On Roman, Global and Restrained Domination in Graphs

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Abstract

In this paper, we present new upper bounds for the global domination and Roman domination numbers and also prove that these results are asymptotically best possible. Moreover, we give upper bounds for the restrained domination and total restrained domination numbers for large classes of graphs, and show that, for almost all graphs, the restrained domination number is equal to the domination number, and the total restrained domination number is equal to the total domination number. A number of open problems are posed.

Keywords: graphs, Roman domination number, global domination number, restrained domination number.

1 Introduction

All graphs will be finite and undirected without loops and multiple edges. If G is a graph of order n, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in G. Let $N(x)$ denote the neighbourhood of a vertex x. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of G, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

A set X is called a *dominating set* if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is called the *domination number* $\gamma(G)$. The following fundamental result for the domination number was proved by many authors [1, 3, 9, 12]:

Theorem 1 ([1, 3, 9, 12]) *For any graph G,*

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.$$

Let H be a k-uniform hypergraph with n vertices and m edges. The transversal number $\tau(H)$ of H is the minimum cardinality of a set of vertices that intersects all edges of H. Alon [2] proved a fundamental result that if $k > 1$, then

$$\tau(H) \leq \frac{\ln k}{k}(n + m).$$

He also showed that this bound is asymptotically best possible, i.e. there exist k-uniform hypergraphs H such that for sufficiently large k,

$$\tau(H) = \frac{\ln k}{k}(n+m)(1+o(1)).$$

Alon [2] gives an interesting probabilistic construction of such a hypergraph H. In fact, H is a random k-uniform hypergraph on $[k \ln k]$ vertices with k edges constructed by choosing each edge randomly and independently according to a uniform distribution on k-subsets of the vertex set. This construction implies that the above bound for the domination number is asymptotically best possible:

Theorem 2 ([2]) *When n is large there exists a graph G such that*

$$\gamma(G) \geq \frac{\ln(\delta+1)+1}{\delta+1} n(1+o(1)).$$

The concept of *global domination* was introduced by Brigham and Dutton [5] and also by Sampathkumar [15]. It is a variant of the domination number. A set X is called a *global dominating set* if X is a dominating set in both G and its *complement* $\overline{G}$. The minimum cardinality of a global dominating set of G is called the *global domination number* $\gamma_g(G)$. There are a number of bounds on the global domination number $\gamma_g(G)$. Brigham and Dutton in [5] give the following bounds on the global domination number in terms of order, minimum and maximum degrees, and the domination number of G:

Theorem 3 ([5]) *If either G or $\overline{G}$ is disconnected, then*

$$\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}.$$

Theorem 4 ([5]) *For any graph G, either*

$$\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\} \text{ or } \gamma_g(G) \leq \min\{\Delta(G), \Delta(\overline{G})\} + 1.$$

Theorem 5 ([5]) *For any graph G, if $\delta(G) = \delta(\overline{G}) \leq 2$, then*

$$\gamma_g(G) \leq \delta(G) + 2;$$

otherwise

$$\gamma_g(G) \leq \max\{\delta(G), \delta(\overline{G})\} + 1.$$

Another variant of the domination number, the *Roman domination number*, was introduced by Stewart [16]. In [16] and [14], a *Roman dominating function* (RDF) of a graph G is defined as a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is defined as the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The Roman domination number of a graph G, denoted $\gamma_R(G)$, is equal to the minimum weight of an RDF on G. In fact, Roman domination is of both historical and mathematical interest. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighbouring location only if there was a second army which would stay and protect the home. Thus, there were two types of armies: stationary and travelling. Each vertex with no army must have a neighbouring vertex with a travelling army. Stationary armies then dominate their own vertices, and a vertex with two armies is dominated by its stationary army, and its open neighbourhood is dominated by the travelling army. Thus, the definition of Roman domination has its historical background and it can be used for the problems of this type, which arise in military and commercial decision making. The following results about the Roman domination number are known:

Theorem 6 ([6]) *For any graph G,*

$$\gamma(G) \le \gamma_R(G) \le 2\gamma(G).$$

Theorem 7 ([6]) *For any graph G of order n and maximum degree Δ,*

$$\gamma_R(G) \ge \frac{2n}{\Delta+1}.$$

Telle and Proskurowski [17] introduced restrained domination as a vertex partitioning problem. A dominating set X of a graph G is called a *restrained dominating set* if every vertex in $V(G)-X$ is adjacent to a vertex in $V(G)-X$. If, in addition, every vertex of X is adjacent to a vertex of X, then X is called a *total restrained dominating set.* The minimum cardinality of a restrained dominating set of G is the *restrained domination number* $\gamma_r(G)$, and the minimum cardinality of a total restrained dominating set of G is the *total restrained domination number* $\gamma_{\rm tr}(G)$. For these parameters, the following upper bounds have been found:

Theorem 8 ([7]) *If $\delta(G) \ge 2$, then*

$$\gamma_r(G) \le n - \Delta.$$

Theorem 9 ([8]) *If G is a connected graph with $n \ge 4$, $\delta \ge 2$ and $\Delta \le n-2$, then*

$$\gamma_{\rm tr}(G) \le n - \frac{\Delta}{2} - 1.$$

In this paper, we present new upper bounds for the global and roman domination numbers, and show that our results are asymptotically best possible. Moreover, we give upper bounds for the restrained domination and total restrained domination numbers for large classes of graphs. A number of open problems are posed.

2 Upper Bounds for the Global Domination Number

The following theorem provides an upper bound for the global domination number. In what follows, we denote $\bar{\delta} = \delta(\overline{G})$ and

$$\delta' = \min\{\delta, \bar{\delta}\}.$$

Theorem 10 *For any graph G with $\delta' > 0$,*

$$\gamma_g(G) \le \left(1 - \frac{\delta'}{2^{1/\delta'}\,(1+\delta')^{1+1/\delta'}}\right) n.$$

Proof: Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

$$p = 1 - \frac{1}{2^{1/\delta'}\,(1+\delta')^{1/\delta'}}.$$

Let us denote $B = V(G) - N[A]$ and $C = \{v_i \in V(G), v_i \,\text{is not dominated by}\, A \,\text{in}\, \overline{G}\}$. It is easy to show that

$$\begin{aligned}\mathbf{P}[v_i \in B] &= (1-p)^{1+\deg(v_i)} &&\le (1-p)^{1+\delta}\\ \mathbf{P}[v_i \in C] &= (1-p)^{1+(n-\deg(v_i)-1)} &&\le (1-p)^{1+\bar{\delta}}.\end{aligned}$$

It is obvious that the set $D = A \cup B \cup C$ is a global dominating set. The expectation of $|D|$ is

$$\begin{aligned}\mathbf{E}[|D|] &\le \mathbf{E}[|A|] + \mathbf{E}[|B|] + \mathbf{E}[|C|]\\ &= pn + \sum_{i=1}^{n} \mathbf{P}[v_i \in B] + \sum_{i=1}^{n} \mathbf{P}[v_i \in C]\\ &\le pn + (1-p)^{1+\delta} n + (1-p)^{1+\bar{\delta}} n\\ &\le pn + 2(1-p)^{1+\min\{\delta,\bar{\delta}\}} n\\ &= pn + 2(1-p)^{1+\delta'} n\\ &= \left(1 - \frac{\delta'}{2^{1/\delta'}(1+\delta')^{1+1/\delta'}}\right) n, \qquad (1)\end{aligned}$$

as required. The proof of the theorem is complete. ∎

The proof of Theorem 10 implies the following upper bound, which is asymptotically same as the bound of Theorem 10.

Corollary 1 *For any graph G,*

$$\gamma_g(G) \le \frac{\ln(\delta'+1) + \ln 2 + 1}{\delta'+1} n.$$

Proof: Using the inequality $1 - p \le e^{-p}$, we obtain the following estimation of the expression (1):

$$\mathbf{E}[|D|] \le pn + 2e^{-p(\delta'+1)} n.$$

If we put $p = \min\{1, \frac{\ln(\delta'+1)+\ln 2}{\delta'+1}\}$, then

$$\mathbf{E}[|D|] \le \frac{\ln(\delta'+1) + \ln 2 + 1}{\delta'+1} n,$$

as required. ∎

We now prove that the upper bound of Corollary 1, and therefore of Theorem 10, is asymptotically best possible.

Theorem 11 *When n is large there exists a graph G such that*

$$\gamma_g(G) \ge \frac{\ln(\delta'+1) + \ln 2 + 1}{\delta'+1} n(1 + o(1)).$$

Proof: Let us modify Alon's probabilistic construction described in the introduction as follows. Let F be a complete graph $K_{[\delta \ln \delta]}$, and let us denote $F = V(F)$. Next, we add a set of new vertices $V = \{v_1, ..., v_\delta\}$, where each vertex v_i is adjacent to δ vertices that are randomly chosen from the set F. Let us add a new component $K_{\delta+1}$ and denote the resulting graph by G, which has $n = [\delta \ln \delta] + 2\delta + 1$ vertices. Note that $\delta' = \delta$ because $\bar{\delta} > \delta$. We will prove that with high probability

$$\gamma_g(G) \geq \frac{\ln \delta'}{\delta'} n(1 + o_{\delta'}(1)) = \frac{\ln \delta}{\delta} n(1 + o_\delta(1)) = \ln^2 \delta(1 + o_\delta(1)).$$

Let us denote by H the graph G without the component $K_{\delta+1}$. It is obvious that

$$\gamma_g(G) = \gamma(H) + 1.$$

Therefore, the result will follow if we can prove that with high probability

$$\gamma(H) > \ln^2 \delta(1 + o_\delta(1)).$$

Without loss of generality we may only consider dominating sets in H that are subsets of F. Let us consider a dominating set X in H such that $X \subseteq F$ and $|X| \leq \ln^2 \delta - \ln \delta \ln \ln^5 \delta$. It is easy to show that the probability of the set X not dominating a vertex $v_i \in V$ is

$$\mathbf{P}[X \text{ does not dominate } v_i] \quad = \quad \frac{\binom{|F| - |X|}{\delta}}{\binom{|F|}{\delta}} \geq \left(\frac{|F| - |X| - \delta}{|F| - \delta}\right)^\delta = \left(1 - \frac{|X|}{|F| - \delta}\right)^\delta.$$

Using the inequality $1 - x \geq e^{-x}(1 - x^2)$ if $x < 1$, we obtain the following estimation:

$$\begin{aligned}
\mathbf{P}[X \text{ does not dominate } v_i] \quad &\geq \quad e^{-\frac{\ln^2 \delta - \ln \delta \ln \ln^5 \delta}{\delta \ln \delta - \delta}\delta} \left(1 - \left(\frac{\ln^2 \delta - \ln \delta \ln \ln^5 \delta}{\delta \ln \delta - \delta}\right)^2\right)^\delta \\
&= \quad e^{\frac{-\ln \delta + \ln \ln^5 \delta}{1 - 1/\ln \delta}} (1 + o_\delta(1)) \\
&= \quad e^{\ln\left(\frac{\ln^5 \delta}{\delta}\right)(1 + o_\delta(1))} (1 + o_\delta(1)) \\
&= \quad \left(\frac{\ln^5 \delta}{\delta}\right)^{1 + o_\delta(1)} (1 + o_\delta(1)) \\
&\geq \quad \frac{\ln^4 \delta}{\delta}.
\end{aligned}$$

Thus, we conclude that

$$\mathbf{P}[X \text{ dominates } V] \leq \left(1 - \frac{\ln^4 \delta}{\delta}\right)^\delta \leq e^{-\ln^4 \delta}.$$

It is obvious that the number of choices for the set X is less than $\sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i}$. We have

$$\sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i} < \ln^2 \delta \binom{\delta \ln \delta}{\ln^2 \delta} < (\delta \ln \delta)^{\ln^2 \delta} < e^{2 \ln^3 \delta}.$$

Now we can estimate the probability that the domination number of the graph H is less than or equal to $\ln^2 \delta - \ln \delta \ln \ln^5 \delta$:

$$\mathbf{P}\left[\gamma(H) \leq \ln^2 \delta - \ln \delta \ln \ln^5 \delta\right] < \sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i} \mathbf{P}[X \text{ dominates } V] < e^{2 \ln^3 \delta - \ln^4 \delta} = o_\delta(1).$$

Therefore, with high probability $\gamma(H) > \ln^2 \delta - \ln \delta \ln \ln^5 \delta = \ln^2 \delta(1 + o_\delta(1))$, as required. The proof of the theorem is complete. ∎

3 Upper Bounds for the Roman Domination Number

The following theorem provides an upper bound for the Roman domination number:

Theorem 12 *For any graph G with $\delta > 0$,*

$$\gamma_R(G) \le 2\left(1 - \frac{2^{1/\delta}\delta}{(1+\delta)^{1+1/\delta}}\right) n.$$

Proof: Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

$$p = 1 - \left(\frac{2}{1+\delta}\right)^{1/\delta}.$$

We denote $B = N[A] - A$ and $C = V(G) - N[A]$. Let us assume that f is a function $f : V(G) \to \{0, 1, 2\}$ and assign $f(v_i) = 2$ for each $v_i \in A$, $f(v_i) = 0$ for each $v_i \in B$ and $f(v_i) = 1$ for each $v_i \in C$. It is obvious that f is a Roman dominating function and $f(V(G)) = 2|A| + |C|$.

It is easy to show that

$$\mathbf{P}[v_i \in C] \;=\; (1-p)^{1+\deg(v_i)} \le (1-p)^{1+\delta}.$$

The expectation of $f(V(G))$ is

$$\begin{aligned}
\mathbf{E}[f(V(G))] &\le 2\mathbf{E}[|A|] + \mathbf{E}[|C|] \\
&= 2pn + \sum_{i=1}^{n} \mathbf{P}[v_i \in C] \\
&\le 2pn + (1-p)^{1+\delta} n \\
&= 2\left(1 - \frac{\delta\, 2^{1/\delta}}{(1+\delta)^{1+1/\delta}}\right) n. \qquad (2)
\end{aligned}$$

Since the expectation is an average value, there exists a particular Roman dominating function of the above order, as required. The proof of the theorem is complete. ∎

Theorem 12 implies the following upper bound.

Corollary 2 *For any graph G with $\delta > 0$,*

$$\gamma_R(G) \le \frac{2\ln(\delta+1) - \ln 4 + 2}{\delta + 1} n.$$

Proof: Using the inequality $1 - p \le e^{-p}$, we obtain the following estimation of the expression (2):

$$\mathbf{E}[f(V(G))] \le 2pn + e^{-p(\delta+1)} n.$$

If we put $p = \frac{\ln(\delta+1) - \ln 2}{\delta+1}$, then

$$\mathbf{E}[f(V(G))] \le \frac{2\ln(\delta+1) - \ln 4 + 2}{\delta + 1} n,$$

as required. ∎

Note that the result of Corollary 2 was also proved in [6], even though the upper bound in [6] contains a misprint.

Now let us prove that the upper bound of Corollary 2, and therefore of Theorem 12, is asymptotically best possible.

Theorem 13 *When n is large there exists a graph G such that*

$$\gamma_R(G) \geq \frac{2\ln(\delta+1) - \ln 4 + 2}{\delta+1} n(1+o(1)).$$

Proof: Let F be a complete graph $K_{[\delta \ln \delta]}$, and let us denote $F = V(F)$. Next, we add a set of new vertices $V = \{v_1, ..., v_\delta\}$, where each vertex v_i is adjacent to δ vertices that are randomly chosen from the set F. The resulting graph is denoted by G and it has $n = [\delta \ln \delta] + \delta$ vertices. We will prove that with positive probability

$$\gamma_R(G) \geq \frac{2\ln\delta}{\delta} n(1+o_\delta(1)) = 2\ln^2\delta(1+o_\delta(1)).$$

Let $f = (D_0, D_1, D_2)$ be a γ_R-function of G, i.e. f is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. It is easy to see that we may assume that $D_2 \subseteq F$ and $D_1 \subseteq V$.

Let us consider two cases. If $|D_2| > \ln^2\delta - \ln\delta \ln\ln^4\delta$, then $f(V(G)) > 2\ln^2\delta(1+o_\delta(1))$, as required. If $|D_2| \leq \ln^2\delta - \ln\delta\ln\ln^4\delta$, then, similar to the proof of Theorem 11, we can prove that the probability of the set D_2 dominating a vertex $v_i \in V$ is

$$\mathbf{P}[D_2 \,\text{dominates}\, v_i] \leq 1 - \frac{\ln^3\delta}{\delta}.$$

Let us consider the random variable $|N(D_2) \cap V|$. The expectation of $|N(D_2) \cap V|$ is

$$\mathbf{E}[|N(D_2) \cap V|] = \sum_{i=1}^{\delta} \mathbf{P}[D_2 \,\text{dominates}\, v_i] \leq \delta - \ln^3\delta.$$

Thus we can conclude that there exists a graph G, for which $|D_1| \geq \ln^3\delta$, i.e. $f(V(G)) \geq \ln^3\delta > 2\ln^2\delta(1+o_\delta(1))$, as required. ∎

4 Restrained and Total Restrained Domination

Theorem 1 implies that when $\delta(G)$ is large, $\gamma(G)/n$ is close to 0 for any graph G. Similar results were proved for the global and Roman domination numbers in the previous sections. However, for the total restrained domination numbers this is not the case, because for any δ there exists (see [8]) an infinite family of graphs G with minimum degree δ, for which $\gamma_{\rm tr}(G)/n \to 1$ when n tends to ∞. The above is also true for the restrained domination number. Thus, for the class of all graphs, it is impossible to provide an upper bound for these parameters similar to the result of Theorem 1. In this section, we will give such upper bounds for large classes of graphs.

Let us first find the restrained domination number of a 'typical' graph. Let $0 < p < 1$ be fixed and put $q = 1 - p$. Denote by $\mathcal{G}(n, \mathbf{P}[edge] = p)$ the discrete probability space consisting of all graphs with n fixed and labelled vertices, in which the probability of each graph with M edges is $p^M q^{N-M}$, where $N = \binom{n}{2}$. Equivalently, the edges of a labelled random graph are chosen independently and with the same probability p. We say that a random graph $\mathbf{G}$ *satisfies a property Q* if

$$\mathbf{P}[\mathbf{G} \text{ has } Q] \to 1 \text{ as } n \to \infty.$$

If a random graph $\mathbf{G}$ has a property Q, then we also say that almost all graphs satisfy Q.

It turns out that, for almost all graphs, the restrained domination number is equal to the domination number, which has two points of concentration, and the total restrained domination number is equal to the total domination number. This is formulated in the following theorem, which is based on the fundamental results of Bollobás [4] and Weber [18]. Note that Weber's result has been generalised in [19]. Remind that a dominating set X is called a *total dominating set* if every vertex of X is adjacent to a vertex of X. The total domination number $\gamma_t(G)$, which is one of the basic domination parameters, is the minimum cardinality of a total dominating set of G.

Theorem 14 *For almost every graph, $\gamma_r(G) = \gamma(G)$ and $\gamma_{\rm tr}(G) = \gamma_t(G)$. Moreover,*

$$\gamma_r(G) = \lfloor \log n - 2\log\log n + \log\log e \rfloor + \epsilon,$$

where $\epsilon = 1$ or 2, and $\log$ denotes the logarithm with base $1/q$.

Proof: Bollobás [4] proved that a random graph $\mathbf{G}$ satisfies

$$\mid \delta(\mathbf{G}) - pn + (2pqn\log n)^{1/2} - \Big(\frac{pqn}{8\log n}\Big)^{1/2} \log\log n \mid \leq C(n)\Big(\frac{n}{\log n}\Big)^{1/2},$$

where $C(n) \to \infty$ arbitrarily slowly. Therefore,

$$\delta(\mathbf{G}) = pn(1 + o(1)).$$

Weber [18] showed that the domination number of a random graph $\mathbf{G}$ is equal to

$$k + 1 \quad \text{or} \quad k + 2,$$

where

$$k = \lfloor \log n - 2\log\log n + \log\log e \rfloor$$

and log denotes the logarithm with base $1/q$. Let us consider a minimum dominating set D of this size. We have

$$|D| = \log n(1 + o(1)).$$

For any vertex $v \in V(\mathbf{G}) - D$ and large n,

$$\deg v \geq \delta = pn(1 + o(1)) > \log n(1 + o(1)) = |D|,$$

since p is fixed. Therefore, the vertex v is adjacent to a vertex in $V(\mathbf{G}) - D$, i.e. D is a restrained dominating set.

Now let us consider a minimum total dominating set T, i.e. $|T| = \gamma_t(\mathbf{G})$. It is not difficult to see that

$$\gamma_t(\mathbf{G}) \leq 2\gamma(\mathbf{G}).$$

Therefore,

$$|T| \leq 2|D| = 2\log n(1 + o(1)).$$

Thus, for any vertex $v \in V(\mathbf{G}) - T$ and large n,

$$\deg v \geq \delta = pn(1 + o(1)) > 2\log n(1 + o(1)) \geq |T|,$$

since p is fixed. Therefore, the vertex v is adjacent to a vertex in $V(\mathbf{G}) - T$, i.e. T is a total restrained dominating set, which is also minimum. The result follows. ∎

However, the property of a 'typical' graph stated in the above theorem cannot be used as a bound for the (total) restrained domination number for a given graph. Let us find such upper bounds for large classes of graphs.

Proposition 1 *If $\delta > 0$ and $n < \delta^2/(\ln \delta + 1)$, then*

$$\gamma_r(G) \le \frac{\ln(\delta+1)+1}{\delta+1} n$$

and

$$\gamma_{\rm tr}(G) \le \frac{\ln \delta + 1}{\delta} n.$$

Proof: Using Theorem 1, let us consider a dominating set D such that

$$|D| \le \frac{\ln(\delta+1)+1}{\delta+1} n.$$

Note that the condition $n < \delta^2/(\ln \delta + 1)$ can be written as follows:

$$\delta > \frac{\ln \delta + 1}{\delta} n.$$

Now, for any vertex $v \in V(G) - D$,

$$\deg v \ge \delta > \frac{\ln \delta + 1}{\delta} n > \frac{\ln(\delta+1)+1}{\delta+1} n \ge |D|.$$

Therefore, the vertex v is adjacent to a vertex in $V(G) - D$, i.e. D is a restrained dominating set.

Using the probabilistic method of the proof of Theorem 1, we can show that for any graph G with $\delta > 0$,

$$\gamma_t(G) \le \frac{\ln \delta + 1}{\delta} n.$$

Let us consider a total dominating set T such that

$$|T| \le \frac{\ln \delta + 1}{\delta} n.$$

For any vertex $v \in V(G) - T$,

$$\deg v \ge \delta > \frac{\ln \delta + 1}{\delta} n \ge |T|.$$

Therefore, the vertex v is adjacent to a vertex in $V(G)-T$, i.e. T is a total restrained dominating set. ∎

Note that the result of Bollobás [4] on the minimum degree implies that the condition $n < \delta^2/(\ln \delta + 1)$ is satisfied for almost all graphs, i.e. Proposition 1 gives upper bounds for a very large class of graphs. Moreover, in the class of graphs with $n < \delta^2/(\ln \delta + 1)$, the upper bounds of Proposition 1 cannot be improved. This can be proved in the same way as Theorem 2.

The *matching number* of a graph G, denoted by $\beta_1(G)$, is the largest number of pairwise non-adjacent edges in G. This number is also called the *edge independence number*.

Theorem 15 *For any graph G with $\delta > 0$,*

$$\gamma_r(G) \le \frac{2\ln(\delta+1)+\delta+3}{\delta+1} n - 2\beta_1$$

and

$$\gamma_{\rm tr}(G) \le \frac{2\ln \delta + \delta + 2}{\delta} n - 2\beta_1.$$

Proof: Let us consider a minimum dominating set $|D|$ of the graph G, i.e. $|D| = \gamma(G)$. Let M be a matching with $\beta_1(G)$ edges:

$$M = (e_1, e_2, ..., e_{\beta_1}).$$

Without loss of generality we may assume that the first k edges of M have at least one end in D, thus $\beta_1 - k$ edges of M have both ends in $V(G) - D$. It is obvious that

$$k \leq |D| = \gamma(G).$$

Therefore, at least $\beta_1(G) - \gamma(G)$ edges in M have both end vertices in $V(G) - D$. Note that $\beta_1(G) \geq \gamma(G)$, because each vertex of a total dominating set S has a private neighbour not in S, thus providing a matching of size $|S|$, which is at least $\gamma(G)$.

Now we form a restrained dominating set D' by adding to D all vertices not belonging to the last $\beta_1 - k$ edges of M. We obtain

$$\gamma_r(G) \leq |D'| = n - 2(\beta_1 - k) \leq n - 2\beta_1 + 2\gamma.$$

By Theorem 1,

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.$$

Therefore,

$$\gamma_r(G) \leq \frac{2\ln(\delta + 1) + \delta + 3}{\delta + 1} n - 2\beta_1,$$

as required.

Let us prove the latter upper bound. Consider a minimum total dominating set T and the above matching M. Using a similar technique, we can construct a total restrained dominating set T' such that

$$\gamma_{\mathrm{tr}}(G) \leq |T'| \leq n - 2\beta_1 + 2\gamma_t.$$

Using the probabilistic method of the proof of Theorem 1, we can show that for any graph G with $\delta > 0$,

$$\gamma_t(G) \leq \frac{\ln\delta + 1}{\delta} n.$$

Therefore,

$$\gamma_{\mathrm{tr}}(G) \leq \frac{2\ln\delta + \delta + 2}{\delta} n - 2\beta_1,$$

as required. ∎

A matching is called *perfect* if it contains all vertices of a graph (or all vertices but one if n is odd). The following corollary follows immediately from the above theorem:

Corollary 3 *If G has a perfect matching, then*

$$\gamma_r(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} 2n + \epsilon$$

and

$$\gamma_{\mathrm{tr}}(G) \leq \frac{\ln\delta + 1}{\delta} 2n + \epsilon,$$

where $\epsilon = 0$ if n is even and $\epsilon = 1$ otherwise.

It may be pointed out that the class of graphs with a perfect matching includes all Hamiltonian graphs. It is well known that almost all graphs are Hamiltonian [11], thus Corollary 3 provides upper bounds for a very large class of graphs.

5 Concluding Remarks and Open Problems

By Theorem 14, the total restrained domination number is equal to the total domination number for almost every graph. However, we do not know exact values of the total domination number for almost all graphs. Such a result for the domination number is known [18].

Problem 1 *For almost all graphs, find points of concentration of the total, global and Roman domination numbers.*

Theorem 1 is formulated for all graphs and it gives an excellent upper bound if δ is big. However, for small values of δ, better (sharp) bounds are known:

Theorem 16 (Ore) *If $\delta(G) \geq 1$, then*

$$\gamma(G) \leq \frac{n}{2}.$$

Theorem 17 ([10]) *If G is a connected graph with $\delta \geq 2$ and it is not isomorphic to one of seven graphs (not shown here), then*

$$\gamma(G) \leq \frac{2}{5}n.$$

Theorem 18 ([13]) *If G is a connected graph with $\delta \geq 3$, then*

$$\gamma(G) \leq \frac{3}{8}n.$$

The above situation is also true for many upper bounds proved in this paper. They are good when δ is not small. Can better upper bounds be found for small values of δ?

Problem 2 *Determine sharp upper bounds for the global and Roman domination numbers of a graph with small minimum degree.*

Problem 3 *Determine sharp upper bounds for the restrained and total restrained domination numbers of a graph with a perfect matching and small minimum degree.*

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