The missing magnetic flux in the HP memristor found

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Abstract

Ever since real-world two terminal memristor devices were created, detractors have derided them as not 'true' Chua memristors because there is no magnetic flux in the system. This report shows that the flux is present and equal to the magnetic flux of the oxygen vacancies. This shows that the HP memristor is a 'true' Chua memristor. The memristor description has been separated out into the memory function, which is the memrsitance described by Chua in 1971 and the conservation function, which is necessary to describe the whole HP memristor. As the memory function deals with the vacancy mobility and the measurable current is mostly electronic, it is apparent that a description of the HP memristor must include two charge carriers. From analysis of Stan William's model in terms of memory and conservation functions, a direct relation to Chua's original memristance equation and an approximation of magnetic flux have both been identified as being present in the original model. With this, the phenomenological model used by experimentalists and the mathematical model beloved by theoreticians have been combined into one.

1 Introduction

There are four fundamental circuit properties which describe a circuit's operation: the electrical potential difference, V, the electronic current, I, the magnetic flux, φ and the charge, q. Three pairs of relationships were known in 1971 which defined the operation of the three (then) fundamental circuit elements: the resistor ($R = V/I$); the capacitor ($C = q/V$); the inductor ($I = \varphi/V$). Analysed like this, there is a missing relationship that should relate charge and magnetic flux, and this was the relationship that Chua proposed [1] would be satisfied by the discovery or creation of a memristor. From electrodynamics it is known that a moving charge has an associated magnetic field, so this idea seemed sound. Nonetheless, between 1971 and 2008, no one who had read Chua's work was able to make a memristor, perhaps because they were investigating magnetic materials used for inductors, and the idea of a memristor languished in the drawer of theoretical curiosities.

Over this time period, many experimentalists had reported 'anomalous' I-V curves with variations on a pinched hysteresis loop. As current is the time differential of charge ($I = dq/dt$) and voltage is the time differential of magnetic flux ($V = d\varphi/dt$), any linear relationship between charge and magnetic flux would manifest itself as a non-linear relationship between current and voltage, such as that which might describe a pinched hysteresis loop. It wasn't until 2008 that Stan William's group at HP published a paper describing a working memristor [2], complete with a phenomenological model for its operation and references to Chua's theoretical work. There were two problems however, the magnetic flux was missing from the model and the device did not have an appreciable magnetic field, as was expected.

In that 2008 paper, Strukov et al stated that the missing magnetic flux did not matter, because all the definition of a memristor requires is the existence of a non-linear relationship between charge and flux to fit the mathematics. This preemptive explanation did not prevent some people [3] [4] from deriding the both the HP memristor as not a proper Chua memristor and the phenomenological model as a 'toy' model seemingly unrelated to Chua's equations.

In this report I will demonstrate that the HP memristor does include magnetic flux and using electrodynamic theory I will derive how the memristance relates this flux to the charge flow. From there, I will extend the description to cover the electronic current and voltage of the memristor and demonstrate that Stan William's 'toy' model actually includes an (admittedly accidental) approximation for the magnetic flux.

2 Memristor behaviour.

Chua asserted that a memristor would be a device governed by the following relationship [1]:

$$d\varphi = M(q)dq \tag{1}$$

where M is the memristance and is a function of q. If q doesn't change over time then the memristance is constant and Ohms law is sufficient to describe the device (ie it is a resistor). This is the theoretical model of a memristor.

The HP memristor is a layer of titanium dioxide sandwiched between two electrodes and it works by being able to reversibly inter-convert between the high resistance, R_{off}, stoichiometric form TiO_2 and the low resistance, R_{on}, doped form, TiO_{2-x}, where x oxygen atoms (per mole) have been lost from the structure leaving a positive (p-type semiconductor) vacancies. Based on these quantities, the Strukov et al put forward the phenomenological model [2] of their memristor's memristance in a form similar to:

$$M(q) = R_{\text{off}}\left(1 - R_{\text{on}}\beta q(t)\right)\ ,$$

where the actual titanium dioxide properties (ion mobility and physical dimensions of the TiO_2 layer) have been gathered into β, the material parameter. NB. Strukov et al never expressed their model in this exact form.

Memristive systems (Chua's generalisation of his ideas in 1976 [5]) have a hysteresis because at a given voltage there is more than one possible current, ie. there is more than one possible resistance. The value of the current measured at a point must be related to the history of the device, if the memristive system is to have a memory. To make a memristor, rather than a memristive system, the device property that causes this change in resistance must be controlled by voltage - this is necessary for the memristor to be a two-terminal device (part of Chua's definition for a fundamental circuit element memristor).

I'm going to approach the question of how the HP memristor can be a true Chua memristor backwards, by asking how a Chua memristor would work using knowledge of its behaviour gleaned from the Chua's equations. We're going to focus on the property in the memristor responsible for its memory, which is also a property that changes in response to voltage which I shall call the 'memory property'. I postulate that for there to be a memory in either a memristive system or a memristor, the memory property must be both separate from and slower to respond to a voltage change than the conducting electrons. This slower response time leads to the lag in current which gives rise to the hysteresis loop and explains the frequency dependence of memristance: if the voltage changes too fast for the memory property, it can't respond quick enough for a measurable change and the size of the hysteresis loop shrinks to a straight line (this is the Ohmic regime). The memory property has to respond to the voltage, which suggests that it either needs to be affected by the potential difference and therefore be charged or to undergo a structural change due to the electrical energy supplied. And, for the memristor to be of any real use, this change in memory property has to be (at least qualitatively) reversible, so the device can switch back and forth.

There are several different possibilities for memory properties, such as charged ions in the PEO-PANI memristive system [6], or the concentration of spin electrons [7] in spintronic systems or the 'thermal' phase change that can be triggered by voltage in a VO_2 thin film [8].

In the HP memristor the memory property is the charged oxygen vacancies in the TiO_{2-x}. This may be a strange way of thinking about hole dopants, as the hole is caused by the lack of an electron, however, there are three important points about these hole dopants: 1, the conducting electrons move much faster than the dopants drift; 2, changing the dopant concentration in a volume of the memristor will change the resistance, conducting electrons won't do this; 3, charged dopants will respond directly to voltage. Therefore, they obey the conditions presented above.

If there is a second charge carrier in the memristor, then we have to ask the question of which charge is related to the magnetic flux (a question the HP group never thought to ask - they assume that q is the electronic charge). I am going to demonstrate that the relevant charge is the charge of the non-electronic charge carriers and therefore it is the relationship between this charge and its associated magnetic flux which defines the memristance. To be explicit, q_v is the charge associated with the non-electronic charge carriers, this is the product of the formal charge on that type of charge carrier and the number of them present. Thus, although it is the effect on the electronic current which is measured (and will be of use in real world devices), the electronic current is irrelevant in the actual process of memristance.

This paper will calculate the magnetic flux of the non-electronic charge carriers in a memristor and show that this fits Chua's definitions for a memristor. We will then demonstrate that this flux is present in both the real world HP memristor and in the model equations, by doing this, Chua's mathematical theory will be united with Strukov et al's phenomenological model. This new description will offer a new way of understanding the workings of real world devices, enabling better predictions of memristive materials.

3 Method

First, the method for calculating the magnetic flux for any memristor where non-electronic charge carriers are responsible for the memory property will be described, then the magnetic flux in the HP memristor will be derived.

We start with the Biot-Savert law for the magnetic field associated with a volume current. This is the most appropriate formulation of the Biot-Savert law because we are going to consider the magnetic flux just above the memristor surface where the memristor is best viewed as a 3-dimensional object. The Biot-Savert law comes from magnetostatics, a branch of theory that describes the magnetic effects due to constant currents, although our current will change, magnetostatics is still a valid approach because the changing current is far slower than that to which such theory is successfully applied (namely mains A.C. (50-60Hz)).

The volume current, $\mathbf{J}$, is given by

$$\mathbf{J} = \frac{q_v \mu_v \mathbf{L}}{Vol}$$

where q_v is the charge in that volume due to the non-electronic charge carriers, μ_v is the ion mobility of the non-electronic charge carriers and L is the average electric field causing the movement of charge. The magnetic field or magnetic flux density, $\mathbf{B}$, at a point, p, associated with with this charge is given by:

$$\mathbf{B}(p) = \frac{\mu_0}{4\pi} \int \frac{J \mathbf{d}\hat{\mathbf{J}} \times \hat{\mathbf{r}}}{r^2} \mathrm{d}\tau \tag{2}$$

where μ_0 is the permittivity of a vacuum, $\mathrm{d}\hat{\mathbf{J}}$ and $\mathrm{d}\hat{\mathbf{r}}$ are the unit vectors for $\mathbf{J}$ and $\mathbf{r}$ where $\mathbf{r}$ is the vector of length r from the volume infinitesimal $\mathrm{d}\tau$ to point p, given by $\mathbf{r} = \{r_x \hat{\imath}, r_y \hat{\jmath}, r_z \hat{\ell}\}$.

To get the magnetic flux through a surface associated with this field, φ, we need to take the surface integral

$$\varphi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{A}$$

where $\mathrm{d}\mathbf{A}$ is the normal vector from the surface infinitesimal $\mathrm{d}A$. This method should work for any memristor where the memory property is related to moving non-electronic charge carriers.

4 Example derivation for HP memristor

In the HP memristor, the non-electronic charge carriers are the vacancies created in the TiO_2 layer. The magnetic field integral in equation 3 is taken over the volume of the device that contains flowing vacancies. This varies with time, but at an instant in time, t, the magnetic field is given by

$$\mathbf{B}(\mathbf{p},\mathbf{t}) = \frac{\mu_0}{4\pi} \int_0^F \int_0^E \int_0^w \frac{\mathbf{J}\times\mathbf{r}}{|\mathbf{r}|^3} \mathrm{d}r_x \mathrm{d}r_y \mathrm{d}r_z \,. \tag{3}$$

where $w(t)$ is the position of the boundary between doped and undoped titanium dioxide at time t and $0 < w < D$.

The x-axis is taken as the direction of current flow between the electrodes, the limit of which is D, the device thickness which is 10nm [2]. The y and z axes are in the plane of the electrodes, with the limits E and F, the length and width of the crossed electrode area and are both 50nm in the crossbar memristor. We take the terms of our integral in the coordinate system for inside the memristor, ie: r_x, r_y and r_z.

The integral is solved using the technique of integration by parts, taking the cross product in the numerator, $\mathbf{J}\times\mathbf{r}$, as $\mathrm{d}g(r_x, r_y, r_z)$ and the denominator, $\frac{1}{(r_x^2+r_y^2+r_z^2)^{\frac{3}{2}}}$ as $f(r_x, r_y, r_z)$, the details are in appendix A.1.

This gives:

$$\mathbf{B}(p) = \frac{\mu_0}{4\pi} L\mu_v q\{0, -xzP_y, xyP_z\}$$

with

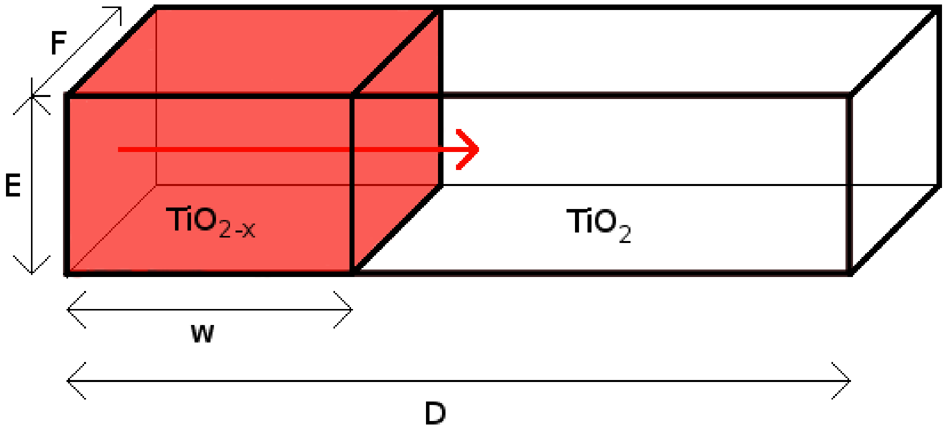

Figure 1: The HP memristor. The shaded area is the doped low resistance titanium dioxide and the unshaded region is the stoichiometric high resistance titanium dioxide. The arrow indicates the direction of vacancy movement through the material, these move in the x direction. The boundary between the two materials is w. The limits of the titanium dioxide layer in the y and z directions is E and F. As the vacancies move to the right along the x axis, the magnetic **B** field associated with them curls around in an anti-clockwise direction (not shown). The surfaces which cross magnetic field lines are those in the x-y and x-z planes. It is these surfaces that the magnetic flux is calculated over.

$$
\begin{aligned}
P_y &= \frac{F}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\
&\quad -\frac{1}{2wEF}\frac{\left(wE\left(F^2\left(E^2+F^2\right)^2+w^4\left(2E^2+F^2\right)+w^2\left(2E^4+5E^2F^2+2F^4\right)\right)\right)}{\left(\left(w^2+F^2\right)\left(E^2+F^2\right)\left(w^2+E^2+F^2\right)^{\frac{3}{2}}\right)} \\
&\quad +F\arctan\left(\frac{wE}{F\sqrt{w^2+E^2+F^2}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
P_z &= \frac{E}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\
&\quad -\frac{1}{2wEF}\frac{\left(wF\left(E^2\left(E^2+F^2\right)^2+w^4\left(E^2+2F^2\right)+w^2\left(2E^4+5E^2F^2+2F^4\right)\right)\right)}{\left(\left(w^2+E^2\right)\left(E^2+F^2\right)\left(w^2+E^2+F^2\right)^{\frac{3}{2}}\right)} \\
&\quad +E\arctan\left(\frac{wF}{E\sqrt{w^2+E^2+F^2}}\right).
\end{aligned}
$$

P_y and P_z contain only the dimensions of the memristor, so even if they are not analytically simple, they are easy to calculate numerically. As expected of a magnetic field, the divergence of the feild is zero. The curl is non-zero as the field curls around the current in an anti-clockwise direction.

For a memristor which is close to being full with the maximum number of vacancies (ie the limit) the field at point p is given by

$$\mathbf{B}(p) = \{0, -6.37qVxz, 6.37qVxy\},$$

where V is the applied voltage, $p = \{x, y, z\}$ and x, y and z refer to a second set of coordinates which are located outside the memristor whose unit vectors are $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ [1].

As $\mathbf{B}$ is also known as the magnetic flux density, to calculate the magnetic flux we need to pick a surface to evaluate over. It makes sense to choose a surface that correlates to one of the surfaces of the device. Picking the surface just above the device ($0 < x < D$, $0 < y < E$, $z = F$ [2]), we use the surface normal area infinitesimal, d$\mathbf{A}$, which is given by d$\mathbf{A} = \{0, 0, \hat{\imath}\hat{\jmath}\}$. As is standard in electromagnetism, we integrate over the entire area. In this case, the limits of the surface are taken to be the dimensions of the device.

The final value for φ over this surface is

$$\varphi = \frac{\mu_0}{4\pi} Lq\mu_v xyP_z \; . \tag{4}$$

where xyP_z is the z component of the magnetic $\mathbf{B}$ field and this is generalised to any surface in the appendix A.2. We've now shown that there is a magnetic flux associated with the HP memristor's operation.

[1] The volume current is constrained within the memristor and can be written in terms of coordinates inside the memristor. The magnetic feild (as caused by the volume current) can only exist outside the memristor and therefore can be written in terms of coordinates from outside the memristor. The two sets are labelled differently here to avoid confusion. If the distinction is not made between the two sets, then it's possible that the inside coordinates might be integrated over twice, which would be wrong. Perhaps confusingly, the limits are the similar. The inside coordinates have the limits: $0 \leq r_x \leq D$; $0 \leq r_y \leq E$; $0 \leq r_z \leq F$;. The outside coordinates can go from $-\infty$ to ∞ but must avoid the volume within the memristor

[2] Actually $z = F + dF$ so the surface is just above the memristor, avoiding any surface effects.

Putting in real-world values for the device characteristics for the HP memristor gives a flux of 2.44×10^{-29}Wb. In contrast, the magnetic flux associated with the conducting electrons through the same surface[3] is -4.07×10^{-24}Wb. This is in the opposite direction and approximately 100 000 times bigger than the vacancies' magnetic flux. This may explain why the magnetic flux associated with memristor function has not been experimentally measured.

5 Relation to other memristor models

5.1 The memristance

Chua's original formulation of memristance, equation 1, is also the instantaneous or chord resistance [9], which is defined as the gradient of the I-V curve at that point in time and this is the ratio between $\mathrm{d}\varphi$ and $\mathrm{d}q$. From this definition and equation 9, we can immediately write the memristance, M as:

$$M(q(t)) = \frac{\mu_0}{4\pi} DEL\mu_v P_z(q(t)) \tag{5}$$

where M is a function of $q(t)$ because the movement of the boundary over time is proportional to the amount of vacancies created: $P_z(w(t)) \propto P_z(q(t))$.

This is the memristance as calculated from the top surface. We get the same numerical value of memristance if we calculate it from the other four surfaces which cut magnetic feild lines, but if we calculate it from a surface parallel to the magnetic field lines we get the answer of zero. This means that the memristance is not calculable from any surface perpendicular to the vacancy current, ie surfaces in the y-z plane.

This is intriguing as it raises the possibility that memristance could be a tensor quantity, or at the very least, some care must be taken in calculating it. For the rest of this discussion, we will concentrate on the memristance through

[3]To get the number of electrons, we've assumed that the TiO_2 is acting like a metal and every titanium atom is giving up a conducting electron. To get the number of oxygens that can be lost, we're assuming that maximum of 3% of available oxygen atoms

the surfaces which do cross magnetic field lines (note that regardless of which of these surfaces we chose, we will get the same answer because the y and z components of the $\mathbf{B}$ field are equal).

Equation 5 can be considered as three separate parts:

1. universal constants: $\frac{\mu_0}{4\pi}$, this term includes the effects of the permittivity of a vacuum on memristance. It's inclusion in the equation clearly demonstrates that magnetism is involved in memristance.

2. experimental constants: DEL, where DE is the surface the flux was calculated over, in this case the top of the device, and the electric field, and it doesn't change over the experiment. L is the root mean squared electric field strength if we are calculating memristance in an A.C. circuit.

3. β, the material parameter: $\mu_v P_z$, this includes the physical dimensions of the device, but it will change throughout the experiment as a result of the moving boundary caused by the movement of vacancies in the device. This is the only term that contains variables.

5.2 Memory Function + Conservation Function view of Memristance

When measuring a memristor it is conventional to measure the electronic current, not the ionic current. As the electronic current is many times larger and faster than the movement of vacancies, we can ignore the vacancy contribution to the total flow of charge. What is therefore needed is the memristance as experienced by the conducting electrons. The component of that memristance which is directly due to the changing resistivity of the doped material is given by

$$M_{e^-}(e^-) = C M_{v^+}(v^+) ,$$

where C is an experimentally determined parameter for the material. Since the ion mobilities for the electrons, μ_{e^-} , and the vacancies μ_{v^+} are experimentally measured quantities, it is predicted that $C = \frac{q_{e^-}\mu_{e^-}}{q_{O^+}\mu_{O^+}}$.

In the HP system, M_{v^+}, is the Chua memristance, this is the direct relationship between magnetic flux and charge and it only holds for that portion of the magnetic flux that is related to the charge carriers that hold the memory, ie the memory property of the device. The memory function is the Chua memristance in a form relevant to the conducting electrons and thus $M_{e^-}(e^-)$ the memory function of the HP memristor.

Now we have a description of the memristance of the non-stoichiometric TiO_{2-x}, in terms of the memory property, we need to account for the rest of the device.

In equation 5 w was allowed to change as a function of time, and this is exactly what happens in the HP memristor if the system is modelled as a moving boundary between two types of interconvertable materials. The model is not complete unless we include a description of the stoichiometric TiO_2. This is necessary to ensure that space is conserved in the memristor model, therefore the function that does that will be called the conservation function, R_{con}.

In the HP memristor, the conservation function is simply the resistance of the stoichiometric TiO_2. From the geometry of the device and resistivity of stoichiometric TiO_2, ρ_{off}, we can write the conservation function as

$$R_{\text{con}} = \frac{(D-w)\rho_{\text{off}}}{EF}.$$

The total resistance as experienced by the conducting electrons, R_{tot} is then

$$R_{\text{tot}}(t) = \frac{(D-w)\rho_{\text{off}}}{EF} + M_{e^-}(q_{e^-}) \quad (6)$$

Where $R_{\text{tot}}(t)$ can be called a memristance because it is a resistance that changes with time due to the action of charge. This is not the Chua memristance, but does contain it. The Chua memristance is the memory function term where the resistivity of the material can change. The conservation function will also change as a result of the charge (because w is dependent on q), this changes the effective size of undoped part of the memristor, but the resistivity of the material within that volume does not change. Essentially, the difference between the two functions is whether the intrinsic resistivity of the material can change, in the memory function it can, in the conservation function it can't.

Plotting this equation with test values gives the pinched hysteresis loop in I-V space indicative of memristance, see figure 2. Separately, both conservation and memory functions are sufficient to give rise to a memristive I-V curve. A plot of the memory function by itself will give the memristance due to the change in size and resistivity of doped part. A plot of the conservation function by itself will show memristance due to the change in size of the undoped part. The two functions can have different magnitudes and fundamental frequencies. For an individual system the terms may be of the same order and interact or either one of them may be responsible for most of the measured memristive effect. That depends on the resistivity of the two materials, however the second case is more likely as appreciable memristance is easier to see if the two materials have very different resistivities. In the HP memristor, the ratio between the undoped and dope resistivity was 160 [2].

We shall call $R_{\mathrm{tot}}(t)$ the total memristance and it is interesting as it is similar to Stan William's formulation of memristance.

$R_{\mathrm{tot}}(t)$ expressed in eqn 6 shows that the HP memristor is a perfect Chua memristor because the entire equation can be written to be a function of a state variable, w [5]. I've now shown that the HP memristor contains magnetic flux, can be described using Chua's equations and is a true Chua memristor, I will now go on to evaluate the model presented in Strukov et al's paper.

5.3 Relation to Strukov et al's model

Strukov et al's model can be expressed in terms of conservation and memory functions:

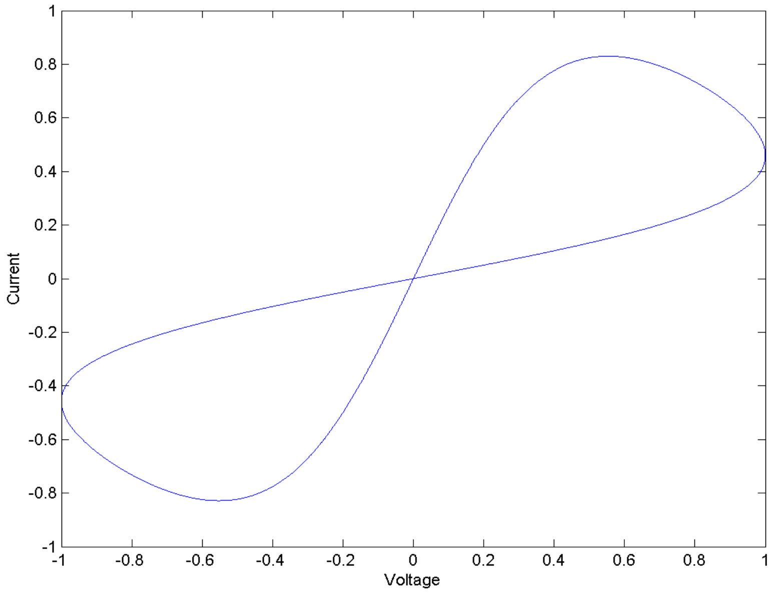

Figure 2: Simulated I-V curve based on the memory-conservation model of the HP memristor

$$R_{\text{tot}}(t) = [R_{\text{on}}^2 + R_{\text{off}} - R_{\text{on}}R_{\text{off}}][\beta q]$$

where the first bracket is the conservation function and the second is the memory function. (NB, the R_{on}^2 term was dropped in their paper [2] because it is many times smaller than the other terms in the description). The conservation function arises from the description of the memristor as two space-conserving variable resistors, see appendix A.4, exactly as in the model presented here.

The memory function in this model is interesting. To understand why, we need to look at the role of the material parameter. Strukov et al never put their material properties into a single parameter in this way, I've done it all the way through this document for a very good reason. Despite Strukov et al's assertions, and many other scientists agreement, that HP model does not contain magnetic flux, it does.

The material parameter, β has the units of the inverse of magnetic flux, ie.

Wb[1]. This makes sense when we examine what it is.

$$\beta = \frac{\mu_v}{D^2}$$

β is the ion mobility divided by an area. We know that movement of charged particles has an associated magnetic flux, and β^{-1} is Strukov et al's accidental approximation for it[4]. If the device was a wide as it was thick, the area, D^2, would be one of the faces of the memristor that the magnetic flux field lines would cross, ie one of the faces memristance is calculable from. Compared to the correct form of the memristance, Strukov et al have missed out both the electric field and the magnetic permittivity. Essentially, if the magnetic flux were on the same order as the flux of charged particles, and we could simplify the geometry-related terms in the theory [5], then β^{-1} could be taken as an approximation to the magnetic flux.

It is possible to find a term that is similar to Strukov et al's β term in the model presented in this paper. This is the material parameter, $\mu_v P_z$, for the top surface of the memristor. P_z is the dimensions parameter, a large term that involves only the three dimensions of the TiO_2 layer (this is the geometry-related term referred to above). However, it is in units of m^{-2} and thus the D^{-2} part of the HP's β is an approximation for the dimensions parameter. It is possible that the Strukov et al's model works because the β term was close enough, D, E and F are all on the order of 10's of nm, therefore the error introduced between using D^2 instead of P_z is easy to miss (for a full discussion of Strukov et al's derivation and their errors, see appendix A.4). Note, as β does not vary in the HP model, the conservation function is responsible for all of the measured memristance.

As an approximation for the magnetic flux, β still quantitatively wrong. For β to be a magnetic flux it must change as the device charges or discharges. This could be fixed by replacing D with w. Far more worrying is that β^{-1} is 1×10^{24}

[4]Strukov et al never claimed that any part of their model contained the magnetic flux.

[5]the magnetic flux is at 90°to the ion flow, so a cross-product should be involved and this is the cause of the complexity of the geometry-related terms

larger than the actual magnetic flux. If we look at the magnetic field that this approximation for the flux would require, we expect a magnetic feild vector of length $\frac{2}{\mu_v}$ which is a field on the order of 1×10^{14} Tesla (see appendix A.5). To put that in perspective, the largest pulsed magnetic feild that has been created was 2.8 kT and the average neutron star puts out a field of 1×10^{7} T. There's no way the HP memristor is more magnetic than 10 million neutron stars.

If we take β^{-1} to be an approximation for the magnetic flux, then it's apparent that the memory function (βq) in their model is $\frac{q}{\varphi}$, which is the inverse of the Chua memristance for that system and thus Strukov et al's equations are built on Chua's memristor equations. Therefore not only is the HP memristor a true Chua memristor, Strukov et al's model is, in form, a true Chua memristor model.

6 Memristors as magnetic resistors

There are four circuit properties that Chua originally used to predict the existence of the memristor, φ, q, V and I. Because electromagnetism contains both electricity and magnetism (the difference between the two fields depends a lot on where you're standing and how fast you're moving) every circuit contains all four properties.

In the memristor, we can use these four circuit properties to describe the system in a novel way. We can designate the charge associated with the memristive magnetic flux as the magnetic charge q_{magnetic}. Note, I am not implying that these charge carriers are in any way more magnetic than any other charge carrier (neither am I claiming that they are magnetic monopoles, although Umul [10] has suggested that electrons in spin-ice memristive material behave like magnetic monopoles). Instead, I am conceptually separating out the charge responsible for the memory function of the memristor from the electrons. In the HP memristor, q_{magnetic} is q_v [6].

[6] We can imagine systems where the relationship is not so simple, for example where the interaction of several ions is involved.

If we've designated the φ and q_v to be 'magnetic' properties of the system, it makes sense (by symmetry) to designate the current I and the voltage V 'electric' properties. In this way, if a resistor is a relationship between V and I, then the memristor can be viewed as a magnetic resistor. The main point of doing this is to underline that there are two separate systems at work: the 'magnetic' system, which is the Chua memristance relation between charge and flux and the 'electric' system, which is the resulting relation between current and voltage. This does mean that to properly characterise a memristor, we must measure the flux response to the charge, because only by knowing that would you be able to predict the memristor's behaviour, measuring V-I is not enough (for a discussion on this point see [9]).

This analysis does require that we start thinking of magnetism in a novel way. Every moving charge, no matter how slow or seemingly insignificant has an associated magnetic flux. The discussion presented in this paper demonstrates that we can not simply ignore these charges or their associated magnetic fluxes, because the proper electromagnetic description of memristance requires them. To identify memristive materials, we need to think less like physicists and electronic engineers who label magnetic materials only those with large responses to magnetic fields, and more like NMR spectroscopists who know that almost every material will respond in some way to a magnetic field. Thus, in this context, we should now expand the label of magnetic materials from the traditional materials to include semi-conductors and other memristive materials.

7 Conclusion

The magnetic flux has been found for the HP memristor, allowing Chua's theoretical framework to be united with Strukov et al's phenomenological model. The standard model for a memristor has been expanded and separated out into the memory function and space-conservation parts of the theory, which will aid greater understanding of how memristive devices work. By underlining the importance of a second slow ion current, it will now be easier to understand and design memristor materials. It has been demonstrated that the HP memristor is a true Chua memristor and magnetic flux was present as an approximation in the original model of this device. The analysis offered here for the first and perhaps archetypical memristor can be applied to other memristor systems. The concepts of memory function and conservation function can be carried across to other systems.

A Appendix of Supplementary Information

A.1 Integrating by parts to get an expression for B.

The actual integration by parts is:

$\iiint f(r_x, r_y, r_z) \left(\frac{\partial}{\partial r_x} \frac{\partial}{\partial r_y} \frac{\partial}{\partial r_z} g(r_x, r_y, r_z) \right) \mathrm{d}r_x \mathrm{d}r_y \mathrm{d}rz = f(r_x, r_y, r_z)\, g(r_x, r_y, r_z) - \iiint \left(\frac{\partial}{\partial r_x} \frac{\partial}{\partial r_y} \frac{\partial}{\partial r_z} f(r_x, r_y, r_z) \right) g(r_x, r_y, r_z) \mathrm{d}r_x \mathrm{d}r_y \mathrm{d}rz$

Which we set equal to the simpler shorthand:

$$\int u dv = uv \int v du \tag{7}$$

To solve the integral in equation 3, which is

$$\int_0^F \int_0^E \int_0^w \frac{\mathbf{J} \times \mathbf{r}}{|\mathbf{r}|^3} \mathrm{d}r_x \mathrm{d}r_y \mathrm{d}r_z \ , \tag{8}$$

we set

$$u = \frac{1}{(r_x^2 + r_y^2 + r_z^2)^{\frac{3}{2}}} \ ,$$

and

$$dv = \mathbf{J} \times \mathbf{r} = \{0, -\frac{L q_v \mu_v r_z \hat{\imath} \hat{k}}{Vol}, \frac{L q_v \mu_v r_y \hat{\imath} \hat{\jmath}}{Vol}\} \ .$$

Thus

$$du = -\frac{105 r_x r_y r_z}{(r_x^2 r_y^2 r_z^2)^{\frac{9}{2}}}$$

and

$$v = \{0, -\frac{L q_v \mu_v r_x r_y r_z^2 \hat{\imath}\hat{k}}{2Vol}, \frac{L q_v \mu_v r_x r_y^2 r_z \hat{\imath}\hat{\jmath}}{2Vol}\} .$$

Substituting for the right hand side of equation (7) and simplifying gives the following vector **F**

$$udv = \mathbf{F} = \{F_x, F_y, F_z\}$$

where

$$F_x = 0 \, ,$$

$$\begin{aligned}
F_y &= \frac{\mu_v Lqxz}{2Vol}\left(-P_y^{indef}|_0^w|_0^E|_0^F\right) , \\
P_y^{indef} &= \frac{r_x r_y r_z^2}{\left(r_x^2 + r_y^2 + r_z^2\right)^{\frac{3}{2}}} \\
&\quad \frac{-\left(r_x r_y \left(r_z^2 \left(r_y^2 + r_z^2\right)^2 + r_x^4 \left(2r_y^2 + r_z^2\right) + r_x^2 \left(2r_y^4 + 5r_y^2 r_z^2 + 2r_z^4\right)\right)\right)}{\left(\left(r_x^2 + r_z^2\right)\left(r_y^2 + r_z^2\right)\left(r_x^2 + r_y^2 + r_z^2\right)^{\frac{3}{2}}\right)} \\
&\quad + r_z \arctan\left(\frac{r_x r_y}{r_z \sqrt{r_x^2 + r_y^2 + r_z^2}}\right)
\end{aligned}$$

and

$$
\begin{aligned}
F_z &= \frac{\mu_v Lqxy}{2Vol}\left(P_z^{indef}|_0^w|_0^E|_0^F\right), \\
P_z^{indef} &= \frac{r_x r_y^2 r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{\frac{3}{2}}} \\
&\quad \frac{-\left(r_x r_z\left(r_y^2\left(r_y^2+r_z^2\right)^2+r_x^4\left(r_y^2+2r_z^2\right)+r_x^2\left(2r_y^4+5r_y^2r_z^2+2r_z^4\right)\right)\right)}{\left(\left(r_x^2+r_y^2\right)\left(r_y^2+r_z^2\right)\left(r_x^2+r_y^2+r_z^2\right)^{\frac{3}{2}}\right)} \\
&\quad +r_y \arctan\left(\frac{r_x r_z}{r_y\sqrt{r_x^2+r_y^2+r_z^2}}\right)
\end{aligned}
$$

We then substitute in the limits for the integration, $r_x|_0^w$, $r_y|_0^E$ and $r_z|_0^F$ and replace Vol by wEF to simplify P_y where $P_y = \frac{1}{Vol}P_y^{indef}|_0^w|_0^E|_0^F$. Thus,

$$
\begin{aligned}
P_y &= \frac{F}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\
&\quad -\frac{1}{2wEF}\frac{\left(wE\left(F^2\left(E^2+F^2\right)^2+w^4\left(2E^2+F^2\right)+w^2\left(2E^4+5E^2F^2+2F^4\right)\right)\right)}{\left(\left(w^2+F^2\right)\left(E^2+F^2\right)\left(w^2+E^2+F^2\right)^{\frac{3}{2}}\right)} \\
&\quad +F\arctan\left(\frac{wE}{F\sqrt{w^2+E^2+F^2}}\right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
P_z &= \frac{E}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\
&\quad -\frac{1}{2wEF}\frac{\left(wF\left(E^2\left(E^2+F^2\right)^2+w^4\left(E^2+2F^2\right)+w^2\left(2E^4+5E^2F^2+2F^4\right)\right)\right)}{\left(\left(w^2+E^2\right)\left(E^2+F^2\right)\left(w^2+E^2+F^2\right)^{\frac{3}{2}}\right)} \\
&\quad +E\arctan\left(\frac{wF}{E\sqrt{w^2+E^2+F^2}}\right).
\end{aligned}
$$

It is not a very elegant solution.

To get the solution for B, **F** is multiplied by the permittivity of a vacuum constants that was outside the integral in equation 3, ie $\frac{\mu_0}{4\pi}$ and

$$B(p) = \frac{\mu_v \mu_0 L q}{8\pi}\{0, -xzP_y, xyP_z\},$$

where the Vol occupied by the volume current has been gathered up into dimensional parameters P_y and P_z so all the dimensions associated with the size of the doped part of the device are in one place.

A.2 Integration over the surface.

The general form of φ is given by

$$\varphi = \frac{\mu_0}{4\pi} Lq\mu_v ijP_k \, . \tag{9}$$

where i and j are the two chosen Cartesian coordinates of the surface you wish to integrate over and P_k refers to the geometric term for the magnetic field component for the other Cartesian axis. This is the part of the **B** field lines that is perpendicular to the chosen surface and therefore the only part that would contribute to the flux. The values are given in table 1.

Device surface	Area infinitesimal	Integral	Value for HP memristor
Top	$\mathrm{d}\hat{\mathbf{A}} = \{0, 0, \hat{\imath}\hat{\jmath}\}$	$\varphi_{\text{top}} = \int_0^E \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}y$	$3.186{\times}10^{-15}q$
Bottom	$\mathrm{d}\hat{\mathbf{A}} = \{0, 0, -\hat{\imath}\hat{\jmath}\}$	$\varphi_{\text{bottom}} = \int_0^E \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}y$	$-3.186{\times}10^{-15}q$
Front	$\mathrm{d}\hat{\mathbf{A}} = \{0, \hat{\imath}\hat{k}\}, 0\}$	$\varphi_{\text{front}} = \int_0^F \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}z$	$-3.186{\times}10^{-15}q$
Back	$\mathrm{d}\hat{\mathbf{A}} = \{0, -\hat{\imath}\hat{k}, 0\}$	$\varphi_{\text{back}} = \int_0^F \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}z$	$3.186{\times}10^{-15}q$
Left	$\mathrm{d}\hat{\mathbf{A}} = \{\hat{\jmath}\hat{k}\}, 0, 0\}$	$\varphi_{\text{left}} = \int_0^F \int_0^E \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}y \mathrm{d}z$	0
Right	$\mathrm{d}\hat{\mathbf{A}} = \{-\hat{\jmath}\hat{k}\}, 0, 0\}$	$\varphi_{\text{right}} = \int_0^F \int_0^E \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}y \mathrm{d}z$	0

Table 1: Table for the magnetic flux as calculated from the different possible surfaces of the memristor. The magnitude is the same for any of the four surfaces which cross the magnetic field, the sign changes dependent on the direction of the field. The end surfaces, labelled left and right, are perpendicular to the flow of vacancies, and this parallel to the magnetic field and therefore have a magnetic flux of 0.

Integrals are done over x, y or z which are coordinates outside the device. It's best to think of the surface as hovering an infinitesimal above the actual surface of the device, to avoid edge effects. Note, that the values of the non-zero φ's will change as the device switches. This dependence is shown above by keeping q as a variable but the limits in the x are taken a D and are thus the limiting case, in actual fact they could be less.

A.3 Memristor theory Comparison

A table to compare the differences between my theory and Strukov et al's model.

Quantity	HP Model	My Model
Memristance	$R_{\text{off}} - \beta R_{\text{off}} R_{\text{on}} q$	$M(q(t)) = \frac{(\mu_0 DEL\mu_v)}{4\pi} P_z$
β	$\frac{\mu_V}{D^2}$	$\mu_v P_z$
$M(\varphi)$	$\frac{D^2}{\mu_v}$	$\frac{1}{\mu_v P_z}$
Memory function, $[Mem]$	$\beta q = \frac{D^2 q}{\varphi} \approx \frac{\mathrm{d}q}{\mathrm{d}\varphi} = M(q)$	$M(q)$
Conservation function	$[R_{\text{on}}^2 + R_{\text{off}} - R_{\text{on}} R_{\text{off}}][Mem]$	$\frac{\rho_{off}(D-w)}{EF} + [Mem]$
$\frac{\mathrm{d}d(t)}{\mathrm{d}t}$	$\mu_v L = \mu_v \frac{V}{D} = \mu_v \frac{iR}{D}$ *	$\mu_v L = \mu_v \frac{V}{D}$

* Note, the equation $\mu_v \frac{V}{D} = \mu_v \frac{iR}{D}$ which is utilized by Strukov et al in their derivation is incorrect, because the current, i, is the electronic current and the movement of the boundary, $\frac{\mathrm{d}w(t)}{dt}$, depends only on the drift velocity of vacancies, which is $\mu_v L$

A.4 Discussion of Strukov et al's phenomenological model derivation [2]

Strukov et al's 2008 Nature paper [2] announcing the manufacture of the TiO_2/TiO_{2-x} at HP included a set of equations which modelled this system as two variable resistors linked by a rule that ensured space conservation. The terseness of these equations, due to the necessity for brevity, obfuscate some assumptions made. For this reason, the derivation is discussed below.

First, they define the position on the x-axis of the boundary between the TiO_2/TiO_{2-x} as being equal to w, where w is restricted to varying between 0 and D, the length of the device, ie, $0 \leq w \leq D$. This makes sure that space is conserved, ie the total amount of Titanium dioxide can not change in an unphysical way.

From their definition of w, they get the following equation for the memristance that relates $I(t)$ and $V(t)$:

$$V(t) = \left(R_{on}\frac{w(t)}{D} + R_{off}\left(1 - \frac{w(t)}{D}\right)\right) I(t),$$

which they call equation 5 in their paper, and the fact that it is a distance divided by the total distance, D, means that the two resistances vary according to fractions of 1, and thus that space is conserved in the model.

They then concern themselves with the change in w over time, $\frac{dw(t)}{dt}$ They expect that the boundary will move over time, dependent on the ion mobility(μ_v), which is the effective speed of an ion, in this case oxygen vacancies, in a unit electric field. The electric field is given by E in these equations, L in the derivations in this paper. Thus,

$$\frac{dw(t)}{dt} = \mu_v E,$$

which they do not state in the paper.

Voltage, V, is equal to the electric feild, measured between two points divided by the distance between those two points. If we look at the voltage across the two ends of the device (which is handily measurable in the lab), then we get, $V = ED$ and so

$$\frac{dw(t)}{dt} = \mu_v \frac{V}{D},$$

which is also not included in the paper.

However, the next step is, and the following equation is given as equation 6, which introduces $\frac{dw(t)}{dt}$ and follows on directly from equation 5 as a statement:

$$\frac{dw(t)}{dt} = \mu_v \frac{iR}{D}$$

In the paper they state that they are assuming ‘ohmic electrical conductivity’, namely that $V = iR$ and so can be replaced with it. I have used the lower case i to indicate the current, mostly to show that it is not necessarily I. If $V = iR$, then this current must be the current that is related to the voltage which is measured between the two ends of the memristor, ie the electronic current, i_{e^-}. In the HP memristor there are actually two currents at work, the electrical current, which is the electrons flowing through the device, and the current due to the oxygen vacancies, i_{O^+}, whose movement changes the resistance.

The second thing that is worrying about this equation is the resistance, R. What is it? There are two resistances in the system, R_{off} and R_{on}. Why do the authors chose R_{on}? The answer is that they have decided that 0 on the x axis corresponds to the side of the device where the doped, (R_{on}) material is. So the oxygen vacancy only passes through R_{on} material to get to the boundary. As most the potential drop is across the TiO_2 material, and because R_{off} is 100 times bigger than R_{on}, by approximating the voltage at the boundary as V they are introducing a small error of about 1 in 100 (as the voltage drop across TiO_2 is 100 times more than TiO_{2-x}.

To get the, now-famous, equation for memristance in terms of the charge transferred, $q(t)$, they now integrate the above expression with respect to time:

$$\int_t \frac{dw(t)}{dt} = \mu_v \frac{R_{on}}{D} i(t)$$

as $\int_t i(t) = q(t)$,

they write the answer as eqn 7, which is:

$$w(t) = \mu_v \frac{R_{on}}{D} q(t).$$

This is odd as $i(t)$ hasn't been explicitly defined but from equation 6 where V is replaced with iR, this must be the electronic current. In that case, $q(t)$ must be the amount of electronic charge that has passed through the device, q_{e^-}. This is fine, except that Chua's formulation seems to suggest that the charge is transferred onto the device somehow. In fact it is, the TiO_2 undergoes a chemical change which turns it to TiO_{2-x} as a result of something in the system and, over time, the amount of TiO_{2-x} increases. This is an increase of charge in the system as each vacancy is a positive charge. These charge carriers are what remembers the state of the memristor and that they remember it with zero power draw and thus can retain a state when the power is off. This argument strongly implies that the charge in the memristor equation should be that of the oxygen vacancies, not the electronic charge.

The equation above is put into eqn 5 in the paper which gives:

$$v(t) = \left(R_{on} \frac{\left[\frac{\mu_v R_{on} q(t)}{D}\right]}{D} + R_{off} \frac{\left[1 - \frac{\mu_v R_{on} q(t)}{D}\right]}{D} \right) i(t)$$

and thus memristance, M, is equal to

$$M = \frac{\mu_v R_{on}^2 q(t)}{D^2} + R_{off} \left(1 - \frac{\mu_v R_{on} q(t)}{D^2}\right),$$

which because R_{on} is 100 times less than R_{off}, they disregard the first term and report the rest in their paper.

A.5 B field associated with HP's 'φ'

If

$$\varphi = \frac{D^2}{\mu_v}$$

and

$$\varphi = \int B.dA,$$

and D^2 taken to be an approximation for the surface of the memristor xy, ie the area we integrate over then

$$B = \frac{d\varphi}{dA}$$

As $dA = dxdy$

$$B = \frac{d^2\varphi}{dxdy} = \frac{d^2\left(\frac{D^2}{\mu_v}\right)}{dxdy} = \frac{d^2\left(\frac{xy}{\mu_v}\right)}{dxdy} = \frac{1}{\mu_v}$$

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