The k-Tuple Domination Number Revisited

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Abstract

The following fundamental result for the domination number $\gamma(G)$ of a graph G was proved by Alon and Spencer, Arnautov, Lovász and Payan:

$$\gamma(G) \le \frac{\ln(\delta+1)+1}{\delta+1} n,$$

where n is the order and δ is the minimum degree of vertices of G. A similar upper bound for the double domination number was found by Harant and Henning [On double domination in graphs. *Discuss. Math. Graph Theory* **25** (2005) 29–34], and for the triple domination number by Rautenbach and Volkmann [New bounds on the k-domination number and the k-tuple domination number. *Applied Math. Letters* **20** (2007) 98–102], who also posed the interesting conjecture on the k-tuple domination number: for any graph G with $\delta \ge k-1$,

$$\gamma_{\times k}(G) \le \frac{\ln(\delta-k+2)+\ln(\widehat{d}_{k-1}+\widehat{d}_{k-2})+1}{\delta-k+2} n,$$

where $\widehat{d}_m = \sum_{i=1}^{n} \binom{d_i}{m} /n$ is the m-degree of G. This conjecture, if true, would generalise all the mentioned upper bounds and improve an upper bound proved in [A. Gagarin and V. Zverovich, A generalised upper bound for the k-tuple domination number. *Discrete Math.* (to appear)].

In this paper, we prove Rautenbach–Volkmann's conjecture.

Keywords: *graphs, domination number, double domination, triple domination, k-tuple domination.*

1 Notation

All graphs will be finite and undirected without loops and multiple edges. If G is a graph of order n, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in G, d_i denotes the degree of v_i and $d = \sum_{i=1}^{n} d_i/n$ is the average degree of G. Let $N(x)$ denote the neighbourhood of a vertex x. Also let $N(X) = \cup_{x\in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of G, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$. A set X is called a dominating set if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is the domination number $\gamma(G)$. A set X is called a k-tuple dominating set of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \ge k$. The minimum cardinality of a k-tuple dominating set of G is the k-tuple domination number $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $\delta \ge k-1$. It is easy to see that $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \le \gamma_{\times k'}(G)$ for $k \le k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is called the double domination number and the 3-tuple domination number $\gamma_{\times 3}(G)$ is called the triple domination number. A number of interesting results on the k-tuple domination number can be found in [3]–[8] and [11].

2 Introduction

The following fundamental result was proved by many authors:

Theorem 1 ([1, 2, 9, 10]) *For any graph G,*

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1} n.$$

A similar upper bound for the double domination number was found by Harant and Henning [4]:

Theorem 2 ([4]) *For any graph G with $\delta \geq 1$,*

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(d+1)+1}{\delta} n.$$

Rautenbach and Volkmann posed the following interesting conjecture for the k-tuple domination number:

Conjecture 1 ([11]) *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2)+\ln\left(\sum_{i=1}^{n}\binom{d_i+1}{k-1}\right)-\ln(n)+1}{\delta-k+2} n.$$

For $m \leq \delta$, let us define the m-degree $\widehat{d}_m$ of a graph G as follows:

$$\widehat{d}_m = \widehat{d}_m(G) = \sum_{i=1}^{n}\binom{d_i}{m}/n.$$

Note that $\widehat{d}_1$ is the average degree d of a graph and $\widehat{d}_0 = 1$. Also, we put $\widehat{d}_{-1} = 0$.

Since

$$\binom{d_i+1}{k-1} = \binom{d_i}{k-1} + \binom{d_i}{k-2},$$

we see that the above conjecture can be re-formulated as follows:

Conjecture 1′ *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2)+\ln(\widehat{d}_{k-1}+\widehat{d}_{k-2})+1}{\delta-k+2} n.$$

It may be pointed out that this conjecture, if true, would generalise Theorem 2 and also Theorem 1 taking into account that $\widehat{d}_{-1} = 0$. Rautenbach and Volkmann proved the above conjecture for the triple domination number:

Theorem 3 ([11]) *For any graph G with $\delta \geq 2$,*

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta-1)+\ln(\widehat{d}_2+d)+1}{\delta-1} n.$$

The next result generalises all the above theorems, but it is still far from Conjecture 1′.

Theorem 4 ([3]) *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2)+\ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m+\epsilon\right)+1}{\delta-k+2} n,$$

where $\epsilon = 1$ if $k = 1$ or 2, and $\epsilon = -d$ if $k \geq 3$.

3 Proof of the Conjecture

The following theorem proves Rautenbach–Volkmann's conjecture.

Theorem 5 *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta - k + 2} n.$$

Proof: Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability p, $0 \leq p \leq 1$. For $m = 0, 1, ..., k-1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, ..., k-2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set A_m, we form a set A'_m in the following way. For every vertex in the set A_m, we take $k-m-1$ neighbours not in A and add them to A'_m. Such neighbours always exist because $\delta \geq k-1$. It is obvious that $|A'_m| \leq (k-m-1)|A_m|$. For each set B_m, we form a set B'_m by taking $k-m-1$ neighbours not in A for every vertex in B_m. We have $|B'_m| \leq (k-m-1)|B_m|$.

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m \right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set D is a k-tuple dominating set. Indeed, if there is a vertex v which is not k-tuple dominated by D, then v is not k-tuple dominated by A. Therefore, v would belong to A_m or B_m for some m, but all such vertices are k-tuple dominated by the set D by construction.

The expected value of $|D|$ is

$$\begin{aligned} E(|D|) &\leq E\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m| \right) \\ &\leq E\left(|A| + \sum_{m=0}^{k-2} (k-m-1)|A_m| + \sum_{m=0}^{k-1} (k-m)|B_m| \right) \\ &= E(|A|) + \sum_{m=0}^{k-2} (k-m-1)E(|A_m|) + \sum_{m=0}^{k-1} (k-m)E(|B_m|). \end{aligned}$$

We have

$$E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn.$$

Also,

$$\begin{aligned} E(|A_m|) &= \sum_{i=1}^{n} P(v_i \in A_m) \\ &= \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1-p)^{d_i - m} \\ &\leq p^{m+1}(1-p)^{\delta - m} \sum_{i=1}^{n} \binom{d_i}{m} \\ &= p^{m+1}(1-p)^{\delta - m} \widehat{d}_m n \end{aligned}$$

and

$$\begin{aligned}
E(|B_m|) &= \sum_{i=1}^{n} P(v_i \in B_m) \\
&= \sum_{i=1}^{n} (1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m} \\
&\leq p^m (1-p)^{\delta - m + 1} \sum_{i=1}^{n} \binom{d_i}{m} \\
&= p^m (1-p)^{\delta - m+1} \widehat{d}_m n.
\end{aligned}$$

Taking into account that $\widehat{d}_{-1} = 0$, we obtain

$$\begin{aligned}
E(|D|) &\leq pn + \sum_{m=0}^{k-2} (k-m-1) p^{m+1} (1-p)^{\delta - m} \widehat{d}_m n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta - m+1} \widehat{d}_m n \\
&= pn + \sum_{m=1}^{k-1} (k-m) p^m (1-p)^{\delta - m+1} \widehat{d}_{m-1} n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta-m+1} \widehat{d}_m n \\
&= pn + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta - m+1} (\widehat{d}_{m-1} + \widehat{d}_m) n \\
&= pn + (1-p)^{\delta - k+2} n \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_{m-1} + \widehat{d}_m).
\end{aligned}$$

Let us denote

$$\mu = \delta - k + 2.$$

Using the inequality $1 - x \leq e^{-x}$, we obtain

$$(1-p)^{\delta - k+2} = (1-p)^{\mu} \leq e^{-p\mu}.$$

Thus,

$$E(|D|) \leq pn + e^{-p\mu} n \Theta,$$

where

$$\Theta = \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_m + \widehat{d}_{m-1}). \tag{1}$$

We will prove that

$$\Theta \leq \widehat{d}_{k-1} + \widehat{d}_{k-2}.$$

We have

$$\begin{aligned}
\Theta &= \sum_{m=0}^{k-1} (k-m)(\widehat{d}_m + \widehat{d}_{m-1}) \sum_{i=0}^{k-m-1} (-1)^i \binom{k-m-1}{i} p^{m+i} \\
&= k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{0} p^0 - k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{1} p^1 + \ldots + k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{k-1} (-1)^{k-1} p^{k-1} \\
&\qquad + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{0} p^1 + \ldots + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{k-2} (-1)^{k-2} p^{k-1} \\
&\qquad\qquad \ldots \\
&\qquad\qquad\qquad \ldots \\
&\qquad\qquad\qquad\qquad \ldots
\end{aligned}$$

$$+(1)(\widehat{d}_{k-1}+\widehat{d}_{k-2})\binom{0}{0}(-1)^0 p^{k-1}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-1}\Big(\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)(\widehat{d}_{k-i-j-1}+\widehat{d}_{k-i-j-2})\Big)p^{k-j-1}\\
&= \sum_{j=0}^{k-1} s_j p^{k-j-1},
\end{aligned}
$$

where

$$
\begin{aligned}
s_j &= \sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)(\widehat{d}_{k-i-j-1}+\widehat{d}_{k-i-j-2})\\
&\quad \text{(taking into account that } \widehat{d}_{-1}=0\text{)}\\
&= \sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-1}+\sum_{i=0}^{k-j-2}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-2}\\
&= \binom{j}{0}(j+1)\widehat{d}_{k-j-1}+\sum_{i=1}^{k-j-1}(-1)^i\binom{i+j}{i}(i+j+1)\widehat{d}_{k-i-j-1}\\
&\qquad +\sum_{i=1}^{k-j-1}(-1)^{i-1}\binom{i+j-1}{i-1}(i+j)\widehat{d}_{k-i-j-1}\\
&= (j+1)\widehat{d}_{k-j-1}+\sum_{i=1}^{k-j-1}(-1)^i(j+1)\binom{i+j}{i}\widehat{d}_{k-i-j-1}\\
&= (j+1)\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\widehat{d}_{k-i-j-1}\\
&= (j+1)\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\sum_{l=1}^{n}\binom{d_l}{k-i-j-1}/n\\
&= (j+1)\sum_{l=1}^{n}\sum_{i=0}^{k-j-1}(-1)^i\binom{i+j}{i}\binom{d_l}{k-i-j-1}/n\\
&= (j+1)\sum_{l=1}^{n}\binom{d_l-j-1}{k-j-1}/n \qquad \text{(by Lemma 3)}\\
&\geq 0.
\end{aligned}
$$

Thus, the function $\Theta(p)=s_0p^{k-1}+s_1p^{k-2}+...+s_{k-1}$ is monotonously increasing in $0\leq p\leq 1$. Therefore, (1) implies

$$\Theta \leq \widehat{d}_{k-1}+\widehat{d}_{k-2}.$$

We obtain

$$E(|D|)\leq pn+e^{-p\mu}n\Theta\leq pn+e^{-p\mu}n(\widehat{d}_{k-1}+\widehat{d}_{k-2}).$$

Let us denote

$$f(p)=pn+e^{-p\mu}n(\widehat{d}_{k-1}+\widehat{d}_{k-2}).$$

For $p\in[0,1]$, the function $f(p)$ is minimised at the point $\min\{1,z\}$, where

$$z=\frac{\ln\mu+\ln(\widehat{d}_{k-1}+\widehat{d}_{k-2})}{\mu}.$$

There are two cases to consider.

If $z \le 1$, then

$$E(|D|) \le f(z) = \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n.$$

Since the expected value is an average value, there exists a particular k-tuple dominating set of order at most $f(z)$, as required.

Suppose now that $z > 1$. Taking into account that $\mu > 0$, we obtain

$$\gamma_{\times k}(G) \le n < \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n,$$

as required. The proof of Theorem 5 is complete. ∎

For $s \ge 1$, let us denote

$$T_t^s = \binom{s}{t} - \binom{s}{t-1} + ... + (-1)^t \binom{s}{0}.$$

Lemma 1

$$T_t^s = \binom{s-1}{t}.$$

Proof: Induction on t:

$$T_t^s = \binom{s}{t} - T_{t-1}^s = \binom{s}{t} - \binom{s-1}{t-1} = \binom{s-1}{t}.$$

∎

Lemma 2 *For $j \ge 1$,*

$$\binom{j-1}{0} + \binom{j}{1} + ... + \binom{j+i-1}{i} = \binom{j+i}{i}.$$

Proof: Induction on i:

$$\binom{j-1}{0} + \binom{j}{1} + ... + \binom{j+i-1}{i} = \binom{j+i-1}{i-1} + \binom{j+i-1}{i} = \binom{j+i}{i}.$$

∎

Lemma 3

$$\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \binom{r-j-1}{l}.$$

Proof: Induction on j. If $j = 0$, then

$$\sum_{i=0}^{l} (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \sum_{i=0}^{l} (-1)^i \binom{r}{l-i} = T_l^r = \binom{r-1}{l},$$

as required.

Suppose that $j \ge 1$ and the equation of Lemma 3 is true for any $j' \le j-1$. Applying Lemmas 1 and 2, we obtain:

$$
\begin{aligned}
\sum_{i=0}^{l}(-1)^i\binom{i+j}{i}\binom{r}{l-i} &= \sum_{i=0}^{l}(-1)^i\left(\binom{j-1}{0}+\binom{j}{1}+\ldots+\binom{j+i-1}{i}\right)\binom{r}{l-i}\\
&= \binom{j-1}{0}\sum_{i=0}^{l}(-1)^i\binom{r}{l-i}+\binom{j}{1}\sum_{i=1}^{l}(-1)^i\binom{r}{l-i}+\ldots\\
&\qquad +\binom{j+l-1}{l}\sum_{i=l}^{l}(-1)^l\binom{r}{0}\\
&= \binom{j-1}{0}T_l^r-\binom{j}{1}T_{l-1}^r+\ldots+\binom{j+l-1}{l}(-1)^lT_0^r\\
&= \sum_{i=0}^{l}(-1)^i\binom{j+i-1}{i}T_{l-i}^r\\
&= \sum_{i=0}^{l}(-1)^i\binom{j+i-1}{i}\binom{r-1}{l-i}\\
&= \binom{r-j-1}{l}. \qquad\qquad \text{(by hypothesis)}
\end{aligned}
$$

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References

[1] N. Alon and J.H. Spencer, *The Probabilistic Method*, John Wiley and Sons, Inc., 1992.

[2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices. *Prikl. Mat. i Programmirovanie* (11)(1974) 3–8.

[3] A. Gagarin and V. Zverovich, A generalised upper bound for the k-tuple domination number. *Discrete Math.* (to appear)

[4] J. Harant and M.A. Henning, On double domination in graphs. *Discuss. Math. Graph Theory* **25** (2005) 29–34.

[5] F. Harary and T.W. Haynes, Double domination in graphs. *Ars Combin.* **55** (2000) 201–213.

[6] F. Harary and T.W. Haynes, Nordhaus-Gaddum inequalities for domination in graphs. *Discrete Math.* **155** (1996) 99-105.

[7] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[8] R. Klasing and C. Laforest, Hardness results and approximation algorithms of k-tuple domination in graphs. *Inform. Processing Letters* **89** (2)(2004) 75–83.

[9] L. Lovász, On the ratio of optimal integral and fractional covers. *Discrete Math.* **13** (1975) 383–390.

[10] C. Payan, Sur le nombre d'absorption d'un graphe simple. *Cahiers Centre Études Recherche Opér.* **17** (1975) 307–317.

[11] D. Rautenbach and L. Volkmann, New bounds on the k-domination number and the k-tuple domination number. *Applied Math. Letters* **20** (2007) 98–102.