 4.1 [43, 44]: When the measured output x_1 of system (9) is free of sensor noise, then the states of the differentiator (10) converge to the states of system (9) in finite-time $T > 0$, i.e. $\hat{z}_i = z_i, i = 1\cdots n$ holds for any $t \ge T$, and the differentiator (10) is Lyapunov stable.

Lemma 4.2 [43, 44]: When the measured output x_1 of system (9) is subject to bounded sensor noise, e.g. $\left|z_1 - x_1\right| \le \kappa$ for $\kappa > 0$, then the states of differentiator (10) converge to a compact set around the states of system (9) in finite-time $T > 0$, such that

$$\left|\tilde{z}\right| = \left|z_i - \hat{z}_i\right| \le \ell_i \kappa^{(n-i+2)/(n+1)}, i = 1\cdots n, \text{for } t \ge T, \tag{11}$$

where ℓ_i are positive constants, which is determined by the parameters $\lambda_i, i = 1,\cdots,n+1$ used in differentiator (10). In general, large gains λ_i can increase the convergence rate of observer, while too large λ_i can trigger oscillations due to the adopted sign function. Thus, a trade-off between the convergence rate and oscillations should be considered when we choose the gains λ_i, where some guidelines have been presented in [43, 44].

The proof of Lemmas 4.1 and 4.2 can be found in [43, 44], which will not be shown here. The differentiator (10) was designed by modifying the super-twisting algorithm, thus it can achieve very fast transient observer response (i.e. finite-time convergence). This finite-time property retains the *separation principle* [50] to be almost true when the observed states are used in the control design. As shown in [43, 44], a control design with the differentiator (10) can preserves major features of the same control with the fully measured states under a practically feasible condition, i.e. the derivatives of x_1 is bounded during arbitrarily short transient period [43, 44]. This property motivates the use of Levant's differentiator (10) in this paper rather than the other observers, e.g. high gain observer [50] and function approximation based observers [16-19].

Remark 4.1: The above Levant's differentiator will be used in the following control designs, because even in the presence of sensor noise, the unknown states $z = [z_1, z_2 \cdots z_n]^T$ can be accurately reconstructed in finite time. As a consequence of Lemma 4.2, there exist positive constants $\hbar$ and t_Δ depending on the bound of noise κ and design parameters, such that the observer error $\tilde{z} = z - \hat{z}$ is bounded by $\left\|\tilde{z}\right\| \le \hbar$ for $t > t_\Delta$.

B. *Filter based unknown dynamics estimator*

To achieve tracking control of system (9), the lumped uncertainties should be compensated. In most of existing results, NNs or FLZs are usually used [4, 5, 7, 8, 10, 11, 15]. However, theses function approximators are only valid in a compact set determined by the system trajectory, leading to the semi-global stability of the control system. Moreover, online adaptive learning must be used in these results to update the unknown weights of NNs or FLZs, where the sluggish transient learning phase could cause a control performance degradation, and the tuning of learning parameters is generally difficult [39]. In this section, we will design a new unknown dynamics estimator by using first-order filters [45, 46] with only one scalar selected by the designers but guaranteed exponential convergence.

As shown in Lemma 3.1, the control function $\chi_n(\bar{x}_n, u^\theta)$ in (9) is positive and bounded, which means that there are positive constants χ_0 and χ_1, such that $0 < \chi_0 \le \chi_n(\bar{x}_n, u^\theta) \le \chi_1$ holds as shown in [4, 5, 10, 15]. Without loss of generality, we define $\chi_m = \sqrt{\chi_0 \chi_1}$ as the nominal value of $\chi_n(\bar{x}_n, u^\theta)$, and thus $\chi_n(\bar{x}_n, u^\theta) = \chi_m \Delta\chi(\bar{x}_n, u^\theta)$ holds, where the uncertainty $\Delta\chi(\bar{x}_n, u^\theta)$ fulfills $b_0 \le \Delta\chi(\bar{x}_n, u^\theta) \le b_1$ for positive constants $b_0 = \chi_0 / \chi_m, b_1 = \chi_1 / \chi_m$, which can be calculated in practice. In this case, the last equation of (9) can be rewritten as

$$\begin{aligned} \dot{z}_n &= \phi_n(\bar{x}_n) + \chi_m(\Delta\chi(\bar{x}_n, u^\theta) - 1)u + \chi_m u \\ &= F(\bar{x}_n, u) + \chi_m u \end{aligned}, \tag{12}$$

where $F(\bar{x}_n, u) = \phi_n(\bar{x}_n) + \chi_m(\Delta\chi(\bar{x}_n, u^\theta) - 1)u$ denotes the lumped

unknown dynamics. To facilitate the design of estimator to handle unknown dynamics, we have the following assumption:
Assumption 4.1: The derivative of the lumped dynamics F is bounded, i.e., $\sup_{t\geq 0}|\dot{F}|\leq \hbar$ for an unknown constant $\hbar>0$.

Remark 4.2: The motivation for using the nominal input gain χ_m in (12) is to facilitate the subsequent control design. The existence of such a stable control for system (1) has been asserted in the literatures, e.g. [4, 5], by means of the implicit function theorem. For practical systems, this nominal value can be calculated based on the hardware configuration, thus can be used in the control design [42]. The error stemming from this nominal value χ_m can be taken into the lumped dynamics $F(\bar{x}_n,u)$ and then addressed via the estimator to be presented. Moreover, Assumption 4.1 is required for the convergence analysis of the proposed estimator, whilst the upper bound $\hbar$ is not used in the control implementation. This condition has been well-recognized in the disturbance observer designs [45, 46] and adaptive NN control designs [4, 5], and can be practically fulfilled when the system is operated with a proper control.

To facilitate the design of unknown dynamics estimator, we define the filtered variables of z_n and u as

$$\begin{cases} k\dot{z}_{nf}+z_{nf}=z_n, & z_{nf}(0)=0 \\ k\dot{u}_f+u_f=u, & u_f(0)=0 \end{cases}, \tag{13}$$

where $k>0$ is a positive constant. As shown in [28], the above operation can be easily implemented by applying a low-pass filter $1/(ks+1)$ on z_n and u.

Then the idea of invariant manifold [51] will be further explored to design the unknown dynamics estimator.

Lemma 4.3: Consider system (12) and filter (13), the variable $\beta=(z_n-z_{nf})/k-\chi_m u_f-F$ is bounded for any $k>0$, and it decreases in an exponential manner. Moreover, we have $\lim_{k\to 0}[\lim_{t\to\infty}\{(z_n-z_{nf})/k-\chi_m u_f-F\}]=0$, which means that $(z_n-z_{nf})/k-\chi_m u_f-F=0$ is an invariant manifold.

Proof: From (12)-(13), we can calculate the derivative $\dot{\beta}$ as

$$\dot{\beta}=\frac{\dot{z}_n-\dot{z}_{nf}}{k}-\chi_m\dot{u}_f-\dot{F}=-\frac{1}{k}(\beta+k\dot{F}). \tag{14}$$

We first need to prove the boundedness of β. Choose a Lyapunov function as $V_\beta=\beta^2/2$, then we have

$$\dot{V}_\beta=-\frac{1}{k}\beta^2+\beta\dot{F}\leq-\frac{1}{k}V_\beta+\frac{k}{2}\hbar^2. \tag{15}$$

By integrating both sides of (15), we can further obtain that $V_\beta(t)\leq e^{-t/k}V_\beta(0)+k^2\hbar^2/2$ holds. Hence, the variable β exponentially converges to a set around the origin given by $|\beta(t)|=\sqrt{2V_\beta(t)}\leq\sqrt{\beta^2(0)e^{-t/k}+k^2\hbar^2}$, whose size depends on the parameters k and $\hbar$, i.e. $\sup_{t\geq 0}|\dot{F}|\leq\hbar$, which vanishes for a sufficiently small k and/or any constant F (i.e. $\hbar$=0). Moreover, for infinitesimal $k\to 0$, it can be verified that $\lim_{k\to 0}\lim_{t\to\infty}\beta(t)=0$ is true, which means that β converges to zero for any finite $\hbar$, such that $\beta=0$ is an invariant manifold for $k>0$. ◇

The invariant manifold given in Lemma 4.3 indicates an implicit mapping from the available variables (z_n,z_{nf},u_f) to the unknown lumped dynamics $F(\bar{x}_n,u)$ given in (12). However, only the estimated state $\hat{z}_n$ is available rather than the true state z_n. Thus, based on the manifold defined in Lemma 4.3, a feasible estimator of F is given by

$$\hat{F}=\frac{\hat{z}_n-\hat{z}_{nf}}{k}-\chi_m u_f, \tag{16}$$

where $\hat{z}_{nf}$ is the filtered version of $\hat{z}_n$ given by

$$k\dot{\hat{z}}_{nf}+\hat{z}_{nf}=\hat{z}_n,\hat{z}_{nf}(0)=0. \tag{17}$$

Now, we can prove that the estimation error $\tilde{F}=F-\hat{F}$ exponentially converges to a small compact set around zero.

Theorem 4.1: For system (12) with estimator (16) and $\hat{z}_n$ given in (10), then the estimation error $\tilde{F}$ can exponentially converge to a set around origin defined by $|\tilde{F}(t)|\leq\sqrt{\tilde{F}^2(0)e^{-t/k}+k^2(\hbar+\bar{\hbar}/k)^2}$ with $|\dot{\tilde{z}}_n|\leq\bar{\hbar}$ being the observer error defined in Lemma 4.2, so that $\hat{F}\to F$ holds for $k\to 0$ and/or $\bar{\hbar}\to 0$.

Proof: We first calculate the dynamics of estimator error $\tilde{F}$. By applying a low-pass filter $1/(ks+1)$ on (12), it follows

$$\dot{z}_{nf}=F_f+\chi_m u_f, \tag{18}$$

where F_f is the filtered version of the nonlinearities F given by $k\dot{F}_f+F_f=F,F_f(0)=0$. Moreover, from the first equation of (13), we can verify that $\dot{z}_{nf}=(z_n-z_{nf})/k$. Then, it follows from (16) and (18) that

$$\hat{F}=\frac{\hat{z}_n-\hat{z}_{nf}}{k}-\chi_m u_f=\dot{z}_{nf}-\chi_m u_f-(\dot{z}_{nf}-\dot{\hat{z}}_{nf})=F_f-\dot{\tilde{z}}_{nf}, \tag{19}$$

which means that the estimator $\hat{F}$ in (16) is the filtered version of the unknown dynamics with a residual error $\dot{\tilde{z}}_{nf}=\dot{z}_{nf}-\dot{\hat{z}}_{nf}$, which is the filtered version of the observer error $\dot{\tilde{z}}_n=\dot{z}_n-\dot{\hat{z}}_n$.

We can calculate the error dynamics in the time-domain as

$$\dot{\tilde{F}}=-\frac{1}{k}\tilde{F}+\dot{F}+\frac{1}{k}\dot{\tilde{z}}_n, \tag{20}$$

where the last term is the observer error of differentiator (10), which is bounded by $|\dot{\tilde{z}}_n|\leq\bar{\hbar}$ for a constant $\bar{\hbar}>0$ by recalling Lemma 4.2.

We select a Lyapunov function as $V_F=\tilde{F}^2/2$, then calculate its derivative $\dot{V}_F$ along (20) as

$$\dot{V}_F=-\frac{1}{k}\tilde{F}^2+\tilde{F}(\dot{F}+\frac{1}{k}\dot{\tilde{z}}_n)\leq-\frac{1}{k}V+\frac{k}{2}(\hbar+\frac{1}{k}\bar{\hbar})^2. \tag{21}$$

Then, similar to the proof of Lemma 4.3, we can obtain that $|\tilde{F}(t)|=\sqrt{2V_F(t)}\leq\sqrt{\tilde{F}^2(0)e^{-t/k}+k^2(\hbar+\bar{\hbar}/k)^2}$, which indicates that $\tilde{F}(t)\to 0$ for $k\to 0$ and/or $\bar{\hbar}\to 0$. ◇

The implementation of the estimator (16) with filters (13) and (17) is straightforward, which can be achieved by applying a low-pass filter on the input u and the observed state $\hat{z}_n$, and then conducting algebraic calculations in (16). This filter based estimation has a linear structure, which is simpler than function approximators. Moreover, only one scalar $k>0$ needs to be selected by the designer, which defines the bandwidth of the low-pass filter given in (13), which determines the convergence speed of the estimation error as shown in (20). It is also shown in Theorem 4.1 that the estimator error $\tilde{F}$ converges to a small

compact set around zero, whose size can be calculated via (21). The effect of the estimator error $\tilde{F}$ will also be studied in the stability analysis of the closed-loop control system.

Remark 4.3: If the studied system has bounded disturbances, based on the above derivations, their effect can be lumped into the unknown dynamics $F(\bar{x}_n, u)$, which can be estimated by the proposed unknown system dynamics estimator, and then compensated in the following presented control.

C. Tracking control design and stability analysis

As shown in the previous subsections, the observer states $\hat{z} = [\hat{z}_1, \hat{z}_2 \cdots \hat{z}_n]^T$ can be obtained from differentiator (10), and the nonlinearities $F(\bar{x}_n, u)$ in (12) are estimated by estimator (16). Thus, we can design a feedback control for system (9) to achieve the output tracking of the original system (1).

To accomplish the control design, we define the tracking error of system (9) as $e = z - \bar{x}_d$, where $\bar{x}_d = [y_d, \dot{y}_d, \cdots, y_d^{n-1}]^T$ is the given bounded desired trajectory, i.e. $\|\bar{x}_d\| \le c_d$. Then, we can further define:

$$s = [\Lambda^T \quad 1]e, \tag{22}$$

where $\Lambda = [\Lambda_1, \Lambda_2 \cdots \Lambda_{n-1}]^T$ is chosen such that $s^{n-1} + \Lambda_{n-1}s^{n-2} + \cdots \Lambda_1$ is a stable polynomial. In this case, the convergence of s implies the convergence of the tracking error e [50][49][49].

However, in practical control implementation, we can only use the observed states $\hat{z}$ instead of the unknown system states z. Thus, we define the practical tracking errors $\hat{e}$ and $\hat{s}$ as

$$\hat{e} = \hat{z} - \bar{x}_d, \qquad \hat{s} = [\Lambda^T \quad 1]\hat{e}, \tag{23}$$

which define the differences between the states of differentiator (10) and the given trajectory $\bar{x}_d$.

Now, we can design the following feedback control

$$u = \frac{1}{\chi_m}\left(-k_1\hat{s} - \hat{F} + y_d^n - [0 \quad \Lambda^T]\hat{e}\right), \tag{24}$$

where $k_1 > 0$ is the feedback gain, $\hat{s}$ and $\hat{e}$ are the tracking errors given in (23), and $\hat{F}$ is the estimation of the lumped unknown nonlinearities F given in (16). The designed control (24) can be implemented by using the Levant's differentiator (10), the estimation $\hat{F}$ given in (16) and the tracking errors $\hat{e}$ and $\hat{s}$ defined in (23).

We will analyze the stability of the proposed control system. For this purpose, we substitute (9) and (12) into (22), and consider (23), then have

$$\begin{aligned} \dot{s} &= [0 \quad \Lambda^T]e + F(\bar{x}_n, u) + \chi_m u - y_d^n \\ &= [0 \quad \Lambda^T]\hat{e} + F(\bar{x}_n, u) + \chi_m u - y_d^n + [0 \quad \Lambda^T]\tilde{z}, \end{aligned} \tag{25}$$

where the fact $\tilde{z} = z - \hat{z} = e - \hat{e}$ can be verified based on the definition $e = z - \bar{x}_d$ and $\hat{e} = \hat{z} - \bar{x}_d$ given in (23).

We further substitute the feedback control (24) into (25) and then derive the closed-loop error dynamics as

$$\dot{s} = -k_1\hat{s} + F - \hat{F} + [0 \quad \Lambda^T]\tilde{z} = -k_1\hat{s} + \tilde{F} + [0 \quad \Lambda^T]\tilde{z}. \tag{26}$$

We now provide the main results of this paper as follows:

Theorem 4.2: Consider system (1) and the transformed system (9). The Levant's differentiator (10), the filter based estimator (16) and feedback control (24) are used, then the closed-loop system is stable. Moreover, the output tracking error s and the estimation error $\tilde{F}$ all exponentially converge to a small set around zero, which are given as

$$\Omega := \left\{ s, \tilde{F} \mid |s| \le \sqrt{2\vartheta/\gamma}, \left|\tilde{F}\right| \le \sqrt{2\vartheta/\gamma} \right\}, \tag{27}$$

where $\gamma = 2\min\{(k_1 - k_1/\eta), (1/k - \eta/k_1)\}$ an $\vartheta = k_1\eta c^2/2 + k_1(\hbar + \bar{\hbar}/k)^2/(2\eta)$ with $\eta > 1$ are positive constants.

Proof: We choose a Lyapunov function as

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{F}^2. \tag{28}$$

From the fact $\hat{s} = [\Lambda^T \quad 1]\hat{e} = [\Lambda^T \quad 1](e - \tilde{z}) = s - [\Lambda^T \quad 1]\tilde{z}$, we can calculate the derivative of V_1 along (20) and (26) as

$$\begin{aligned} \dot{V} &= s(-k_1\hat{s} + \tilde{F} + [0 \quad \Lambda^T]\tilde{z}) + \tilde{F}(-\frac{1}{k}\tilde{F} + \dot{F} + \frac{1}{k}\dot{\tilde{z}}_n) \\ &= -k_1 s^2 - k_1 s\left([\Lambda^T \quad 1] + [0 \quad \frac{1}{k_1}\Lambda^T]\right)\tilde{z} + s\tilde{F} - \frac{1}{k}\tilde{F}^2 + \tilde{F}(\dot{F} + \frac{1}{k}\dot{\tilde{z}}_n). \end{aligned} \tag{29}$$

We recall Lemma 4.2 and know that the observer error $\tilde{z}$ is bounded even when the measured output x_1 is subject to noise, i.e. $\|\hat{z} - z\| \le \hbar$ holds in finite-time. Thus, there exists a constant $c = \max\{[\Lambda^T \quad 1] + [0 \quad 1/k_1\Lambda^T]\}\hbar$, so that $\left|([\Lambda^T \quad 1] + [0 \quad 1/k_1\Lambda^T])\tilde{z}\right| \le c$ is fulfilled. In this case, by applying the Young's inequality, then (29) can be further reformulated as

$$\begin{aligned} \dot{V} &\le -k_1 s^2 + k_1 c|s| + |s|\left|\tilde{F}\right| - \frac{1}{k}\tilde{F}^2 + \left|\tilde{F}\right|(\hbar + \frac{1}{k}\bar{\hbar}) \\ &\le -k_1 s^2 + \frac{k_1}{2\eta}s^2 + \frac{k_1\eta}{2}c^2 + \frac{k_1}{2\eta}s^2 + \frac{\eta}{2k_1}\tilde{F}^2 - \frac{1}{k}\tilde{F}^2 \\ &\quad + \frac{\eta}{2k_1}\tilde{F}^2 + \frac{k_1}{2\eta}(\hbar + \frac{1}{k}\bar{\hbar})^2 \\ &= -(k_1 - \frac{k_1}{\eta})s^2 - (\frac{1}{k} - \frac{\eta}{k_1})\tilde{F}^2 + \frac{k_1\eta}{2}c^2 + \frac{k_1}{2\eta}(\hbar + \frac{1}{k}\bar{\hbar})^2 \\ &\le -\gamma V + \vartheta \end{aligned}, \tag{30}$$

where $\gamma = 2\min\{(k_1 - k_1/\eta), (1/k - \eta/k_1)\}$ and $\vartheta = k_1\eta c^2/2 + k_1(\hbar + \bar{\hbar}/k)^2/(2\eta)$ are constants. If we set the parameters such that $\eta > 1$, $k \le k_1/\eta$, then γ and ϑ are all positive. From Lyapunov theorem, we can claim that V and thus the tracking error s and the estimation error $\tilde{F}$ are all uniformly ultimately bounded. This together with the fact that $\tilde{z}$ is bounded further implies that $\hat{s}$, e, $\hat{e}$ are bounded, and thus the control u and the system states z_i and x_i are all bounded.

Finally, we can calculate the ultimate bounds of s and $\tilde{F}$ by integrating (30) over $[0, t]$, such that

$$V(t) \le V(0)e^{-\gamma t} + \frac{\vartheta}{\gamma}(1 - e^{-\gamma t}) \le V(0)e^{-\gamma t} + \frac{\vartheta}{\gamma}. \tag{31}$$

Consequently, the errors s, $\tilde{F}$ will exponentially converge to a compact set given in (27) for $t \to \infty$. The size of this compact set depends on the observer error $\tilde{z}$, filter parameter k and feedback gain k_1. Specifically, it is shown that a large gain k_1 can increase the convergence rate of tracking error s defined by γ, while a too large k_1 can lead to a large residual error bound denoted by ϑ. Thus, it can be set as a trade-off between the error convergence rate and the steady-state

performance. ◇

Remark 4.4: The above control (24) has a simple feedback linearization structure, and is clearly simpler than backstepping or DSC schemes (e.g. [4, 5, 7, 8, 10, 11, 15]) for system (1), since the recursive procedure involved in the backstepping is not needed. This will contribute to improving the transient control convergence, which will be shown in the simulations.

To implement the proposed control (24) with Levant's differentiator (10) and filter based estimator (16), there are only several parameters to be selected by the designers. The number of these parameters and the associated tuning procedure are simpler than other adaptive control schemes. In general, the observer gains λ_i in (10) are chosen as a trade-off between the convergence response of the observer error $\tilde{z}$ and the smoothness of the observer states $\hat{z}_i, i = 1 \cdots n$. Detailed discussion on these parameters are given in [43, 44]. The filter coefficient k in (16) defines the bandwidth of the low-pass filter $1/(ks+1)$, thus it should be set to compromise the error response $\tilde{F}$ and the robustness. In general, k cannot be set sufficiently small. Finally, as shown in the above discussion, the feedback gain k_1 in the control (24) is included in both γ and ϑ in (31), thus it needs to be set as a trade-off between the convergence rate and the steady-state response.

V. Simulations

Consider the following benchmark non-affine pure-feedback system, which has been widely used in the literatures (e.g. [10, 15]) to verify various control designs

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + \dfrac{x_2^3}{5} \\ \dot{x}_2 = x_1 x_2 + u + \dfrac{u^3}{7} \end{cases} . \tag{32}$$

In the simulation, the command signal to be tracked is given as $y_d = \sin(t) + \cos(0.5t)$, and the initial condition is set as $x(0) = [0.8, 0.3]^T$. The parameters of differentiator (10) are set as $\lambda_1 = 10, \lambda_2 = 15, \lambda_3 = 20$, such that

$$\begin{cases} \dot{\hat{z}}_1 = \omega_1 \\ \dot{\hat{z}}_2 = \omega_2 \\ \dot{\hat{z}}_3 = \omega_3 \\ \omega_1 = -10\left|\hat{z}_1 - x_1\right|^{2/3} sign(\hat{z}_1 - x_1) + \hat{z}_2 \\ \omega_2 = -15\left|\hat{z}_2 - \omega_1\right|^{1/2} sign(\hat{z}_2 - \omega_1) + \hat{z}_3 \\ \omega_3 = -20 sign(\hat{z}_3 - \omega_2) \end{cases} . \tag{33}$$

The tracking errors used in the control implementation are given as $\hat{e} = \hat{z} - \bar{x}_d$, $\hat{s} = [\Lambda \;\; 1]^T \hat{e}$ with $\Lambda = 2$, and the feedback control gain in (24) is $k_1 = 20$. Finally, the unknown dynamics estimator (16) is carried out with the filter constant $k = 0.01$. It should be noted that in the proposed control, only the system output x_1 is used.

For comparison, the backstepping control with multiple NN approximation initially proposed in [10] is also simulated. This beckstepping control with two NNs can be given as follows

$$\begin{aligned} & z_1 = x_1 - y_d, \\ & \alpha_1 = -c_1 z_1 - \hat{W}_1^T S_1(Z_1), \\ & \dot{\hat{W}}_1 = \Gamma_1 (S_1(Z_1) z_1 - \sigma_1 \hat{W}_1), \\ & z_2 = x_2 - \alpha_1, \\ & u = -z_1 - c_2 z_2 - \hat{W}_2^T S_2(Z_2), \\ & \dot{\hat{W}}_2 = \Gamma_2 (S_2(Z_2) z_2 - \sigma_2 \hat{W}_2) \end{aligned} , \tag{34}$$

where $Z_1 = [x_1, \dot{y}_d]^T$ and $Z_2 = [x_1, x_2, \partial\alpha_1 / \partial x_1, \phi_1]^T$ with $\phi_1 = \dfrac{\partial \alpha_1}{\partial y_d} \dot{y}_d + \dfrac{\partial \alpha_1}{\partial \dot{y}_d} \ddot{y}_d + \dfrac{\partial \alpha_1}{\partial \hat{W}_1} \dot{\hat{W}}_1$, and $c_1, c_2 > 0$ are the feedback gains, $\Gamma_1, \Gamma_2 > 0$ are the adaptive learning gains, which should be carefully selected to tradeoff the convergence speed and the error response. The constants $\sigma_1, \sigma_2 > 0$ are the leakage coefficient to guarantee the boundedness of NN weights $\hat{W}_1, \hat{W}_2$. It is clear that two NNs are used in the backstepping control (34), such that it imposes demanding computational costs in the control implementation. Specifically, the terms $\partial\alpha_1 / \partial x_1, \phi_1$ are used as the inputs of NN, resulting in the 'explosion of complexity' issue [8]. Moreover, as shown in (34), there are many parameters to be set by the designers, and the compact set within which the NN approximation is valid should be properly set based on the system trajectory, which are not trivial tasks. Nevertheless, the full system states x_1, x_2 are required in the backstepping control. In this sense, this paper provides an alternative yet more efficient output feedback control for the studied system.

Simulation results are provided in Fig. 2-Fig. 6. It is shown in Fig. 2-Fig. 4 that fairly good tracking control response can be achieved with this output-feedback control. The system states and control signal are bounded and smooth. In particular, the observed state z_2 tracks the given command x_{d2} as stated in Theorem 4.2. This control response can be explained by the fact that the Levant's differentiator can achieve sufficiently small observer error and fast convergence (Fig. 5). Moreover, the proposed unknown dynamics estimator (16) can capture the unknown dynamics $F(\bar{x}_n, u)$ well as shown in Fig.6.

Comparative tracking errors of the backstepping control (34) and the presented control (24) are given in Fig.7. One may find from Fig.7 that the proposed control (24) has faster error convergence speed than the backstepping control (34), though their steady-state error bounds are comparable. This is because the online learning for NNs is not needed in the control (24), and the suggested estimator (16) for unknown dynamics can achieve exponential convergence as proved in Theorem 4.1, while the scalar k can be easily set in advance. However, the backstepping control (34) uses two NNs to handle unknown dynamics, and thus it requires fairly long online learning phase before it achieves convergence, even full states x_1, x_2 are used.

From above simulations, one can find that the control (24) suggested in this paper only requires x_1 to implement the feedback control, without using any NN approximators and backstepping, while better control response can be achieved.

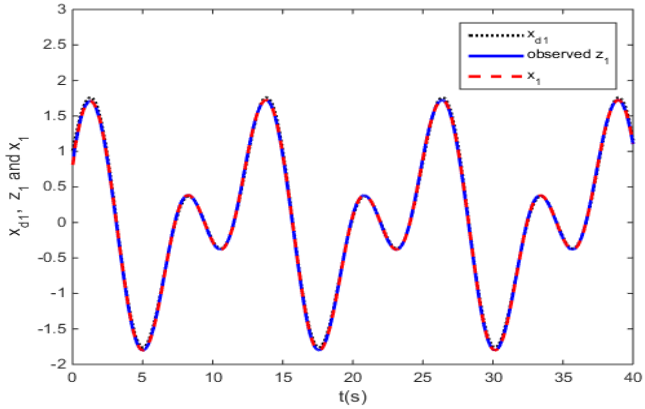

Fig. 2 Output tracking response and observer output.

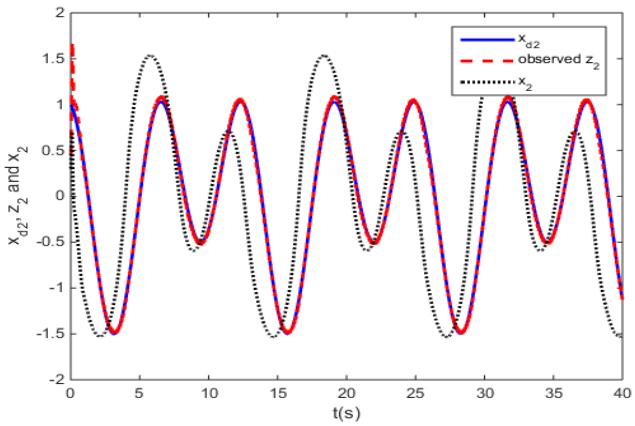

Fig. 3 System state and observer state.

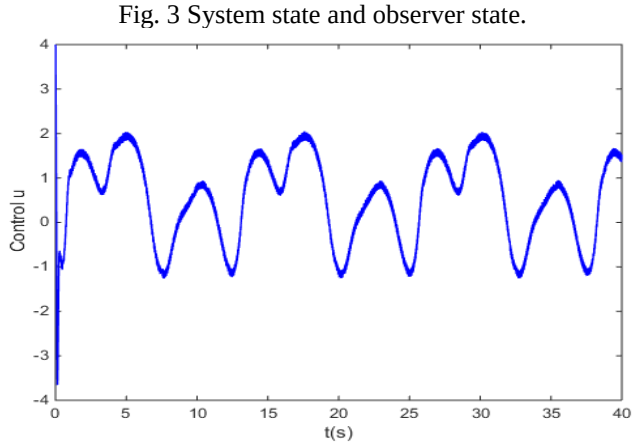

Fig. 4 Profile of control signal.

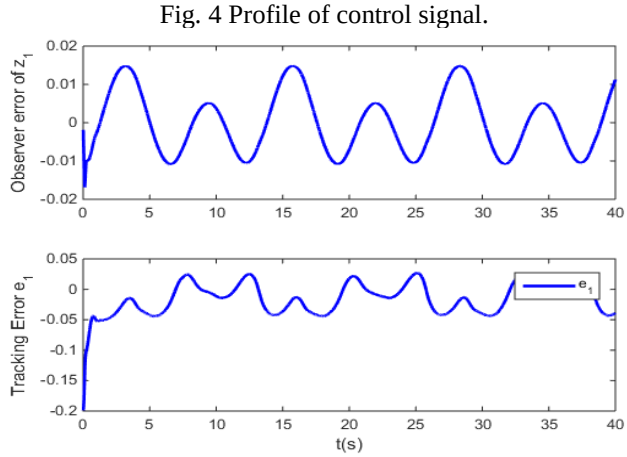

Fig. 5 Observer error $\tilde{z}_1$ and tracking error e_1.

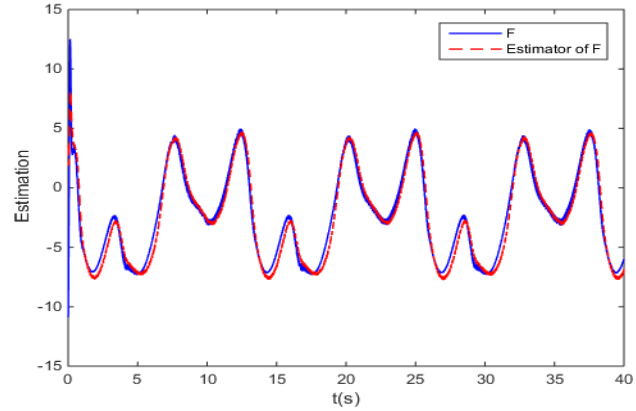

Fig. 6 Estimation of unknown dynamics of $F(\bar{x}_n, u)$.

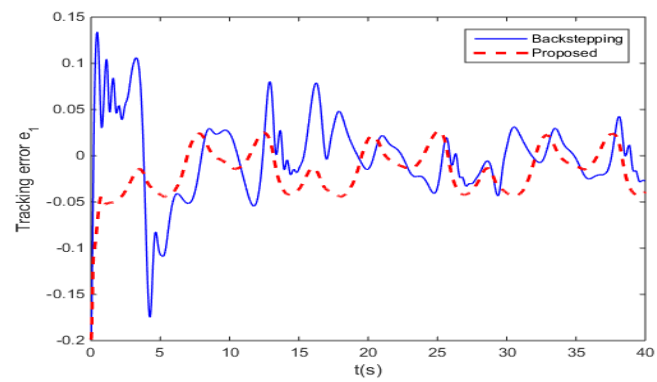

Fig. 7 Comparative tracking errors.

VI. PRACTICAL EXPERIMENTS

To validate the effectiveness of the proposed control scheme, a turntable servo system is used as the experimental test-rig, whose schematic is shown in Fig.8. This experiment platform consists of a PMSM (HC-UFS13), a DSP (TMS3202812), and a PWM amplifier in the motor drive card (MR-J2S-10A). The control algorithm is implemented by C++ program coded in CCS3.0. An encoder with a sampling rate 10ms is used to measure the rotation angular of motors. The aim is to control to rotation position to track a given command. The detailed description of this test-rig can be found in [46]. According to the modeling work given in [52, 53], the rotation motion behavior of this servo system is described as

$$\begin{cases} J\ddot{q} + f(q,\dot{q}) + T_f + T_l + T_d = T_m \\ K_E\dot{q} + L_a\dfrac{dI_a}{dt} + R_a I_a = u \\ T_m = K_T I_a \end{cases}, \tag{35}$$

where

q, $\dot{q}$, angular position and velocity;
J, motor inertia;
$f(q,\dot{q})$, unknown resonances and modeling uncertainties;
T_f, friction torque;
T_l, load torque;
T_d, disturbance torque;
u, input voltage;
K_T, torque constant;
K_E, electronmotive force coefficient.

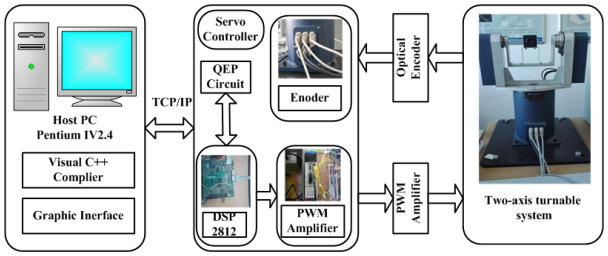

Fig. 8 Diagram of servo system driven by PMSM.

The nominal values of these parameters are listed in Table I. Select $x=[x_1,x_2]=[q,\dot{q}]$, then system (35) is rewritten as

$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{J}\left(K_1 u - K_2 x_2 - f(x_1,x_2) - T_l - T_d - T_f\right)\end{cases}, \tag{36}$$

where $K_1 = K_T / R_a$, $K_2 = K_E K_T / R_a$. Clearly, system (36) can be taken as a specific form of pure-feedback system (1), thus the proposed control can be directly used to achieve the angular tracking control.

Table I. Nominal system parameters

Parameters	Values
J	0.025
R	10
L	0.043
K_T	1.25
K_E	0.1

In order to illustrate the effectiveness of the proposed control method and compare its performance with other controllers, the following two control algorithms are also implemented:

1) The proposed control (24) with feedback gains $k_1 = 40$, $\Lambda = 15$, and $k = 0.01$ for estimator (16). The parameters of differentiator (10) are the same as those used in the simulations.

2) Adaptive neural control (ANC): This controller was presented in [54], which is given as $u = ks + \hat{W}^T\Phi + u_1$ with $u_1 = \sigma s/|s|$ for $s \neq 0$ or $u_1 = 0$ for $s = 0$. $s = \Lambda e_1 + e_2$ is the tracking error with $e_1 = x_1 - y_d$ and $e_2 = x_2 - \dot{y}_d$. The adaptive law is $\dot{\hat{W}} = \Gamma s \Phi$, and the control parameters are $\Lambda = 15, k = 1$, $\Gamma = 0.5$, and $\sigma = 0.005$.

Case 1- Sinusoidal Trajectory Tracking: In this case, a sinusoidal command $y_d = 0.6\sin(2\pi/5)$ is used to test the proposed control method. Experiment results are depicted in Fig. 9, where the tracking performance, tracking error and control signal are all given. It can be found that the proposed control can achieve fairly good tracking response, and fast error convergence can be retained. This is due to that the suggested unknown dynamics estimator can capture and compensate the unknown dynamics in the system, which is given in Fig. 10.

For comparison, Fig. 11 shows the response of the above shown ANC method. From Fig. 9 and Fig. 11, it is clearly shown that the proposed control with filter based estimator can achieve faster transient error convergence and smaller steady-state tracking error than the ANC method, since the exponential convergence of estimation error for (16) can be guaranteed without using any sluggish online learning for NN approximators.

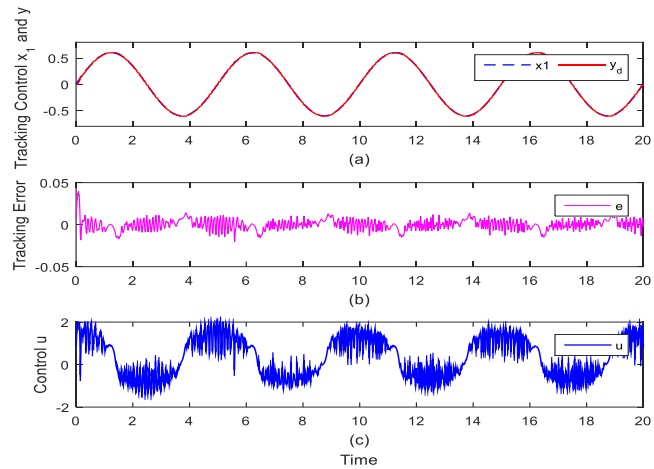

Fig. 9 Response of proposed control for $y_d = 0.6\sin(2\pi/5)$.

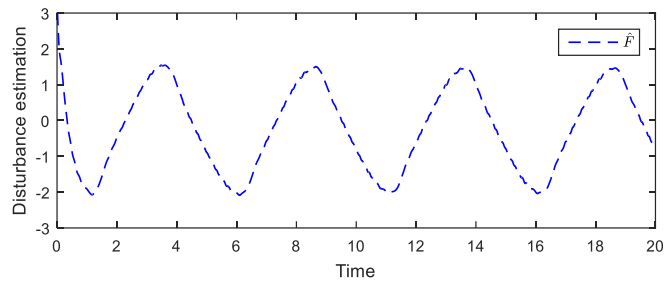

Fig. 10 Estimated dynamics with $y_d = 0.6\sin(2\pi/5)$.

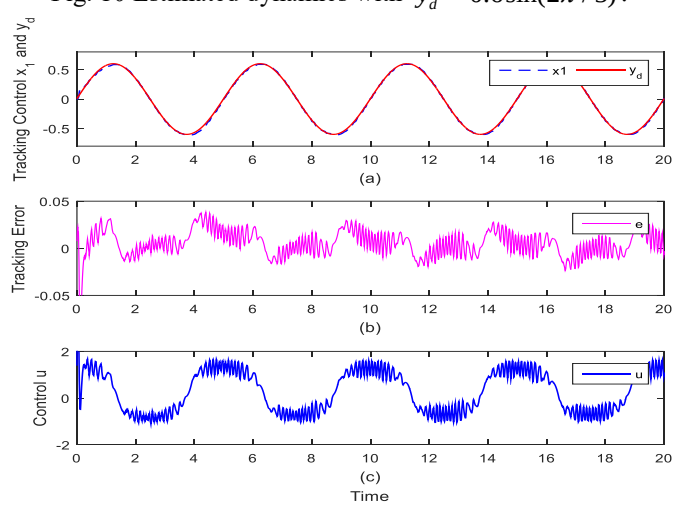

Fig. 11 Response of ANC [54] for $y_d = 0.6\sin(2\pi/5)$.

Case 2- Saw Tooth Trajectory Tracking: To further validate the proposed control scheme under sudden varying conditions, a saw tooth signal with jumps is used. The parameters used in this experiment are the same as those used in Case 1. Comparative results are depicted in Figs. 12-14. From Fig. 12, it is found that the presented control method can also achieve a satisfactory tracking performance, owing to its capability to cope with unknown time-varying dynamics in terms of the developed unknown dynamics estimator. In fact, it is shown in Fig. 13 that this unknown dynamics estimator (16) can capture the lumped uncertainties even under the tooth signal. However, the ANC with NN approximation requires fair transient time to achieve convergence of the used online learning. Moreover, the peaks in the tracking error when the position trajectory changes its moving direction are larger than that of the proposed control.

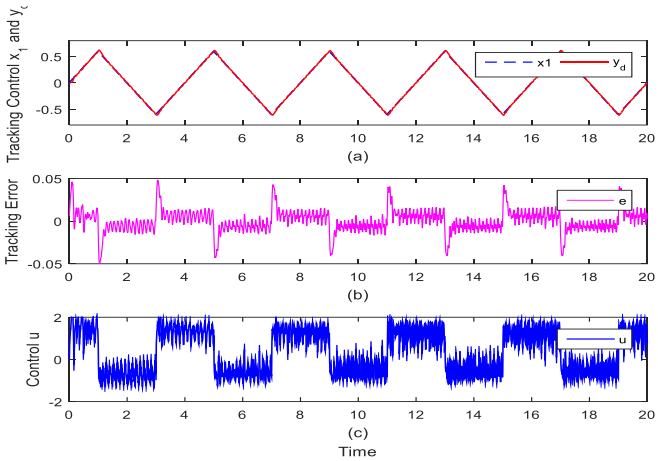

Fig. 12 Response of proposed controller for tooth signal.

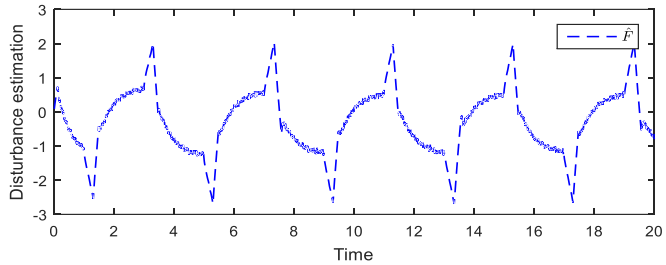

Fig. 13 Estimated dynamics for tooth signal.

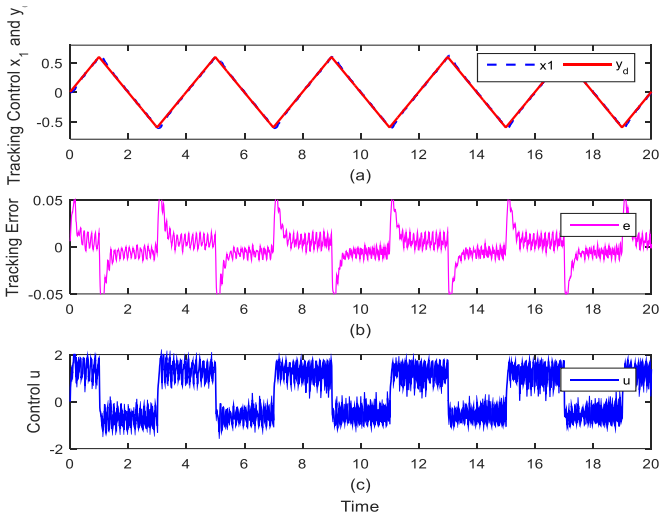

Fig. 14 Response of ANC [54] for tooth signal.

All these experimental results illustrate how the proposed filter based estimator captures and then compensates the time-varying nonlinearities including modeling uncertainties, frictions and disturbances, so as to improve both the transient and steady-state tracking response. In particular, compared with the ANC, the main advantage of the suggested control is that the parameter tuning procedure is simpler, while the convergence speed is faster. Moreover, only the system output is required in this control implementation.

VII. Conclusion

This paper presents a new output-feedback control for uncertain nonlinear pure-feedback systems, where the widely used backstepping or DSC scheme and function approximators are avoided. We first use a coordinate transform to reformulate the pure-feedback system into a canonical form. Then the Levant's differentiator with attractive finite-time convergence is used to reconstruct the immeasurable states of the derived system. Finally, we introduce a novel filter based estimator to address the unknown dynamics in the system, where only one constant needs to be selected. In this case, the online learning of function approximators (e.g. NNs, FLSs) with potential sluggish transient and increased computational costs are all remedied. Moreover, the backstepping procedure is avoided in this control framework. Consequently, the implementation and the analysis of this proposed control is simpler than the existing backstepping based methods. Nevertheless, only the system output needs to be measured for the control implementation. Simulations with a benchmark pure-feedback system model and experiments on a servo system are carried out to verify the efficacy of this method. This output-feedback control design methodology can also be further explored for strict-feedback systems, which deserves further investigations.

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