View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by UWE Bristol Research Repository

AIP | Chaos
An Interdisciplinary Journal of Nonlinear Science

A consistent approach for the treatment of Fermi acceleration in time-dependent billiards

A. K. Karlis, F. K. Diakonos, and V. Constantoudis

Citation: Chaos **22**, 026120 (2012); doi: 10.1063/1.3697399

View online: http://dx.doi.org/10.1063/1.3697399

View Table of Contents: http://chaos.aip.org/resource/1/CHAOEH/v22/i2

Published by the American Institute of Physics.

Related Articles

Enhanced diffusion in conic channels by means of geometric stochastic resonance
J. Chem. Phys. 135, 174102 (2011)

Entropic noise-induced nonequilibrium transition
J. Chem. Phys. 133, 204102 (2010)

Diffusion dynamics and first passage time in a two-coupled pendulum system
Chaos 20, 033120 (2010)

Communications: Drift and diffusion in a tube of periodically varying diameter. Driving force induced intermittency
J. Chem. Phys. 132, 221104 (2010)

Communications: Drift and diffusion in a tube of periodically varying diameter. Driving force induced intermittency
JCP: BioChem. Phys. 4, 06B401 (2010)

Additional information on Chaos

Journal Homepage: http://chaos.aip.org/

Journal Information: http://chaos.aip.org/about/about_the_journal

Top downloads: http://chaos.aip.org/features/most_downloaded

Information for Authors: http://chaos.aip.org/authors

ADVERTISEMENT

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

A consistent approach for the treatment of Fermi acceleration in time-dependent billiards

A. K. Karlis,[1,2,a)] F. K. Diakonos,[1,b)] and V. Constantoudis[3,c)]
[1]*Department of Physics, University of Athens, GR-15771 Athens, Greece*
[2]*Physikalisches Institut, Universität Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany*
[3]*Institute of Microelectronics, NCSR Demokritos, P.O. Box 60228, Attiki, Greece*

(Received 14 December 2011; accepted 7 March 2012; published online 25 June 2012)

The standard description of Fermi acceleration, developing in a class of time-dependent billiards, is given in terms of a diffusion process taking place in momentum space. Within this framework, the evolution of the probability density function (PDF) of the magnitude of particle velocities as a function of the number of collisions n is determined by the Fokker-Planck equation (FPE). In the literature, the FPE is constructed by identifying the transport coefficients with the ensemble averages of the change of the magnitude of particle velocity and its square in the course of one collision. Although this treatment leads to the correct solution after a sufficiently large number of collisions have been reached, the transient part of the evolution of the PDF is not described. Moreover, in the case of the Fermi-Ulam model (FUM), if a standard simplification is employed, the solution of the FPE is even inconsistent with the values of the transport coefficients used for its derivation. The goal of our work is to provide a self-consistent methodology for the treatment of Fermi acceleration in time-dependent billiards. The proposed approach obviates any assumptions for the continuity of the random process and the existence of the limits formally defining the transport coefficients of the FPE. Specifically, we suggest, instead of the calculation of ensemble averages, the derivation of the one-step transition probability function and the use of the Chapman-Kolmogorov forward equation. This approach is generic and can be applied to any time-dependent billiard for the treatment of Fermi-acceleration. As a first step, we apply this methodology to the FUM, being the archetype of time-dependent billiards to exhibit Fermi acceleration. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3697399]

Fermi acceleration, which is the increase of the mean energy of an ensemble of particles due to random collisions off moving scatterers, is clearly one of the most interesting physical mechanisms linked to time-dependent billiards. Despite this fact, the standard approach in the literature for its analytical treatment can at best describe Fermi acceleration in the asymptotic time limit. Herein, we propose a methodology, which describes the evolution of Fermi acceleration at all times and, even more, obviates any unclear or *ad hoc* assumptions, which can lead to inconsistencies or unreliable results. We exemplify the proposed approach in the prototype of billiards exhibiting Fermi acceleration; The Fermi-Ulam model.

I. INTRODUCTION

More than 60 years ago, Fermi[1] proposed an intuitive mechanism for the explanation of the origin of the highly energetic cosmic ray particles and ever since it has been a subject of intense study. The mechanism consists in the increase of the mean energy of particles as a result of random collisions with moving scatterers. Soon after his seminal paper, his co-worker Ulam introduced a simple mechanical model for testing Fermi's idea,[2] known as the Fermi-Ulam model (FUM), linking for the first time Fermi acceleration with the study of time-dependent billiards.

Since the introduction of the FUM, the standard description of Fermi acceleration developing in a class of time-dependent billiards is given in terms of a diffusion process taking place in momentum space.[3–5] Within this framework, the evolution of the probability density function (PDF) of the magnitude of particle velocities as a function of the number of collisions n is determined by the Fokker-Planck equation (FPE). In the literature, the FPE is constructed by identifying the transport coefficients with the ensemble averages of the change of the magnitude of particle velocity and its square in the course of one collision.[3–6] Although this treatment leads to the correct solution after a sufficiently large number of collisions have been reached, the transient part of the evolution of the PDF is not described. Moreover, in the case of the FUM, if a standard simplification is employed—known as the static wall approximation (SWA) or the simplified Fermi Ulam Model (SFUM)—the solution of the FPE is even inconsistent with the values of the transport coefficients used for its derivation.[7]

The aim of the work presented is to provide a self-consistent methodology for the derivation of the PDF of particle velocities for all times. The proposed approach obviates any assumptions for the continuity of the random process and the existence of the limits formally defining the transport coefficients of the FPE. Specifically, we suggest, instead of the calculation of ensemble averages, the derivation of the one-step transition probability function (TPF) and the use of

[a)]Current address: Department of Physics, University of Athens, GR-15771 Athens, Greece. Electronic mail: akkarlis@gmail.com.
[b)]Electronic mail: fdiakono@phys.uoa.gr.
[c)]Electronic mail: vconst@imel.demokritos.gr.

 © 2012 American Institute of Physics

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

the Chapman-Kolmogorov (forward) equation (CKE). This approach is generic and can be applied to any time-dependent billiard for the treatment of Fermi-acceleration. As a first step, we apply this methodology to the FUM, being the archetype of time-dependent billiards to exhibit Fermi acceleration. In this context, we show that the FPE reported in the literature[5] describing the evolution of the PDF of the magnitude of particle velocities is not valid, and that the observed agreement for $n \gg 1$ between the analytical and numerical results, in this case, should be regarded as accidental, i.e., due to the validity of the central limit theorem (CLT).

II. STATISTICAL DESCRIPTION OF FERMI ACCELERATION

Fermi acceleration developing in a time-dependent billiard can be described in terms of a stochastic process taking place in the velocity space. Let $W(v, z)$ denote the probability of a particle being at the velocity z to perform a jump to velocity v in the course of a single collision and $\rho(v, n|v', n')$ the probability of a particle to possess velocity v after n collisions given that at the n'th collision it had velocity v'. This jump process can be described by the following equation:

$$\rho(v, n|v', n') = \int dz \rho(z, n-1|v', n') W(v, z). \tag{1}$$

Equation (1) is exact, on the condition that the process is Markovian. From a physical point of view, this means that the probability of a particle to experience a velocity jump equal to Δv upon the nth collision depends only on the velocity it had at the previous step, i.e. at the nth-1 collision.

A. The Fokker-Planck approximation

The standard approach in the literature for the determination of the asymptotic behaviour of the PDF of particle velocities, is the approximation of the jump process with a diffusion process, described by the FPE.[3–5] This approximation has also been used for the analytical treatment of Fermi acceleration developing in higher-dimensional billiards, such as the simplified periodic Lorentz gas,[6] i.e., the oscillating hard circular scatterers oscillate only in the velocity space. An equation of the form of the FPE can be derived from Eq. (1) as follows:[8]

If we introduce $\Delta v \equiv v - z$, then the integrand in Eq. (1) can be rewritten as,

$$\rho(v, n|v', n') = \int d(\Delta v) \rho(v - \Delta v, n-1|v', n') \times W(v - \Delta v + \Delta v, v - \Delta v). \tag{2}$$

Expanding the distribution function $\rho(v, \Delta v, v', n')$ and the TPF $W(v; \Delta v)$ in a Taylor series—known in the literature as the Kramers-Moyal expansion[8]—yields,

$$\rho(v, n|v', n') = \int d(\Delta v) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (\Delta v)^m \times \frac{\partial^m}{\partial v^m} \rho(v, n-1|v', n') W(v + \Delta v, v). \tag{3}$$

Integrating now Eq. (3) over Δv we obtain,

$$\rho(v, n|v', n') = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial v^m} M_m(v) \rho(v, n-1|v', n'), \tag{4}$$

where $M_m(v)$ stands for the mth moment of the TPF, i.e.,

$$M_m(v) = \int (\Delta v)^m W(v + \Delta v, z) d(\Delta v).$$

Therefore,

$$\rho(v, n|v', n') - \rho(v, n-1|v', n') = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial v^m} M_m(v) \rho(v, n|v', n'). \tag{5}$$

By truncating the above series to the second order, and further by approximating the discrete derivative

$$\Delta_k \rho(v, n|v', n') = [\rho(v, n+k|v', n') - \rho(v, n|v', n')]/k,$$

$(k = 1)$ with the continuous derivative $\partial \rho(v, n|v', n')/\partial n$, for $n \gg 1$ one obtains an equation resembling the FPE.

$$\frac{\partial}{\partial n} \rho(v, n|v', n') = -\frac{\partial}{\partial v} [B \rho(v, n|v', n')] + \frac{1}{2} \frac{\partial^2}{\partial v^2} [D \rho(v, n|v', n')], \tag{6}$$

where the coefficient B and D is the ensemble average of the change of particle velocities and its square, respectively, in one mapping period.

The approximations applied above for the construction of the FPE are valid on the condition that only very small jumps are probable and further that the solution $\rho(v, n|v', n')$ varies slowly with v so that one can perform the expansion in a Taylor series. More formally,[9] we demand that there exists a $\delta > 0$,

$$W(z + \Delta z, z) \approx 0, \quad \text{for} \quad |\Delta z| > \delta \tag{7a}$$

$$\rho(v + \Delta v, n|v', n') \approx \rho(v, n|v', n'), \quad \text{for} \quad |\Delta v| < \delta. \tag{7b}$$

In the literature,[5] the derivation of an FPE from Eq. (1) for the statistical description of Fermi acceleration is carried out on an *ad hoc* basis. As a consequence, as shown in the following, it has produced contradictory results. Moreover, by construction, the description of Fermi acceleration with a continuous stochastic process, can at best describe the statistics only for $n \gg 1$. Hence, a full description of Fermi acceleration (FA) in a time-dependent billiard can only be given in the context of a jump process and consequently by Eq. (1).

B. A complete description: The Chapman-Kolmogorov equation

The study of the transient statistics can only be accomplished by means of the Chapman-Kolmogorov equation, i.e., Eq. (1). Assuming that initially particle velocities are distributed according to $\rho(v, 0) = \delta(v - z)$, Eq. (1) can be rewritten in respect with the one-step TPF. $W(v, v')$ as,

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

$$\rho(v_n, n|z, 0) = \int \ldots \int W(v_n, v_{n-1}) \ldots W(v_1, z)\mathbf{dv}, \quad (8)$$

where $\mathbf{dv} = \Pi_{i=1}^{n-1} dv_i$. The derivation of the one-step TPF can be achieved by determining the PDF $p(\mathbf{q})$ of the variables $\mathbf{q} \equiv \{x_i\}$ appearing in the dynamical equation defining the velocity of a particle after a collision with the moving boundary of the time-dependent billiard, $v_n = f(v_{n-1}, \mathbf{q})$. Then, the TPF is

$$W(v_n, v_{n-1}) = \int p(\mathbf{q})\delta[v_n - f(\mathbf{q}, v_{n-1})]\mathbf{dq}. \quad (9)$$

If the resulting TPF is a function of the difference of velocities at successive steps $W(v_n, v_{n-1}) = W(v_n - v_{n-1})$, Eq. (8) can be easily solved in the Fourier space. Specifically, if this condition is met, then by taking the Fourier transform of Eq. (8) we find,

$$\mathcal{F}[\tilde{\rho}(v, n|z, 0)] = (2\pi)^{\frac{n-1}{2}} e^{-ikz}\{\mathcal{F}[W(v)]\}^n, \quad (10)$$

where $\mathcal{F} = 1/(\sqrt{2\pi}) \int_{-\infty}^{\infty} \exp[-ikv]\, dv$.

Moreover, in this case, an approximate solution can be obtained directly in the velocity space, using the saddlepoint approximation technique.[10] Specifically, from Eq. (8), one can derive the moment generating function

$$\begin{aligned}\phi(t, n) &= \int_{-\infty}^{\infty} e^{tx} \rho(v, n|z, 0) dv \\ &= \left(\int_{-\infty}^{\infty} e^{tv} W(v) dv\right)^n e^{tz}\end{aligned} \quad (11)$$

of the velocity PDF. To find the saddlepoint $\hat{t}(v, n)$, we solve the equation $\kappa'(t, n) = v$, where $\kappa(t, n) = \log(\phi(t, n))$. Provided that $\kappa''(\hat{t}(v, n)) > 0$, the PD\F is approximately,

$$\begin{aligned}\rho(v, n|z, 0) \approx &\sqrt{\frac{1}{2\pi\kappa''(\hat{t}(v, n))}} \\ &\times \exp[\kappa(\hat{t}(v, n)) - \hat{t}(v, n)v] + \mathcal{O}(n^{-3/2}).\end{aligned} \quad (12)$$

Equation (12) can always be employed in this context, since the particle velocity after the nth collision can be viewed as the sum of n identically and independently distributed random numbers $v_n = \sum_{i=1}^{n} \Delta v_i + v_0$. In this case, it has been proven that $\kappa''(\hat{t}(v, n)) > 0$.[10,11] Finally, we note that the approximation can be improved by renormalizing the result, i.e., $\int \rho(v, n|z, 0) dv = 1$.

In Sec. III, we implement the proposed methodology in the prototype of time-dependent billiards exhibiting Fermi acceleration; FUM.

III. FERMI ACCELERATION IN THE STOCHASTIC SIMPLIFIED FUM

The Fermi-Ulam model, originally proposed for testing the feasibility of gaining energy through scattering off moving targets, i.e., Fermi acceleration, consists of one harmonically oscillating and one fixed infinitely heavy hard wall and an ensemble of non-interacting particles bouncing between them. Ever since, many different versions of the original model have been suggested and investigated, such as variants of the FUM with dissipation,[12–16] different deterministic or random drivings of the moving wall[17,18] the quantum-mechanical version[19–24] and the, so called, bouncer model,[25] where a particle performs elastic[26] or inelastic[27–33] collisions with an oscillating infinitely heavy platform under the influence of a gravitational field. Recently, a hybrid version of the FUM and the bouncer model has also been investigated.[34,35]

The equations defining the dynamics of the FUM are of implicit form with respect to the collision time, which complicates numerical simulations and hinders an analytical treatment. A simplification[5]—known as the SWA (Refs. 17 and 36)—consists in treating the oscillating wall as immobile, located at its equilibrium position, yet allowing the transfer of momentum upon impact with a particle as if the wall were harmonically oscillating. This simplification has become over the time the standard approximation for studying the FUM.[37] The SWA speeds-up the numerical simulations and facilitates the analytical treatment of the problem, while it has been generalized to higher-dimensional billiards with time-dependent boundaries, such as the time-dependent Lorentz Gas.[6,17]

Let us consider, without loss of generality, a FUM consisting of a fixed wall on the right and a moving wall on the left, oscillating with frequency ω. If we further assume that the positive direction of particle velocities is towards the right, then the dynamics of the billiard within the framework of the SWA is defined by the following set of dimensionless difference equations:

$$t_n = t_{n-1} + \frac{2}{v_{n-1}}, \quad (13a)$$

$$v_n = |v_{n-1} + 2u_n|, \quad (13b)$$

$$u_n = \epsilon \cos(t_n + \eta_n), \quad (13c)$$

where u_n is the velocity of the "oscillating" wall, v_n is the algebraic value of the particle velocity immediately *after* the nth collision with the "oscillating" wall measured in units of ωw (w denoting the spacing between the walls), t_n the time when the nth collision occurs measured in units of $1/\omega$, η_n a random variable uniformly distributed in the interval $[0, 2\pi)$ updated immediately after each collision between a particle and the fixed wall and ϵ the dimensionless ratio of the amplitude of oscillation to the spacing between the "oscillating" and the fixed wall. It is noted that in all numerical simulations ϵ was fixed at 1/10.

The absolute value in Eq. (13b) is introduced in order to avoid the occurrence of positive particle velocities after a collision with the "oscillating" wall, which would lead to the escape of the particle from the area between the walls. It should be stressed that such a collision, within the framework of the exact model, corresponds to a particle experiencing at least one second consecutive collision with the "oscillating" wall. Therefore, if $|V_{n-1}| < 2|u_n|$ and $u_n \leq 0$, in order to prevent the particle from escaping the region between the walls the velocity is reversed artificially. The presence of the absolute value function in Eq. (13b), nevertheless, complicates the analytical treatment of the acceleration problem. For this reason, it has become a standard

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

practice in the treatment of the FUM to remove the absolute value function, thereby neglecting the set of collision events upon which the particle direction is not reversed after its collision with the "oscillating" wall. However, this further simplification gives rise to a fundamental inconsistency: the ensemble mean of the absolute velocity obtained analytically does not change through collisions with the "*oscillating*" wall, despite the well-established numerical result that Fermi acceleration does take place in the phase-randomized FUM.

A. The asymptotics of the PDF of particle velocities

1. Application of the central limit theorem

In this section, we will discuss the asymptotic behaviour of the PDF of particle velocities in the SFUM. Evidently, after n collisions, the velocity of a particle evolving in the SFUM is the sum of the velocity jumps, it has experienced up to this point, i.e., $v_n = \sum_{m=1}^{n} \Delta v_n + v_0$. Furthermore, due to Fermi-acceleration developing in the SFUM, after $n \gg 1$ collisions, the vast majority of the particles has acquired velocities much greater than the maximum wall velocity, irrespective of the initial distribution. Therefore, most of the collisions, after a sufficiently large "time," take place in the high velocity regime. In this limit, the absolute value function can be neglected and we immediately obtain[3,7] $\langle \Delta v \rangle = 0$ and $\langle (\Delta v)^2 \rangle = 2\epsilon^2$. Therefore, the velocity jumps are completely uncorrelated, i.e., do not depend on the velocity of the particle had at the previous step. Thus, for $n \gg 1$ and $v \gg \epsilon$, the CLT dictates that the PDF of particle velocities tends to a Gaussian distribution, with a mean value equal to $\sum_{i=1}^{n} \langle \Delta v_i \rangle$ and variance $\sigma^2 = \sum_{i=1}^{n} [\langle (\Delta v_i)^2 \rangle - \langle \Delta v_i \rangle^2]$.

Hence, the PDF of particle velocities for $n \gg 1$ is

$$\rho(v,n) = \frac{1}{\epsilon\sqrt{\pi n}} \exp\left[-\frac{v^2}{4\epsilon^2 n}\right]. \tag{14}$$

In Fig. 1, Eq. (14) is plotted along with the histogram of particle velocities obtained from the simulation of 1.2×10^6 trajectories for $n = 10^5$ collisions. The ensemble was initially distributed according to the delta function $\delta(v - \epsilon)$. The analytical result obtained from the application of the CLT is in perfect agreement with the numerically computed PDF.

2. FPE equation in the SFUM

As mentioned in Sec. III A 1, assuming that for $n \gg 1$, the probability measure of the events occurring in the low-velocity regime is negligible, $B \equiv \langle \Delta v \rangle \simeq 0$, $D \equiv \langle (\Delta v)^2 \rangle \simeq 2\epsilon^2$. In this limit, Eq. (6) obtains the form of a standard diffusion equation, which for a delta initial distribution of velocities $v = z$ together with reflecting conditions at $v = 0$ has as a solution the sum of two spreading Gaussians

$$\rho(v,n|z,0) = \frac{1}{2\sqrt{\pi n \epsilon^2}} \left\{ \exp\left[-\frac{(v-z)^2}{4n\epsilon^2}\right] + \exp\left[-\frac{(v+z)^2}{4n\epsilon^2}\right] \right\}, \tag{15}$$

which for $n \geq z^2/(4\epsilon^2 \ln 2)$ transforms to Eq. (14).

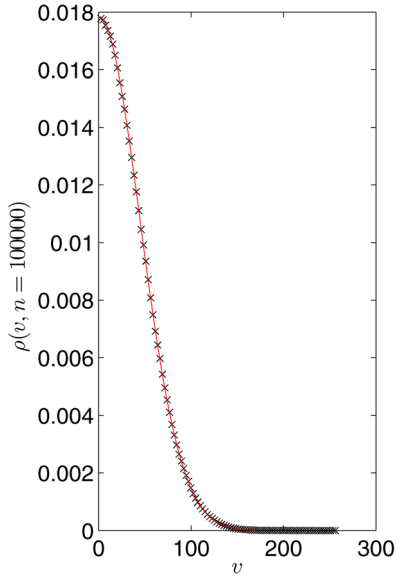

FIG. 1. Histogram—diagonal crosses—of particle velocities after $n = 10^5$ collisions, obtained by the iteration of Eq. (13), on the basis of an ensemble of 1.2×10^6 particles initially distributed as $\rho(v,0) = \delta(v - \epsilon)$. The analytical result derived through the application of the CLT [Eq. (14)] is also plotted—solid (red) line.

3. Remarks

Although the solution derived by means of the FPE is in agreement with the one obtained from the application of the CLT, the methodology used for the derivation of Eq. (6) stands on very shaky ground, since the termination of the series at the second term in Eq. (5) is completely arbitrary.[9] In general, a jump process can be approximated by a diffusion process, on the condition that a scaling assumption for the transition probability holds. Namely, in the limit of infinitely small time intervals, the jumps should become smaller and more frequent, such that the random process can be viewed as a continuous one.[38] An intuitive way to examine this is to consider the average square of the jump size $\langle (\Delta v)^2 \rangle$ a particle makes having a velocity v prior to the collision.

Given that the SWA treats the moving wall as fixed in the configuration space, all phases upon collision are possible, independently of the velocity v of particles before a collision. As a result, the average jump size is not reduced as $v \to 0$. In contrast, within the exact model, as the velocity of the particle prior to a collision decreases, it becomes increasingly probable to collide with the wall at the turning points, where its velocity is close to zero. Moreover, if the velocity of the particle before a collision is small, then successive collisions are likely to occur, the exact dynamics result to higher exit velocities. Consequently, successive collisions render small particle velocities improbable, as opposed to the SFUM, where as shown in Sec. I, $v = 0$ is the most probable velocity.

Summarizing, the application of the CLT for the determination of the long-time statistics is much more straightforward and renders the solution of a differential equation redundant. More importantly, the assumption of continuity of the stochastic process describing Fermi acceleration, which is essential for the construction of an FPE, is not required.

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

B. Short-time statistics in the SFUM

From Eq. (13), the particle velocity after the nth collision given that it had velocity z is,

$$v = -z - 2u - 2(z + 2u)\,\Theta\,(2u + z), \tag{16}$$

where $\Theta(x)$ is the Heaviside unit-step function.

According to Eq. (13b), the wall velocity u_n is determined by the phase $\xi_n \equiv t_n + \eta_n$ of oscillation at the instant of the nth collision. Due to the fact that in the stochastic SFUM, the phase is randomly shifted through the addition of a random number η_n—distributed uniformly in the interval $(0, 2\pi)$—after each collision, the oscillation phase ξ_n is completely uncorrelated between collisions, following a uniform distribution. Furthermore, given that in the context of the SFUM the wall remains fixed in the configuration space, the wall velocity upon collision does not depend on the velocity of the particle, therefore, u_n and v_{n-1} are also uncorrelated. From the fundamental transformation law of probabilities, the PDF of the wall velocity upon collision is,

$$p(u) = \frac{1}{\pi\sqrt{\epsilon^2 - u^2}}. \tag{17}$$

For the single-step TPF $W(v, z)$, we can write,

$$W(v, z) = \int_{-\epsilon}^{\epsilon} p(u)\delta[v - v(u, z)]du. \tag{18}$$

Substituting Eqs. (16) and (17) into Eq. (18), we obtain after integrating over u,

$$W(v,z) = \frac{1}{\pi}\left[\frac{\Theta(2\epsilon - v - z, 2\epsilon - z)}{\sqrt{4\epsilon^2 - (v+z)^2}} + \frac{\Theta(2\epsilon - v + z, 2\epsilon + v - z)}{\sqrt{4\epsilon^2 - (v-z)^2}}\right]. \tag{19}$$

In Fig. 2, the analytical result of Eq. (19) is compared with the histogram of particle velocities after a single collision, obtained numerically using Eq. (13) and an ensemble of 1.2×10^6 particles, with initial velocity $z = 0.1$. Clearly, the numerical and analytical results are in agreement.

The analytical result of Eq. (19) reveals that the TPF depends only on the most immediate history of a particle, which is on the velocity it had at the previous step. Consequently, the stochastic process is indeed Markovian. Even more, if the particle before a collision has velocity $z > 2\epsilon$, then the velocity jump $\Delta v = v - z$ it undergoes is completely independent on its history. Therefore, changes in velocity in the high-velocity regime are completely uncorrelated.

In more detail, Eq. (19) consists of two parts, one of which does indeed depend only on the jump size. However, the other branch of the TPF, taking effect for $v < 2\epsilon$—relating to the set of rare events[7]—depends also on the velocity at the last step. Nevertheless, the action of both branches of W allows of a simple geometrical interpretation: At each step, the second branch of the TPF stretches the PDF $\rho(v, n|z, 0)$, resulting to a probability flux towards negative

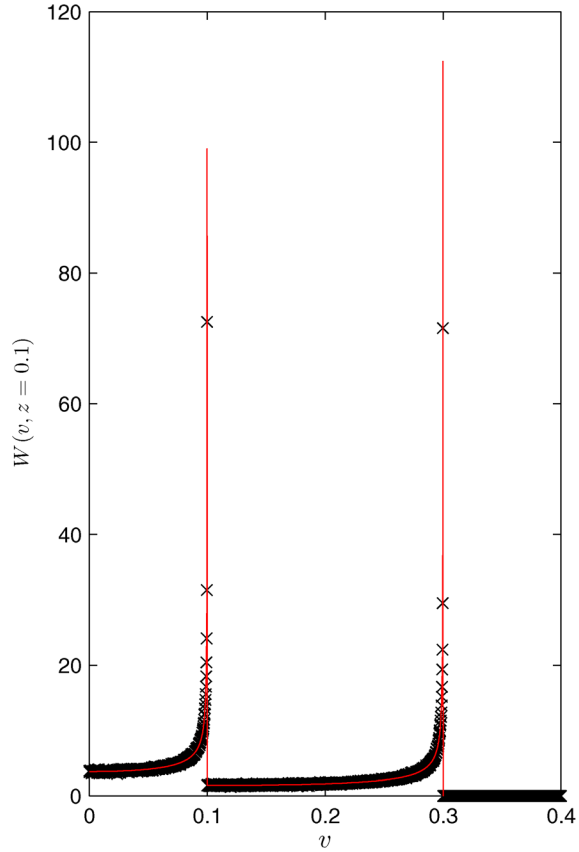

FIG. 2. Histogram—diagonal crosses—of particle velocities after a single collision with the "moving" wall, obtained using Eq. (13) and an ensemble of 1.2×10^6 particles with initial velocity $z = 0.1$. The analytical result [Eq. (19)] for the one-step transition probability is also plotted for the sake of comparison—solid line

values of velocity. This unphysical result caused by the stretching is negated by the first branch, which folds the part of the ρ density over the vertical line at $v = 0$. Therefore, the solution of Eq. (8) can be obtained by extending the domain of $\rho(v, n|z, 0)$ to the whole real line and applying the method of images. Thus, for any number of collisions, we have

$$\rho(v, n|z, 0) = \tilde{\rho}(v, n|z, 0) + \tilde{\rho}(v, n| - z, 0), \tag{20}$$

where $\tilde{\rho}$ is the solution of the unrestricted problem. Substituting Eq. (19) into Eq. (10) we obtain,

$$\tilde{\rho}(k, n|z, 0) = \frac{1}{\sqrt{2\pi}}\exp(-ikz)\,J_0\,(2\epsilon|k|)^n. \tag{21}$$

Equation (21) cannot be inverted analytically. To obtain an analytical result into the velocity space, we use the saddlepoint approximation [Eq. (12)]. The moment generating function of $\rho(v, n|z, 0)$ is,

$$\phi(t, n) = (I_0(2t\epsilon))^n e^{tz}, \tag{22}$$

where I_0 is the modified Bessel function of the first kind. Consequently, the characteristic function is $\kappa(t, n) = \log\phi(t, n) = n\log(I_0(2t\epsilon)) + tz$. The saddlepoint is the point $\hat{t}(v, n)$ that satisfies

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

$$\kappa'(t,n) = v \Rightarrow \frac{2n\epsilon I_1(2t\epsilon)}{I_0(2t\epsilon)} + z = v. \tag{23}$$

Equation (23) is implicit and cannot be solved analytically. To derive an explicit equation, we expand $\kappa'(t,n)$ in powers of ϵ to second order. Doing so we get,

$$\hat{t}(v,n) = \frac{v-z}{2n\epsilon^2}. \tag{24}$$

Substituting Eq. (24) into Eq. (12) we have,

$$\rho(v,n|z,0) \approx \frac{1}{2\epsilon} e^{-\frac{(v-z)^2}{2n\epsilon^2}} I_0\left(\frac{v-z}{n\epsilon}\right)^n \times \left[\frac{I_0\left(\frac{v-z}{n\epsilon}\right)^2}{\pi n I_0\left(\frac{v-z}{n\epsilon}\right)^2 + \pi n I_2\left(\frac{v-z}{n\epsilon}\right) I_0\left(\frac{v-z}{n\epsilon}\right) - 2\pi n I_1\left(\frac{v-z}{n\epsilon}\right)^2}\right]^{1/2}. \tag{25}$$

To improve the accuracy of the approximation, we renormalize Eq. (25) by integrating numerically over v for a given n.

In Fig. 3, we present the exact numerical solution of Eq. (8) [red solid line] as well as the approximate one given by Eq. (25) [blue solid line] for $n = \{3, 5, 10, 31\}$, using only the second branch of the one-step TPF [Eq. (19)], followed by the application of the method of images [Eq. (20)]. The numerical solution is in total agreement with the histogram of particle velocities obtained by the iteration of the dynamical equations (13a)–(13c) [upright crosses], for all times. Even more, we see that the saddlepoint approximative solution describes very accurately the evolution of the PDF for $n \geq 5$. As can be observed, the PDF of particle velocities quickly approaches to a Gaussian distribution, in accordance with the prediction of the CLT. This is attributed to the fact that the TPF can be reduced to a difference kernel. Consequently, the additional assumption we made for the application of the CLT in Sec. I, namely, that the statistical weight of the rare events[7] is negligible, is redundant. This can be circumvented, as aforementioned, by extending the domain of particle velocities. Thus, if one applies the CLT on the whole real line, then all the conditions for its applications are met exactly. As a final remark, we would like to stress that the success of the Fokker-Planck type of equation reported in the literature for even short times is attributed to the validity of the CLT, guaranteeing that the PDF will converge to a normal distribution, allowing for the use of a diffusion equation. If, however, the reduction of the TPF to a difference kernel is not feasible, then the transient can be arbitrarily long, a point demonstrated via an example in Sec. III C.

C. Long transients

In the last section, we showed that the specific choice made for treating negative velocities after a collision, i.e., reflection with respect to the $v = 0$, reduces the TPF to an even function of the jump size. As a consequence, the PDF of particle velocities approaches rapidly to a sum of two spreading Gaussians. Clearly, after a number of collisions, the system will "forget" its initial distribution, and the sum will converge to a single half-Gaussian centered at $v = 0^+$. Therefore, the most probable velocity for a particle evolving in the phase-randomized SFUM will eventually be $v_p = 0^+$, in clear contrast with the results given by the numerical simulation and analytical results derived using the exact dynamical mapping,[17,36] according to which as $v \to 0, \rho(v,n) \to 0$. From a physical point of view, this happens because if the motion of the wall in the configuration space is taken into account, as $v \to 0$ collisions resulting in an energy loss can occur only in a small neighborhood around the wall's extreme positions, where its velocity is zero, resulting to a minimal energy loss. Furthermore, if the particle velocity is comparable with the wall velocity, consecutive collisions can take place, resulting in a higher exit velocity from the interaction region within the exact model.

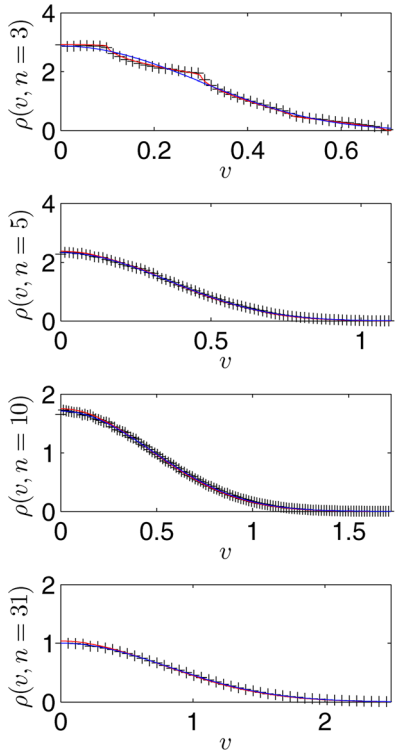

FIG. 3. Histogram—upright crosses— of particle velocities after $n = \{3, 5, 10, 31\}$ collisions, obtained by the iteration of Eq. (13), on the basis of an ensemble of 1.2×10^6 particles initially distributed as $\rho(v,0) = \delta(v-\epsilon)$. The exact numerical solution of Eq. (8) [red solid line] as well as the approximate one given by Eq. (25) [blue solid line] for $n = \{3, 5, 10, 31\}$, using only the second branch of the one-step TPF [Eq. (19)], followed by the application of the method of images [Eq. (20)] are also plotted for the sake of comparison line.

On account of these properties of the collision process in the exact model, the reflection of negative velocities is not realistic. To gap the difference between the results of the simplified and the exact FUM, we propose instead of the inversion of negative particle velocities, the inversion of the direction of the wall's velocity, if the collision would lead to a negative particle velocity. This would lead in a greater energy gain in comparison with the reflection, as $|v+u| \leq |v| + |u|$. Therefore, Eq. (13) change to,

$$t_n = t_{n-1} + \frac{2}{V_{n-1}}, \tag{26a}$$

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

$$V_n = V_{n-1} + 2|u_n|, \tag{26b}$$

$$u_n = \epsilon \cos(t_n + \eta_n). \tag{26c}$$

Let us now derive the TPF. From Eqs. (17), (18), and (26b) we obtain,

$$W(v,z) = \frac{\Theta(2\epsilon + z - v)}{\pi\sqrt{4\epsilon^2 - (v-z)^2}}\{\Theta(2\epsilon + z - v) + \Theta(\epsilon - z/2)[\Theta(z - v - 2\epsilon) + \Theta(v - 2z)]\}. \tag{27}$$

A comparison of the analytical result given by Eq. (27) and the histogram of velocities obtained on the basis of an ensemble of 1.2×10^6 particles after 1 iteration of Eq. (26) is presented in Fig. 4, proving the validity of the derived result.

As expected, the TPF has two branches, one taking effect only in the low velocity regime, i.e., $z < 2\epsilon$ and another which is nonzero for any velocity z prior to a collision. As was also the case in the SFUM with reflection of negative velocities, the part of the TPF that is relevant to the low-velocity regime depends on the jump size as well as on the velocity of a particle prior to a collision. However, due to the fact that this branch of W does not have a simple geometrical interpretation, the single-step transition function cannot be reduced to a difference propagator by an extension of the domain of W to the whole real line. Thus, the conditions for the application of the CLT are not met exactly. However, due to the acceleration of the particles, as $n \to \infty$, the probability measure of particles having velocity $z < 2\epsilon$ becomes negligible. Therefore, for $n \gg 1$ and $v \gg \epsilon$, the PDF of particle velocities tends to a Gaussian distribution [Eq. (14)].

The study of the transient behaviour of the PDF requires the solution of the CKE [Eq. (8)]. The numerical solution of Eq. (8) at times $n = \{3, 5, 17, 316\}$ is presented in Fig. 5. The histograms of particle velocities for the same times, calculated by iterating an ensemble of 1.2×10^6 particles for up to $n = 10^5$ collisions, are also plotted for the sake of comparison. It can be seen that the solution of the CKE is in agreement with the results of the simulation for all times presented. In Fig. 6, the histogram of velocities for $n = 10^5$ collisions is plotted. The solution obtained from the application of the CLT—on the assumption that the statistical weight of collisions happening in the region $v < 2\epsilon$ is negligible, is also plotted, and is full agreement with the PDF in this velocity region. However, a blow-up of the low-velocity region shows that even after 10^5 collisions, the PDF diverges from the Gaussian profile. This is clear evidence that even after a very large number of collisions, the PDF in the whole velocity domain cannot be described by an FPE, in contrast to the standard version of the SFUM [Eq. (13)] even though the argumentation used in both cases was the same, i.e., the particles are accelerated. This exemplifies the potential pitfalls of a diffusion approximation of Fermi-acceleration in time-dependent billiards.

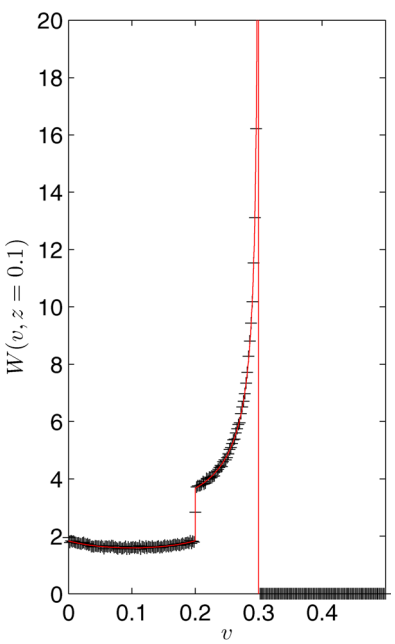

FIG. 4. Histogram—diagonal crosses—of particle velocities after a single collision with the "moving" wall, obtained using Eq. (26) and an ensemble of 1.2×10^6 particles with initial velocity $z = 0.1$. The analytical result [Eq. (27)] for the one-step transition probability is also plotted for the sake of comparison—solid line

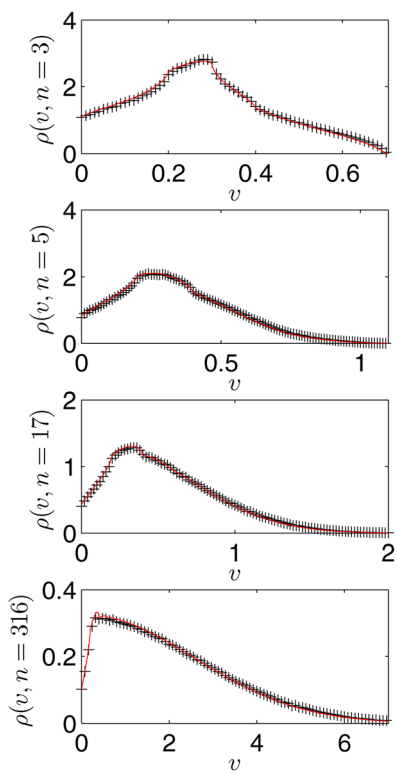

FIG. 5. Histogram—upright crosses—of particle velocities after $n = \{3, 5, 17, 316\}$ collisions, obtained by the iteration of Eq. (26), on the basis of an ensemble of 1.2×10^6 particles initially distributed as $\rho(v, 0) = \delta(v - \epsilon)$. The solution obtained by numerically solving the forward CKE [Eq. (8)] using as the TPF of the modified version of the SFUM [Eq. (27)] is also plotted—solid (red) line.

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

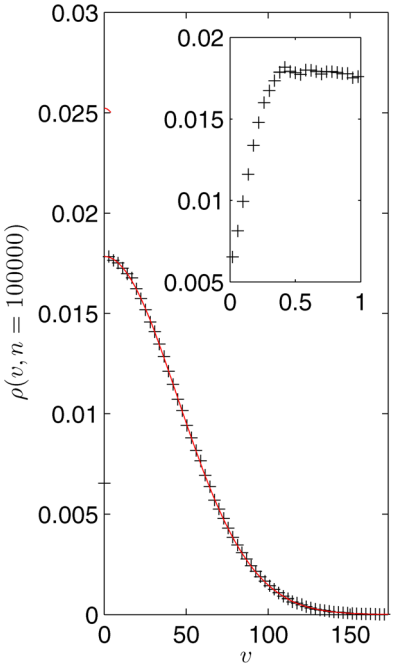

FIG. 6. Histogram—upright crosses—of particle velocities after $n = 10^5$ collisions, obtained by the iteration of Eq. (26), on the basis of an ensemble of 1.2×10^6 particles initially distributed as $\rho(v, 0) = \delta(v - \epsilon)$. The asymptotic Gaussian distribution [Eq. (14)] predicted by the CLT is also plotted—solid (red) line. A blow-up of the numerically obtained histogram at the low-velocity region is illustrated in the inset.

D. Fermi acceleration in the exact FUM

The simplification employed to the treatment of Fermi acceleration in the FUM—treating the wall as fixed in real space—however, widespread and thoroughly studied, prohibits the study of the details of Fermi acceleration. In Refs. 7, 17 and 36, it was proved that small additional fluctuations of the time of collision due to dynamical correlations induced by the displacement of the scatterer upon impact, quantitatively as well as qualitatively change the evolution of the PDF of velocities, increasing the efficiency of Fermi acceleration. Moreover, the development of correlations causes the CLT to break down and the asymptotic PDF ceases to be a normal distribution.

In Ref. 36 utilizing a novel simplification, the so-called hopping approximation, which succeeds into retaining all the essentials of the exact dynamics, an analytical solution for the asymptotic behaviour of the PDF of particle velocities in the exact FUM, which was in excellent agreement with the numerical simulation of the exact FUM. Specifically, it was shown by means of a Fokker-Planck type of equation that, in contrast with the SFUM, the attractor of the PDF of velocities in the function space is a Maxwell-Boltzman like distribution, i.e., independently of the initial distribution of velocities, the PDF converges to a Maxwell-Boltzman like distribution. Therefore, in the case of the exact FUM for $v \to 0^+$, $\rho(v, n|z, 0) \to 0$, in contrast to the SFUM where $\rho(v, n|z, 0)$ attains its maximum value for $v \to 0^+$. This difference between the simplified and the exact FUM can be understood as follows: If the velocity of a particle after a collision with the moving wall is small, then multiple successive collisions are likely to occur within the exact FUM, resulting into higher exit velocities, as opposed to the simplified model, within which successive collisions cannot be realized.

Another subtle difference between the simplified and the exact FUM [see Sec. III] is that within the exact FUM particles with low velocity are more likely to collide with the oscillating wall near its turning points, where the wall velocity is close to 0. Hence, the velocity jump performed by a particle due to a collision with the wall $\Delta v \to 0$ as $v \to 0$. As a result, Fermi acceleration when using the exact dynamics can be better approximated by a continuous stochastic process, or equivalently, by the FPE. Still, the transient statistics in the system can only be studied by means of the CKE.

As aforementioned, the movement of the wall in the configuration space described by the exact dynamics results into a more efficient energy transfer from the moving wall to the particles upon collision, when compared to the SFUM. In mathematical terms, this causes the PDF of the oscillation phase on collisions to deviate from the uniform distribution, reflecting the fact that head-on collisions are more preferable than head-tail collisions. However, the phase of oscillation of the moving wall when a particles collides with the fixed wall—or when it passes through any fixed point within the area between the two walls comprising the FUM—is uniformly distributed. The map describing the exact dynamics is,

$$d_{n,} = \epsilon \sin(\delta t_n + t_{n-1} + \eta_n), \tag{28a}$$

$$u_n = \epsilon \cos(\delta t_n + t_{n-1} + \eta_n), \tag{28b}$$

$$v_n = v_{n-1} + 2u_n, \tag{28c}$$

where $d_{n,}$ stands for the position of the moving wall in the instant of the nth collision, u_n for the wall velocity, η_n for the random phase component, and v_n for the particle velocity after the nth collision. The time of free flight δt_n is obtained by solving the implicit equation

$$x_{n-1} + v_{n-1}\delta t_n = d_n, \tag{29}$$

where x_n stands for the position of the particle in the instant of the nth collision.

If we denote the phase of oscillation of the moving wall when a particle collides with the fixed wall with ψ, then

$$\psi_n = \cos^{-1}\left(\frac{u_n}{\epsilon}\right) + \frac{1}{z}\left(1 + \sqrt{\epsilon^2 - u_n^2}\right). \tag{30}$$

For Eq. (30), we obtain for the distribution of the wall velocity upon collision,

$$p_e(u) = \frac{u + z}{\pi z\sqrt{\epsilon^2 - u^2}}. \tag{31}$$

Installing Eq. (31) into Eq. (18), we obtain the one-step TPF for the exact model

$$W_e(v, z) = \frac{\Theta(2\epsilon - |v - z|)(v + z)}{2\pi z\sqrt{4\epsilon^2 - (v - z)^2}}. \tag{32}$$

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

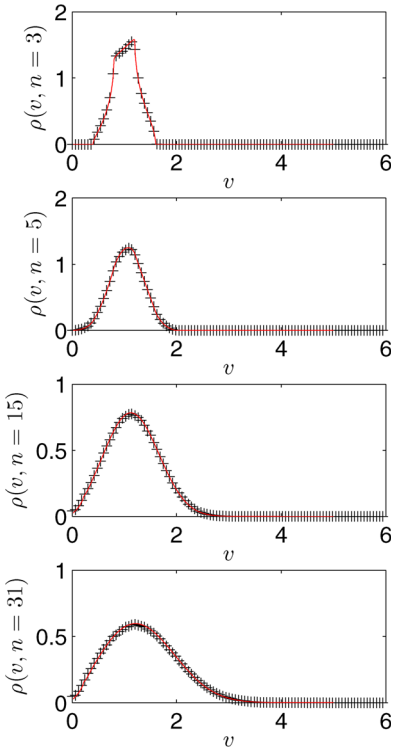

FIG. 7. Histogram—upright crosses—of particle velocities after $n = \{3, 5, 17, 316\}$ collisions, obtained by the iteration of Eq. (28), on the basis of an ensemble of 1.2×10^6 particles initially distributed as $\rho(v, 0) = \delta(v - 10)$. The solution obtained by numerically solving the forward CKE [Eq. (8)] using as the TPF of the exact model [Eq. (32)] is also plotted—solid (red) line.

Inserting Eq. (32) into the CKE we can numerically compute the evolution of the PDF of particle velocities. In Fig. 7, the numerical solution of the CKE is compared with the histogram of particle velocities, obtained by simulating 1.2×10^6 trajectories using Eq. (28). The particles were initially distributed according to $\rho(v, 0) = \delta(v - 10)$. Once more, the solution of the CKE is in complete agreement with the results of the simulation, proving that the method can also be successfully employed when the exact dynamics are taken into account.

IV. SUMMARY AND CONCLUSIONS

Fermi acceleration is one of the most interesting aspects of time-dependent billiards, as it has been understood over the years, that it is a fundamental acceleration mechanism, playing a key role in a variety of phenomena, far beyond its original scope, i.e., cosmic-ray particle acceleration.

Until now, the investigation of Fermi acceleration—in the class of time-dependent billiards in which it develops—has been carried out via its approximation with a diffusion process. Within this framework, the evolution of the density of particle velocities was determined by a Fokker-Planck equation. However, its derivation is always based on assumptions and approximations that rarely can be justified. Moreover, its prediction power is limited in the long-term statistics of the system and no information is given for the transient behaviour. Even more, its use in the SFUM, which is the first system that was successfully investigated with the use of the FPE, is completely redundant, as the CLT yields the same results in a far more straightforward manner.

Herein, we proposed a consistent methodology, which obviates unclear assumptions and even more, can give an accurate description of the transient evolution of particle velocities. The cornerstone of this methodology is the use of the Chapman-Kolmogorov equation. The fundamental difference in comparison with the traditional approach using the FPE, is that no assumption for the continuity of the stochastic process describing Fermi acceleration needs not to be made. Another advantage of the proposed approach is that all collision events can be taken into account, which cannot be done in the construction of the FPE, and even when possible, it can lead to less accurate results.

The method was successfully applied to the FUM, which is the prototype of time-dependent billiards exhibiting Fermi acceleration. In specific, we studied the standard SFUM, within which collisions leading to a potential escape from the system are handled by artificially inverting the particle exit velocity, as well as a variant of the SFUM, where if a collision would lead to a particle still moving towards the wall, the velocity of the wall is inverted before the collision takes place. Finally, we showed how this method can be applied to the exact model, showing how the effect of the motion of the wall in the configuration space can be included in the description of Fermi acceleration through the CKE. In all three cases, the CKE yielded accurate results for all times. As a final remark, we would like to stress that this methodology can be applied to higher-dimensional billiards[39] and therefore is generic.

ACKNOWLEDGMENTS

This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Heracleitus II. Investing in knowledge society through the European Social Fund.

This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada.

[1]E. Fermi, Phys. Rev. **75**, 1169 (1949).
[2]S. Ulam, in *Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics, and Probability* (California University Press, Berkeley, 1961), Vol. 3, p. 315.
[3]M. A. Lieberman and A. J. Lichtenberg, Phys. Rev. A **5**, 1852 (1972).
[4]A. J. Lichtenberg, M. A. Lieberman, and R. H, Cohen, Physica D **1**, 291 (1980).
[5]A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1992), Vol. 38.
[6]A. Yu. Loskutov, A. B. Ryabov, and L. G. Akinshin, J. Exp. Theor. Phys. **89**, 966 (1999); J. Phys. A **33**, 7973 (2000).
[7]A. K. Karlis, F. K. Diakonos, V. Constantoudis, and P. Schmelcher, Phys. Rev. E **78**, 046213 (2008).
[8]H. Risken, *The Fokker-Planck Equation, Methods of Solution and Applications* (Springer-Verlag, Berlin/Heidelberg, 1989), Vol. 18.
[9]N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, Amsterdam, 2007).

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions

[10]C. Goutis and G. Casella, *Statistics and Econometrics Series* 27, Working Paper No. 95–63 (1995).
[11]H. E. Daniels, Ann. Math. Stat. **25**, 631 (1954).
[12]G. A. Luna-Acosta, Phys. Rev. A **42**, 7155 (1990).
[13]E. D. Leonel and R. E. de Carvalho, Phys. Lett. A **364**, 475 (2007).
[14]E. D. Leonel and P. V. E. McClintock, Phys. Rev. E **73**, 066223 (2006).
[15]E. D. Leonel, J. Phys. A: Math. Theor. **40**, F1077 (2007).
[16]D. F. M. Oliveira and E. D. Leonel, Braz. J. Phys. **38**, 62 (2008).
[17]A. K. Karlis, P. K. Papachristou, F. K. Diakonos, V. Constantoudis, and P. Schmelcher, Phys. Rev. E **76**, 016214 (2007).
[18]E. D. Leonel and E. P. Marinho, Physica A **388**, 4927 (2009).
[19]J. V. José and R. Cordery, Phys. Rev. Lett. **56**, 290 (1986).
[20]W. M. Visscher, Phys. Rev. A **36**, 5031 (1987).
[21]A. J. Makowski and S. T. Dembiński, Phys. Lett. A **154**, 217 (1991).
[22]M. Razavy, Phys. Rev. A **44**, 2384 (1991).
[23]S. R. Jain, Phys. Rev. Lett. **70**, 3553 (1993).
[24]M. L. Glasser, J. Mateo, J. Negro, and L. M. Nieto, Chaos, Solitons Fractals **41**, 2067 (2009)
[25]L. D. Pustylnikov, Trans. Mosc. Math. Soc. **2**, 1 (1978).
[26]D. G. Ladeira and J. K. da Silva, J. Phys. A: Math. Theor. **40**, 11467 (2007).
[27]P. J. Holmes, J. Sound Vib. **84**, 173 (1982).
[28]R. M. Everson, Physica D **19**, 355 (1986).
[29]S. Celaschi and R. L. Zimmerman, Phys. Lett. A **120**, 447 (1987).
[30]Z. J. Kowalik, M. Franaszek, and P. Pierański, Phys. Rev. A **37**, 4016 (1988).
[31]A. Mehta and J. M. Luck, Phys. Rev. Lett. **65**, 393 (1990).
[32]E. D. Leonel and A. L. P. Livorati, Physica A **387**, 1155 (2008).
[33]A. L. P. Livorati, D. G. Ladeira, and E. D. Leonel, Phys. Rev. E **78**, 056205 (2008).
[34]E. D. Leonel and P. V. E. McClintock, J. Phys. A **38**, 823 (2005).
[35]D. G. Ladeira and E. D. Leonel, Chaos **17**, 013119 (2007).
[36]A. K. Karlis, P. K. Papachristou, F. K. Diakonos, V. Constantoudis, and P. Schmelcher, Phys. Rev. Lett. **97**, 194102 (2006).
[37]E. D. Leonel, P. V. E. McClintock, and J. K. da Silva, Phys. Rev. Lett. **93**, 014101 (2004).
[38]G. W. Gardiner, *Handbook of Stochastic Methods, for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, New York, 1985).
[39]A. K. Karlis, F. K. Diakonos, and V. Constantoudis (unpublished).

Downloaded 23 Jul 2012 to 137.205.50.42. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/about/rights_and_permissions