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SEQUENTIAL AND CONTINUUM BIFURCATIONS IN DEGENERATE ELLIPTIC EQUATIONS

R. E. BEARDMORE AND R. LAISTER

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ABSTRACT. We examine the bifurcations to positive and sign-changing solutions of degenerate elliptic equations. In the problems we study, which do not represent Fredholm operators, we show that there is a critical parameter value at which an infinity of bifurcations occur from the trivial solution. Moreover, a bifurcation occurs at each point in some unbounded interval in parameter space. We apply our results to non-monotone eigenvalue problems, degenerate semi-linear elliptic equations, boundary value differential-algebraic equations and fully non-linear elliptic equations.

1. INTRODUCTION

In this paper we consider the non-linear, degenerate eigenvalue problem

$$(1) \quad Lg(u) = \lambda u, \quad x \in \Omega := (0, 1),$$

$$(2) \quad u = 0, \quad x \in \partial\Omega,$$

where $Lu := -(a(x)u_x)_x + b(x)u$ and the coefficients $a, b \in C^1(\bar{\Omega})$ satisfy $a > 0$ and $b \geq 0$ on $\bar{\Omega}$. Consequently L is uniformly elliptic, but the non-linear function $g \in C^1(\mathbb{R})$ is assumed to *degenerate* at zero with $g(0) = g'(0) = 0$.

Let us define $\gamma(u) = g(u)/u$ with $\gamma(0) = 0$ and begin with the statement of our assumptions on g :

- G1. g is an odd, strictly increasing function on \mathbb{R} ,
- G2. $u > 0$ implies $\gamma'(u) > 0$,
- G3. $\gamma(u) \rightarrow \infty$ as $|u| \rightarrow \infty$.

These are all satisfied if, for instance, $g(u) = u|u|^m$, where $m > 0$.

Definition 1.1. Let X, Y be Banach spaces, $F : X \times \mathbb{R} \rightarrow Y$ be continuous and satisfy $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. Let $\Sigma \subset X \times \mathbb{R}$ denote the set of all non-trivial ($u \neq 0$) solutions of $F(u, \lambda) = 0$. We say that λ_0 is a *sequential bifurcation point* from the trivial solution for $F(u, \lambda) = 0$ if there is a sequence $(u_n, \lambda_n) \in \Sigma$ such that $(u_n, \lambda_n) \rightarrow (0, \lambda_0)$ in $X \times \mathbb{R}$ as $n \rightarrow \infty$. If such a sequence (u_n, λ_n) lies in some connected set $C \subset \Sigma$, then λ_0 is said to be a *continuum bifurcation point*.

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We prove the following for (1)-(2). To each $\lambda > 0$ there is a sequence $u_n(\lambda) \in C^0(\bar{\Omega})$ of solutions of (1)-(2) such that (i) the number of zeros of $u_n(\lambda)$ in Ω is n , (ii) $u_n(\lambda) \rightarrow 0$ in $C^0(\bar{\Omega})$ as $n \rightarrow \infty$, (iii) $u_n(\lambda) \rightarrow 0$ in $C^0(\bar{\Omega})$ as $\lambda \rightarrow 0$, (iv) every $\lambda > 0$ is a sequential bifurcation point but *not* a continuum bifurcation point and (v) $\lambda = 0$ is a continuum bifurcation point.

We remark that the theory in [1] could be used to obtain *local* versions of some of the results proved here. However, our results are complementary to [1] in that they are global and impose no conditions on the growth of g^{-1} near zero. Furthermore, we establish the existence of an unbounded interval of sequential bifurcation points. For the special case $g(u) = u|u|^m$, we note that a global branch of positive solutions was shown to exist in [2] in a study of flows in porous media.

The remainder of the paper is structured as follows. Section 2 introduces some notation and preliminary results. The main results of the paper appear in Section 3. Finally, in Section 4 we apply our results to non-monotone degenerate eigenvalue problems, degenerate semi-linear elliptic equations, boundary value differential-algebraic equations and fully non-linear elliptic equations.

2. PRELIMINARIES

Throughout we write \bar{U} for the closure of U in a given metric space. We denote by $C^k(\bar{\Omega})$ the space of k -times differentiable functions on $\bar{\Omega}$, henceforth written simply as C^k when there is no ambiguity. We note here that the imbedding $C^k \hookrightarrow C^r$ is compact if $k > r$. For any $u \in C^0$ with finitely many zeros we shall denote the number of zeros of u in Ω by $\zeta(u)$.

It is well known that $L : C^2 \rightarrow C^0$ together with the Dirichlet boundary condition (2) has positive, simple eigenvalues, henceforth denoted by μ_j for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where the principal eigenvalue μ_0 has an associated positive eigenfunction ϕ_0 . Furthermore, L has a continuous inverse $K : C^0 \rightarrow C^2$ which induces a compact linear map $K : C^0 \rightarrow C^0$.

The problem of finding continuous solutions of (1)-(2) with $g(u(\cdot)) \in C^2$ is therefore equivalent to

$$(3) \quad F(u, \lambda) := g(u) - \lambda Ku = 0, \quad u \in C^0,$$

where $g : C^0 \rightarrow C^0$ is the C^1 Nemytskii operator for g defined by $(g(u))(x) = g(u(x))$. Our approach to solving (3) will be based on the regularized problem

$$(4) \quad F(u, \lambda; \varepsilon) := g(u) + (\varepsilon I - \lambda K)u = 0, \quad \varepsilon \geq 0.$$

We define some solution sets. Throughout $E := C^0 \times \mathbb{R}$ is endowed with the norm $\|(u, \lambda)\|_E = \|u\| + |\lambda|$, where $\|\cdot\|$ denotes the sup-norm on C^0 . The symbol $\langle \cdot, \cdot \rangle$ denotes the usual L^2 inner product. For $\varepsilon \geq 0$, $\Sigma(\varepsilon) \subset E$ will denote the set of non-trivial solutions (u, λ) of $F(u, \lambda; \varepsilon) = 0$ in E . For $j \in \mathbb{N}_0$ we write $\Sigma_j(\varepsilon)$ for the subset of $\Sigma(\varepsilon)$ consisting of functions with j zeros in Ω . By $\Sigma_j^+(\varepsilon)$ ($\Sigma_j^-(\varepsilon)$) we denote the subset of $\Sigma_j(\varepsilon)$ of functions u such that $g(u)_x(0) > 0$ ($g(u)_x(0) < 0$). For notational convenience we will simply write Σ instead of $\Sigma(0)$ and Σ_j^\pm for $\Sigma_j^\pm(0)$. We note here that since g is odd, $(u, \lambda) \in \Sigma(\varepsilon)$ if and only if $(-u, \lambda) \in \Sigma(\varepsilon)$. Consequently $\Sigma_j^-(\varepsilon) = -\Sigma_j^+(\varepsilon)$.

Remark 1. The map $F : C^0 \times \mathbb{R} \rightarrow C^0$ is C^1 and has partial Fréchet derivative $d_u F(u, \lambda)[h] = g'(u)h - \lambda Kh$ which is not a Fredholm mapping at $u = 0$ since $g'(0) = 0$. Consequently, one cannot use reduction methods based on the implicit

function theorem to study bifurcations of (3) from the trivial solution. See also [3, 4, 15]. Moreover, $d_u F(0, \lambda) - \lambda K$ which, for $\lambda \neq 0$, has point spectrum and zero in the essential spectrum, but when $\lambda = 0$, the spectrum consists only of zero.

Lemma 2.1. *Fix $\varepsilon \geq 0$. If $(u, \lambda) \in \Sigma(\varepsilon)$, then $\lambda > 0$; that is, $\Sigma(\varepsilon) \subset C^0 \times (0, \infty)$.*

Proof. Multiplying the relation $F(u, \lambda; \varepsilon) = 0$ by u and integrating over Ω gives, after setting $v = Ku$,

$$\int_{\Omega} \varepsilon u^2 + ug(u) \, dx = \lambda \int_{\Omega} uKu \, dx = \lambda \int_{\Omega} vLv \, dx.$$

Noting that $ug(u) \geq 0$ and $\langle v, Lv \rangle \geq 0$, the result follows. □

Lemma 2.2. *For $\varepsilon \in [0, 1]$ the following a priori bound applies: to each $\ell > 0$ there is an $M(\ell) > 0$, independent of ε , such that if $\lambda \in [0, \ell]$, then $\|u\| \leq M(\ell)$ whenever $(u, \lambda) \in \Sigma(\varepsilon)$.*

Proof. Suppose that $\varepsilon u + g(u) = \lambda Ku$, where $0 \leq \varepsilon \leq 1, 0 \leq \lambda \leq \ell$ and let $x_0 \in \Omega$ satisfy $\|u\| = |u(x_0)|$. Then

$$\left| |g(u(x_0))| - |\varepsilon u(x_0)| \right| \leq |g(u(x_0)) + \varepsilon u(x_0)| \leq \lambda \|K\| \|u(x_0)\|,$$

where $\|K\|$ denotes the operator norm of $K \in BL(C^0)$. We therefore obtain $\gamma(\|u\|) \leq \lambda \|K\| + \varepsilon \leq \ell \|K\| + 1$. Noting that $\gamma : [0, \infty) \rightarrow [0, \infty)$ is surjective (by G3) and non-decreasing (by G2), the result follows on defining $M(\ell)$ to be any positive solution of $\gamma(M) = \ell \|K\| + 1$. □

Since $\varepsilon + g'(u) \geq \varepsilon > 0$ for all $u \in \mathbb{R}$ and $\varepsilon > 0$, the algebraic equation $\varepsilon u + g(u) = v$ has a unique solution $u = G(v; \varepsilon)$, where $G(\cdot; \varepsilon) \in C^1(\mathbb{R})$. When $\varepsilon = 0$ we simply have $G(v; 0) = g^{-1}(v)$, which is continuous. Moreover, $G : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is continuous. We shall use this notation throughout and in the following theorem, which is a consequence of global bifurcation theory.

Theorem 2.3. *For each $\varepsilon > 0$ and $j \in \mathbb{N}_0$, there are open, connected and unbounded sets $C_j^\pm(\varepsilon) \subset \Sigma_j^\pm(\varepsilon)$ such that $(0, \varepsilon \mu_j) \in \overline{C_j^\pm(\varepsilon)}$. Furthermore, for every $\lambda > \varepsilon \mu_j$ there exist $(\pm u_{j,\varepsilon}, \lambda) \in C_j^\pm(\varepsilon)$, so that $(\varepsilon \mu_j, \infty) \subset \Pi(C_j^\pm(\varepsilon))$, where $\Pi : E \rightarrow \mathbb{R}$ is the natural projection.*

Proof. For each fixed $\varepsilon > 0$, apply global bifurcation results [13] to $v = \lambda KG(v; \varepsilon)$ and use the nodal properties of solutions to regular elliptic equations to demonstrate the existence of disjoint, unbounded continua $C_j^\pm(\varepsilon)$ with the stated properties. The existence of $(\pm u_{j,\varepsilon}, \lambda)$ for $\lambda > \varepsilon \mu_j$ follows from the unboundedness of $C_j^\pm(\varepsilon)$ in E , Lemma 2.1 and Lemma 2.2. □

If u is a non-trivial solution of (4) with $\varepsilon > 0$, then the zeros of the function $\varepsilon u + g(u)$ are transverse. The following result shows that transversality persists when $\varepsilon = 0$.

Theorem 2.4 (see [9, Theorem 2.2]). *Suppose that $f \in C^0(\mathbb{R})$ is strictly increasing and $f(0) = 0$. If $u \in C^2(\overline{\Omega})$ is a solution of the initial value problem $Lu = f(u)$ on $\overline{\Omega}$ with $u(\alpha) = u_x(\alpha) = 0$ for some $\alpha \in \overline{\Omega}$, then $u \equiv 0$ on $\overline{\Omega}$. Furthermore, u has a finite number of zeros in $\overline{\Omega}$.*

Corollary 2.5. *If $(u, \lambda) \in \Sigma$, then $\zeta(u) = \zeta(g(u)) < \infty$ and all zeros of $g(u)$ in $\overline{\Omega}$ are transverse. In particular, $\Sigma = \bigcup_{j=0}^{\infty} (\Sigma_j^+ \cup \Sigma_j^-)$.*

Proof. If $(u, \lambda) \in \Sigma$ and $v := g(u)$, then $Lv = \lambda g^{-1}(v)$. The result follows from Lemma 2.1 and Theorem 2.4 with $f(v) = \lambda g^{-1}(v)$. \square

3. THE MAIN RESULTS

In this section we prove the main results on the existence of non-trivial solutions of (3) and the nature of bifurcation points.

3.1. Existence of non-trivial solutions. We begin with an existence and uniqueness result for elliptic equations.

Lemma 3.1. *Suppose $Au := -(\alpha(x)u_x)_x + \beta(x)u$, where α and β satisfy the same assumptions as a and b . Let $\lambda > 0$ and $\varepsilon \geq 0$ be fixed. If there exists a positive subsolution ψ of the elliptic problem*

$$(5) \quad Av = \lambda G(v; \varepsilon), \quad v(0) = v(1) = 0,$$

then there exists a unique non-trivial, non-negative solution v of (5). Moreover, $v \geq \psi$.

Proof. By assumption G3, $\lim_{v \rightarrow \infty} G(v; \varepsilon)/v = 0$ for fixed $\varepsilon \geq 0$. In particular this implies that $\limsup_{v \rightarrow \infty} \lambda G(v; \varepsilon)/v < \kappa_0$, where κ_0 denotes the principal eigenvalue of A . It is well known [6, 11] that non-negative solutions of the associated parabolic problem

$$(6) \quad v_t = -Av + \lambda G(v; \varepsilon), \quad v(0, t) = v(1, t) = 0$$

(with continuous initial condition $v(x, 0) = v_0(x)$) have non-empty omega-limit sets $\omega(v_0)$ contained in the equilibrium set, comprising of solutions of (5). In particular, since ψ is also a subsolution of (6), there exists a solution v of (5) such that $v \geq \psi$. It therefore remains only to establish the uniqueness of v .

Suppose w is any non-trivial, non-negative solution of (5). By G1 and the maximum principle, $w > 0$ in Ω . Now, $\int_0^1 vAw - wAv \, dx = 0$ so that

$$\lambda \int_0^1 vG(w; \varepsilon) - wG(v; \varepsilon) \, dx = \int_0^1 \lambda vw \left(\frac{G(w; \varepsilon)}{w} - \frac{G(v; \varepsilon)}{v} \right) \, dx = 0.$$

By G2, $s \mapsto G(s; \varepsilon)/s$ is decreasing for all $s > 0$. Hence if v and w are ordered in C^0 , then $v = w$ and v is unique. If v and w are not ordered in C^0 , then, for any $v_0 \geq \max\{v, w\}$, $\omega(v_0)$ must contain a solution z of (5) such that $z \geq \max\{v, w\}$, whence $z \neq v$ and $z \neq w$. Hence z and v are ordered in C^0 and the above argument (with w replaced by z) yields $z = v$, a contradiction. \square

The following result is crucial, showing that non-trivial j -zero solutions of the regularized problem (4) cannot accumulate on the trivial branch as $\varepsilon \rightarrow 0$, except possibly at the origin.

Proposition 3.2. *Let $j \in \mathbb{N}_0$ be fixed and $0 \leq \varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. If $(u_n, \lambda_n) \in \Sigma_j^+(\varepsilon_n)$ satisfies $(u_n, \lambda_n) \rightarrow (0, \lambda)$ in E as $n \rightarrow \infty$, then $\lambda = 0$. An analogous result holds for $\Sigma_j^-(\varepsilon_n)$.*

Proof. Necessarily $\lambda \geq 0$ by Lemma 2.1, so suppose that $\lambda > 0$. We first consider the case $j = 0$ (positive solutions). Fix $\lambda_* \in (0, \lambda)$ and choose n_0 such that $\varepsilon_n < \min\{\mu_0, (\lambda_*/\mu_0)\}$ and $\lambda_n > \lambda_*$ for all $n > n_0$. By the degeneracy of g there is a $U > 0$ (independent of n) such that $g(u) + \varepsilon_n u \leq (\lambda_*/\mu_0)u$ for all $u \in [0, U]$ and $n > n_0$. Hence there is a $V > 0$ (independent of n) such that $G(v; \varepsilon_n) \geq (\mu_0/\lambda_*)v$

for all $v \in [0, V]$ and $n > n_0$. Let us normalise the principal eigenfunction of L , ϕ_0 , so that $\|\phi_0\| = V$. Since $G(\phi_0, \varepsilon_n) \geq (\mu_0/\lambda_*)\phi_0$ it follows that

$$-L\phi_0 + \lambda_n G(\phi_0; \varepsilon_n) \geq -L\phi_0 + \lambda_n(\mu_0/\lambda_*)\phi_0 \geq 0$$

and so ϕ_0 is a subsolution of

$$(7) \quad Lv = \lambda_n G(v; \varepsilon_n), \quad v(0) = v(1) = 0.$$

Hence by Lemma 3.1 there exists a unique positive solution w_n of (7) and $w_n \geq \phi_0$. Now, $v_n := \varepsilon_n u_n + g(u_n)$ is also a positive solution of (7) and so by uniqueness $v_n = w_n$. But since $v_n = \lambda_n K u_n$ and $u_n \rightarrow 0$ in C^0 as $n \rightarrow \infty$, it follows that $v_n \rightarrow 0$ in C^2 . In particular, by Hopf's boundary point lemma [12] applied to ϕ_0 , there exists an $n_1 > n_0$ such that $v_n < \phi_0$ in Ω for all $n > n_1$, a contradiction. This proves the result for $j = 0$. The result for $\Sigma_j^-(\varepsilon_n)$ is a trivial consequence of the symmetry of g .

Now suppose that $j \geq 1$. If ξ_n^i ($i = 0, \dots, j + 1$) denote the zeros of u_n in $\bar{\Omega}$ in increasing order, let $\delta_n^i = \xi_n^{i+1} - \xi_n^i$ ($i = 0, \dots, j$). Then $v_n := \varepsilon_n u_n + g(u_n)$ (suitably restricted) is a constant sign solution of

$$(8) \quad Lv = \lambda_n G(v; \varepsilon_n), \quad v(\xi_n^i) = v(\xi_n^{i+1}) = 0.$$

Since $\sum_{i=0}^j \delta_n^i \equiv 1$, we can assume for some i that δ_n^i ($=: \delta_n$) remains uniformly bounded away from zero. Passing to a subsequence if necessary we may assume that $\delta_n \rightarrow \delta_\infty \in (0, 1]$ as $n \rightarrow \infty$. Now rescale the spatial variable x in (8) according to $x \mapsto (x - \xi_n^{i+1})/\delta_n$ and, without loss of generality by the symmetry of g , we obtain a sequence v_n of positive solutions of

$$(9) \quad L_n v \lambda_n G(v; \varepsilon_n), \quad v(0) = v(1) = 0,$$

with $v_n \rightarrow 0$ in C^2 , where $L_n v := -\delta_n^{-2}(a(x)v_x)_x + b(x)v$. If we denote by $\{\mu_0^n, \phi_0^n\}$ the principal eigenpair of the operator L_n , then spectral perturbation results for simple eigenvalues [7] show that $\mu_0^n \rightarrow \mu_0^\infty$, the principal eigenvalue of L_∞ , and $\phi_0^n \rightarrow \phi_0^\infty$ in C^2 , where ϕ_0^∞ is the corresponding principal eigenfunction.

Note that there is a $V > 0$ (independent of n) and an $n_2 > n_0$ such that $G(v; \varepsilon_n) \geq (\mu_0^\infty + 1)v/\lambda_*$ for all $v \in [0, V]$ and $n > n_2$. If ϕ_0^n is normalised so that $\|\phi_0^n\| = V$, then

$$(10) \quad -L_n \phi_0^n + \lambda_n G(\phi_0^n; \varepsilon_n) \geq -L_n \phi_0^n + \lambda_n(\mu_0^\infty + 1)\phi_0^n/\lambda_* \geq (\mu_0^\infty + 1 - \mu_0^n)\phi_0^n \geq 0,$$

for all $n > n_2$ and so ϕ_0^n is a positive subsolution of (9) for all such n . An identical argument to the $j = 0$ case then leads to a contradiction as before. \square

We can now prove the following existence result for (3).

Theorem 3.3. *Let $\lambda > 0$ and $j \in \mathbb{N}_0$ be given. Then there exist $(\pm u_j, \lambda) \in \Sigma_j^\pm$; that is, $\Pi(\Sigma_j^\pm) = (0, \infty)$.*

Proof. Let $\varepsilon_n \rightarrow 0$ be any positive sequence. From Lemma 2.2 and Proposition 3.2 with $\lambda_n \equiv \lambda$, there is a sequence u_n of C^2 solutions of (4) which is C^0 -bounded and bounded away from zero in C^0 . Since $K u_n$ is therefore C^2 -bounded we may pass to a subsequence if necessary and assume that there is a $z \in C^1$ such that $K u_n \rightarrow z$ in C^1 . Hence, it follows that $\varepsilon_n u_n + g(u_n) \rightarrow \lambda z$ in C^1 , from where $\varepsilon_n u_n \rightarrow 0$ in C^0 , so that $g(u_n) \rightarrow \lambda z$ in C^0 . Consequently, $u_n \rightarrow g^{-1}(\lambda z) =: u$ in C^0 . Therefore,

$$\begin{aligned} \|g(u) - \lambda K u\| &= \|(g(u) - g(u_n)) + (g(u_n) - \lambda K u_n) + (\lambda K u_n - \lambda K u)\| \\ &\leq \|g(u) - g(u_n)\| + \varepsilon_n \|u_n\| + \lambda \|K\| \|u_n - u\| \rightarrow 0. \end{aligned}$$

Hence u is a solution of (3). Since z is a C^1 -limit of functions with exactly j transverse zeros we have $\zeta(z) = j$, whence $\zeta(u) = \zeta(g(u)) = \zeta(\lambda z) = j$. \square

3.2. Sequential and continuum bifurcations. We may now establish the existence of an unbounded interval of sequential bifurcation points.

Theorem 3.4. *For each $\lambda > 0$ there exists a sequence $(u_j, \lambda) \in \Sigma$ such that $\zeta(u_j) = j$ and $u_j \rightarrow 0$ in C^0 as $j \rightarrow \infty$. In particular, every $\lambda \geq 0$ is a sequential bifurcation point for (3).*

Proof. Clearly, for each fixed $\lambda > 0$ there are infinitely many solutions of (3), u_j , parameterised by the number of zeros $j \in \mathbb{N}_0$. Recall that the corresponding zeros of $g(u_j)$ are transverse. We claim that $\lim_{j \rightarrow \infty} u_j = 0$ in C^0 . Using the bound $\|u_j\| \leq M(\lambda)$ from Lemma 2.2, we may assume (on passing to a subsequence) that there is a $z \in C^1$ such that $Ku_j \rightarrow z$ in C^1 , so that $g(u_j) \rightarrow \lambda z$ in C^1 and therefore $u_j \rightarrow g^{-1}(\lambda z)$ in C^0 . If $u := g^{-1}(\lambda z)$, then u is a solution of (3). Since $\zeta(g(u_j)) = j$, $g(u)$ cannot have finitely many zeros in Ω . Hence by Theorem 2.4 $g(u) = 0$, from where $z = 0$. Hence $g(u_j) \rightarrow 0$ in C^1 and therefore $u_j \rightarrow 0$ in C^0 .

In turn, this implies that $\lambda = 0$ is a sequential bifurcation point, simply by setting $\lambda_n = 1/n$ and choosing any $(\bar{u}_n, \lambda_n) \in \Sigma$ with $\|\bar{u}_n\| \leq 1/n$. \square

Next we examine the question of which $\lambda \geq 0$ are continuum bifurcation points.

Lemma 3.5. *If $\mathcal{C} \subset \Sigma$ is connected and $(u, \lambda), (u', \lambda') \in \mathcal{C}$, then $\zeta(u) = \zeta(u')$.*

Proof. Let $(u, \lambda) \in \mathcal{C}$ and suppose that $(u_n, \lambda_n) \in \mathcal{C}$ satisfies $(u_n, \lambda_n) \rightarrow (u, \lambda)$ as $n \rightarrow \infty$. Using $g(u_n) \equiv \lambda_n Ku_n$ we find that $g(u_n) \rightarrow g(u)$ in C^1 and because $g(u)$ has finitely many transverse zeros, $\zeta(u_n)\zeta(g(u_n)) = \zeta(g(u)) = \zeta(u)$ for all n sufficiently large. This shows that $\zeta(\cdot)$ is an integer-valued continuous function on \mathcal{C} and is therefore constant on \mathcal{C} . \square

Corollary 3.6. *For all $\lambda > 0$, λ is not a continuum bifurcation point.*

Proof. If $\lambda > 0$ is a continuum bifurcation point, then there exists a connected set $\mathcal{C} \subset \Sigma$ and a sequence $(u_n, \lambda_n) \in \mathcal{C}$ such that $(u_n, \lambda_n) \rightarrow (0, \lambda)$ in E . By Lemma 3.5 there exists a $j \in \mathbb{N}_0$ such that $(u_n, \lambda_n) \in \Sigma_j$ for all n . Passing to a subsequence if necessary, we may assume without loss of generality that $(u_n, \lambda_n) \in \Sigma_j^+$ for all n . By Proposition 3.2 with $\varepsilon_n \equiv 0$ it follows that $\lambda = 0$, a contradiction. \square

Theorem 3.7. *$\lambda = 0$ is a continuum bifurcation point for (3).*

Proof. For each $\lambda > 0$ there is a unique $(u^+, \lambda) \in \Sigma_0^+$ by Theorem 3.3 and Lemma 3.1. We prove that the map $\lambda \mapsto u^+(\lambda)$ (with $u^+(0) = 0$) from $[0, \infty) \rightarrow C^0$ is continuous.

Fix $\lambda \geq 0$ and let $\lambda_n > 0$ be any sequence satisfying $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Let $u_n^+ := u^+(\lambda_n)$. Suppose that $u^+(\cdot)$ is not continuous at λ ; then there is a $\delta > 0$ such that $\|u_n^+ - u^+(\lambda)\| \geq \delta$ for all n . By Lemma 2.2, u_n^+ is bounded in C^0 . From $u_n^+ = \lambda_n Kg^{-1}(u_n^+)$ and the compactness of K , there exists a convergent subsequence, say $u_{n_j}^+ \rightarrow u^*$ in C^0 . Hence u^* is a solution of $Lu^* = \lambda g^{-1}(u^*)$. By Proposition 3.2, if $\lambda > 0$, then $u^* = u^+(\lambda)$, while if $\lambda = 0$, then $u^* = 0$. Either way this contradicts the above δ -bound. \square

We now utilise a theorem from topological analysis to obtain connectedness results for the sets of non-trivial sign-changing solutions.

Definition 3.8. Suppose that (Z, d) is a complete metric space and that $\{S_n\}_{n=0}^\infty$ is a family of connected subsets of Z . For $S \subset Z$ define $d(z, S) := \inf_{s \in S} d(z, s)$,

$$S_{\text{inf}} := \left\{ z \in Z : \lim_{n \rightarrow \infty} d(z, S_n) = 0 \right\},$$

$$S_{\text{sup}} := \left\{ z \in Z : \liminf_{n \rightarrow \infty} d(z, S_n) = 0 \right\}.$$

Theorem 3.9 (see [17]). *Suppose that $\bigcup_{n=0}^\infty S_n$ is relatively compact in Z . If $S_{\text{inf}} \neq \emptyset$, then S_{sup} is a non-empty, closed and connected subset of Z .*

Theorem 3.10. *Let $j \in \mathbb{N}_0$ be given. There exist unbounded, closed and connected sets $C_j^\pm \subset \Sigma_j^\pm \cup \{(0, 0)\}$ such that $(0, 0) \in C_j^\pm$. In particular, $\Pi(C_j^\pm) = [0, \infty)$.*

Proof. Let $\varepsilon_n \rightarrow 0$ be any positive sequence. For fixed $\nu > 0$ let $S_n^{+,j}(\nu)$ be the maximal connected component of $C_j^+(\varepsilon_n) \cap (C^0 \times [0, \nu])$ which contains $(u, \lambda) = (0, \varepsilon_n \mu_j)$ in its closure, where $C_j(\varepsilon)$ is defined in Theorem 2.3. Note that by Theorem 2.3, $S_n^{+,j}(\nu)$ contains non-trivial elements of the form (u, λ) for all $\lambda \in [\varepsilon_n \mu_j, \nu]$, provided n is sufficiently large and $(0, \varepsilon_n \mu_j) \in S_n^{+,j}(\nu)$. By the compactness of $[0, \nu]$ and of the operator $K : C^0 \rightarrow C^0$ it follows that $\bigcup_{n=0}^\infty S_n^{+,j}(\nu)$ is relatively compact in E . Clearly $(0, 0) \in S_{\text{inf}}^{+,j}(\nu)$ and so $S_{\text{inf}}^{+,j}(\nu)$ is non-empty. Hence by Theorem 3.9 $S_{\text{sup}}^{+,j}(\nu)$ is non-empty, closed and connected in E .

Now, by the construction of solutions in Theorem 3.3 it follows that

$$\{(u_j, \lambda) \in \Sigma_j^+ : \lambda \in (0, \nu]\} \cup \{(0, 0)\} \subset S_{\text{inf}}^{+,j}(\nu) \subset S_{\text{sup}}^{+,j}(\nu).$$

Moreover, if $(u, \lambda) \in S_{\text{sup}}^{+,j}(\nu)$ there exists a sequence $(u_n, \lambda_n) \in S_n^{+,j}(\nu)$ such that $(u_n, \lambda_n) \rightarrow (u, \lambda)$ in E . Then,

$$\begin{aligned} \|g(u) - \lambda K u\| &\leq \|g(u) - g(u_n)\| + |\lambda_n - \lambda| \|K u_n\| \\ &\quad + \lambda \|K(u_n - u)\| + \varepsilon_n \|u_n\| \rightarrow 0, \end{aligned}$$

so that (u, λ) is a solution of (3). By Proposition 3.2 and Theorem 2.4 either $(u, \lambda) = (0, 0)$ or $(u, \lambda) \in \Sigma_j^+$ for some $j \in \mathbb{N}_0$.

Clearly, $S_{\text{sup}}^{+,j}(\nu) \subset S_{\text{sup}}^{+,j}(\nu')$ if $\nu < \nu'$ and it follows that $C_j^+ : \bigcup_{\nu > 0} S_{\text{sup}}^{+,j}(\nu)$ has the stated properties. The result for C_j^- follows similarly. \square

Example 1. Consider a semi-linear, degenerate elliptic equation $\Delta \varphi(v) + \lambda f(v) = 0$ with Dirichlet boundary conditions on an annulus $R_1 < |y| < R_2$ in \mathbb{R}^n [8]. Suppose that φ and f are strictly increasing, odd functions satisfying $\varphi(0) = f(0) = 0$. Setting $u = f(v)$ one obtains $\Delta g(u) + \lambda u = 0$, where $g(u) := \varphi(f^{-1}(u))$. Suppose that φ and f are such that g satisfies G1-G3. Now, radially symmetric solutions satisfy $(r^{n-1}g(u)_r)_r + \lambda r^{n-1}u = 0$, where $r = |y|$. Setting $x = r^n/n$ then yields the equivalent problem $-(a(x)g(u)_x)_x = \lambda u$ for $x \in (R_1^n/n, R_2^n/n)$, where $a(x) := (nx)^{2(1-1/n)}$, to which the results of this section apply. Such a situation occurs when $\varphi(v) = v|v|^{m-1}$ and $f(v) = v|v|^{p-1}$ for $m > p > 0$.

4. APPLICATIONS

4.1. Non-monotone eigenvalue problems. Here we apply our main results to problems where g is only *locally* monotonic near zero. We still obtain infinitely many solution sets in E parameterised by zeros together with an unbounded interval of sequential (but not continuum) bifurcation points.

Lemma 4.1. *Let $\delta > 0$ and suppose that $g : [0, \delta] \rightarrow [0, \infty)$ is a strictly increasing C^1 function which is C^2 on $(0, \delta]$ with $g(0) = g'(0) = 0$ and $g''(\delta) > 0$. If $\gamma(u) = g(u)/u$ satisfies $\gamma'(u) > 0$ on $(0, \delta]$, then there exists an odd, strictly increasing C^1 extension $\bar{g} : \mathbb{R} \rightarrow \mathbb{R}$ such that $g|_{[0, \delta]} = \bar{g}|_{[0, \delta]}$. Moreover, if $\bar{\gamma}(u) := \bar{g}(u)/u$, then $\bar{\gamma}'(u) > 0$ for all $u > 0$ and $\bar{\gamma}(u) \rightarrow \infty$ as $|u| \rightarrow \infty$.*

Proof. Since $u^2\gamma'(u) = ug'(u) - g(u)$ we have $g'(\delta) > 0$. Now define \bar{g} to be the odd extension of the function

$$\begin{cases} g(u) & : 0 \leq u \leq \delta, \\ g(\delta) + (u - \delta)g'(\delta) + \frac{1}{2}(u - \delta)^2g''(\delta) & : u \geq \delta, \end{cases}$$

and then for $|u| \geq \delta$ we have $u^2\bar{\gamma}'(u) = \delta^2\gamma'(\delta) + \frac{1}{2}g''(\delta)(u^2 - \delta^2) > 0$. □

We can now deduce the following result when g is only *locally* monotonic.

Theorem 4.2. *For some $\delta > 0$ suppose that $g : [-\delta, \delta] \rightarrow \mathbb{R}$ is a strictly increasing, odd, C^1 function which is C^2 on $[-\delta, \delta] \setminus \{0\}$ and $g(0) = g'(0) = 0, g''(\delta) > 0$. If $\gamma'(u) \geq 0$ on $(0, \delta]$, then there exist closed, connected sets $C_j^\pm \subset \Sigma_j^\pm \cup \{(0, 0)\}$ such that $(0, 0) \in C_j^\pm$. At least one, but possibly both, of the following is true:*

- (1) C_j^\pm is unbounded,
- (2) there exists a $(u, \lambda) \in C_j^\pm$ such that $\|u\| = \delta$.

Furthermore, for each $\lambda > 0$ there exists a sequence $u_j \in \Sigma$ such that $\zeta(u_j) \rightarrow \infty$ and $u_j \rightarrow 0$ in C^0 as $j \rightarrow \infty$. In particular, every $\lambda \geq 0$ is a sequential bifurcation point and $\lambda = 0$ is a continuum bifurcation point for (3).

Proof. Use Lemma 4.1 to replace (3) by $\bar{g}(u) = \lambda Ku$ to which Theorems 3.10 and 3.4 apply. The result follows from the fact that solutions of $\bar{g}(u) = \lambda Ku$ with $\|u\| \leq \delta$ also satisfy (3). □

4.2. Degenerate diffusion equations. Consider a quasi-linear parabolic equations of the form

$$(11) \quad v_t - (a(x)D(v)_x)_x + b(x)D(v) = \lambda f(v),$$

supplied with Dirichlet boundary conditions and given initial data. Such equations arise naturally in many branches of the physical and biological sciences [5, 14]. Upon setting $u = f(v)$ and defining $g(u) = D(F(u))$ (see below) one may use Theorem 4.2 to obtain information on the existence of equilibrium solutions of (11) whenever f and D are monotonic near zero. We omit the trivial proof.

Theorem 4.3. *Suppose that $D, f \in C^1(\mathbb{R})$ are odd, strictly increasing functions such that $D(0) = D'(0) = f(0) = 0$ and $f'(0) > 0$. Let F denote the local C^1 inverse of f near 0. If there exists a $\delta^* > 0$ such that $D \in C^2(0, \delta^*)$ and $uF'(u)D'(F(u)) - D(F(u)) > 0$ on $(0, \delta^*)$, then the conclusions of Theorem 4.2 hold for equilibrium solutions of (11) for each $\delta \leq \delta^*$ for which $(D(F))''(\delta) > 0$. In particular, the latter conditions hold for all sufficiently small $\delta > 0$ whenever $D, f \in C^3(\mathbb{R}), D''(0) = 0$ and $D'''(0) > 0$.*

Example 2. Theorem 4.3 applies to a degenerate form of the Chafée-Infante problem (see [6])

$$v_t - (v|v|^m)_{xx} = \lambda v(1 - v^2), \quad m > 0.$$

Example 3. Consider the *slow diffusion* problem

$$u_t - (a(x)[\exp(-1/u)]_x)_x = \lambda u$$

with Dirichlet boundary conditions, where $g(u) := [\exp(-1/u)]$ denotes the odd extension of $\exp(-1/u)$ for $u > 0$. Theorem 4.3 applies to the associated steady-state problem. Note however, that the global results of Section 3 do not apply even though g is globally monotonic due to the failure of the coercivity condition G3. Due to the *flat* nature of g at $u = 0$, the results of [1] do not apply to this equation.

4.3. Boundary value differential-algebraic equations. We can also use the above results to find steady-states of parabolic systems

$$\begin{aligned} u_t + Lu &= \lambda F(u, v), & u(0, t) = u(1, t) = 0, \\ v_t &= G(u, v), \end{aligned}$$

or equivalently, the boundary value differential-algebraic equation (DAE)

$$(12) \quad Lu = \lambda F(u, v), \quad G(u, v) = 0, \quad u(0) = u(1) = 0.$$

Problems of this nature are considered in [10], motivated by interactions between diffusive and non-diffusive species. We have the following theorem regarding solutions of (12).

Theorem 4.4. *Suppose that F and G are C^r functions with $r \geq 4$ such that $F(0, 0) = G(0, 0) = 0$, $G_v(0, 0) = G_{vv}(0, 0) = 0$, $F(-u, -v) = -F(u, v)$ and $G(u, -v) = -G(-u, v)$. If $G_u F_v G_{vvv} < 0$ at $(0, 0)$, then $\lambda = 0$ is a continuum bifurcation point to a branch of positive solutions of (12). There are countably many sets of non-trivial solutions $\mathcal{C}_j \subset C^2(\bar{\Omega}) \times C^0(\bar{\Omega}) \times \mathbb{R}$ such that $\mathcal{C}_j \cup \{(0, 0, 0)\}$ is connected, and if $(u, v, \lambda) \in \mathcal{C}_j$, then u and v have j zeros in Ω . Every $\lambda \in (0, \infty)$ is a sequential bifurcation point, but no element of $(0, \infty)$ is a continuum bifurcation point.*

Proof. Apply the implicit function theorem to $G(u, v) = 0$ and solve this constraint as $u = U(v)$, where $U(0) = U'(0) = U''(0) = 0$ and $U'''(0) = -G_{vvv}(0, 0)/G_u(0, 0) \neq 0$. Then (12) is reduced to $LU(v) = \lambda F(U(v), v)$, so now set $w = F(U(v), v)$. This can be solved by the inverse function theorem for $v = V(w)$ such that $V(0) = 0, V'(0) = 1/F_v(0, 0)$ and $V''(0) = -F_{vv}(0, 0)/F_v(0, 0)^3$. Now, (12) is locally equivalent to $LU(V(w)) = \lambda w$, so we set $g(w) = U(V(w))$.

Now, the hypotheses on F and G ensure that U and V are odd functions, so that $g(w)$ is also odd; now set $\gamma(w) = g(w)/w$. Differentiating, we see that $g(w) = \xi w^3 + o(w^3)$ where $\xi = -G_{vvv} G_u F_v / (G_u^2 F_v^4) > 0$ and where each of these derivatives is evaluated at $(u, v) = (0, 0)$. Hence there is a $\delta > 0$ such that $g(w) > 0, \gamma'(w) > 0$ on $(0, \delta]$ and $g''(\delta) > 0$. One can now apply Theorem 4.2 to $Lg(w) = \lambda w$. \square

Example 4. The hypotheses of Theorem 4.4 are satisfied by the steady-state problem for the reaction-diffusion system

$$\begin{aligned} u_t - u_{xx} &= \lambda \sin v, & u(0, t) = u(1, t) = 0, \\ v_t &= u + u^2 v - v^3. \end{aligned}$$

Remark 2. Fully non-linear elliptic equations of the form

$$(13) \quad Lu = f(u, Lu), \quad u(0) = u(1) = 0,$$

can be written as a boundary value DAE by setting $v = Lu$, $F(u, v) = v$ and $G(u, v) = f(u, v) - v$. Problems of this type are studied, for instance, in [16]. A

solution of (12) when $\lambda = 1$ provides a solution of (13) and these can be obtained using Theorem 4.4 with suitable restrictions on f .

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