The Ratio of the Irredundance Number and the Domination Number for Block-Cactus Graphs

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Abstract

Let $\gamma(G)$ and $ir(G)$ denote the domination number and the irredundance number of a graph G, respectively. Allan and Laskar [1] and Bollobás and Cockayne [2] proved independently that $\gamma(G) < 2\,ir(G)$ for any graph G. For a tree T, Damaschke [4] obtained the sharper estimation $2\gamma(T) < 3\,ir(T)$. Extending Damaschke's result, Volkmann [11] proved that $2\gamma(G) \leq 3\,ir(G)$ for any block graph G and for any graph G with cyclomatic number $\mu(G) \leq 2$. Volkmann [11] also conjectured that $5\gamma(G) < 8\,ir(G)$ for any cactus graph. In this article we show that if G is a block-cactus graph having $\pi(G)$ induced cycles of length $2 \pmod 4$, then $\gamma(G)(5\pi(G)+4) \leq ir(G)(8\pi(G)+6)$. This result implies the inequality $5\gamma(G) < 8\,ir(G)$ for a block-cactus graph G, thus proving the above conjecture. *J. Graph Theory 29 (1998), 139-149*

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1 Introduction and Preliminary Results

All graphs will be finite and undirected, without loops and multiple edges. If G is a graph, $V(G)$ denotes the set of vertices in G. The edge set of G is denoted by $E(G)$. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. A connected graph with no cut vertex is called a *block*. A *block of a graph* G is a subgraph of G which is itself a block and which is maximal with respect to that property. A block H of G is called an *end block* of G if H has at most one cut vertex of G. A graph G is a *block graph* if every block of G is complete, and G is a *block-cactus graph* if every block of G is either a complete graph or a cycle. Block-cactus graphs generalize the known class of cactus graphs. Recall that G is a *cactus graph* if each edge of G belongs to at most one cycle. If $k(G)$ denotes the number of components of G, then $\mu(G) = |E(G)| - |V(G)| + k(G)$ is the *cyclomatic number* of G.

A set X is called a *dominating set* if $N[X] = V(G)$. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. A set $X \subseteq V(G)$ is *irredundant* if for every vertex $x \in X$,

$$P_G(x, X) = P(x, X) = N[x] - N[X - \{x\}] \neq \emptyset.$$

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The minimum cardinality taken over all maximal irredundant sets of G is the *irredundance number* $ir(G)$.

It is well known [3] that for any graph G,

$$ir(G) \leq \gamma(G).$$

Allan and Laskar [1] and Bollobás and Cockayne [2] proved independently that $\gamma(G) < 2\,ir(G)$ for any graph G. For a tree T, Damaschke [4] obtained the sharper estimation $2\gamma(T) < 3\,ir(T)$. Extending Damaschke's result, Volkmann [11] proved that $2\gamma(G) \leq 3\,ir(G)$ for any block graph G and for any graph G with cyclomatic number $\mu(G) \leq 2$. Volkmann [11] also posed the following conjecture.

Conjecture 1 (Volkmann [11]) *If G is a cactus graph, then*

$$5\gamma(G) < 8\,ir(G).$$

In this article, we find the strict ratio of the irredundance and domination numbers for block-cactus graphs having $\pi(G)$ induced cycles of length $2\,(\text{mod}\,4)$. This result implies the above conjecture. The ratio of related parameters was studied in [5, 7]. Interesting results for block-cactus graphs can be found in [6, 8, 9, 10].

Proposition 1 (Bollobás and Cockayne [2]) *Let I be a maximal irredundant set of the graph G. Suppose that the vertex u is not dominated by I. Then for some $x \in I$,*

a) $P(x, I) \subseteq N(u)$, and

b) for $x_1, x_2 \in P(x, I)$ such that $x_1 \neq x_2$, either $x_1x_2 \in E(G)$ or there exist $y_1, y_2 \in I - \{x\}$ such that x_1 is adjacent to each vertex of $P(y_1, I)$ and x_2 is adjacent to each vertex of $P(y_2, I)$.

Let G be a block-cactus graph, $F \subseteq V(G)$ and $W = V(G) - F$. A cycle C in G is called *alternating* if the sets F and W do not contain edges of C. An *alternating path* is defined analogously.

Lemma 1 *Let G be a block-cactus graph and $F \subseteq V(G)$ such that $|N(w) \cap F| \geq 2$ for all $w \in W = V(G) - F$. If G does not contain an alternating cycle C_{4k+2} as an induced subgraph, then there exists a subset $F' \subseteq F$ such that $W \subseteq N(F')$ and $2|F'| \leq |F|$.*

Proof. Without loss of generality we may assume that G is a connected graph and $W \neq \emptyset$. We prove the lemma by induction on the number of vertices in G. The lemma is obvious if G contains few vertices. Suppose that G consists of one block. If G is a complete graph, then the lemma is obvious. Suppose that G is a cycle and consider a maximal alternating path P between F and W in G. Let $\langle P \rangle$ be a path. We have

$$P = f_1w_1f_2w_2 \ldots f_{t-1}w_{t-1}f_t, \quad t \geq 2,$$

where $f_i \in F$, $1 \leq i \leq t$, and $w_i \in W$, $1 \leq i \leq t-1$. The set

$$D = \{f_2, f_4, f_6, ...\}$$

dominates the set $P \cap W$ and $|D| \leq t/2$. The maximal alternating paths in the cycle G are vertex disjoint and hence it is easy to construct the set F'. Now suppose that $\langle P \rangle$ is a cycle. We have $\langle P \rangle = G$ and there are two possibilities. If

$$P = f_1 w_1 f_2 w_2 \dots f_{t-1} w_{t-1} f_t, \quad t \geq 2,$$

where $f_1 f_t \in E(G)$, $f_i \in F$, $1 \leq i \leq t$, and $w_i \in W$, $1 \leq i \leq t-1$, then the set

$$F' = \{f_2, f_4, f_6, ...\}$$

satisfies the necessary properties. If

$$P = f_1 w_1 f_2 w_2 \dots f_t w_t, \quad t \geq 2,$$

where $f_1 w_t \in E(G)$, $f_i \in F$ and $w_i \in W$, $1 \leq i \leq t$, then $\langle P \rangle$ is an alternating cycle. Since $2t = |P| \neq 4k+2$, it follows that t is even and the set

$$F' = \{f_1, f_3, ..., f_{t-1}\}$$

gives the desired result.

Suppose now that the statement of Lemma 1 holds for any block-cactus graph having fewer vertices than G, and let G consist of at least two blocks. Then there exists an end block B of G with only one cut vertex v of G.

Case 1. The block B is a complete graph.

Subcase 1.1. The cut vertex v is an element of F. Assume that $V(B) \subseteq F$. If we consider the block-cactus graph $G' = G - (V(B) - \{v\})$, then $|V(G')| < |V(G)|$, and $|N_{G'}(w) \cap F| \geq 2$ for all $w \in V(G') - F$. Hence, by the induction hypothesis, the desired result easily follows.

Let $V(B) \cap W \neq \emptyset$. Now $G' = G - (V(B) \cup (N_G(v) \cap W)) \neq \emptyset$ is a block-cactus graph such that $|N_{G'}(w) \cap F| \geq 2$ for all $w \in V(G') - F$. Again, by the induction hypothesis we obtain the statement of the lemma.

Subcase 1.2. The cut vertex v is an element of W. If $|F \cap V(B)| \geq 2$, then the block-cactus graph $G' = G - V(B)$ together with the induction hypothesis (as well as using v) yields the desired result.

If $|F \cap V(B)| \leq 1$, then $|F \cap V(B)| = 1$ so let $F \cap V(B) = \{b\}$. Since $|N(v) \cap F| \geq 2$, it follows that there exists a further neighbor $a \in F$ of v in $G - V(B)$. Now we define $G' = G - (V(B) \cup \{a\} \cup (N_G(a) \cap W))$. If $G' = \emptyset$, then $F' = \{a\}$ fulfills the statement of Lemma 1. Finally, if $G' \neq \emptyset$, then by the induction hypothesis there exists a set $F^* \subseteq F - \{a, b\}$ with $W \cap V(G') \subseteq N_{G'}(F^*)$ and $2|F^*| \leq |F| - 2$. Consequently, for $F' = F^* \cup \{a\} \subseteq F$, we deduce that $W \subseteq N_G(F')$ and $2|F'| \leq |F|$.

Case 2. The block B is a cycle.

Subcase 2.1. Suppose that $v \in F$. If $N_B(v) \cap W = \emptyset$, then the graphs $B - \{v\}$ and $G - (V(B) - \{v\})$ together with the induction hypothesis yield the desired result. Therefore we can assume that there is $w_1 \in N_B(v) \cap W$. Let P' be the maximal alternating path in the graph $B - \{v\}$ such that $w_1 \in P'$. Consider the path $P = P' \cup \{v\}$. Suppose firstly that P has the following form:

$$P = v w_1 f_1 \dots w_t f_t, \quad t \geq 1,$$

where $f_i \in F$ and $w_i \in W$, $1 \le i \le t$. The graph $\langle P \rangle$ is either a path or a cycle depending on the existence of the edge vf_t. If t is even, then the set $\{f_1, f_3, ..., f_{t-1}\}$ dominates $P' \cap W$ and the graph $G - P'$ together with the induction hypothesis gives the desired result. If t is odd, then the set $\{v, f_2, f_4, ..., f_{t-1}\}$ dominates the set $(P \cup N(v)) \cap W$. By the induction hypothesis, the statement of Lemma 1 holds for the graph $G - P - (N(v) \cap W)$, and the result easily follows.

Now suppose that

$$P = vw_1f_1 \dots w_tf_tw_{t+1}, \quad t \ge 1,$$

where $f_i \in F$, $1 \le i \le t$, and $w_i \in W$, $1 \le i \le t+1$. We have $vw_{t+1} \in E(G)$, i.e., $\langle P \rangle$ is an alternating cycle and $\langle P \rangle = B$. Now $2t + 2 = |P| \ne 4k + 2$ and hence t is odd. The set $\{v, f_2, f_4, ..., f_{t-1}\}$ dominates $(P \cup N(v)) \cap W$, and the graph $G - P - (N(v) \cap W)$ together with the induction hypothesis gives the desired result.

Subcase 2.2. Suppose that $v \in W$. The set $V(B) \cap W$ does not contain edges of G and therefore $|N(v) \cap V(B) \cap F| \ge 2$. The graphs B and $G - B$ satisfy the conditions of Lemma 1. By the induction hypothesis, there are corresponding dominating sets and the union of these sets yields the set F'. The proof is complete. ∎

2 Main Result

Let $\pi(G)$ denote the number of induced cycles of length $2 \pmod 4$ in a graph G. The following theorem gives the ratio of the irredundance and domination numbers for block-cactus graphs in terms of $\pi(G)$. We will see later that this ratio is strict.

Theorem 1 *If G is a block-cactus graph, then*

$$\frac{ir(G)}{\gamma(G)} \ge \frac{5\pi(G) + 4}{8\pi(G) + 6}.$$

Proof. Let I be an ir-set of G, i.e., I is a maximal irredundant set and $|I| = ir(G)$, and denote $U = V(G) - N[I]$.

We say that G contains an *S-subgraph* if there exist sets $\{v_1, v_2, v_3\} \subseteq I$ and $S = \{u, v_i', v_i'', i = 1, 2, 3\}$ satisfying the following conditions:

$$P(v_i, I) = \{v_i', v_i''\},\ i = 1, 2, 3, \quad \{v_1', v_1''\} \subseteq N(v_2'), \quad \{v_3', v_3''\} \subseteq N(v_2''),$$

and

$$\{v_2', v_2''\} \subseteq N(u)$$

for $u \in U$. Note that the vertices v_1, v_2, v_3 are not isolated in the graph $\langle I \rangle$, since $v_i \notin P(v_i, I)$ for $i = 1, 2, 3$.

Now suppose that G contains an S-subgraph. Remove from G the vertices of S together with incident edges, and add the set $S' = \{w_1, w_2, p_i, u_i, i = 1, 2, 3\}$ together with edges $w_1v_1, w_1v_2, w_2v_2, w_2v_3$ and v_ip_i, p_iu_i, $i = 1, 2, 3$. Denote the resulting graph by G'. The vertices v_1, v_2, v_3 belong to different connected components of the graph $G - S$, since otherwise G is not a block-cactus graph. Therefore G' is a block-cactus graph and G' does not contain new cycles, i.e., $\pi(G') \le \pi(G)$. Furthermore,

$$P_{G'}(x, I) = P_G(x, I) \ne \emptyset \quad \text{for each} \quad x \in I - \{v_1, v_2, v_3\},$$

and

$$P_{G'}(v_i, I) = \{p_i\} \quad \text{for} \quad i = 1, 2, 3.$$

Consequently, the set I is irredundant in G'. Let z be a vertex of $V(G) - I - S$. The set $I \cup \{z\}$ is redundant in G, since I is a maximal irredundant set in G. By definition, either

$$N_G[z] \subseteq N_G[I]$$

or

$$P_G(y, I) \subseteq N_G[z]$$

for some vertex $y \in I$. Note that $y \notin \{v_1, v_2, v_3\}$, since otherwise G is not a block-cactus graph. Therefore $I \cup \{z\}$ is a redundant set in G'. If $z \in S'$, then it is straightforward to see that $I \cup \{z\}$ is a redundant set in G'. Thus I is a maximal irredundant set in G'. Now let D denote a minimum dominating set in G'. We have $|D \cap (S' \cup \{v_1, v_2, v_3\})| \geq 4$ and the set $(D - (S' \cup \{v_1, v_2, v_3\})) \cup \{u, v_1, v_2, v_3\}$ is a dominating set of G. Hence $\gamma(G') \geq \gamma(G)$.

Applying the above construction to G we can obtain a block-cactus graph H such that I is a maximal irredundant set in H and H does not contain S-subgraphs with respect to I. Moreover,

$$\pi(H) \leq \pi(G) \quad \text{and} \quad \gamma(H) \geq \gamma(G). \tag{1}$$

Consider now the graph H and denote $U = V(H) - N[I]$. By Proposition 1, for any vertex $u \in U$ there is a vertex $f(u) \in I$ such that $P(f(u), I) \subseteq N(u)$. Put

$$A = \{f(u) : u \in U\} \cup \{v \in I : |P(v, I)| = 1 \text{ and } v \notin P(v, I)\}.$$

Form the set B choosing for each vertex $a \in A$ one vertex from $P(a, I)$ by the following rule. If $P(a, I) = \{p\}$, then we add p into B. If $|P(a, I)| > 1$, then there is a vertex $u \in U$ such that $P(a, I) \subseteq N(u)$. Since H is a block-cactus graph, we have $P(a, I) = \{p_1, p_2\}$ and $p_1 p_2 \notin E(H)$. By Proposition 1, p_i dominates $P(y_i, I)$, where $y_i \in I - \{a\}$, $i = 1, 2$. It is evident that $y_i \notin P(y_i, I)$. We have $y_1 \neq y_2$ and $|P(y_i, I)| \leq 2$, since otherwise H is not a block-cactus graph. The graph H has no S-subgraph, and so without loss of generality $|P(y_1, I)| = 1$. Now add p_2 into B. Note that $y_1 \in A$. Therefore $P(y_1, I) \in B$ and the vertex p_1 is dominated by B.

Thus the set B dominates $A \cup U \cup \{P(a, I) : a \in A\}$ and $|B| = |A|$. Let $C = (I - A) \cup B$ and let C dominate $V(H) - W$. We have $|C| = |I| = ir(G)$ and for each $w \in W$,

$$|N_H(w) \cap A| \geq 2.$$

Denote

$$D = \{u \in I : N_{\langle I \rangle}(u) \neq \emptyset\}.$$

Clearly, for each vertex $d \in D$,

$$\deg_{\langle D \rangle} d \geq 1. \tag{2}$$

By the definitions of A and D,

$$A \subseteq D \subseteq I$$

and therefore for each $w \in W$,

$$|N_H(w) \cap D| \geq 2. \tag{3}$$

Suppose that the graph $\langle D \cup W \rangle$ has no D-W-alternating cycle of length $2\,(\mathrm{mod}\ 4)$. Then, by Lemma 1,

$$\gamma(G) \leq \gamma(H) \leq \frac{3}{2}|I| = \frac{3}{2}\, ir(G)$$

which implies the desired inequality. Define now the graph F with the vertex set D as follows. Replace in the graph $\langle D \cup W \rangle$ all alternating cycles $C^1, C^2, ..., C^k$ $(k \geq 1)$ of length $2\,(\mathrm{mod}\,4)$ by complete graphs and denote the resulting graph by H_1. Let F be the subgraph of H_1 induced by D. It is obvious that H_1 and F are block-cactus graphs and the sets $K^i = C^i \cap D$, $1 \leq i \leq k$, induce complete subgraphs in F. Moreover, the K^i are blocks in F and $|K^i| \geq 3$ for all $i = 1, 2, ..., k$. Call the blocks K^i *special*.

We will add a set of extra edges in the set D of the graph H_1 in such a way that the resulting graph H^* possesses Property A.

Property A. For any vertex $w \in W$ in the graph H^* there exist vertices $u, v \in N(w) \cap D$ such that either

$$\deg_{\langle D \rangle} u \geq 2 \quad \text{and} \quad \deg_{\langle D \rangle} v \geq 2,$$

or

$$N_{\langle D \rangle}(u) = \{v\}.$$

Construct the sequence of block-cactus graphs

$$H_1, H_2, ..., H_m$$

in accordance with the following rule. Suppose that we have the block-cactus graph H_i and it contains the vertex $w_i \in W \cap V(H_i)$ and the vertices $u_i, v_i \in N_{H_i}(w_i) \cap D$ satisfying

$$\deg_{\langle D \rangle} u_i = 1 \quad \text{and} \quad u_i v_i \notin E(H_i).$$

If the vertices u_i and v_i belong to different connected components of the graph $H_i - \{w_i\}$, then

$$H_{i+1} = (H_i - \{w_i\}) \cup u_i v_i$$

is a block-cactus graph. If the vertices u_i and v_i belong to one connected component of the graph $H_i - \{w_i\}$, then the vertices u_i, w_i, v_i in the graph H_i belong to a block which is a cycle. Again, the graph H_{i+1} is a block-cactus graph.

Thus, the graph H_m is a block-cactus graph. Taking into account (2) and (3) we see that for any vertex $w \in W \cap V(H_m)$ there exist vertices $u, v \in N_{H_m}(w) \cap D$ such that either $\deg_{\langle D \rangle} u \geq 2$ and $\deg_{\langle D \rangle} v \geq 2$, or $N_{\langle D \rangle}(u) = \{v\}$. Moreover, in the graph H_m, $\deg_{\langle D \rangle} u_i \geq 2$ and $\deg_{\langle D \rangle} v_i \geq 2$ for all $i = 1, 2, ..., m-1$. Put

$$F^* = F \cup_{i=1}^{m-1} u_i v_i \quad \text{and} \quad H^* = H_1 \cup_{i=1}^{m-1} u_i v_i.$$

The graph F^* is a block-cactus graph, since it is an induced subgraph of H_m. Furthermore, $H^* - \cup_{i=1}^{m-1} w_i = H_m$, and therefore the graph H^* satisfies Property A.

For the above alternating cycles C^i, the sets $C_i \cap D$, $i = 1, 2, .., k$, do not contain edges in the graph H. By the definitions of the set D and the graph F^*, we obtain the following property.

Property B. For any vertex $u \in V(F^*)$ there is the edge $uv \in E(F^*)$ not belonging to any block K^i, $1 \le i \le k$.

Let the graph F^* contain $r \in \{0, 1, ..., k\}$ special blocks K^i satisfying Property C. Without loss of generality we may assume that the following blocks possess this property:

$$K^1, K^2, ..., K^r.$$

Property C. The block K^i contains the vertex v_i such that

$$N_{F^*}(v_i) - K^i = \{p_i\} \quad \text{and} \quad \deg_{F^*} p_i = 1.$$

Lemma 2 *For the graph F^*,*

$$|V(F^*)| = |D| \ge 5k - 3r + 4. \tag{4}$$

Proof. We prove (4) by induction on the number k of special blocks. Let $k = 1$. Taking into account Properties B and C, we obtain $|V(F^*)| \ge 3|K^1| \ge 9$ if $r = 0$, and $|V(F^*)| \ge 2|K^1| \ge 6$ if $r = 1$. Now suppose that (4) holds for any block-cactus graph having fewer special blocks K^i with $|K^i| \ge 3$ and satisfying Property B. If F^* is not a connected graph, then the result easily follows. Let F^* be a connected graph and denote

$$K = \cup_{i=1}^{k} K^i.$$

For the vertex $u \in K$ denote by B_u all connected components of the graph $F^* - \{u\}$ which do not contain vertices of the set K. The graph

$$F^* - \cup_{u \in K} B_u$$

has an end block K^t with only one cut vertex x. By Property B, there is the edge $xy \in E(F^*)$ such that $xy \notin K^i$ for any $i = 1, 2, ..., k$. Consider the graph

$$F' = F^* - \cup_{u \in K^t - \{x\}} (B_u \cup \{u\}).$$

It is evident that F' is a block cactus graph having $k - 1$ special blocks and satisfying Property B. Suppose that some block K^i, $r < i \le k$, satisfies Property C in the graph F'. The first possibility is that $\deg_{F'} x = 1$ and $y \in K^i$. The second possibility is that $\deg_{F'} y = 1$ and $x \in K^i$. In either of these two cases we add to F' the new vertex z and the edge xz. The block K^i does not satisfy Property C in the resulting graph, and this operation evidently does not produce new special blocks satisfying Property C. Thus, if $i \in \{r+1, r+2, ..., k\}$, then the block K^i does not satisfy Property C in the graph F' and

$$V(F^*) = \cup_{u \in K^t - \{x\}} (B_u \cup \{u\}) \cup V(F') - \{z\}.$$

Case 1. If $t \le r$, then F' contains exactly $r - 1$ special blocks satisfying Property C. Using the induction hypothesis, Property B and the inequality $|K^t| \ge 3$, we see that

$$\begin{aligned} |V(F^*)| &\ge |\cup_{u \in K^t - \{x\}} B_u \cup \{u\}| + |V(F')| - 1 \\ &\ge 2(|K^t| - 1) + 5(k-1) - 3(r-1) + 3 \\ &\ge 5k - 3r + 5. \end{aligned}$$

Case 2. If $t > r$, then F' contains exactly r special blocks satisfying Property C. The block K^t does not satisfy Property C and hence $|B_u| \geq 2$ for each $u \in K - \{x\}$. We obtain

$$\begin{aligned} |V(F^*)| &\geq |\cup_{u \in K^t - \{x\}} B_u \cup \{u\}| + |V(F')| - 1 \\ &\geq 3(|K^t| - 1) + 5(k-1) - 3r + 3 \\ &\geq 5k - 3r + 4. \end{aligned}$$

The proof of Lemma 2 is complete. ∎

Now consider the sets

$$V = \{v_1, v_2, ..., v_k\}$$

and

$$P = \{p_1, p_2, ..., p_r\},$$

where v_i and p_i are the vertices defined in Property C if $i \leq r$, and v_i is some vertex of K^i if $i > r$. We have, $v_i p_i \in E(F^*)$ and $\deg_{F^*} p_i = 1$ for $i = 1, 2, ..., r$. Note that the set $\{v_1, ..., v_r\}$ contains different vertices by Property C, while the set $\{v_{r+1}, ..., v_k\}$ does not necessarily contain different vertices. Therefore,

$$|V| - |P| = |\{v_{r+1}, ..., v_k\}| \leq k - r. \quad (5)$$

Denote

$$X = D - (V \cup P).$$

Lemma 3 *For each vertex $w \in W - N_H(V)$ in the graph H,*

$$|N_H(w) \cap X| \geq 2.$$

Proof. Denote by H' the induced subgraph $\langle D \cup W \rangle$ in the graph H. By definitions, the graph H^* is obtained from H' by adding edges in the sets $C^1, C^2, ..., C^k$ and the set D. Therefore, $N_{H'}(w) \subset N_{H^*}(w)$ if $w \in C^i \cap W$, and $N_{H'}(w) = N_{H^*}(w)$ if $w \in W - \cup_{i=1}^k C^i$. Now assume that $w \in W - N_{H'}(V)$ and

$$|N_{H'}(w) \cap X| \leq 1.$$

Consider the case $w \in C^i \cap W$ where $i \in \{1, 2, ..., k\}$. Since C^i is an alternating cycle and $w \notin N_{H'}(V)$, it follows that there are vertices $c', c'' \in N_{H'}(w) \cap C^i \cap D$ and $c', c'' \notin V$. In the graph H^* we have $c', c'' \in K^i$ and therefore $c', c'' \notin P$. Thus, $c', c'' \in X$ and $|N_{H'}(w) \cap X| \geq 2$, a contradiction. Now consider the case $w \in W - \cup_{i=1}^k C^i$. Since $N_{H'}(w) = N_{H^*}(w)$, we have $N_{H^*}(w) \cap V = \emptyset$. Thus, in the graph H^* the vertex w is adjacent only to vertices of P and to at most one vertex of X, contrary to Property A. The proof of Lemma 3 is complete. ∎

In the graph H consider the induced subgraph $X \cup W'$, where $W' = W - N_H(V)$. This graph is a block-cactus graph having no alternating cycles of length 2 (mod 4) as induced subgraphs. By Lemma 3, $|N_{\langle X \cup W' \rangle}(w) \cap X| \geq 2$ for each vertex $w \in W'$. By Lemma 1, there exists $X' \subseteq X$ such that X' dominates W' and $2|X'| \leq |X|$. Thus, the set $T = V \cup X'$

dominates W in the graph H, and $C \cup T$ is a dominating set of H. Using (4), (5), and the inequality $k \geq r \geq 0$, we obtain

$$|T| = |V| + |X'| \leq |V| + \frac{1}{2}(|D| - |V| - |P|) = \frac{1}{2}(|D| + |V| - |P|)$$

$$\leq \frac{|D| + k - r}{2|D|}|I| \leq \frac{3k - 2r + 2}{5k - 3r + 4}|I| \leq \frac{3k + 2}{5k + 4}|I|.$$

Using (1) and the inequality $k \leq \pi(H)$, we finish the proof of Theorem 1

$$\gamma(G) \leq \gamma(H) \leq |C| + |T| \leq \frac{8k + 6}{5k + 4}|I| \leq \frac{8\pi(G) + 6}{5\pi(G) + 4}\, ir(G).$$

∎

The following corollaries follow directly from Theorem 1.

Corollary 1 *If G is a block-cactus graph, then* $ir(G)/\gamma(G) > 5/8$.

Since any cactus graph is a block-cactus graph, Corollary 1 proves Conjecture 1. The example below shows that the bound 5/8 is best possible for cactus graphs and, consequently, for block-cactus graphs.

Corollary 2 (Volkmann [11]) *If G is a block graph, then* $ir(G)/\gamma(G) \geq 2/3$.

The bound 2/3 is best possible for block graphs (see [11]).

In conclusion we show that the bounds in Theorem 1 and Corollary 1 are sharp. Let $C^i = a_i b_i c_i d_i e_i f_i a_i$, $i = 1, 2, ..., k$ be simple cycles of length 6 and let $T^i = x_i y_i z_i$, $i = 0, 1, ..., k+1$ be cycles of length 3. Add the edges

$$\{e_i a_{i+1} :\ 1 \leq i \leq k-1\}, \quad \{c_i x_i :\ 1 \leq i \leq k\}, \quad \text{and} \quad \{x_0 a_1, e_k x_{k+1}\}.$$

Put

$$I = \{a_i, c_i, e_i :\ 1 \leq i \leq k\} \cup \{x_i, y_i :\ 0 \leq i \leq k+1\}.$$

Also add the paths $P_u = uu'u''$ for each vertex $u \in I$. Denote the resulting graph by G. The graph G is both a block-cactus graph and a cactus graph.

Every maximal irredundant set of the graph G contains at least one vertex of the set $\{u, u', u'' :\ u \in I\}$. Therefore, $ir(G) \geq |I| = 5k + 4$. On the other hand, I is a maximal irredundant set of G and hence $ir(G) = 5k + 4$. It is not difficult to see that

$$\{u' :\ u' \in P_u, u \in I\} \cup \{a_i, c_i :\ 1 \leq i \leq k\} \cup \{x_i :\ 0 \leq i \leq k+1\}$$

is a minimum dominating set and therefore $\gamma(G) = 8k + 6$. Thus,

$$ir(G)/\gamma(G) = (5k + 4)/(8k + 6) \quad \text{and} \quad \lim_{k \to \infty} (5k + 4)/(8k + 6) = 5/8.$$

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