Spectral properties of reversible one-dimensional cellular automata

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Abstract

Reversible cellular automata are invertible dynamical systems characterized by discreteness, determinism and local interaction. This article studies the local behavior of reversible one-dimensional cellular automata by means of the spectral properties of their connectivity matrices. We use the transformation from every one-dimensional cellular automaton to another of neighborhood size 2 to generalize the results exposed in this paper. In particular we prove that the connectivity matrices have a single positive eigenvalue equal to 1; based on this result we also prove the main result of this paper: the idempotent behavior of these matrices. This property is an important feature for detecting which one-dimensional cellular automata are reversible. Hence we present a procedure using the eigenvectors of these matrices to find the inverse rule for a given reversible one-dimensional cellular automaton. Finally illustrative examples are provided.

Key words: Cellular automata, spectrum of graphs, idempotent behavior

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1 Introduction

Cellular automata were invented by John von Neumann to prove the existence of self-reproducing systems [17]. The study of reversible automata began with the papers by Edward F. Moore [11] and John Myhill [13] about the existence of the “Garden of Eden” states for a given cellular automaton. Reversibility in one-dimensional cellular automata was straightforward treated by the paper of Gustav A. Hedlund [4], in particular this paper studies the automorphisms of the shift system. The main contribution of this paper is defining in a combinatorial way the local properties that a one-dimensional cellular automaton must hold to be reversible. Further papers discuss the calculation of reversible automata [1], analyze them using graphic tools [14] [9] [10] [16] and presents their deterministic characterization by means of block permutations [6].

In this sense, a fundamental problem is detecting if the local behavior of a cellular automaton holds the properties to be reversible, therefore the analysis of the local behavior is important for understanding the global one. For this reason the goal of this paper is to characterize the local behavior of reversible one-dimensional cellular automata by means of matrix representations to obtain a procedure for detecting the reversible behavior.

The idea is to use a relevant result presented in the paper of Tim Boykett [2] which shows that any one-dimensional automaton can be transformed into another of neighborhood size 2. In this way it is just necessary to study this case to understand the rest. With this transformation a matrix presentation of the local behavior is obtained, where each state has a connectivity matrix which presents how the state is formed.

Experimental observations [15] [12] [8] suggest that such connectivity matrices have a single positive eigenvalue equal to 1 and also that these matrices are idempotent. The last feature is very important because it establishes an important condition for deciding either a one-dimensional cellular automaton is reversible or not. This paper proves that the previous characteristics are fulfilled for all reversible one-dimensional cellular automaton. As an additional result the eigenvectors associated with the single positive eigenvalue 1 are used to find the invertible behavior of a given reversible automaton.

The paper is organized as follows. Section 2 presents the basic concepts and the features of reversible one-dimensional cellular automata, and how these features define the structure of the connectivity matrices. Section 3 proves that these matrices have a single positive eigenvalue equal to 1; with this result and using other ones from graph theory, the idempotent behavior of these matrices is proved. Based on these features, section 4 establishes a procedure for calculating the inverse rule by means of the eigenvectors of the single positive eigenvalue in the connectivity matrices. Section 5 present illustratives example of the previous results and the final section provides the concluding remarks of the paper.

2 Basic concepts of cellular automata

A one-dimensional cellular automaton consists of a one-dimensional array of cells, every cell takes a single value from a finite set K of states. The assignation from states to cells of the array is called a configuration of the automaton.

Let k be the cardinality of K. For $n \in \mathbb{Z}^+$, let K^n be the set of sequences with n states. For some $n \in \mathbb{Z}^+$ a mapping $\varphi : K^n \to K$ is defined. The mapping is applied to all sequence with n cells or neighborhood in a given configuration. Each neighborhood overlaps with its contiguous neighborhoods in $n-1$ cells.

The local mapping yields a new configuration, i.e. it induces a global mapping between configurations of the automaton. For some state $a \in K$, the neighborhoods in K^n which evolve into a are the ancestors of a and they have $n-1$ more cells than a. Thus, for each $m \in \mathbb{Z}^+$, the ancestors of each sequence $w \in K^m$ belong to K^{m+n-1}. This property is important because it allows to transform any automaton of neighborhood size n into another of neighborhood size 2. This transformation is exposed in the paper of Boykett [2] and is useful to generalize results since properties fulfilled in automata of neighborhood size 2 are fulfilled also in all the other cases. We present now a brief description for this transformation.

Let n be the number of cells of a neighborhood in a one-dimensional cellular automaton. For the sequences in K^{n-1}, their ancestors belong to K^{2n-2} and the evolution rule defines a mapping $\varphi : K^{2n-2} \to K^{n-1}$.Take a new set S of states with cardinality equal to k^{n-1}. Then each sequence in K^{n-1} can be associated with a single state in S and the evolution rule also defines a mapping $\tau : S^2 \to S$. This mapping presents the same behavior that the evolution of the sequences in K^{2n-2}, but τ is also an evolution rule of neighborhood size 2. In this way the original automaton is simulated by another, of course, the new automaton has a greater number of states than the original one.

For a cellular automaton with evolution rule $\varphi : K^2 \to K$, the rule is presented by a matrix where its indices are the states in K and the coordinates of each entry represent a whole neighborhood of the automaton. Thus the value of each entry is the evolution of the neighborhood formed by the coordinates of the entry.

2.1 Reversible one-dimensional cellular automata

A cellular automaton is reversible if its global mapping is invertible by another evolution rule. That is, for the original evolution rule $\varphi : K^n \to K$ there is another rule $\varphi^{-1} : K^m \to K$ (possibly $m \neq n$) such that φ^{-1} defines the inverse global mapping of the automaton.

The study of reversible one-dimensional cellular automata is relevant because they represent systems which conserve information through time, besides they offer a model for simulating several physical, chemical and biological reversible systems. In this paper we only study reversible one-dimensional cellular automata with neighborhood size 2 since all the other cases can be transformed to this one.

2.2 Properties of reversible one-dimensional cellular automata

The properties of reversible one-dimensional cellular automata are widely discussed by Gustav A. Hedlund [4]. In his work Hedlund proves two fundamental properties of these systems. A reversible one-dimensional cellular automaton of k states and neighborhood size 2 in both invertible rules holds that:

1. Each finite sequence of states has k ancestors.
2. The ancestors of each finite sequence have L initial states, a common central part and R final states, with $LR = k$.

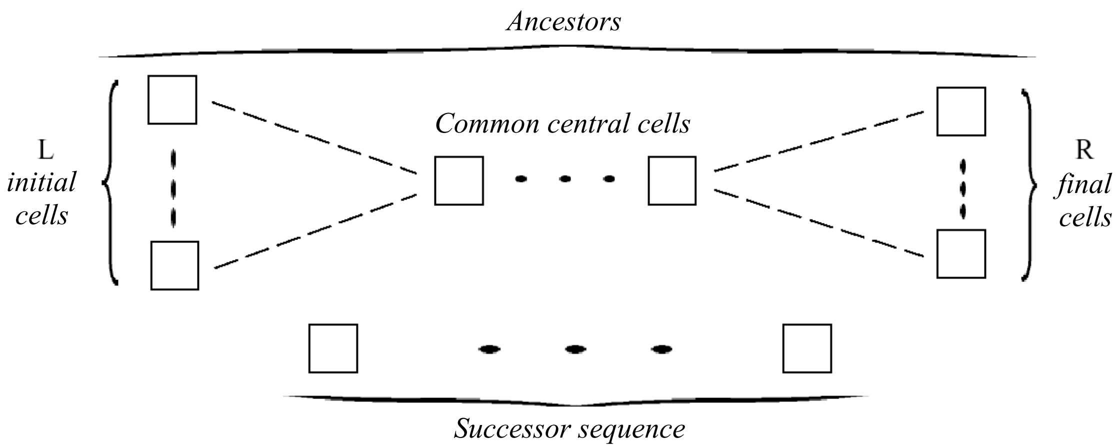

Figure 1: Form of the ancestors in a reversible automaton of neighborhood size 2.

Property 1 is called the uniform multiplicity of ancestors and the values L and R in Property 2 are known as Welch indices. Another relevant result is provided by Masakazu Nasu [14], proving that the ancestors of a given finite sequence have one initial state equal to one and only one final state. In a reversible one-dimensional cellular automaton of neighborhood size 2, for each $s \in K$ a new matrix A_s is obtained from the matrix presentation of φ. In A_s the row and column indices are the states of the automaton. The value of each entry a_{ij} in A_s is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad \varphi(ij) = s \\ 0 & \text{any other case} \end{cases} \tag{1}$$

In this way, A_s is a $0-1$ matrix which presents the neighborhoods evolving into s. A_s is called the connectivity matrix of s. First of all, there are k connectivity matrices (one for each state in K). Based on the papers developed by Hedlund and Nasu, we prove the next result.

Theorem 1. *For each $s \in K$ let A_s be the connectivity matrix associated with s, let a_{ij} be an entry of A_s and let A_s^n be the $n-th$ power of A_s. Then A_s holds the following properties:*

1. *The sum of elements in* A_s^n *is equal to* k.

2. *There is a single positive entry equal to* 1 *in the main diagonal of* A_s^n.

3. *For* $i \neq j$, *if* $a_{ij} = 1$ *then* $a_{ji} = 0$ *in* A_s^n.

Proof.

1. This property is a straightforward result from the uniform multiplicity of ancestors because every state must have the same number of ancestors that all its extensions. Hence the number of ancestors is equal to the average of ancestors per state, in this case $k^2/k = k$.

2. The single diagonal value equal to 1 assures a unique ancestor for the sequence composed by repetitions of s. If $a_{ll} = 1$ then the sequence s^n formed by n repetitions of s has a unique ancestor defined by l^{n+1}.

3. Suppose that A_s^n holds that $a_{ji} \neq 0$ in A_s^n, then the sequence s^n has two ancestors in K^{n+1}, one defined as jvi and another as iwj where $v, w \in K^{n-1}$. But a consequence is that the sequence s^{2n} has two cyclic ancestors ($jviwj$ and $iwjvi$) and the automaton is not reversible which is a contradiction.

□

Every connectivity matrix has a graphical representation, the indices of the matrix are the set of nodes and the entries of the matrix define the arcs between nodes. If an entry a_{ij} is equal to 1 then there is an arc from node i to node j, in opposite case there is no arc between i and j. The graphical representation of a connectivity matrix has k arcs by the property 1 of Theorem 1, and this graph also has a single cycle of one node by the properties 2 and 3 of Theorem 1.

The graphical representation of a connectivity matrix is useful for obtaining a better understanding of its spectral properties by means of theory of graphs. In the next sections we discusse these properties and how they reflect the local behavior of reversible one-dimensional cellular automata.

3 Eigenvalues and idempotent behavior

In this section we use some results in theory of graphs for characterizing the spectrum of the connectivity matrices, let us begin with the definition of liner directed graphs.

Definition 1. *A linear directed graph is a digraph in which the indegree and the outdegree of each node is* 1.

In other words a linear directed graph consists only of cycles. The next theorem shows that the linear directed subgraphs of a given graph define the coefficients of the characteristic polynomial corresponding with the adjacency matrix of the graph (in this case with the connectivity matrix). These coefficients will be useful to know the features of the spectrum of these matrices (i.e. the set of eigenvalues).

Theorem 2. *(Milić, Sachs, Spialter [3]) Let G be a digraph and let A be its connectivity matrix with the following characteristic polynomial:*

$$p(A) = \mid A - I\lambda \mid = (-\lambda)^n + (-\lambda)^{n-1} p_1 + \cdots + p^n$$

Then:

$$p_i = \sum_{B \in \mathcal{B}_{\rangle}} (-1)^{per(B)} \quad (i = 1, 2, \cdots, n)$$

where $\mathcal{B}_i$ is the set of all the linear directed subgraphs B of G with exactly i nodes. $per(B)$ is the number of components of B, i.e. the number of cycles defining B.

Theorem 2 says that each coefficient p_i depends only of the set $\mathcal{B}_{\rangle}$ of linear directed subgraphs of G with exactly i nodes. The contribution from each $B \in \mathcal{B}_{\rangle}$ to p_i is 1 if B has an even number of cycles and -1 if B contains an odd number of cycles.

A formal proof of Theorem 2 is not provided in this paper (it can be consulted in the book by Cvetkovic, Doob and Sachs [3]), but its details related with reversible automata are explained. For a reversible one-dimensional cellular automaton of k states and neighborhood size 2, let A be a connectivity matrix with entries a_{ij} and let G be its associated graph. Take the coefficient p_k from $p(A)$ given by the determinant of A, then p_k has the following form:

$$p_k = \sum_{n!} \pm a_{1\alpha} a_{2\beta} \ldots a_{k\gamma} \tag{2}$$

where the indices of each term $a_{1\alpha} a_{2\beta} \ldots a_{k\gamma}$ are defined by some permutation of k elements (Equation 3).

$$\begin{matrix} \text{First index} \\ \\ \text{Second index} \end{matrix} \begin{pmatrix} 1 & 2 & \cdots & k \\ & & & \\ \alpha & \beta & \cdots & \gamma \end{pmatrix} \tag{3}$$

If some term $a_{1\alpha} a_{2\beta} \cdots a_{k\gamma}$ is different form zero, then there are arcs $(1, \alpha), (2, \beta), \ldots, (k\gamma)$ in G. But these arcs define a set of disjoint cycles, in this way the coefficient p_k is defined by the linear directed subgraph with k nodes in G. For $1 \leq i \leq k$, each coefficient p_i of $p(A)$ depends on the minors or subdeterminants of order i, i.e. p_i depends on the linear directed subgraphs of i nodes in G (Equation 4).

$$p(A) = (-\lambda)^k + (-\lambda)^{k-1} \sum_{i=1}^{k} a_{ii} + (-\lambda)^{k-2} \sum_{i<j}^{k} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} + (-\lambda)^{k-3} \sum_{i<j<m}^{k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} + \cdots + |\, A \,| \quad (4)$$

By Theorem 1 the graph associated with a connectivity matrix has only a single cycle of one node. There are not bigger cycles because they imply multiple ancestors for undefined repetitions of the state represented by the connectivity matrix. Hence a straightforward result from Theorem 2 is the following one:

Corollary 3. *Let A be the connectivity matrix of some state in a reversible one-dimensional cellular automaton of k states and neighborhood size 2, then the following inequality is fulfilled:*

$$p(A) = \lambda^k - \lambda^{k-1}$$

Proof. There is a single cycle of one node in the graphic representation of A, then there is a unique linear directed subgraph of one element. Hence $p_1 = -1$ and $p_i = 0$ for $i = 2 \cdots k$ by Theorem 2. □

Corollary 3 has another important implication for the spectrum of a given connectivity matrix.

Corollary 4. *Let A be the connectivity matrix of some state in a reversible one-dimensional cellular automaton of k states and neighborhood size 2, then the spectrum of A has a single positive eigenvalue $\lambda_A = 1$ and another eigenvalue 0 of multiplicity $k-1$.*

Proof. By Corollary 3 $p(A) = \lambda^k - \lambda^{k-1}$, hence the eigenvalues are calculated as follows:

$$\lambda^k - \lambda^{k-1} = 0 \Rightarrow \lambda^{k-1}(\lambda - 1) = 0 \quad (5)$$

Equation 5 shows a single positive eigenvalue $\lambda_A = 1$ and another eigenvalue $\lambda = 0$ of multiplicity $k-1$. □

Corollary 3 also presents another property which is useful to determine either a one-dimensional cellular automaton is reversible or not, this property is the idempotent behavior of these matrices and it is proved in the next corollary.

Corollary 5. *Let A be the connectivity matrix of some state in a reversible one-dimensional cellular automaton of k states and neighborhood size 2, then A is idempotent.*

Proof. Since $p(A) = \lambda^k - \lambda^{k-1}$ by Corollary 3, Cayley-Hamilton theorem says that $A^k - A^{k-1} = 0$. In this way $A^k = A^{k-1}$ proving the idempotent behavior of A. □

Given the idempotent behavior of the connectivity matrices, one question is the shape that a connectivity matrix A^{k-1} has such that $A^k = A^{k-1}$. The answer is in the properties of reversible automata, the matrix has k elements equal to 1 by the uniform multiplicity principle. Since the Welch indices holda that $LR = k$ then the matrix must have L initial states, each connected with the same R final states. In this way the matrix has L identical rows, each with R entries equal to 1. A^{k-1} has a single diagonal element equal to 1 showing a single initial state equal to a single final state and the other initial states are different from the other final states; therefore A^{k-1} has L nonzero identical rows and R nonzero identical columns, where $L-1$ rows are in different positions that $R-1$ columns. Hence the shape of A^{k-1} is in Equation 6.

$$A^{k-1} = \underbrace{\begin{pmatrix} 1 & \cdots & 1 & \cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ \vdots & & \vdots & & \vdots & \\ 1 & \cdots & 1 & \cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \end{pmatrix}}_{R \text{ equal columns}} \left.\vphantom{\begin{pmatrix} 1\\ \vdots\\ \vdots\\ 1\\ \vdots \end{pmatrix}}\right\} L \text{ equal rows} \tag{6}$$

This shape of the connectivity matrices yielded by their idempotent behavior provides a way both for detecting either a cellular automaton is reversible or not. For $n \in \mathbb{Z}^+$, calculate the connectivity matrices for all the sequences of n states, if all the matrices have the shape depicted in Equation 6 then the automaton is reversible, in other case we have to check the sequences with $n+1$ states. The connectivity matrix of a given sequence of states is obtained by the product of the connectivity matrices corrsponding with the states which form the sequence, for instance, let a, b be states in K; the connectivity matrix A_{ab} corresponding with the sequence ab is obtained by the following product:

$$A_{ab} = A_a A_b \tag{7}$$

One problem is to define the maximum length of the sequences to review, this problem is resolved with a length equal to $k-1$ for reversible one-dimensional cellular automata with a Welch index equal to 1 [14, 5]. This problem is not resolved for reversible automata with both Welch indices different from 1 and up to now the best bound in this case is equal to k^2 [1, 10, 16]. In the next section we explain how the inverse rule of a reversible automaton is calculated by means of the eigenvectors corresponding with the single positive eigenvalue equal to 1 of the connectivity matrices.

4 Eigenvectors and inverse rule

In a reversible one-dimensional cellular automaton a connectivity matrix shows both the number of ancestors of a given sequence and their initial and final states. Once defined the reversible shape of the connectivity matrices, an interesting question is to know either this shape is useful or not to find the inverse rule of the automaton. To answer this question we use the eigenvectors corresponding with the eigenvalue equal to 1 associated with the reversible shape of the connectivity matrices.

4.1 Calculating the eigenvectors of the connectivity matrices

For a connectivity matrix A in the reversible shape and its positive eigenvalue $\lambda_A = 1$, an important part in the study of the spectrum of A is the form of the vectors $< e|$ and $|e >$ such that:

$$< e|A =< e|\lambda_A \;\; and \;\; A|e >= \lambda_A|e > \tag{8}$$

The vectors $< e|$ and $|e >$ are the eigenvectors of A associated with λ_A. We solve the system of scalar equations presented by $(A - I\lambda_A) = 0$to obtain $< e|$, analogously $|e >$ is obtained solving $(A - I\lambda_A)^T = 0$. Since A has L equal rows and R equal columns, $(A - I\lambda_A)$ takes the shape presented in Equation 9.

$$\begin{pmatrix} 1-\lambda_A & \cdots & \cdots & 1 & \cdots \\ \vdots & -\lambda_A & & & \\ 1 & \cdots & -\lambda_A & 1 & \cdots \\ \vdots & & & \ddots & \vdots \end{pmatrix} \tag{9}$$

Replacing $\lambda_A = 1$ in the system of scalar equations the next equation is obtained.

$$\left(\begin{array}{ccccc|c} 0 & \cdots & \cdots & 1 & \cdots & 0 \\ \vdots & -1 & & & & \vdots \\ 1 & \cdots & -1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots \end{array}\right) \tag{10}$$

The eigenvector $|e >$ must represent a linear combination which resolves all the equations in Equation 10, therefore we must analize the shape of the system for knowing the eigenvector $|e >$:

- Observing the system we can see that $|e >$ can not take into account columns with entries 1 and diagonal entries -1 because the i-th row where $a_{ii} = 0$ represents a scalar equation in which $|e >$ would take positive values which could not be nullified. This avoids that the linear combination of the elements in the row is equal to zero.
- $|e >$ can not take the $j - th$ column with a diagonal entry -1 such that the j-th row in A is null, because the scalar equation presented by the j-th row could not be zero.
- $|e >$ can take the $j - th$ column where both the entry $a_{jj} = -1$ and the j-th row is nonzero in A, because the diagonal element can be nullified by the element $a_{ji} = 1$ such that $a_{ii} = 1$ in A.

The previous explanation implies that $|e>$ takes also the i-th column where $a_{ii} = 0$ in $(A - I\lambda)$. It does not affect the solution of the system because $|e>$ does not take any positive value in the i-th row of $(A - I\lambda)$. The form of $|e>$ is presented in Equation 11.

$$
\begin{aligned}
|e> &= \{ \begin{array}{ccccc} 1 & 0 & 1 & 0 & \cdots \end{array} \}^T \\
(A - I\lambda) = 0 &= \left(\begin{array}{cccccc|c} 0 & \cdots & \cdots & 1 & \cdots & & 0 \\ \vdots & -1 & & & & & \vdots \\ 1 & \cdots & -1 & 1 & \cdots & & 0 \\ \vdots & & & \ddots & \vdots & & \vdots \end{array} \right)
\end{aligned} \tag{11}
$$

Hence $|e>$ takes only the nonzero rows of A, that is, for $|e>= \{a_i\}^T$ with $i = 0 \ldots k-1$, if $a_i = 1$ then the $i - th$ row in A is nonzero. In other words $|e>$ represents the L initial states of the ancestors presented by the connectivity matrix A in the reversible shape, the set of these initial states is called a left Welch subset. The previous characteristics are analogous for the eigenvector $< e|$ calculated by means of resolving the system represented by $(A_I \lambda_A)^T$. In this case for $< e| = \{a_i\}$, $i = 0 \ldots k-1$, if $a_i = 1$ then the i-th column in A is nonzero (Equation 12). That is, $< e|$ takes the R final states of the ancestors presented by the connectivity matrix in the reversible shape, this final states form a right Welch subset.

$$
\begin{aligned}
< e| &= \{ \begin{array}{ccccc} 1 & 0 & 0 & 1 & \cdots \end{array} \} \\
(A - I\lambda)^T = 0 &= \left(\begin{array}{cccccc|c} 0 & \cdots & 1 & 0 & \cdots & & 0 \\ \vdots & -1 & \vdots & & & & \vdots \\ \vdots & & -1 & & & & \vdots \\ 1 & \cdots & 1 & -1 & \cdots & & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \end{array} \right)
\end{aligned} \tag{12}
$$

4.2 Finding the inverse rule

For a connectivity matrix A in the reversible shape corresponding with a given sequence, its eigenvectors $|e>$ and $< e|$ associated with $\lambda_A = 1$ represent the initial and final states of the ancestors presented by A; these sets of ending states are the Welch subsets of the automaton. The indices of the entries different from 0 in $|e>$ and $< e|$ are the indices of the nonzero rows and columns in A; since these indices are states of the automaton then the initial and final states of the ancestors represented by A are the indices of the positive entries in $|e>$ and $< e|$. In this way we can use the eigenvectors from all the connectivity matrices to obtain the inverse rule of the reversible automaton, this process has the following implementation:

1. For each sequence w with n states, take its connectivity matrix A_w.

2. If any connectivity matrix does not have the reversible shape and $n > k$ then the automaton is discarded. But if $n \leq k$ then repeat step one for sequences of $n+1$ states.

3. If all the matrices have the reversible shape then take the eigenvectors $< e_w|$ and $|e_w >$ from each matrix A_w.

4. For each sequence w take the eigenvector $< e| = \{a_i\}$ and replace each entry $a_i = 1$ with $a_i = i$.

5. For the sequences v, w with n states, take the product $< e_v|e_w >= i$ for some $i \in K$, then $\varphi^{-1}(vw) = i$.

The explanation of the procedure is that the product $< e_v|e_w >$ yields the common state which is both right ancestor state of v and left ancestor state of w, i.e. the state in which the sequence vw evolves backwards. The use of eigenvectors offers a way for calculating the inverse rule by means of vector products, which makes the procedure easier for computational implementation.

5 Illustrative examples

In this section we present two examples which show the idempotent behavior of the connectivity matrices and how they are useful (in their reversible shape) for obtaining the inverse evolution rule. Both examples have 4 states and neighborhood size 2, the first example presents a reversible automaton with left Welch index equal to 1 and the second corresponds with a reversible automaton with both Welch indices equal to 2.

5.1 Reversible automaton with $L = 1$

Take the one-dimensional cellular automaton with $k = 4$ and neighborhood size 2 in Table 1.

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

Table 1: Evolution rule of a reversible automaton.

Calculating the connectivity matrices for the ancestors of the sequences with two states, the automaton in Table 1 has four kinds of connectivity matrices showing Welch indices $L = 1$ and $R = 4$. The connectivity matrices of the automaton are presented in Table 2.

The connectivity matrices show that the automaton is reversible with Welch indices $L = 1$ and $R = 4$. In every row eigenvector, each value equal to 1 shall be replaced by the value of its index. The inner products of the whole set of eigenvectors are presented in Table 3.

Matrices and eigenvectors			
$A = \begin{pmatrix} 1111 \\ 0000 \\ 0000 \\ 0000 \end{pmatrix}$	$B = \begin{pmatrix} 0000 \\ 1111 \\ 0000 \\ 0000 \end{pmatrix}$	$C = \begin{pmatrix} 0000 \\ 0000 \\ 1111 \\ 0000 \end{pmatrix}$	$D = \begin{pmatrix} 0000 \\ 0000 \\ 0000 \\ 1111 \end{pmatrix}$
$< e_A\| = \{1,1,1,1\}$	$< e_B\| = \{1,1,1,1\}$	$< e_C\| = \{1,1,1,1\}$	$< e_D\| = \{1,1,1,1\}$
$\|e_A >= \{1,0,0,0\}^T$	$\|e_B >= \{0,1,0,0\}^T$	$\|e_C >= \{0,0,1,0\}^T$	$\|e_D >= \{0,0,0,1\}^T$
Connectivity matrices			
$A_{00} = A$	$A_{01} = A$	$A_{02} = A$	$A_{03} = B$
$A_{10} = B$	$A_{11} = B$	$A_{12} = B$	$A_{13} = C$
$A_{20} = C$	$A_{21} = C$	$A_{22} = C$	$A_{23} = A$
$A_{30} = D$	$A_{31} = D$	$A_{32} = D$	$A_{33} = D$

Table 2: Connectivity matrices associated with the automaton in Table 1.

Replacing indices: $< e_A| = \{0,1,2,3\}$ $< e_B| = \{0,1,2,3\}$ $< e_C| = \{0,1,2,3\}$ $< e_D| = \{0,1,2,3\}$

	e_A	e_B	e_C	e_D
e_A	0	1	2	3
e_B	0	1	2	3
e_C	0	1	2	3
e_D	0	1	2	3

Table 3: Inner products of the eigenvectors presented by a matrix, the entry (e_A, e_B) is the inner product $< e_A|e_B >$.

In this way we take the inner products in Table 3 to obtain the invertible rule of the reversible automaton; for instance, the sequence 00 has associated the row eigenvector $< e_A| = \{0,1,2,3\}$ and the sequence 03 has associated the column eigenvector $< e_B| = \{0,1,0,0\}^T$. In this way the inverse evolution of the sequence 0003 is obtained by means of the inner product $< e_A|e_B >= 1$; following the same procedure for all sequence of four states, then inverse evolution rule of the automaton is obtained (Table 4). In this table, the indices by rows and columns represents all the sequences ab of two states and each entry $(ab, a'b')$ of the table shows the inverse evolution of the sequence $(aba'b')$ of four sates.

	00	01	02	03	10	11	12	13	20	21	22	23	30	31	32	33
00	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
01	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
02	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
03	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
10	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
11	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
12	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
13	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
20	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
21	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
22	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
23	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
30	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
31	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
32	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3
33	0	0	0	1	1	1	1	2	2	2	2	0	3	3	3	3

Table 4: Inverse evolution rule of the reversible automaton with Welch index $L = 1$.

An example of the evolutiuon of the automaton is presented in Figure 2.

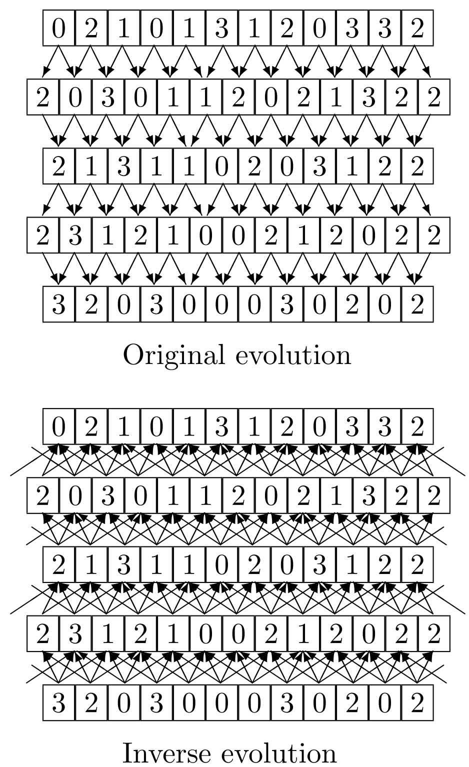

Figure 2: Evolution of the reversible automaton with Welch index $L = 1$.

6 Concluding remarks

The spectral properties and the eigenvectors of the connectivity matrices are very useful to obtain important properties of the local behavior for reversible one-dimensional cellular automata. A relevant tool in this sense is the presentation of every automaton by another of neighborhood size 2, this simulation yields that the connectivity matrices have a very suitable shape to analyze them.

Jarkko Kari [7] has made an analysis of the behavior in additive reversible automata using their matrix representation. Connectivity matrices, de Bruijn diagrams and their relations with symbolic dynamics is established by the paper of Benjamin Weiss about sofic systems [18]. However, the results is sofic systems have not been widely applied over connectivity matrices, therefore more work is needed in this direction.

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