A Semi-Induced Subgraph Characterization of Upper Domination Perfect Graphs

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Abstract

Let $\beta(G)$ and $\Gamma(G)$ be the independence number and the upper domination number of a graph G, respectively. A graph G is called Γ*-perfect* if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G. The class of Γ-perfect graphs generalizes such well-known classes of graphs as strongly perfect graphs, absorbantly perfect graphs, and circular arc graphs. In this article, we present a characterization of Γ-perfect graphs in terms of forbidden semi-induced subgraphs. Key roles in the characterization are played by the odd prism and the even Möbius ladder, where the prism and the Möbius ladder are well-known 3-regular graphs [2]. Using the semi-induced subgraph characterization, we obtain a characterization of $K_{1,3}$-free Γ-perfect graphs in terms of forbidden induced subgraphs. *J. Graph Theory 31 (1999), 29-49*

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1 Introduction

All graphs will be finite and undirected without multiple edges. Unless otherwise stated, all graphs have no loops. If G is a graph, $V(G)$ denotes the set, and $|G|$ the number, of vertices in G. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x\in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ the minimal degree of vertices in G. A path, a cycle and a complete graph of order n will be denoted by P_n, C_n and K_n, respectively.

A set $I \subseteq V(G)$ is called *independent* if no two vertices of I are adjacent. A set X is called a *dominating set* if $N[X] = V(G)$. An *independent dominating set* is a vertex subset that is both independent and dominating, or equivalently, is maximal independent. The *independence number* $\beta(G)$ is the maximum cardinality of a (maximal) independent set of G, and the *independent domination number* $i(G)$ is the minimum cardinality taken over all maximal independent sets of G. The *domination number* $\gamma(G)$ is the minimum

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cardinality of a (minimal) dominating set of G, and the *upper domination number* $\Gamma(G)$ is the maximum cardinality taken over all minimal dominating sets of G. For $x \in X$, the set

$$PN(x, X) = N[x] - N[X - \{x\}]$$

is called the *private neighborhood* of x. If $PN(x, X) = \emptyset$, then x is said to be *redundant* in X. A set X containing no redundant vertex is called *irredundant.* The *irredundance number* $ir(G)$ is the minimum cardinality taken over all maximal irredundant sets of G, and the *upper irredundance number* $IR(G)$ is the maximum cardinality of a (maximal) irredundant set of G.

The following relationship among the parameters under consideration is well-known [7, 9]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Definition 1 *A graph G is called* irredundance perfect (ir-perfect) *if $ir(H) = \gamma(H)$, for every induced subgraph H of G.*

Definition 2 *A graph G is called* domination perfect (γ-perfect) *if $\gamma(H) = i(H)$, for every induced subgraph H of G.*

Definition 3 *A graph G is called* upper domination perfect (Γ-perfect) *if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G.*

Definition 4 *A graph G is called* upper irredundance perfect (IR-perfect) *if $\Gamma(H) = IR(H)$, for every induced subgraph H of G.*

The classes of upper domination perfect graphs and upper irredundance perfect graphs in a sense are dual to the classes of domination perfect graphs and irredundance perfect graphs, respectively. A lot of interesting results on domination perfect graphs [1, 3, 12, 16, 17, 24, 27, 28, 29, 31, 33] and irredundance perfect graphs [3, 4, 10, 20, 21, 25, 26, 32] are known. A finite induced subgraph characterization of the entire class of domination perfect graphs was recently obtained in [33], while the problems of characterizing the entire class of irredundance perfect graphs and upper irredundance perfect graphs are still open. For a short survey on domination perfect graphs, see also [33].

We summarize the known results on Γ-perfect and IR-perfect graphs. The following important theorem gives the relationship between the class of Γ-perfect graphs and the class of IR-perfect graphs.

Theorem A (Gutin and Zverovich [14]) *Any Γ-perfect graph is IR-perfect.*

Thus, Γ-perfect graphs form a subclass of IR-perfect graphs. On the other hand, a number of well-known classes of graphs are subclasses of Γ-perfect graphs, and consequently, IR-perfect graphs. Cockayne et al. [7] proved that bipartite graphs are Γ-perfect, and Jacobson and Peters [23] showed that chordal graphs are Γ-perfect. The next theorem generalizes these results, since bipartite graphs and chordal graphs are strongly perfect graphs. Recall that a graph G is called *strongly perfect* if every induced subgraph H of G has a stable transversal, where a *stable transversal* S of H is a vertex subset of H such that $|S \cap C| = 1$ for any maximal clique C of H.

Theorem B (Cheston and Fricke [5], Jacobson and Peters [22]) *A strongly perfect graph is* Γ*-perfect.*

It may be pointed out that, besides bipartite and chordal graphs, strongly perfect graphs contain comparability graphs, perfectly orderable graphs, peripheral graphs, complements of chordal graphs, Meyniel graphs, parity graphs, i-triangulated graphs, permutation graphs, cographs, and hence all these classes are subclasses of Γ-perfect graphs. Hammer and Maffray [15] defined a graph G to be *absorbantly perfect* if every induced subgraph H of G contains a minimal dominating set that meets all maximal cliques of H. It turned out that absorbantly perfect graphs are Γ-perfect (Theorem C). Since every strongly perfect graph is absorbantly perfect, we see that Theorem B follows from the more general Theorem C.

Theorem C (Gutin and Zverovich [14]) *An absorbantly perfect graph is* Γ*-perfect.*

A graph is called *circular arc* if it can be represented as the intersection graph of arcs on a circle.

Theorem D (Golumbic and Laskar [13]) *A circular arc graph is* Γ*-perfect.*

The following theorem gives a sufficient condition for a graph to be IR-perfect.

Theorem E (Cockayne, Favaron, Payan and Thomason [7]) *If a graph* G *does not contain* P_5*,* C_5*,* $Pr_3 - v_1$ *and* $Pr_3 - v_1 - v_2v_3$ *as induced subgraphs, where* Pr_3 *is shown in Fig. 1, then* G *is* IR*-perfect.*

Using Theorem A we see that Theorem F improves Theorem E.

Theorem F (Gutin and Zverovich [14]) *If a graph* G *does not contain* P_5 *and* Pr_3 *in Fig. 1 as induced subgraphs, then* G *is* Γ*-perfect.*

Other sufficient conditions for a graph to be Γ-perfect or IR-perfect can be found in [10, 14, 23, 30], one of them is stated in Corollary 2. A number of authors [6, 7, 8, 11] investigated graphs G having $\beta(G) = \Gamma(G)$, i.e., these parameters are not necessarily equal for a proper induced subgraph of G. Cockayne et al. [8] proved that

$$\beta(G) = \Gamma(G) = IR(G)$$

for the representative graph of any hereditary hypergraph. Cheston et al. [6] showed that this equality is valid for upper bound graphs which extend the class of representative graphs of hereditary hypergraphs, while Fellows et al. [11] proved that the same equality holds for trestled graphs. It was also shown by Cheston et al. [6] that $\beta(G) = \Gamma(G)$ for simplicial graphs, which generalize the class of upper bound graphs.

In this article, we introduce the concept of a semi-induced subgraph (Definition 8), and we present two characterizations of the entire class of Γ-perfect graphs in terms of forbidden semi-induced subgraphs (Theorems 1 and 2). Key roles in the characterizations are played by the odd prism and the even Möbius ladder, where the prism and the Möbius ladder are well-known 3-regular graphs [2]. Using the semi-induced subgraph characterization of Γ-perfect graphs, we obtain a result of Jacobson and Peters [22] on Γ-perfect graphs (Corollary 1) and also a characterization of $K_{1,3}$-free Γ-perfect graphs in terms of forbidden induced subgraphs. The latter result implies a known sufficient condition for a $K_{1,3}$-free graph to be Γ-perfect (Corollary 2). Notice here that $K_{1,3}$-free graphs are γ-perfect [1], and that $K_{1,3}$-free ir-perfect graphs were characterized by Favaron [10], who also found a sufficient condition for a $K_{1,3}$-free graph to be IR-perfect.

2 Basic Definitions

We need the following definitions.

Definition 5 *Two vertex subsets A, B of a graph G* independently match *each other if $A \cap B = \emptyset$, $|A| = |B|$, and all edges between A and B form a perfect matching in $\langle A \cup B \rangle$.*

Definition 6 *A graph G of order $2k$ is called a* W-graph *if there is a partition $V(G) = A \cup B$ such that A and B independently match each other. Clearly, $|A| = |B| = k$. The sets A and B are called* parts*, and the graph G is denoted by $G(A, B)$.*

It is not difficult to see that a W-graph may have several partitions into parts. Hence a W-graph is considered in Sections 2 and 3 together with a fixed partition into parts.

Definition 7 *Let G be a W-graph, $G = G(A, B)$. Edges between the parts A and B are called* b-edges *and denoted in our figures by bold lines. Edges which are not b-edges are called* l-edges *and denoted by thin lines.*

We can understand the above partition of the edge set as a coloring of the edge set with two colors 'b' and 'l'. Note that if the set E_b of b-edges of a connected W-graph G is given, then there is only one partition $V(G) = A \cup B$ such that A and B independently match each other, i.e., $G = G(A, B)$ and E_b is the set of b-edges with respect to this partition.

Definition 8 *Let $H = H(A, B)$ be a W-graph with parts A and B. The graph H is called a* semi-induced subgraph *of a graph G if H is a subgraph of G, and in the graph G the sets A and B independently match each other.*

In other words, let A and B independently match each other in G and let P be the perfect matching between A and B in $\langle A \cup B \rangle$. If $E_1 \subseteq E\langle A \rangle$ and $E_2 \subseteq E\langle B \rangle$, then the graph H having $V(H) = A \cup B$ and $E(H) = E_1 \cup E_2 \cup P$ is a semi-induced subgraph of G. Thus, any semi-induced subgraph of a graph is a W-graph, and if H is not a W-graph, then G cannot contain H as a semi-induced subgraph.

Definition 9 *A graph G is called a* bl-graph *if a partition of the set $E(G)$ into the set of b-edges (bold) and l-edges (thin) is given, provided that the set of b-edges forms a matching in G. If the b-edges form a perfect matching, then G is called a* perfect bl-graph. *For example, any W-graph is a perfect bl-graph. An* even (odd) bl-graph *has the even (odd) number of b-edges.*

Definition 10 *A simple bl-chain P is called* alternating *if for any two consecutive edges of P one of them is a b-edge and another is an l-edge. The alternating simple chain P is called a* b-chain (l-chain) *if the end edges of P are b-edges (l-edges). Clearly, b-chains and l-chains always have even order. If we identify the end vertices u_1 and u_{2n} in the l-chain $(u_1, u_2, ..., u_{2n})$, where $n \geq 2$, then we obtain the simple cycle $(u_1, u_2, ..., u_{2n-1})$ which is called an* l-cycle starting with u_1.

Definition 11 *For a perfect bl-graph G we define the* operation of W-reducibility *as follows. Each vertex $u \in V(G)$ is labeled by $c(u) \in \{A, B\}$. Further, each edge $e = vw \in E(G)$ is replaced by an alternating bl-chain P_e with end vertices v, w in accordance with the next rule:*

- *If e is an l-edge and $c(v) = c(w)$, then P_e is an even l-chain.*
- *If e is an l-edge and $c(v) \neq c(w)$, then P_e is an odd l-chain.*
- *If e is a b-edge and $c(v) = c(w)$, then P_e is an even b-chain.*
- *If e is a b-edge and $c(v) \neq c(w)$, then P_e is an odd b-chain.*

Definition 12 *The* prism Pr_n $(n \geq 3)$ *consists of two disjoint cycles*

$$C_1 = (u_1, u_2, ..., u_n), \quad C_2 = (v_1, v_2, ..., v_n),$$

and the remaining edges are of the form $u_i v_i$, $1 \leq i \leq n$. The prism Pr_1 *is two loops connected by the edge $u_1 v_1$, this is the only case where loops are permitted. If the prism Pr_n is considered as a perfect bl-graph, then its set of b-edges is $\{u_i v_i : 1 \leq i \leq n\}$.*

Definition 13 *The* Möbius ladder Ml_n *is constructed from the cycle $C = (u_1, u_2, ..., u_{2n})$ by adding the edges $u_i u_{n+i}$ $(1 \leq i \leq n)$ joining each pair of opposite vertices of C. If the Möbius ladder Ml_n is considered as a perfect bl-graph, then its set of b-edges is $\{u_i u_{n+i} : 1 \leq i \leq n\}$.*

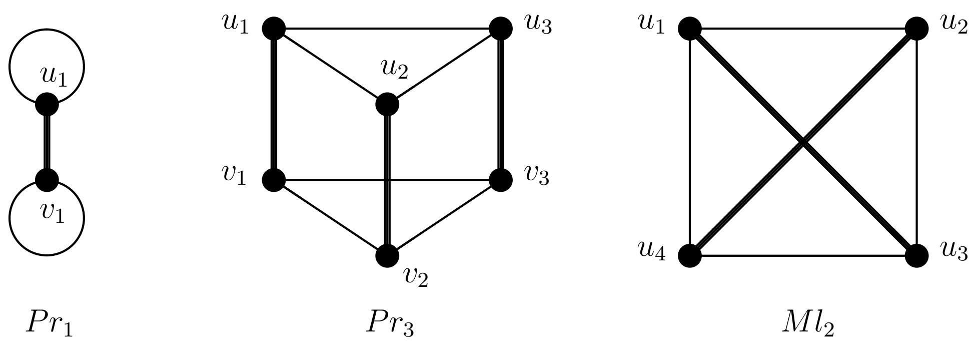

FIGURE 1. Odd prisms Pr_1, Pr_3 and even Möbius ladder Ml_2.

The odd prisms Pr_1 and Pr_3 and the even Möbius ladder Ml_2 are shown in Fig. 1. The odd prisms and the even Möbius ladders play a key role in the definition of basis graphs.

Definition 14 *A graph G without loops is called a* basis *if it can be obtained from the odd prism Pr_{2n+1} $(n \geq 0)$ or the even Möbius ladder Ml_{2m} $(m \geq 1)$ by the operation of W-reducibility.*

We will prove later that a basis graph is a W-graph whose perfect matching between the parts consists of b-edges determined by the operation of W-reducibility. A basis graph G cannot have loops. Hence, if G is obtained from Pr_1, then every loop (l-edge) of Pr_1 must be replaced in accordance with Definition 11 by an alternating even l-cycle (l-chain with equal end vertices) having at least two b-edges.

3 Characterization of Γ-Perfect Graphs

The following theorem gives a characterization of upper domination perfect graphs in terms of forbidden semi-induced subgraphs.

Theorem 1 *A graph G is a Γ-perfect graph if and only if G does not contain any basis graph as a semi-induced subgraph.*

Proof: The proof of Theorem 1 is based on 11 lemmas.

Lemma 1 *If G is a W-graph of order $2k$, then*

$$\Gamma(G) = k \geq \beta(G).$$

Proof: Any independent set of G contains at most one vertex of each b-edges, and hence $\beta(G) \leq k$. Since A is a minimal dominating set, we have $\Gamma(G) \geq k$. Let us prove that $\Gamma(G) \leq k$. Let D be a minimal dominating set of G of cardinality $\Gamma(G)$. If $\deg_{\langle D \rangle} d > 0$ for $d \in D$, then there is a vertex $f \in V(G) - D$ such that $N(f) \cap D = \{d\}$. If $\deg_{\langle D \rangle} d = 0$ for $d \in D$, then there is a vertex f such that df is a b-edge. Obviously $f \in V(G) - D$. Thus, for each vertex $d \in D$ we can indicate a vertex f from $V(G) - D$ and evidently that different vertices of D result in different vertices of $V(G) - D$, i.e., $|D| \leq |V(G) - D|$. We have $\Gamma(G) = |D| \leq k$. ∎

Definition 15 *A W-graph G of order $2k$ is called* strong *if*

$$\beta(G) < k.$$

Lemma 2 *A graph G is Γ-perfect if and only if G does not contain any strong graph as an induced subgraph.*

Proof: The necessity follows from the fact that for a strong graph H, $\beta(H) < \frac{1}{2}|H|$, while $\Gamma(H) = \frac{1}{2}|H|$ by Lemma 1. To prove the sufficiency, let G' be an arbitrary induced subgraph of G, and let D be a minimal dominating set in G' of cardinality $\Gamma(G')$. Denote by J the set of all isolated vertices in $\langle D \rangle$, i.e. $J = \{v \in D : \deg_{\langle D \rangle} v = 0\}$, and let $A = D - J$. Since D is a minimal dominating set, it follows that for each vertex $a \in A$ there is a vertex $b \notin D$ such that $N(b) \cap D = \{a\}$. Taking such a vertex b for each $a \in A$, we define B as the union of these vertices. The graph $H = \langle A \cup B \rangle$ is obviously a W-graph with parts A and B, and $|A| = |B| = k$. Since H cannot be strong, we have $\beta(H) \geq k$. Let I be an independent set of H of cardinality k. It is evident that the set $I \cup J$ is independent in G'. Consequently,

$$\beta(G') \geq |I| + |J| = |A| + |J| = |D| = \Gamma(G').$$

Since $\beta(G') \leq \Gamma(G')$, we have $\beta(G') = \Gamma(G')$. Thus, the graph G is Γ-perfect. ∎

Definition 16 *A connected strong graph $G(A, B)$ of order $2k$ is called* critical *if $\delta(G) \geq 2$ and for any l-edge $e \in E(G)$,*

$$\beta(G - e) = k.$$

Lemma 3 *Any strong graph G with parts A and B contains a critical subgraph G^* with parts $A^* \subseteq A$ and $B^* \subseteq B$.*

Proof: Let E' be the maximum set of l-edges in G such that $\beta(G') < k$, where $V(G') = V(G)$ and $E(G') = E(G) - E'$. Since E' is maximum and deleting all l-edges from G produces the graph with independence number k, we obtain $\beta(G' - e) = k$ for any l-edge $e \in E(G')$. Suppose that G' contains a vertex u of degree 1, and denote by uv the b-edge incident to u. Let us show that $\deg_{G'} v = 1$. Suppose to the contrary that there is an l-edge vw in G'. Since $\beta(G' - vw) = k$, there is an independent set I in $G' - vw$ of cardinality k. We have $v, w \in I$, for otherwise I is independent in G', contrary to the fact that $\beta(G') < k$. Now the set $I' = (I - \{v\}) \cup \{u\}$ is independent in G' and $|I'| = |I| = k$, a contradiction again. Consequently, $\deg_{G'} v = 1$. Thus, if $\deg_{G'} u = 1$, then the b-edge incident to u is an isolated edge in G'.

Consider now the connected component $G^*(A^*, B^*)$ of the graph G' such that $A^* \subseteq A$, $B^* \subseteq B$ and $\beta(G^*) < k^*$, where $k^* = |A^*| = |B^*|$. Such a component does exist, for otherwise $\beta(H) = \frac{1}{2}|H|$ for each connected component H of G' and hence $\beta(G') = \frac{1}{2}|G'| = k$, a contradiction. We see that G^* is a connected strong graph of order $2k^*$. If $\delta(G^*) = 1$, then G^* is an isolated b-edge in G' and so $\beta(G^*) = k^* = 1$, a contradiction. Hence $\delta(G^*) \geq 2$. If there exists an l-edge e in G^* such that $\beta(G^* - e) < k^*$, then obviously $\beta(G' - e) < k$, contrary to the maximality of E'. Thus, $\beta(G^* - e) = k^*$ for any l-edge $e \in E(G^*)$. We conclude that G^* is a critical graph. ∎

Lemma 4 *A graph G is Γ-perfect if and only if G contains no critical graph as a semi-induced subgraph.*

Proof: Let G be a Γ-perfect graph and suppose that G contains a critical graph $H(A, B)$ as a semi-induced subgraph. We have $\beta(H) < k$, where $k = |A| = |B|$. Consider in the graph G the induced subgraph $F = \langle A \cup B \rangle$. This graph is obtained from H by adding some edges in the parts A, B. Therefore, $\beta(F) < k$ and F is a W-graph, i.e., F is a strong graph. This is a contradiction, since, by Lemma 2, the graph G does not contain any strong graph as an induced subgraph.

Now let G contain no critical graph as a semi-induced subgraph, and suppose that G is not Γ-perfect. By Lemma 2, the graph G contains a strong graph H as an induced subgraph. Now, by Lemma 3, $H(A, B)$ contains a critical subgraph $H^*(A^*, B^*)$ such that $A^* \subseteq A$ and $B^* \subseteq B$, i.e., H^* is a semi-induced subgraph of H. Therefore, the critical graph H^* is a semi-induced subgraph of G, a contradiction. ∎

In the remaining part of the proof we give a description of the class of critical graphs. In fact we prove that a graph is critical if and only if it is a basis. This result together with Lemma 4 will provide the characterization of Γ-perfect graphs.

Lemma 5 *If G' is obtained from a perfect bl-graph G by the operation of W-reducibility, then G' is a W-graph whose perfect matching between the parts consists of b-edges determined by the operation of W-reducibility.*

Proof: Let G' be obtained from a perfect bl-graph G by the operation of W-reducibility. Note that the graph G may have loops only if $G = Pr_1$. In that case the loops (l-edges) of

G are replaced in accordance with Definition 11 by alternating even l-cycles. If $u \in V(G')$ is an old vertex, i.e. $u \in V(G)$, then u is labeled by $c(u)$ in G'. If $u \in V(G')$ is a new vertex, then u is a non-end vertex of some chain P_e. We label all vertices from $V(G') - V(G)$ by the following inductive rule. If $e = uv$ is an edge of G' such that u has a label but v has no label yet, then we put:

- $c(v) = A$ if $c(u) = A$ and e is an l-edge.
- $c(v) = B$ if $c(u) = A$ and e is a b-edge.
- $c(v) = A$ if $c(u) = B$ and e is a b-edge.
- $c(v) = B$ if $c(u) = B$ and e is an l-edge.

Now, the vertices of G' with label A form the part A, the vertices with label B form the part B, and the set of b-edges of G' forms a perfect matching between A and B, i.e., the sets A, B independently match each other in G'. Thus, the graph G' is a W-graph. ∎

Lemma 6 *Let G be a perfect bl-graph of order $2k$ and let $C = (u_1, u_2, ..., u_{2n+1})$ be an l-cycle in G starting with u_1. If $\beta(G) = k$, then the vertex u_1 belongs to no maximum independent set of G.*

Proof: By definition, the edge $u_{2i}u_{2i+1}$ is a b-edge for any i, $1 \leq i \leq n$. Suppose that there is a maximum independent set I containing u_1. Since $\beta(G) = k$, the set I contains exactly one vertex of each b-edge. We have $u_1 \in I$ and hence $u_2 \notin I$. Therefore, $u_3 \in I$. If we continue this process, we finally arrive at $u_{2n+1} \in I$. This is a contradiction, since the set I contains two adjacent vertices u_1 and u_{2n+1}. ∎

Definition 17 *A perfect bl-graph G is called a* semi-basis *if G consists of two l-cycles C and C' starting with u and u' ($u \neq u'$), respectively, and also of a b-chain P connecting u and u'. Note that C, C' and P do not necessarily contain different vertices. However, any of the graphs C, C' or P has no self-intersections, since it is simple.*

Lemma 7 *If a perfect bl-graph G of order $2k$ contains a semi-basis subgraph, then*

$$\beta(G) < k.$$

Proof: Suppose to the contrary that G has an independent set I of cardinality k. Then, obviously, $\beta(G) = k$. By Lemma 6, the starting vertices u, u' of the l-cycles C, C' do not belong to the set I. Let $P = (u_1, u_2, ..., u_{2m})$ be a b-chain connecting $u = u_1$ and $u' = u_{2m}$. The edges $u_{2i-1}u_{2i}$ ($1 \leq i \leq m$) are b-edges and the set I contains exactly one vertex of each b-edge, since $\beta(G) = k$. The vertex $u = u_1$ does not belong to I, and so $u_2 \in I$. Hence $u_3 \notin I$ and $u_4 \in I$. Going on in the same way, we obtain $u_{2i} \in I$ for all i, $1 \leq i \leq m$. This is a contradiction, since the vertex $u' = u_{2m}$ does not belong to the set I. ∎

Lemma 8 *A critical graph G is a semi-basis. The graph $G - e$ does not contain a semi-basis subgraph for any l-edge $e \in E(G)$.*

Proof: Let v be an arbitrary vertex of a critical graph $G(A, B)$, say $v \in A$. Put $X_0 = \{v\}$ and $X'_0 = N(v) \cap A$. For $i \geq 0$, we define the sets X_{i+1} and X'_{i+1} as follows:

$$X_{i+1} = \{x \in V(G) : xy \text{ is a b-edge}, y \in X'_i\},$$

$$X'_{i+1} = N(X_{i+1}) - (\cup_{j=0}^{i} X'_j \cup \{v'\}),$$

where $v' \in B$ and vv' is a b-edge. The construction of the sequence

$$X_0, X'_0, X_1, X'_1, ..., X_n, X'_n$$

is finished for minimal n such that $X'_n = \emptyset$. Clearly, the above sets are pairwise disjoint. Put

$$X = \cup_{j=0}^{n} X_j,$$

$$X' = \cup_{j=0}^{n-1} X'_j \cup \{v'\}.$$

Let us show that the set X is not independent. The graph G is critical, and so $\delta(G) \geq 2$. Hence there is a vertex $w \in A$ adjacent to v. Moreover, $\beta(G - vw) = k = |A| = |B|$. Let I be an independent set of $G - vw$ of cardinality k. Since I is not independent in G, we have $v, w \in I$. Put

$$\begin{aligned} A_1 &= A - (X \cup X'), \\ B_1 &= B - (X \cup X'), \\ I_1 &= A_1 \cap I, \\ I_2 &= B_1 \cap I. \end{aligned}$$

By the definitions, no vertex of X is adjacent to a vertex of $A_1 \cup B_1$. The set $I' = X \cup I_1 \cup I_2$ has cardinality k, and hence I' is not independent in G. On the other hand, $I_1 \cup I_2$ is independent in G and there is no edge between $I_1 \cup I_2$ and X in G. We conclude that X is not independent, and hence $x_s \in X_s$ is adjacent to $y_t \in X_t$. Clearly, s and t have the same parity. If $s < t$, then we have a contradiction, since y_t must belong to X'_s but $X'_s \cap X_t = \emptyset$. Thus, $s = t$, i.e., $x_s \in X_s$ is adjacent to $y_s \in X_s$. Now we construct two alternating simple chains. Put

$$P_1 = (x_s, x'_{s-1}, x_{s-1}, x'_{s-2}, ..., x'_0, x_0 = v),$$

$$P_2 = (y_s, y'_{s-1}, y_{s-1}, y'_{s-2}, ..., y'_0, y_0 = v),$$

where $x_i, y_i \in X_i$ and $x'_i, y'_i \in X'_i$ $(0 \leq i \leq s)$. Let z be the first common vertex of P_1 and P_2 if we go from x_s to v (possibly, $z = v$). Obviously, $z \in X$. Now, the edge $x_s y_s$, the (x_s, z)-subchain of P_1 and the (y_s, z)-subchain of P_2 form the l-cycle C starting with z. If $z \neq v$, then the (z, v)-subchain of P_1 is an alternating (z, v)-chain in which z is incident to a b-edge and v is incident to an l-edge.

In fact we proved the following lemma.

Lemma 9 *For any vertex v of a critical graph G, there exists an l-cycle C starting with z and such that if $v \neq z$, then there is an alternating (v, z)-chain in which v is incident to an l-edge, and z is incident to a b-edge, and moreover, z is the only common vertex of this chain and C.*

We go on with the proof of Lemma 8. Denote by zz_1 the b-edge incident to the vertex z, and apply Lemma 9 to the vertex z_1. Let C' be the l-cycle starting with z'. If $z_1 \neq z'$, let P be the alternating (z', z_1)-chain in which z_1 is incident to an l-edge and z' is incident to a b-edge. If $z_1 = z'$, then put $P = \emptyset$. Let $P^+ = P \cup z_1 z$, thus P^+ is the b-chain connecting the starting vertices z and z' of the l-cycles C and C'. The union of the cycles C, C' and the chain P^+ produces a semi-basis subgraph G' of the graph G. The semi-basis graph G' is shown in Fig. 2 provided that it has no self-intersections.

Let e be an l-edge of the graph G and suppose that the graph $G - e$ contains a semi-basis subgraph. By Lemma 7, $\beta(G - e) < k$. On the other hand, G is critical, and so $\beta(G - e) = k$, a contradiction. Thus, the graph $G - e$ does not contain a semi-basis subgraph for any l-edge $e \in E(G)$. Therefore, the semi-basis subgraph G' of G contains all l-edges of G. Since G is critical, we have $\delta(G) \geq 2$. Hence $V(G') = V(G)$. Taking into account that any semi-basis graph is a perfect bl-graph, we conclude that G' must contain all b-edges of G. Thus, $G' = G$. The proof of Lemma 8 is complete. ∎

Remark 1 *The proof of Lemma 8 implies that the cycle C' and the chain $P - \{z_1\}$ may intersect the set $V(C) - \{z\}$ in the critical graph G, all other intersections are impossible.*

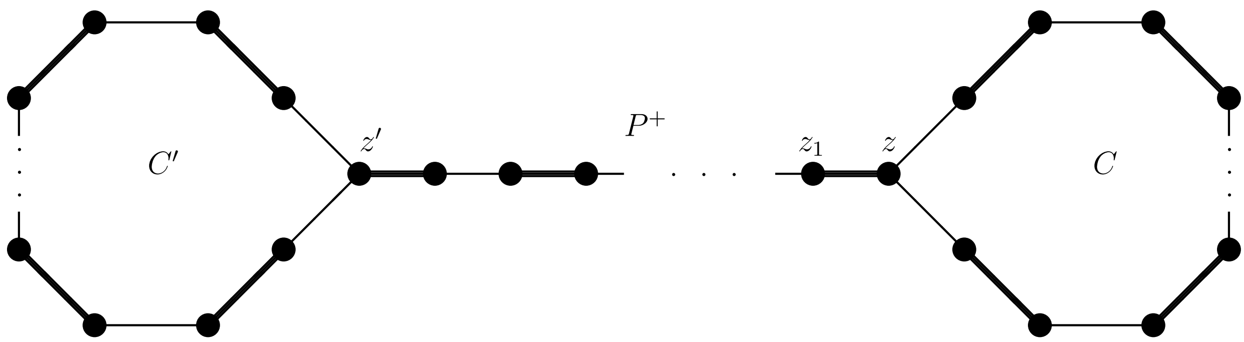

FIGURE 2. Semi-basis graph without self-intersections.

Lemma 10 *Any critical graph is a basis.*

Proof: Let G be a critical graph. By Lemma 8, the graph G is a semi-basis. Employing the notation used in Lemma 8 and taking into account Remark 1, we consider all possible intersections of $V(C') \cup (V(P) - \{z_1\})$ and $V(C) - \{z\}$. Suppose that the intersection of these sets is empty. The graph G is a W-graph, since G is critical. Hence the cycles C and C' have the even number of b-edges. Therefore, G can be obtained from Pr_1 by the operation of W-reducibility, i.e., G is a basis.

Suppose that u is a common vertex of $C' \cup P$ and C. Let Q be a maximal common subchain of $C' \cup P$ and C such that Q contains u. Since every vertex in G is incident to exactly one b-edge, it follows that Q is a b-chain. Maximal common b-chains will be called *intersection intervals*. Obviously, $\deg_G v = 2$ for any non-end vertex v of any intersection interval.

Let us show that the set $V(P) - \{z_1\}$ does not intersect the set $V(C) - \{z\}$. Suppose to the contrary that $u \in (V(P) - \{z_1\}) \cap (V(C) - \{z\})$ and u is the nearest vertex to the vertex z in the chain P^+. Denote by (u, u') the corresponding intersection interval. Let L be the (z, u)-subchain of C such that $u' \in L$, and let $y \in L$ be adjacent to z. Thus, zy is an l-edge. The (z, u)-subchain of P^+ and the (u, z)-subchain of C not containing u' form the l-cycle C'' starting with u. The (u, z')-subchain P' of the chain P^+ connects C'' with the l-cycle C' starting with z'. Thus, for the l-edge zy the graph $G - zy$ contains a semi-basis subgraph formed by C', C'' and P', contrary to Lemma 8. Therefore, $V(P) \cap V(C) = \emptyset$.

Now consider possible intersections of $C' - \{z'\}$ and $C - \{z\}$. Passing round the cycle C' from the vertex z', denote all intersection intervals by

$$(u_1, u_1'), (u_2, u_2'), ..., (u_t, u_t'),$$

where u_i, u_i' $(1 \leq i \leq t)$ are end vertices of the intervals. Since $C' \cap C \neq \emptyset$, we have $t \geq 1$. In what follows it is always supposed that we pass round the cycle C in the direction from u_1' to u_1. We will prove that, passing round the cycle C in this direction, the end vertices of the above intersection intervals are arranged in the following sequence:

$$u_1', u_1, u_2', u_2, ..., u_t', u_t, \tag{1}$$

and moreover the vertex z belongs to the (u_t, u_1')-subchain of C. These statements hold for $t = 1$, and so we may assume that $t \geq 2$.

Suppose that passing round C we arrive to the vertex u_1' from u_s $(s > 1)$, i.e. the (u_s, u_1')-subchain R of C contains no end vertices of any intersection interval excepting u_s and u_1', and the vertex z does not belong to the chain R. We see that R is an l-chain. Hence the chain R and the (u_s, u_1')-subchain of C' containing z' form the l-cycle C'' starting with z'. Let $e = vu_1'$ be the l-edge of C'. Obviously, $e \notin C''$ and $e \notin C$, since (u_1', u_1) is a maximal common b-chain of C and C'. Thus, $G - e$ contains a semi-basis subgraph consisting of C, C'' and P^+, contrary to Lemma 8. Now suppose that passing round C we arrive to u_1' from u_s', and z does not belong to the (u_s', u_1')-subchain R of C. Therefore, R is an l-chain. The chain R and the (u_1', u_s')-subchain of C' not containing z' form the l-cycle C'' starting with u_1'. Let $e = vu_s'$ be the l-edge of C'. Obviously, $e \notin C''$ and $e \notin C$. Thus, $G - e$ contains a semi-basis subgraph consisting of C, C'', P^+ and the (z', u_1')-subchain of C' not containing u_s', contrary to Lemma 8. Therefore, passing round C we arrive to the vertex u_1' from z, and the (z, u_1')-subchain of C contains no end vertex of any intersection interval excepting u_1'.

Suppose now that passing round C we arrive to the vertex u_r and the next end vertex of the intersection intervals is u_s. It is evident that we arrived to the vertex u_r from the vertex u_r'. Since $u_s \neq u_1'$, the (u_r, u_s)-subchain of C does not contain z and hence it is an l-chain. Let $s > r$. Let us define the cycle C'' consisting of the (u_r, u_s)-subchain of C and the (u_r, u_s)-subchain of C' containing u_r'. The cycle C'' is an l-cycle starting with u_s. Let P' be the (u_s, z')-subchain of C' containing u_s'. Thus, $P' \cup P^+$ is a b-chain from u_s to z. Let $e = z'w$ be the l-edge of C' such that $e \notin P'$. Since $z' \notin C$, we have $e \notin C$. Also, $e \notin C''$. We conclude that the graph $G - e$ contains a semi-basis subgraph consisting of C, C'' and $P' \cup P^+$, contrary to Lemma 8. Now let $s < r$. Let us define the cycle C'' consisting of the (u_r, u_s)-subchain of C and the (u_s, u_r)-subchain of C' containing u_s'. The cycle C'' is an l-cycle starting with u_r. Let P' be the (u_r, z')-subchain of C' containing u_r'. Thus,

$P' \cup P^+$ is a b-chain from u_r to z. Let $e = z'w$ be the l-edge of C' such that $e \notin P'$. We have, $e \notin C$ and $e \notin C'''$. We conclude that the graph $G - e$ contains a semi-basis subgraph consisting of C, C'' and $P' \cup P^+$, contrary to Lemma 8. Thus, if we arrive to the vertex u_r passing round C, then the next end vertex of the intersection intervals must be u'_s, i.e., passing round the cycle C the end vertices of the intersection intervals are arranged in the following sequence:

$$u'_1, u_1, u'_{f(2)}, u_{f(2)}, ..., u'_{f(t)}, u_{f(t)},$$

where $f : \{2, 3, .., t\} \to \{2, 3, ..., t\}$ is a bijection. Clearly, z belongs to the $(u_{f(t)}, u'_1)$-subchain of C. Assume that there is $j \in \{2, 3, ..., t\}$ such that $f(j-1) > f(j)$. Denote $r = f(j-1)$ and $s = f(j)$. Thus, $r > s$. Let L be the (u_r, u'_s)-subchain of C. We see that $z \notin L$ and hence L is an l-chain. Let L' be the (u'_s, u_r)-subchain of C' not containing z'. Obviously, L' is an l-chain. Replace the chain L' in C' by the chain L and denote the resulting cycle by C'''. The cycle C''' is an l-cycle starting with z'. Let $e = u_r w$ be the l-edge of L'. It is evident that $e \notin C'''$ and $e \notin C$. We deduce that the graph $G - e$ contains a semi-basis subgraph consisting of C, C''' and P^+, contrary to Lemma 8. Consequently, the end vertices of intersection intervals while passing round C are arranged in accordance with (1).

The graph $G(A, B)$ is a W-graph, since G is critical. We label all vertices of G as follows. Put $c(v) = A$ if $v \in A$, and $c(v) = B$ if $v \in B$. Denote $u_0 = z$ and $u'_0 = z'$. Furthermore, we construct the graph G^* by the following rule. Let

$$\mathcal{P}_b = \{(u_i, u'_i) : 0 \le i \le t\}$$

be the set of b-chains of G consisting of the chain P^+ and the intersection intervals. Replace each chain (u_i, u'_i) from $\mathcal{P}_b$ by the b-edge $u_i u'_i$. Note that the chain (u_i, u'_i) is an even b-chain if u_i and u'_i have the same label, and this chain is an odd b-chain otherwise. Now let

$$\mathcal{P}_l = \{(u_t, u_0), (u'_t, u'_0), (u_{i-1}, u'_i), (u'_{i-1}, u_i) : 1 \le i \le t\}$$

be the set of l-chains of G. Replace each chain L from $\mathcal{P}_l$ by the l-edge connecting the end vertices of L. Note that L is an even l-chain if its end vertices have the same label, and L is an odd l-chain otherwise. The resulting graph G^* is the odd prism Pr_{t+1} whenever $t \ge 1$ is even, and G^* is the even Möbius ladder Ml_{t+1} whenever $t \ge 1$ is odd. Moreover, using the mapping $c : V(G^*) \to \{A, B\}$ constructed above, the graph G is obtained from G^* by the operation of W-reducibility. Therefore, G is a basis graph. The proof of Lemma 10 is complete. ∎

Lemma 11 *All basis graphs are critical.*

Proof: Let G be a basis graph, i.e., G is obtained from Pr_{2n+1} $(n \ge 0)$ or Ml_{2n} $(n \ge 1)$ by the operation of W-reducibility. By Lemma 5, G is a W-graph, $G = G(A, B)$. Obviously, $\delta(G) \ge 2$ and G is a connected graph. Let us show that $\beta(G) < k$. By Lemma 7, it is sufficient to find a semi-basis subgraph in G. If G is obtained from Pr_1, then G is evidently a semi-basis subgraph. Now let G be obtained from Pr_{2n+1} or Ml_{2n}, $n \ge 1$. Since the operation of W-reducibility preserves semi-basis subgraphs, it is sufficient to find such a graph in Pr_{2n+1} or Ml_{2n}, $n \ge 1$. In fact we will show that both Pr_{2n+1} and Ml_{2n} are semi-basis graphs.

Recall that in Pr_{2n+1}, the cycles $(u_1, u_2, ..., u_{2n+1})$ and $(v_1, v_2, ..., v_{2n+1})$ consist of l-edges, and $\{u_i v_i : 1 \le i \le 2n+1\}$ is the set of b-edges. Define two cycles as follows:

$$C = (u_1, u_2, v_2, v_3, u_3, u_4, ..., u_{2n}, v_{2n}, v_{2n+1}, u_{2n+1}, u_1),$$

and

$$C' = (v_1, v_2, u_2, u_3, v_3, v_4, ..., v_{2n}, u_{2n}, u_{2n+1}, v_{2n+1}, v_1).$$

The cycle C starting with u_1 and the cycle C' starting with v_1 are l-cycles connected by the b-chain $(u_1, v_1) = u_1 v_1$, i.e., Pr_{2n+1} is a semi-basis graph for $n \ge 1$.

Consider now the Möbius ladder Ml_{2n}. Recall that the cycle $(u_1, u_2, ..., u_{4n})$ in Ml_{2n} consists of l-edges, and $\{u_i, u_{2n+i} : 1 \le i \le 2n\}$ is the set of b-edges of Ml_{2n}. Define two cycles as follows:

$$C = (u_1, u_2, u_{2n+2}, u_{2n+3}, u_3, u_4, u_{2n+4}, u_{2n+5}, ..., u_{2n}, u_{4n}, u_1),$$

and

$$C' = (u_{2n+1}, u_{2n+2}, u_2, u_3, u_{2n+3}, u_{2n+4}, u_4, u_5, ..., u_{4n}, u_{2n}, u_{2n+1}).$$

The cycle C starting with u_1 and the cycle C' starting with u_{2n+1} are l-cycles connected by the b-chain $(u_1, u_{2n+1}) = u_1 u_{2n+1}$, i.e., Ml_{2n} is a semi-basis graph.

Thus, $\beta(G) < k$. It remains to prove that $\beta(G - e) = k$ for each l-edge $e \in E(G)$. Let G be obtained from Pr_1 by the operation of W-reducibility. Obviously for any l-edge $e \in E(G)$, the graph $G - e$ contains 1 or 2 vertices of degree 1. Starting with a vertex (vertices) of degree 1, it is easily to construct the desired independent set of cardinality k.

Now let G be obtained from $H = \{Pr_{2n+1}, Ml_{2n} : n \ge 1\}$ by the operation of W-reducibility, i.e., b-edges are replaced by alternating b-chains and l-edges are replaced by alternating l-chains. There are two cases to consider.

Case 1. The l-edge e belongs to an l-chain P_f, where f is an l-edge of H. If $H = Pr_{2n+1}$, then without loss of generality we may suppose that $f = u_1 u_{2n+1}$. Put

$$I = \{u_1, u_{2i+1}, v_{2i} : 1 \le i \le n\}.$$

If $H = Ml_{2n}$, then we may assume that $f = u_1 u_2$. In that case put

$$I = \{u_1, u_{2i}, u_{2n}, u_{2n+2i+1} : 1 \le i \le n-1\}.$$

The set I is an independent set of $H - f$ of cardinality $\frac{1}{2}|H|$. Now it is not difficult to construct an independent set of $G - e$ of cardinality $k = \frac{1}{2}|G|$. Indeed, let $uv \in E(H)$ be a b-edge replaced by a b-chain P. Since $|I| = \frac{1}{2}|H|$, we have $|\{u, v\} \cap I| = 1$, say $u \in I$. From each b-edge of P we add in I one vertex which is nearer to the vertex u in the chain P. If $uv \ne f$ is an l-edge of H replaced by an l-chain P, then $|\{u, v\} \cap I| \le 1$ and we can add vertices in I in the same way as above. Now suppose that $uv = f$, and let $e = xy$. Then, from each b-edge of $P_f - e$ we add in I one vertex which is nearer to the vertices x, y in the chain P_f. The constructed set I' is an independent set of $G - e$. Since I' contains one vertex of each b-edge in $G - e$, we have $|I'| = k$. Consequently, $\beta(G - e) = k$.

Case 2. The l-edge e belongs to a b-chain P_f, where f is a b-edge of H. If $H = Pr_{2n+1}$, then without loss of generality we may suppose that $f = u_1 v_1$. Put

$$I = \{u_{2i}, v_{2i+1} : 1 \le i \le n\}.$$

If $H = Ml_{2n}$, then we may assume that $f = u_1 u_{2n+1}$. In that case put

$$I = \{u_{2i}, u_{2n}, u_{2n+2i+1} : 1 \leq i \leq n-1\}.$$

The set I is an independent set of $H - f$ such that each b-edge of $H - f$ has one vertex in I and $|I| = \frac{1}{2}|H| - 1$. Note also that the end vertices of f do not belong to I. Adding vertices in the set I in the same way as in Case 1, we obtain the set I' such that I' contains one vertex of each b-edge of the graph $G - e$. Therefore, $|I'| = k = \frac{1}{2}|G|$, i.e., $\beta(G - e) = k$. ∎

Thus, Lemmas 10 and 11 imply that a graph is critical if and only if it is a basis. Now the proof of Theorem 1 follows from Lemma 4. ∎

4 Corollaries

In this section we illustrate some applications of the characterization of Γ-perfect graphs in terms of forbidden semi-induced subgraphs. We say that a graph G is *2-homeomorphic to* H if G can be obtained from H by replacing edges of H by chains of even order $2k$, $k \geq 1$. Let the *family* $\mathcal{H}$ consist of graphs 2-homeomorphic to the odd prism Pr_{2n+1} ($n \geq 0$) or the even Möbius ladder Ml_{2m} ($m \geq 1$).

Proposition 1 *If H belongs to $\mathcal{H}$, then $\beta(H) < \frac{1}{2}|H|$.*

Proof: For the odd prism we have $\beta(Pr_{2n+1}) = 2n$, i.e., $\beta(Pr_{2n+1}) < \frac{1}{2}|Pr_{2n+1}|$. For the even Möbius ladder we have $\beta(Ml_{2m}) = 2m - 1$, i.e., $\beta(Ml_{2m}) < \frac{1}{2}|Ml_{2m}|$. Let F' be obtained from a graph F by the single 2-partition of the edge uv, i.e., uv is replaced by the chain $P = (u, x, y, v)$. Let U be a maximum independent set of F'. Obviously, $1 \leq |U \cap P| \leq 2$. If $|U \cap P| = 1$, then $U - P$ is an independent set of F of cardinality $|U| - 1 = \beta(F') - 1$. If $|U \cap P| = 2$, then at least one vertex from $\{u, v\}$ belongs to U, say $u \in U$. Now $U - \{x, y, v\}$ is an independent set of F of cardinality $|U| - 1 = \beta(F') - 1$. In any case, $\beta(F) \geq \beta(F') - 1$. Thus, if $\beta(F) < \frac{1}{2}|F|$, then

$$\beta(F') \leq \beta(F) + 1 < \frac{1}{2}|F| + 1 = \frac{1}{2}|F'|.$$

Since H is obtained from the odd prism or the even Möbius ladder by applying the operation of 2-partition, we conclude that $\beta(H) < \frac{1}{2}|H|$. ∎

In our next theorem, the graphs from the family $\mathcal{H}$ are forbidden as semi-induced subgraphs for a graph to be Γ-perfect. Using the fact that a semi-induced subgraph of a graph is a W-graph, we see that the class of forbidden semi-induced subgraphs of Theorem 2 actually consists of W-graphs from the family $\mathcal{H}$. Note that the class of W-graphs from $\mathcal{H}$ is larger than the class of basis graphs used in Theorem 1. For example, $Ml_4 = Ml_4(\{u_2, u_3, u_6, u_7\}, \{u_1, u_4, u_5, u_8\})$ is a W-graph from $\mathcal{H}$ and hence it is forbidden in Theorem 2. On the other hand, Ml_4 is not a basis graph. Another difference between Theorem 1 and Theorem 2 is that a basis graph has a fixed partition into parts determined by the set of its b-edges, while for a W-graph from $\mathcal{H}$ the partition into parts is not fixed.

Theorem 2 *A graph G is Γ-perfect if and only if G does not contain a semi-induced subgraph 2-homeomorphic to the odd prism Pr_{2n+1} ($n \geq 0$) or the even Möbius ladder Ml_{2m} ($m \geq 1$).*

Proof: Let G be a Γ-perfect graph and let H belong to $\mathcal{H}$. If H is not a W-graph, then H cannot be a semi-induced subgraph of G. Suppose now that $H = H(A, B)$ is a W-graph and H is a semi-induced subgraph of G. By Proposition 1, $\beta(H) < \frac{1}{2}|H|$. Let $H' = \langle A \cup B \rangle$. Evidently, $\beta(H') \leq \beta(H)$ and H' is a W-graph. Therefore, by Lemma 1,

$$\Gamma(H') = \frac{1}{2}|H'| = \frac{1}{2}|H| > \beta(H) \geq \beta(H').$$

Thus, $\Gamma(H') > \beta(H')$. This is a contradiction, since G is a Γ-perfect graph.

Suppose that G does not contain any graph from $\mathcal{H}$ as a semi-induced subgraph. Any basis graph F is obtained from the odd prism or the even Möbius ladder by replacing its edges by alternating chains of even order, and the partition into parts of F is determined by the set of its b-edges. Thus, the graph F is 2-homeomorphic to the odd prism or the even Möbius ladder and F has a fixed partition into parts. For a W-graph H from $\mathcal{H}$, the partition into parts of H is not fixed, and hence we may take any partition $V(H) = A \cup B$ such that A and B independently match each other. Therefore, $F \in \mathcal{H}$ and the graph G does not contain any basis graph as a semi-induced subgraph. The result now follows from Theorem 1. ∎

Jacobson and Peters [22] considered the class of graphs G having $\beta(H) = IR(H)$ for all induced subgraphs H of G. By Theorem A, this class is exactly the class of Γ-perfect graphs.

Corollary 1 (Jacobson and Peters [22]) *A graph G is Γ-perfect if and only if for any vertex subsets $A, B \subset V(G)$ that independently match each other, the graph $\langle A \cup B \rangle$ has an independent set of order $|A|$.*

Proof: Let G be a Γ-perfect graph and A, B independently match each other. The set A is minimal dominating in $F = \langle A \cup B \rangle$. Hence, $\beta(F) = \Gamma(F) \geq |A|$. To prove the sufficiency, suppose that G is not Γ-perfect. By Theorem 2, G contains a semi-induced subgraph $H = H(A, B) \in \mathcal{H}$. By Proposition 1, $\beta(H) < |A|$. Thus, the sets A, B independently match each other in G and $\beta\langle A \cup B \rangle < |A|$, a contradiction. ∎

Now we turn to the problem of characterizing Γ-perfect graphs in terms of forbidden induced subgraphs. A graph G is called *minimal* Γ*-imperfect* if G is not Γ-perfect and $\beta(H) = \Gamma(H)$, for every proper induced subgraph H of G.

Proposition 2 *If G is a minimal Γ-imperfect graph, then G contains a basis graph $F(A, B)$ of order $2k$ as a semi-induced subgraph, $G = G(A, B)$ is a connected W-graph of order $2k$, $\delta(G) \geq 2$, and $\beta(G) = k - 1$.*

Proof: By Theorem 1, G contains a basis graph F as a semi-induced subgraph. Since G is minimal, we have $V(G) = V(F)$. By Lemma 11, F is critical, i.e., $F = F(A, B)$ is a connected W-graph of order $2k$, $\delta(F) \geq 2$ and $\beta(F) < k$. The graph G is obtained from F by adding edges in the parts A, B. Therefore, $G(A, B)$ is a connected W-graph of order $2k$,

$\delta(G) \geq 2$ and $\beta(G) < k$. Let uv be a b-edge of G. The graph $G' = G - \{u, v\}$ is Γ-perfect. Hence, using Lemma 1, $\beta(G') = \Gamma(G') = k - 1$. We obtain $\beta(G) \geq \beta(G') = k - 1$. Thus, $\beta(G) = k - 1$. ∎

By Proposition 2, every minimal Γ-imperfect graph has even order $n \geq 6$. Let μ_n denote the number of nonisomorphic minimal Γ-imperfect graphs of order n. It was proved in [14] that $\mu_6 = 1$ and $\mu_8 = 14$. Using a computer search, we discovered that $\mu_{10} = 228$ and the number μ_{12} considerably exceeds μ_{10}. Therefore, it seems unlikely to obtain an explicit list of all minimal Γ-imperfect graphs, i.e., to provide an induced subgraph characterization of the entire class of Γ-perfect graphs. However, for $K_{1,3}$-free Γ-perfect graphs Theorem 1 enables us to obtain such a characterization.

We define the *family* $\mathcal{S}$ consisting of the following classes $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$. Let $C = C_{4m}$ and $C' = C_{4n}$ $(m, n \geq 1)$ be two cycles, let $uv \in E(C)$ and $xy \in E(C')$, and let $(z_1, z_2, ..., z_{2l})$ $(l \geq 1)$ be a chain. Add the edges uz_1, vz_1 and xz_{2l}, yz_{2l}. The resulting graph belongs to $\mathcal{S}_1$. Now let $(u_1, ..., u_k)$, $(v_1, ..., v_l)$ and $(w_1, ..., w_m)$ be three chains such that $k, l, m \geq 2$ and either $k, l, m \equiv 0 (\mathrm{mod}\, 4)$ or $k, l, m \equiv 2 (\mathrm{mod}\, 4)$. Adding the edges u_1v_1, v_1w_1, w_1u_1 and u_kv_l, v_lw_m, w_mu_k, we obtain a graph of the class $\mathcal{S}_2$. Lastly, let $C = C_{4m}$ and $C' = C_{4n}$ $(m, n \geq 1)$ be two cycles and let $uv \in E(C)$ and $xy \in E(C')$. Add the edges ux, uy, vx, vy. The resulting graph belongs to $\mathcal{S}_3$.

Theorem 3 *A $K_{1,3}$-free graph G is Γ-perfect if and only if G does not contain any member of $\mathcal{S}$ as an induced subgraph.*

Proof: Any graph H from the family $\mathcal{S}$ contains a semi-induced subgraph 2-homeomorphic to Pr_1, Pr_3 or Ml_2. By Theorem 2, H is not Γ-perfect. To prove the sufficiency, let G be a minimal counterexample, i.e., G is a $K_{1,3}$-free graph not containing any member of $\mathcal{S}$ as an induced subgraph, G is not Γ-perfect and G has minimal order. Obviously, G is a minimal Γ-imperfect graph. By Proposition 2, G contains a basis graph $F = F(A, B)$ as a semi-induced subgraph, $G = G(A, B)$ is a connected W-graph of order $2k$, $\delta(G) \geq 2$, and $\beta(G) = k - 1$. If the induced subgraph $\langle A \rangle$ or $\langle B \rangle$ of the graph G contains the induced chain P_3, then G has the induced $K_{1,3}$, a contradiction. Hence both $\langle A \rangle$ and $\langle B \rangle$ are disjoint unions of complete graphs.

Lemma 12 *Let $G(A, B)$ be a minimal Γ-imperfect graph of order $2k$. If $\langle A \rangle$ and $\langle B \rangle$ are disjoint unions of complete graphs, then the following statements hold:*

1. *If k is odd, then $\langle A \rangle \cong \langle B \rangle \cong \frac{k-3}{2} K_2 \cup K_3$ $(k \geq 3)$.*
2. *If k is even, then one of the graphs $\langle A \rangle$, $\langle B \rangle$ is $\frac{k}{2} K_2$ and the other is either $\frac{k-4}{2} K_2 \cup K_4$ $(k \geq 4)$ or $\frac{k-6}{2} K_2 \cup 2K_3$ $(k \geq 6)$.*

Proof: Let $\langle A \rangle$ be a disjoint union of the complete graphs $H_1, ..., H_p$. Since $\delta(G) \geq 2$, we have $|H_i| \geq 2$ for any $i \in \{1, ..., p\}$, and hence $p \leq k/2$. Let I be an independent set in G of cardinality $k - 1 = \beta(G)$. Put $I_A = I \cap A$ and $I_B = I \cap B$. The set I contains at most one vertex of each H_i, and hence $|I_A| \leq p \leq k/2$. Analogously, $|I_B| \leq k/2$. Further, $|I_A| = |I| - |I_B| \geq k - 1 - k/2 = k/2 - 1$. Thus,

$$k/2 - 1 \leq |I_A| \leq k/2. \quad (2)$$

Analogously,

$$k/2 - 1 \le |I_B| \le k/2. \tag{3}$$

Put

$$s = \sum_{i=1}^{p} (|H_i| - 2) \ge 0.$$

We have,

$$k = |A| = \sum_{i=1}^{p} |H_i| = s + 2p \ge s + 2|I_A|.$$

Therefore, using (2),

$$s \le k - 2|I_A| \le 2.$$

Thus,

$$s \in \{0, 1, 2\}.$$

If $k = |A|$ is odd, then s is also odd, since $s = k - 2p$. Hence $s = 1$, $k \ge 3$, and $\langle A \rangle \cong \frac{k-3}{2} K_2 \cup K_3$. Analogously, $\langle B \rangle \cong \frac{k-3}{2} K_2 \cup K_3$.

Now let k be even. Using (2) and (3), we see that one of the sets I_A and I_B has cardinality $k/2 - 1$ and the other has cardinality $k/2$. Without loss of generality, let $|I_A| = k/2 - 1$ and $|I_B| = k/2$. Since $\langle B \rangle$ is a disjoint union of complete graphs and $\delta(G) \ge 2$, we have $\langle B \rangle \cong \frac{k}{2} K_2$. Further, $s = k - 2p$ and k is even. Hence s is even and $s = 0$ or 2. If $s = 0$, then $\langle A \rangle \cong \frac{k}{2} K_2$ and therefore G is a disjoint union of even simple cycles. We obtain $\beta(G) = k$, a contradiction. Thus, $s = 2$. Hence $k \ge 4$ and $\langle A \rangle \cong \frac{k-4}{2} K_2 \cup K_4$ or $k \ge 6$ and $\langle A \rangle \cong \frac{k-6}{2} K_2 \cup 2K_3$. The proof of Lemma 12 is complete. ∎

By Lemma 12, G has either exactly 6 vertices of degree 3 or exactly 4 vertices of degree 4, and all other vertices have degree 2. The basis graph F is a spanning subgraph of G. Therefore, either F has at most 6 vertices of degree 3 and all other vertices have degree 2, or F has at most 4 vertices of degree 3 and 4 and all other vertices have degree 2. Consequently, F is obtained from Pr_1, Pr_3 or Ml_2 (see Fig. 1) by the operation of W-reducibility. By Lemma 5, any l-edge of F belongs to A or B.

Suppose that F is obtained from Pr_1. Let uu_1, u_1u' be l-edges of one l-cycle of F and let vv_1, v_1v' be l-edges of the other l-cycle of F. The vertices u, u', u_1 belong to the same part, and v, v', v_1 belong to the same part. We have, $uu' \in E(G)$ and $vv' \in E(G)$, since G is a $K_{1,3}$-free graph. The restrictions on the degrees of vertices of G imply $G = F \cup \{uu', vv'\}$. Since G is a W-graph, we see that the cycle C of G such that $u, u' \in C$ and $u_1 \notin C$ has length $4m$, and the cycle C' of G such that $v, v' \in C'$ and $v_1 \notin C'$ has length $4n$. Thus, $G \in \mathcal{S}_1$, a contradiction.

Assume that F is obtained from Pr_3. The prism Pr_3 has 6 vertices of degree 3. Hence, $G = F$. Suppose that an l-edge of Pr_3 was replaced by an l-chain having more than 2 vertices. Then F has the induced $K_{1,3}$, a contradiction. Therefore, only b-edges of Pr_3 could be replaced by b-chains to obtain F. These chains must be odd b-chains if $C_1 = (u_1, u_2, u_3)$ and $C_2 = (v_1, v_2, v_3)$ belong to different parts of F, and they must be even b-chains if C_1 and C_2 belong to the same part of F. Any odd b-chain has $4k + 2$ vertices, and any even b-chain has $4m$ vertices. Therefore, $G = F \in \mathcal{S}_2$, a contradiction.

Finally, suppose that F is obtained from Ml_2 by the operation of W-reducibility. It is easy to see that Ml_2 has 4 different labelings of $V(Ml_2)$ by $c(u) \in \{A, B\}$ up to replacing A by B. Hence there are 4 cases to consider.

Case 1: $c(u_1) = c(u_4) = A$ and $c(u_2) = c(u_3) = B$. By the definition of W-reducibility, the l-edges u_1u_2 and u_3u_4 had to be replaced by the odd l-chains $(u_1, v_1, ..., v_k, u_2)$, $k \geq 2$, and $(u_3, w_1, ..., w_m, u_4)$, $m \geq 2$. Each of the vertices v_1, v_k, w_1, w_m is an end vertex of P_3 or P_4 consisting of l-edges and hence belonging to A or B. Since any part of G is a disjoint union of complete graphs, we see that each of the above vertices will have degree at least 3 in G. Thus, G has at least 8 vertices of degree at least 3, a contradiction.

Case 2: $c(u_1) = c(u_3) = A$ and $c(u_2) = c(u_4) = B$. This case is analogous to Case 1, since the l-edges u_1u_2 and u_3u_4 had to be replaced by odd l-chains.

Case 3: $c(u_1) = c(u_2) = c(u_3) = A$ and $c(u_4) = B$. The l-edges u_1u_4 and u_3u_4 had to be replaced by the odd l-chains $(u_1, v_1, ..., v_k, u_4)$, $k \geq 2$, and $(u_3, w_1, .., w_m, u_4)$, $m \geq 2$. Each of the vertices v_1, v_k, w_1, w_m is an end vertex of P_r $(r \geq 3)$ consisting of l-edges. Hence, each of these vertices has degree at least 3 in G. Thus, G has at least 8 vertices of degree at least 3, a contradiction.

Case 4: $c(u_i) = A$, $1 \leq i \leq 4$. If some two l-edges from $\{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ were replaced by even l-chains having at least two b-edges, then we derive a contradiction in the same way as above. Suppose that only one l-edge, say u_1u_2, was replaced by the even l-chain $(u_1, v_1, ..., v_k, u_2)$, $k \geq 4$. Then $\langle v_k, u_2, u_3, u_4, u_1, v_1 \rangle$ is a P_6 in F consisting of l-edges. Therefore, G contains K_6, a contradiction. Thus, only the b-edges u_1u_3 and u_2u_4 of Ml_2 were replaced by even b-chains and $\langle u_1, u_2, u_3, u_4 \rangle$ is a C_4 in F consisting of l-edges. Hence $\langle u_1, u_2, u_3, u_4 \rangle$ is a K_4 in G, and so all other vertices in G must have degree 2, i.e., $G = F \cup \{u_1u_3, u_2u_4\}$. Since even b-chains have $4m$ vertices $(m \geq 1)$, we have $G \in \mathcal{S}_3$. This contradiction completes the proof of Theorem 3. ∎

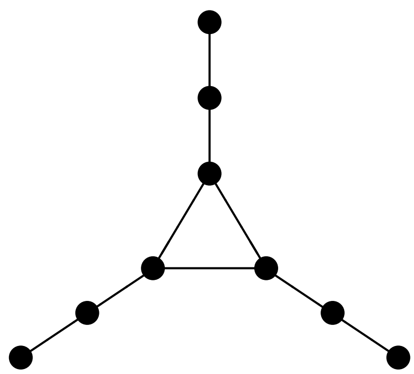

FIGURE 3. Graph H of Corollary 2.

The next result follows directly from Theorem 3 and Theorem A, since each graph from $\mathcal{S}$ contains either C_4 or the graph H of Fig. 3 as an induced subgraph.

Corollary 2 (Jacobson and Peters [23]) *If a graph G does not contain either $K_{1,3}$, C_4 or the graph H of Fig. 3 as an induced subgraph, then G is Γ-perfect and IR-perfect.*

Note in conclusion that using properties of minimal Γ-imperfect graphs stated in Proposition 2, it is not difficult to prove Theorems B, C, D, E, or F from Section 1.

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