A Generalised Upper Bound for the k-Tuple Domination Number

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Abstract

In this paper, we provide an upper bound for the k-tuple domination number that generalises known upper bounds for the double and triple domination numbers. We prove that for any graph G,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m + \epsilon\right) + 1}{\delta - k + 2} n,$$

where $\gamma_{\times k}(G)$ is the k-tuple domination number; δ is the minimal degree; $\widehat{d}_m$ is the m-degree of G; $\epsilon = 1$ if $k = 1$ or 2 and $\epsilon = -d$ if $k \geq 3$; d is the average degree.

We consider finite and undirected graphs without loops and multiple edges. If G is a graph, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in G, d_i denotes the degree of v_i and $d = (\sum_{i=1}^{n} d_i)/n$ is the average degree of G. Let $N(x)$ denote the neighbourhood of a vertex x. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$, where X is a vertex set of G. Denote by $\delta(G)$ the minimal vertex degree of G, and put $\delta = \delta(G)$.

A set $X \subseteq V(G)$ is called a *dominating set* if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is the *domination* *number* $\gamma(G)$. A set X is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k-tuple dominating set of G is the *k-tuple domination number* $\gamma_{\times k}(G)$. It is easy to see that the k-tuple domination number is only defined for graphs with $\delta \geq k-1$. Also, $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \leq \gamma_{\times k'}(G)$ for $k \leq k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ and the 3-tuple domination number $\gamma_{\times 3}(G)$ are called the *double domination number* and the *triple domination number*, respectively. A number of interesting results on the k-tuple domination number can be found in [3]–[9] and [11].

Alon and Spencer [1], Arnautov [2] and Payan [10] independently proved the following fundamental result:

Theorem 1 ([1, 2, 10]) *For any graph G,*

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1} n.$$

Harant and Henning [3] found an upper bound for the double domination number:

Theorem 2 ([3]) *For any graph G with $\delta \geq 1$,*

$$\gamma_{\times 2}(G) \leq \frac{\ln\delta + \ln(d+1)+1}{\delta} n.$$

An interesting upper bound for the triple domination number was given by Rautenbach and Volkmann [11]:

Theorem 3 ([11]) *For any graph G with $\delta \geq 2$,*

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta-1) + \ln(d+\widehat{d}_2)+1}{\delta-1} n,$$

where $\widehat{d}_2 = \frac{1}{n}\sum_{i=1}^{n} \binom{d_i}{2}$.

The following theorem generalises this bound for the k-tuple domination number. For $m \leq \delta$, let us define the m-degree $\widehat{d}_m$ of a graph G as follows:

$$\widehat{d}_m = \widehat{d}_m(G) = \frac{1}{n}\sum_{i=1}^{n} \binom{d_i}{m}.$$

Note that $\widehat{d}_1$ is the average degree d of a graph and $\widehat{d}_0 = 1$.

Theorem 4 *For any graph G with $3 \leq k \leq \delta+1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln\left((k-2)d + (2k-5)\widehat{d}_2 + \sum_{m=3}^{k-1}(k-m)\widehat{d}_m\right)+1}{\delta-k+2} n.$$

Proof: Let A be formed by an independent choice of vertices of G, where each vertex is selected with probability p, $0 \leq p \leq 1$. For $m = 0, 1, ..., k-1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, ..., k-2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set A_m, we form a set A'_m in the following way. For every vertex in the set A_m, we take $k-m-1$ neighbours not in A. Such neighbours always exist because $\delta \geq k-1$. It is obvious that $|A'_m| \leq (k-m-1)|A_m|$. For each set B_m, we form a set B'_m by taking $k-m-1$ neighbours not in A for every vertex in B_m. We have $|B'_m| \leq (k-m-1)|B_m|$.

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m \right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set D is a k-tuple dominating set. Indeed, if there is a vertex v which is not k-tuple dominated by D, then v is not k-tuple dominated by A. Therefore, v would belong to A_m or B_m for some m, but all such vertices are k-tuple dominated by the set D by construction.

The expectation of $|D|$ is

$$\begin{aligned}
E(|D|) &\leq E(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|) \\
&\leq E(|A| + \sum_{m=0}^{k-2} (k-m-1)|A_m| + \sum_{m=0}^{k-1} (k-m)|B_m|) \\
&= E(|A|) + \sum_{m=0}^{k-2} (k-m-1)E(|A_m|) + \sum_{m=0}^{k-1} (k-m)E(|B_m|).
\end{aligned}$$

We have

$$E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn.$$

Also,

$$\begin{aligned}
E(|A_m|) &= \sum_{i=1}^{n} P(v_i \in A_m) = \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1-p)^{d_i - m} \\
&\leq p^{m+1}(1-p)^{\delta - m} \sum_{i=1}^{n} \binom{d_i}{m} = p^{m+1}(1-p)^{\delta - m} \widehat{d}_m n
\end{aligned}$$

and

$$\begin{aligned}
E(|B_m|) &= \sum_{i=1}^{n} P(v_i \in B_m) = \sum_{i=1}^{n} (1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m} \\
&\leq p^{m}(1-p)^{\delta - m + 1} \sum_{i=1}^{n} \binom{d_i}{m} = p^{m}(1-p)^{\delta - m + 1} \widehat{d}_m n.
\end{aligned}$$

Therefore,

$$\begin{aligned}E(|D|) &\le pn + (k-1)E(|A_0|) + (k-2)E(|A_1|)\\ &+(k-3)E(|A_2|) + \sum_{m=3}^{k-2}(k-m-1)E(|A_m|)\\ &+kE(|B_0|) + (k-1)E(|B_1|) + (k-2)E(|B_2|) + \sum_{m=3}^{k-1}(k-m)E(|B_m|).\end{aligned}$$

Let us denote

$$\mu = \delta - k + 2.$$

Since $k \ge 3$, we have

$$(1-p)^{\delta-1} \le (1-p)^{\delta-k+2} = (1-p)^{\mu}.$$

Using the inequality $1 - x \le e^{-x}$, we obtain

$$(1-p)^{\delta-1} \le e^{-p\mu}.$$

For the second and third terms of the above bound for $E(|D|)$, we have:

$$\begin{aligned}(k-1)E(|A_0|) + (k-2)E(|A_1|) &\le (k-1)p(1-p)^{\delta}n + (k-2)p^2(1-p)^{\delta-1}\widehat{d}_1 n\\ &\le (k-1)p(1-p)e^{-p\mu}n + (k-2)p^2 d e^{-p\mu}n.\end{aligned}$$

Let us consider the fourth term $(k-3)E(|A_2|)$. We may assume that $k \ge 4$, for otherwise $k = 3$ and all the inequalities in (1) are true. Note that for $k \ge 4$,

$$(1-p)^{\delta-2} \le (1-p)^{\delta-k+2} = (1-p)^{\mu} \le e^{-p\mu}.$$

We obtain

$$(k-3)E(|A_2|) \le (k-3)p^3(1-p)^{\delta-2}\widehat{d}_2 n \le (k-3)p^3\widehat{d}_2 e^{-p\mu}n. \tag{1}$$

Furthermore,

$$\begin{aligned}\sum_{m=3}^{k-2}(k-m-1)E(|A_m|) &\le \sum_{m=3}^{k-2}(k-m-1)p^{m+1}(1-p)^{\delta-m}\widehat{d}_m n\\ &\le (1-p)^{\mu} n \sum_{m=3}^{k-2}(k-m-1)p\widehat{d}_m\\ &\le e^{-p\mu} n \sum_{m=3}^{k-2}(k-m-1)p\widehat{d}_m.\end{aligned}$$

For the next three terms, we obtain

$$\begin{aligned}kE(|B_0|) &+ (k-1)E(|B_1|) + (k-2)E(|B_2|)\\ &\le k(1-p)^{\delta+1}n + (k-1)p(1-p)^{\delta}\widehat{d}_1 n + (k-2)p^2(1-p)^{\delta-1}\widehat{d}_2 n\\ &\le k(1-p)^2 e^{-p\mu}n + (k-1)p(1-p)d e^{-p\mu}n + (k-2)p^2\widehat{d}_2 e^{-p\mu}n.\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{m=3}^{k-1}(k-m)E(|B_m|) &\leq \sum_{m=3}^{k-1}(k-m)p^m(1-p)^{\delta-m+1}\widehat{d}_m n \\ &\leq \sum_{m=3}^{k-2}(k-m)(1-p)(1-p)^{\delta-m}\widehat{d}_m n + (1-p)^{\mu}\widehat{d}_{k-1}n \\ &\leq e^{-p\mu}n\Big(\sum_{m=3}^{k-2}(k-m)(1-p)\widehat{d}_m + \widehat{d}_{k-1}\Big).\end{aligned}$$

Thus,

$$E(|D|) \leq pn + e^{-p\mu}n\Omega,$$

where

$$\begin{aligned}\Omega &= (k-1)p(1-p) + (k-2)p^2 d + (k-3)p^3\widehat{d}_2 + \sum_{m=3}^{k-2}(k-m-1)p\widehat{d}_m \\ &\quad +k(1-p)^2 + (k-1)p(1-p)d + (k-2)p^2\widehat{d}_2 + \sum_{m=3}^{k-2}(k-m)(1-p)\widehat{d}_m + \widehat{d}_{k-1} \\ &\leq p^3(k-3)\widehat{d}_2 + p^2(\widehat{d}_2(k-2) - d + 1) + p(d(k-1) - k - 1) + k + \sum_{m=3}^{k-1}(k-m)\widehat{d}_m.\end{aligned}$$

Taking into account that $k \geq 3$ and $\delta \geq 2$, we obtain

$$\begin{aligned}\widehat{d}_2(k-2) - d + 1 \geq \widehat{d}_2 - d + 1 &= \Big(\sum_{i=1}^{n} 0.5d_i(d_i - 1) - \sum_{i=1}^{n} d_i + n\Big)/n \\ &= \sum_{i=1}^{n}(0.5d_i^2 - 1.5d_i + 1)/n = \sum_{i=1}^{n}((0.5d_i - 1)(d_i - 1))/n \geq 0\end{aligned}$$

and

$$d(k-1) - k - 1 = k(d-1) - d - 1 \geq 3(d-1) - d - 1 = 2d - 4 \geq 0.$$

Hence

$$\begin{aligned}\Omega &\leq (k-3)\widehat{d}_2 + \widehat{d}_2(k-2) - d + 1 + d(k-1) - k - 1 + k + \sum_{m=3}^{k-1}(k-m)\widehat{d}_m \\ &= (k-2)d + (2k-5)\widehat{d}_2 + \sum_{m=3}^{k-1}(k-m)\widehat{d}_m.\end{aligned}$$

If we denote the last expression by Ψ, then

$$E(|D|) \leq f(p) = pn + e^{-p\mu}n\Psi.$$

For $p \in [0,1]$, the function $f(p)$ is minimised at the point $\min\{1, z\}$, where

$$z = \frac{\ln\mu + \ln\Psi}{\mu}.$$

If $z > 1$, then $f(p)$ is minimised at the point $p = 1$ and the result easily follows. If $z \le 1$, then

$$E(|D|) \le f(z) = \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln \Psi + 1}{\mu} n.$$

Since the expectation is an average value, there exists a particular k-tuple dominating set of order at most $f(z)$, as required. The proof of Theorem 4 is complete. ∎

By a simple modification of the proof of Theorem 4, we obtain the following result:

Corollary 1 *For any graph G with $3 \le k \le \delta + 1$,*

$$\gamma_{\times k}(G) \le \frac{\ln(\delta - k + 2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m - d\right) + 1}{\delta - k + 2} n.$$

Proof: If $k = 3$, then the result follows from Theorem 4. Thus, we may assume that $4 \le k \le \delta + 1$.

Using the notation of the proof of Theorem 4, we obtain:

$$\begin{aligned} E(|D|) &\le pn + (k-1)E(|A_0|) + (k-2)E(|A_1|) + \sum_{m=2}^{k-2}(k-m-1)E(|A_m|) \\ &\quad + kE(|B_0|) + (k-1)E(|B_1|) + \sum_{m=2}^{k-1}(k-m)E(|B_m|). \end{aligned}$$

Therefore,

$$E(|D|) \le pn + e^{-p\mu} n \Omega,$$

where

$$\begin{aligned} \Omega &= (k-1)p(1-p) + (k-2)p^2 d + \sum_{m=2}^{k-2}(k-m-1)p\widehat{d}_m \\ &\quad + k(1-p)^2 + (k-1)p(1-p)d + \sum_{m=2}^{k-2}(k-m)(1-p)\widehat{d}_m + \widehat{d}_{k-1} \\ &\le p^2(1-d) + p(d(k-1) - k - 1) + k + \sum_{m=2}^{k-1}(k-m)\widehat{d}_m. \end{aligned}$$

If $k \ge 4$, then $d \ge \delta \ge 3$ and the function $p^2(1-d) + p(d(k-1)-k-1)$ is monotonically increasing from 0 to 1. Therefore,

$$\Omega \le \Psi = \sum_{m=1}^{k-1}(k-m)\widehat{d}_m - d.$$

The remaining part of the proof is similar to the final part of the proof of Theorem 4. ∎

The next result summarizes all the above theorems and corollaries.

Corollary 2 *For any graph G with $k \le \delta + 1$,*

$$\gamma_{\times k}(G) \le \frac{\ln(\delta - k + 2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m + \epsilon\right) + 1}{\delta - k + 2} n,$$

where $\epsilon = 1$ if $k = 1$ or 2, and $\epsilon = -d$ if $k \ge 3$.

Acknowledgment The authors are thankful to the anonymous referees for their valuable comments and suggestions.

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