Gaussian lower bounds on the Dirichlet heat kernel and non-existence of local solutions for semilinear heat equations of Osgood type

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Abstract

We give a simple proof of a lower bound for the Dirichlet heat kernel in terms of the Gaussian heat kernel. Using this we establish a non-existence result for semilinear heat equations with zero Dirichlet boundary conditions and initial data in $L^q(\Omega)$ when the source term f is non-decreasing and $\limsup_{s\to\infty} s^{-\gamma}f(s) = \infty$ for some $\gamma > q(1+2/n)$. This allows us to construct a locally Lipschitz f satisfying the Osgood condition $\int_1^\infty 1/f(s)\ \mathrm{d}s = \infty$, which ensures global existence for bounded initial data, such that for every q with $1 \le q < \infty$ there is an initial condition $u_0 \in L^q(\Omega)$ for which the corresponding semilinear problem has no local-in-time solution.

Keywords: Semilinear heat equation, Dirichlet problem, non-existence, instantaneous blow-up, Osgood condition, Dirichlet heat kernel.

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1. Introduction

In a previous paper [5] we showed that for locally Lipschitz f with $f > 0$ on $(0, \infty)$, the Osgood condition

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s = \infty, \tag{1}$$

which ensures global existence of solutions of the scalar ODE $\dot{x} = f(x)$, is not sufficient to guarantee the local existence of solutions of the Cauchy problem

$$u_t = \Delta u + f(u) \tag{2}$$

for initial data in $L^q(\mathbb{R}^n)$, $1 \le q < \infty$. This is in stark contrast to the case of bounded initial data, for which (1) implies that any solution of (2) exists globally in time; see [6], for example.

In [5] we considered the PDE (2) on the whole space $\mathbb{R}^n$, which allowed us to use in our calculations the explicit form of the Gaussian heat kernel,

$$G_n(x, y; t) = (4\pi t)^{-n/2}\mathrm{e}^{-|x-y|^2/4t}. \tag{3}$$

The main result there was that for each q with $1 \le q < \infty$ one can find a non-negative, locally Lipschitz and Osgood f such that there are initial data in $L^q(\mathbb{R}^n)$ for which there is no local-in-time integrable solution of (2).

In this paper we obtain a similar result for the equation posed with Dirichlet boundary conditions on a bounded domain, by using Gaussian lower bounds on the Dirichlet heat kernel. Indeed, in Section 2 (Theorem 2.1) we give a lower bound for the Dirichlet heat kernel on a bounded domain Ω:

$$K_\Omega(x, y; t) \ge \beta^n G_n(x, y; t) \qquad \text{for} \quad t \le \epsilon^2/n, \tag{4}$$

whenever $[x, y]$, the line segment joining x and y, is contained in the interior of Ω and is always at least a distance ϵ from the boundary of Ω. Here, $\beta > 0$ is an explicit constant. Based on the argument of van den Berg [8] we also provide in the appendix a proof of a result valid for all $t > 0$

$$K_\Omega(x, y; t) \ge \mathrm{e}^{-n^2 t/4\epsilon^2} G_n(x, y; t),$$

but (4) is sufficient for our purposes and has a significantly simpler proof.

More explicitly, we focus throughout the paper on the following problem (P), posed on a smooth bounded domain $\Omega \subset \mathbb{R}^n$:

$$(\mathrm{P}) \quad \begin{cases} u_t = & \Delta u + f(u) \quad \text{in} \quad \Omega, \\ u = & 0 \quad \text{on} \quad \partial\Omega, \\ u(x,0) = & u_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

The source term $f : [0,\infty) \to [0,\infty)$ is non-decreasing and satisfies the asymptotic growth condition

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty. \tag{5}$$

We show in Theorem 4.1 that if (5) holds for some $\gamma > q(1+2/n)$ then one can find a non-negative $u_0 \in L^q(\Omega)$ such that there is no solution of (P) that is in $L^1_{\mathrm{loc}}(\Omega)$ for $t > 0$.

We finish (see Corollary 5.1) by constructing a function f that grows quickly enough such that (5) holds for every $\gamma \geq 0$, but nevertheless still verifies the Osgood condition (1). This example shows that there are functions f for which (P) is well posed in $L^\infty(\Omega)$ but not in any $L^p(\Omega)$ with $1 \leq p < \infty$.

One can see this result as in some sense dual to that of Fila et al. [3] (see also Section 19.3 of [7]), who show that there exists an f such that all positive solutions of $\dot{x} = f(x)$ blow up in finite time while all solutions of (P) with Dirichlet boundary conditions are global and bounded.

2. A Gaussian lower bound for the Dirichlet heat kernel

For any smooth domain D in $\mathbb{R}^n$ (i.e. D is smooth, open, and connected), we denote by $K_D(x,y,t)$ the Dirichlet heat kernel associated with the Dirichlet heat semigroup $S_D(t)$, i.e.

$$w_D(x,t) = (S_D(t)w_0)(x) := \int_D K_D(x,y;t)w_0(y)\,\mathrm{d}y \tag{6}$$

is the classical solution of the linear heat equation

$$\begin{aligned} w_t = \Delta w \quad &\text{in} \quad D, \\ w = 0 \quad &\text{on} \quad \partial D, \\ w = w_0 \quad &\text{in} \quad D, \end{aligned}$$

In the special case where $D = \mathbb{R}^n$, we will denote the Gaussian heat kernel on the whole space by $G_n(x,y;t)$, as given by (3).

In this section we provide a proof of a particular case of a result due to van den Berg [8], which shows that away from the boundary the Dirichlet heat kernel is bounded below by a multiple of the Gaussian kernel for the heat equation on the whole space. In this context the result for $\Omega \subset \mathbb{R}^n$ is an easy corollary of the result in $\mathbb{R}$; in the one-dimensional case our proof significantly simplifies that of [8].

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^n$, and denote by $K_\Omega(x,y;t)$ the Dirichlet heat kernel on Ω. Suppose that*

$$\epsilon := \inf_{z\in[x,y]} \operatorname{dist}(z,\partial\Omega) > 0, \tag{7}$$

where $[x,y]$ denotes the line segment joining x and y (so in particular $[x,y]$ is contained in the interior of Ω). Then for $t \le \epsilon^2/n$

$$K_\Omega(x,y;t) \ge \beta^n G_n(x,y;t),$$

where $\beta = 1 - 2/\mathrm{e} > 0$.

Note that if Ω is convex then ϵ in (7) is simply given by

$$\epsilon = \min(\operatorname{dist}(x,\partial\Omega), \operatorname{dist}(y,\partial\Omega)).$$

We delay the proof of Theorem 2.1 for a moment. Following [8] we begin with the corresponding result for an interval in $\mathbb{R}$. Our proof is somewhat simpler than that of Lemma 8 in [8], and non-probabilistic (cf. [10]), since we are able to write down directly the Dirichlet kernel in terms of a sum of Gaussian kernels on the whole line.

We write K_a for the one-dimensional heat kernel on $(-a,a)$.

Lemma 2.1. *Take $a > 0$. Then for any $x, y \in \Omega = (-a,a)$*

$$K_a(x,y;t) \ge G_1(x,y;t)\left[1 - 2\mathrm{e}^{-\epsilon^2/t}\right],$$

where $\epsilon = \operatorname{dist}([x,y],\partial\Omega)$. In particular for $t \le \epsilon^2$

$$K_a(x,y;t) \ge \beta G_1(x,y;t), \tag{8}$$

where $\beta = 1 - 2/\mathrm{e} > 0$.

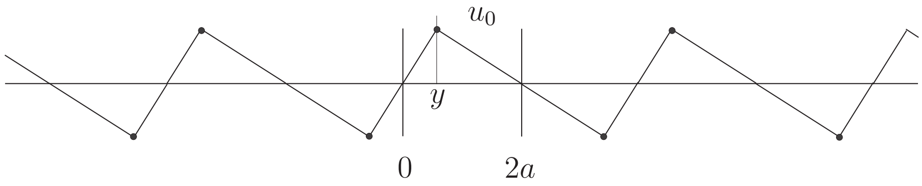

Figure 1: For a particular u_0 defined on $[0, 2a]$, an illustration of the periodic extension that is anti-symmetric about $x = 0$ and $x = 2a$. Dots indicate positions and signs of the delta functions that give rise to $K_{(0,2a)}(x, y; t)$.

Note that Corollary 6.1 in the Appendix improves the lower bound in (8) to $\mathrm{e}^{-\pi^2 t/4\epsilon^2} G_1(x, y; t)$ for all $t > 0$, but the result of this lemma is sufficient for the arguments in the main body of this paper.

Proof. For notational reasons it is simpler to treat the problem on $(0, 2a)$ rather than $(-a, a)$, but since the equations and the resulting lower bound are translation invariant this does not effect the result. We write down the Dirichlet heat kernel on $(0, 2a)$ by reflection. The essential idea is shown in Figure 1: the action of the heat equation on $[0, 2a]$ with initial data u_0 is the same as the action of the heat equation on $\mathbb{R}$ with the periodically extended initial data as illustrated, since this extension is antisymmetric about 0 and a and the Gaussian kernel $G_1(x, y; t)$ is symmetric about x for any $x \in \mathbb{R}$.

The contribution to the heat kernel for $x \in [0, 2a]$ from a source at $y \in (0, 2a)$ will be the sum of the Gaussian kernels with positive point sources at $y + 4ka$ and negative point sources at $-y + 4ka$ (see Figure 1, again), yielding

$$K_{(0,2a)}(x, y; t) = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} \mathrm{e}^{-|x-(y+4ka)|^2/4t} - \mathrm{e}^{-|x-(-y+4ka)|^2/4t}, \tag{9}$$

see Figure 2. [Even if one has doubts about the above derivation, it is clear that $K(x, y, t)$ in (9) satisfies the heat equation, $K(0, y; t) = K(2a, y; t) = 0$, and $K(x, y; 0) = \delta(y)$ for $x, y \in (0, 2a)$.]

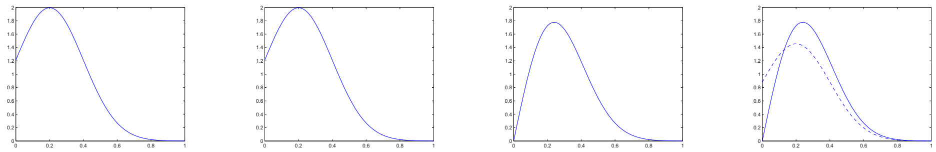

Figure 2: Dirichlet heat kernel on $[0,1]$ as a sum of Gaussians for $y=0.2$, $t=0.02$. From left to right: Gaussian kernel on $\mathbb{R}$; one subtraction ($k=1$) to enforce boundary condition at $x=1$ (little change); second subtraction ($k=0$) towards satisfying the boundary condition at $x=0$; the heat kernel on $[0,1]$ (additional terms make little difference), with the lower bound from Lemma 2.1 indicated by a dashed line.

Now we simply rewrite the sum:

$$\begin{aligned}\sqrt{4\pi t}K_{(0,2a)}(x,y;t) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{-|x-(y+4ka)|^2/4t} - \mathrm{e}^{-|x-(-y+4ka)|^2/4t} \\ &= \mathrm{e}^{-|x-y|^2/4t} - \mathrm{e}^{-|x+y|^2/4t} - \mathrm{e}^{-|x+y-4a|^2/4t} \\ &\quad + \sum_{k=1}^{\infty} \Big\{ \mathrm{e}^{-|x-(y+4ka)|^2/4t} + \mathrm{e}^{-|x-(y-4ka)|^2/4t} - \mathrm{e}^{-|(x-(-y+4a(k+1))|^2/4t} \\ &\qquad - \mathrm{e}^{-|(x-(-y-4ka)|^2/4t} \Big\} \\ &= \mathrm{e}^{-|x-y|^2/4t}\left[1-\mathrm{e}^{-xy/t}-\mathrm{e}^{-(2a-x)(2a-y)/t}\right] \\ &\quad + \sum_{k=1}^{\infty} \mathrm{e}^{-|x-y-4ka|^2/4t} + \mathrm{e}^{-|x-y+4ka|^2/4t} - \mathrm{e}^{-|x+y-4(k+1)a|^2/4t} - \mathrm{e}^{-|x+y+2ka|^2/4t}.\end{aligned}$$

Noting that

$$|x+y+4ka| > |x-y+4ka| \qquad \text{and} \qquad |4(k+1)a-(x+y)| > |4ka-(x-y)|$$

for $k \geq 1$ and $x, y \in (0, 2a)$, it follows that

$$\sqrt{4\pi t}K_{(0,2a)}(x,y;t) \geq \mathrm{e}^{-|x-y|^2/4t}\left[1-2\mathrm{e}^{-\epsilon^2/t}\right].$$

Finally note that for $t \leq \epsilon^2$ the term in the square brackets is at least $\beta = 1 - 2/\mathrm{e}$. □

The idea of the proof of Theorem 2.1, inspired by that of Lemma 9 in [8], is illustrated in Figure 3. We bound the Dirichlet heat kernel on Ω below by the kernel on the parallelepiped Π, which is simply the product of one-dimensional kernels which we can control using Lemma 2.1. In this way the proof uses the monotonicity of the Dirichlet heat kernel with respect to the domain:

$$\Omega \subset U \qquad \Rightarrow \qquad K_\Omega(x, y; t) \leq K_U(x, y; t).$$

A probabilistic proof can be found in [8]; an analytic proof using the theory of semigroups can be found in the notes by Arendt [1].

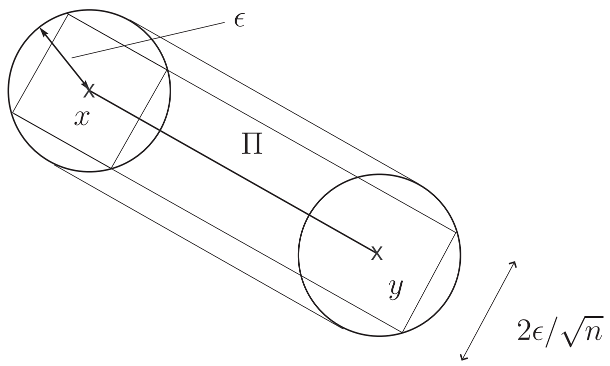

Figure 3: The parallelepiped Π with $n-1$ sides of length $2\epsilon/\sqrt{n}$ when $\mathrm{dist}(x, \partial\Omega) = \epsilon$.

Proof of Theorem 2.1. By the definition of ϵ, the line segment joining x and y is entirely contained in a parallelepiped Π that lies entirely within $\bar{\Omega}$, with one side of length $|x-y|+2\epsilon/\sqrt{n}$ and $n-1$ sides of length $2\epsilon/\sqrt{n}$, see Figure 3. Note that x and y are at least a distance $\epsilon/\sqrt{n}$ from all faces of Π. By monotonicity of the Dirichlet heat kernel with respect to the domain

$$K_\Omega(x, y, t) \geq K_\Pi(x, y, t).$$

If we now refer points in Π to coordinate axes aligned along $[x, y]$ and in the perpendicular directions, so that $x = (\tilde{x}, 0, \ldots, 0)$ and $y = (\tilde{y}, 0, \ldots, 0)$, we can use the separation of variables property to write

$$\begin{aligned} K_\Pi(x,y;t) &= K_{\frac{1}{2}|x-y|+\epsilon/\sqrt{n}}(\tilde{x},\tilde{y},t)[K_{\epsilon/\sqrt{n}}(0,0,t)]^{n-1} \\ &\geq \beta(4\pi t)^{-1/2}\mathrm{e}^{-|\tilde{x}-\tilde{y}|^2/4t}\beta^{n-1}(4\pi t)^{-(n-1)/2} \\ &= (4\pi t)^{-n/2}\mathrm{e}^{-|x-y|^2/4t}\beta^n \\ &= \beta^n G_n(x,y,t), \end{aligned}$$

for all $t \leq \epsilon^2/n$, using the one-dimensional lower bound from Lemma 2.1. □

3. A lower bound for the heat equation

Without loss of generality we henceforth assume that Ω contains the origin. For $r > 0$, $B(x,r)$ will denote the Euclidean ball in $\mathbb{R}^n$ of radius r centred at x, and in an abuse of notation we write $B(r)$ for $B(0,r)$.

As an ingredient in the proof of our main result, we want to show that the action of the heat equation on the singular initial condition

$$w_0(x) = |x|^{-\alpha}\chi_R := \begin{cases} |x|^{-\alpha}, & |x| \leq R, \\ 0, & |x| > R \end{cases} \tag{10}$$

does not have too pronounced an effect for short times. It is easy to see that

$$w_0(x) > \phi \qquad \text{for} \quad |x| < \phi^{-1/\alpha};$$

we now show that such a lower bound holds for a similar set of x for sufficiently small times.

Proposition 3.1. *Fix $\alpha \in (0,n)$ and pick $R > 0$ such that $\overline{B(R)} \subset \Omega$. If $w_\Omega(x,t)$ denotes the solution of the linear heat equation on Ω with initial condition*[1] *$w_0 = |x|^{-\alpha}\chi_R$, as represented by (6), then there exist constants $\sigma = \sigma(R,\alpha,n) > 0$ and $\phi_* = \phi_*(R,\alpha,n) > 0$ such that*

$$w_\Omega(x,t) \geq \phi \qquad \text{for all} \quad |x| \leq \sigma\phi^{-1/\alpha} \quad \text{and} \quad 0 \leq t \leq \sigma\phi^{-2/\alpha} \tag{11}$$

for any $\phi > \phi_*$.

[1] Strictly speaking w_0 is defined on the whole of $\mathbb{R}^n$; we take as initial condition the function in (10) restricted to Ω.

Proof. Let w denote the solution of the linear heat equation on $\mathbb{R}^n$ with the same initial condition $w_0 = |x|^{-\alpha}\chi_R$. Let $\epsilon = \inf_{x\in B(R)} \mathrm{dist}(x, \partial\Omega) > 0$; then it follows from Theorem 2.1 that with $T = \epsilon^2/n$ we have

$$K_\Omega(x, y, t) \geq \beta^n G_n(x, y, t), \qquad \forall x, y \in B(R), \quad t \in (0, T].$$

From here on c will denote any generic constant, and may change from line to line.

Taking $|\hat{x}| = R$, $t \in (0, T]$ and $\psi > 1$, we have

$$\begin{aligned} w_\Omega(\hat{x}/\psi, t) &= \int_\Omega K_\Omega(\hat{x}/\psi, y, t) w_0(y)\,\mathrm{d}y = \int_{B(R)} K_\Omega(\hat{x}/\psi, y, t)|y|^{-\alpha}\,\mathrm{d}y \\ &\geq c \int_{B(R)} G_n(\hat{x}/\psi, y, t)|y|^{-\alpha}\,\mathrm{d}y \\ &\geq c(4\pi t)^{-n/2} \int_{B(R)} \mathrm{e}^{-|(\hat{x}/\psi)-y|^2/4t}|y|^{-\alpha}\,\mathrm{d}y \\ &= c(4\pi t)^{-n/2} \int_{B(R)} \mathrm{e}^{-|\hat{x}-\psi y|^2/4\psi^2 t}|y|^{-\alpha}\,\mathrm{d}y \\ &= c\psi^\alpha (4\pi\psi^2 t)^{-n/2} \int_{B(\psi R)} \mathrm{e}^{-|\hat{x}-z|^2/4\psi^2 t}|z|^{-\alpha}\,\mathrm{d}z \geq c\psi^\alpha w(\hat{x}, \psi^2 t). \end{aligned}$$

Defining $M = M(R, \alpha, n) > 0$ by

$$M = \inf\{w(x, t) \;:\; |x| = R,\ 0 \leq t \leq T\} \tag{12}$$

it follows that $w_\Omega(x, t) \geq cM\psi^\alpha$ for all $|x| = R\psi^{-1}$ and $t \in (0, \psi^{-2}T]$. Furthermore, $w_\Omega(x, 0) = |x|^{-\alpha} \geq R^{-\alpha}\psi^\alpha$ for all $|x| \leq R\psi^{-1}$. Consequently, by the parabolic maximum principle,

$$w_\Omega(x, t) \geq \phi_*\psi^\alpha \qquad \text{for all} \quad |x| \leq R\psi^{-1} \quad \text{and} \quad 0 \leq t \leq \psi^{-2}T,$$

where $\phi_* := \min(cM, R^{-\alpha}) > 0$. With $\sigma = \min(R\phi_*^{1/\alpha}, T\phi_*^{2/\alpha}) > 0$ and $\phi = \phi_*\psi^\alpha > \phi_*$, one therefore obtains

$$w_\Omega(x, t) \geq \phi \qquad \text{for all} \quad |x| \leq \sigma\phi^{-1/\alpha} \quad \text{and} \quad 0 \leq t \leq \sigma\phi^{-2/\alpha}. \qquad \square$$

4. Non-existence of local solutions

In this section we prove the non-existence of local solutions, taking the following as our (essentially minimal) definition of such a solution. Note that the non-existence of a solution in the sense of Definition 4.1 implies the non-existence of mild solutions and of classical solutions [7, p. 77–78].

Definition 4.1. *[7, p. 78] Given $f \geq 0$ and $u_0 \geq 0$ we say that u is a* local integral solution *of* (P) *on $[0,T)$ if $u : \Omega \times [0,T) \to [0,\infty]$ is measurable, finite almost everywhere, and satisfies*

$$u(t) = S_\Omega(t)u_0 + \int_0^t S_\Omega(t-s)f(u(s))\,\mathrm{d}s \tag{13}$$

almost everywhere in $\Omega \times [0,T)$.

We now prove our main result, in which we obtain instantaneous blow-up in $L^1_{\text{loc}}(\Omega)$ for certain initial data in $L^q(\Omega)$, $1 \leq q < \infty$, under the asymptotic growth condition (14) when f is non-decreasing.

Theorem 4.1. *Let $q \in [1,\infty)$. Suppose that $f : [0,\infty) \to [0,\infty)$ is non-decreasing. If*

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty \tag{14}$$

for some $\gamma > q(1+\frac{2}{n})$, then there exists $u_0 \in L^q(\Omega)$ such that (P) *possesses no local integral solution. Indeed, any solution $u(t)$ that satisfies (13) is not in $L^1_{\text{loc}}(\Omega)$ for any $t > 0$.*

Proof. We show that for small $t > 0$, $u(t) \notin L^1_{\text{loc}}(\Omega)$ and hence, arguing as in [5, Theorem 4.1], there can be no local integral solution of (P).

Choose $B(R)$ as in Proposition 3.1. Set $\alpha = (n+2)/\gamma < n/q$, so that

$$\limsup_{s\to\infty} s^{-(n+2)/\alpha} f(s) = \infty.$$

Then in particular there exists a sequence $\phi_i \to \infty$ such that

$$f(\phi_i)\phi_i^{-(n+2)/\alpha} \to \infty \qquad \text{as} \quad i \to \infty.$$

Now take $u_0 = |x|^{-\alpha}\chi_R \in L^q(\Omega)$. Defining T as in the proof of Proposition 3.1, fix $t < \min(T,1)$ and choose i sufficiently large such that $\phi_i > \phi_*$, $\sigma\phi_i^{-1/\alpha} < R/2$ and $\sigma\phi_i^{-2/\alpha} \leq t$. Clearly, by comparison, $u \geq w_\Omega \geq 0$. Hence by monotonicity of f and Theorem 2.1,

$$
\begin{aligned}
I &:= \int_{B(R)} u(t)\,\mathrm{d}x \geq \int_{B(R)} \int_0^t [S_\Omega(t-s) f(w_\Omega(\cdot, s))](x)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_0^t \int_{B(R)} \int_\Omega K_\Omega(t-s, x, y) f(w_\Omega(y, s))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&\geq c \int_0^{\sigma\phi_i^{-2/\alpha}} \int_{B(R)} \int_{B(\sigma\phi_i^{-1/\alpha})} G_n(t-s, x, y) f(\phi_i)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&= c f(\phi_i) \int_0^{\sigma\phi_i^{-2/\alpha}} \int_{B(\sigma\phi_i^{-1/\alpha})} (4\pi(t-s))^{-n/2} \int_{B(R)} \mathrm{e}^{-|x-y|^2/4(t-s)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

Let $z = x - y$. Since $|y| \leq \sigma\phi_i^{-1/\alpha} < R/2$, it follows that

$$
\{z = x - y : x \in B(R)\} \supset B(R/2).
$$

Thus

$$
\begin{aligned}
I &\geq c f(\phi_i) \int_0^{\sigma\phi_i^{-2/\alpha}} \int_{B\left(\sigma\phi_i^{-1/\alpha}\right)} (4\pi(t-s))^{-n/2} \int_{B(R/2)} \mathrm{e}^{-|z|^2/4(t-s)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \\
&= c f(\phi_i) \int_0^{\sigma\phi_i^{-2/\alpha}} \int_{B\left(\sigma\phi_i^{-1/\alpha}\right)} \int_{B\left(R/\sqrt{t-s}\right)} \mathrm{e}^{-|v|^2}\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}s \quad \left(v = z/2\sqrt{t-s}\right) \\
&\geq c f(\phi_i) \int_0^{\sigma\phi_i^{-2/\alpha}} \int_{B\left(\sigma\phi_i^{-1/\alpha}\right)} \int_{B(R)} \mathrm{e}^{-|v|^2}\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}s \quad (\sqrt{t-s} \leq 1) \\
&= c f(\phi_i) \int_0^{\sigma\phi_i^{-2/\alpha}} (\sigma\phi_i^{-1/\alpha})^n\,\mathrm{d}s \\
&= c f(\phi_i) \phi_i^{-(n+2)/\alpha} \to \infty \text{ as } i \to \infty. \qquad \square
\end{aligned}
$$

Note that this result also guarantees instantaneous blow-up of solutions of

$$
u_t = \Delta u + g(u)
$$

for any g such that $g(s) \geq f(s)$, where f satisfies the conditions of Theorem 4.1, even if g is not monotonic. In particular, for the canonical Fujita equation

$$
u_t = \Delta u + u^p, \tag{15}
$$

our argument shows the non-existence of local solutions when $p > q(1 + \frac{2}{n})$. The sharp result in this case is known to be $p > 1 + \frac{2}{n}q$ [12, 13] with equality allowed if $q = 1$ [2].

The existence of a finite limit in (14) implies that $f(s) \le c(1 + s^\gamma)$, and hence by comparison with (15) is sufficient for the local existence of solutions provided that $\gamma < 1 + \frac{2}{n}q$ [11]. We currently, therefore, have an indeterminate range of γ,

$$1 + \frac{2}{n}q \le \gamma \le q(1 + \frac{2}{n})$$

for which we do not know whether (14) characterises the existence or non-existence of local solutions.

5. A very 'bad' Osgood f

To finish, using a variant of the construction in [5], we provide an example of an f that satisfies the Osgood condition (1) but for which

$$\limsup_{s\to\infty} s^{-\gamma} f(s) = \infty, \qquad \text{for every} \quad \gamma \ge 0. \tag{16}$$

Theorem 5.1. *There exists a locally Lipschitz function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$, f is non-decreasing, and f satisfies the Osgood condition*

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s = \infty,$$

but nevertheless (16) holds. Consequently, for this f, for any $1 \le q < \infty$ there exists a $u_0 \in L^q(\Omega)$ such that (P) *has no local integral solution.*

Proof. Fix $\phi_0 = 1$ and define inductively the sequence ϕ_i via

$$\phi_{i+1} = \mathrm{e}^{\phi_i}.$$

Clearly, $\phi_i \to \infty$ as $i \to \infty$. Now define $f : [0, \infty) \to [0, \infty)$ by

$$f(s) = \begin{cases} (\mathrm{e}-1)s, & s \in J_0 := [0, 1], \\ \phi_i - \phi_{i-1}, & s \in I_i := [\phi_{i-1}, \phi_i/2],\ i \ge 1, \\ \ell_i(s), & s \in J_i := (\phi_i/2, \phi_i),\ i \ge 1, \end{cases} \tag{17}$$

where ℓ_i interpolates linearly between the values of f at $\phi_i/2$ and ϕ_i. By construction $f(0) = 0$, f is non-decreasing, and f is Osgood since

$$\int_1^\infty \frac{1}{f(s)}\,\mathrm{d}s \ge \sum_{i=1}^\infty \int_{I_i} \frac{1}{f(s)}\,\mathrm{d}s = \sum_{i=1}^\infty \frac{\phi_i/2 - \phi_{i-1}}{\phi_i - \phi_{i-1}} = +\infty.$$

However, $f(\phi_i) = \mathrm{e}^{\phi_i} - \phi_i$, and so for any $\gamma \geq 0$

$$\lim_{i\to\infty} \phi_i^{-\gamma} f(\phi_i) \to \infty \qquad \text{as} \quad i \to \infty,$$

which shows that (16) holds. □

This example shows that there exist semilinear heat equations that are globally well-posed in $L^\infty(\Omega)$, yet ill-posed in every $L^q(\Omega)$ for $1 \leq q < \infty$.

6. Appendix: Gaussian lower bound on the heat kernel for all $t > 0$

For the sake of completeness we now follow [8] (see also [9]) and use the result of Lemma 2.1 to obtain a lower bound on[2] $K_a(x, y; t)$ in terms of $K_\epsilon(0, 0; t)$. We then bound $K_\epsilon(0, 0; t)$ below by supplementing the bound from Lemma 2.1 with information from the eigenfunction expansion of the kernel. This will allow us a simple proof of a similar form of lower bound on a general domain Ω when $[x, y] \subset \Omega$.

The main idea in the proof is to use repeatedly the semigroup property of the heat semigroup in the form

$$K_a(x, y; t) = \int_{(-a,a)} K_a(x, z; t) K_a(z, y; t) \,\mathrm{d}z.$$

Proposition 6.1. *The one-dimensional heat kernel on $\Omega = (-a, a)$ satisfies*

$$K_a(x, y; t) \geq \mathrm{e}^{-|x-y|^2/4t} K_\epsilon(0, 0, t) \tag{18}$$

for all $x, y \in (-a, a)$ and $t > 0$, where $\epsilon = \operatorname{dist}([x, y], \partial\Omega)$.

Proof. Take $x, y \in (-a, a)$, $t > 0$, and $m \in \mathbb{N}$ with m sufficiently large that $1 - 2\mathrm{e}^{-m\epsilon^2/t} > 0$. For $j = 0, 1, \ldots, m$ set $x_j = x + jz$, where $z = (y - x)/m$. Then using the semigroup property

$$K_a(x, y; t)$$
$$= \int_{\Omega^{m-1}} K_a(x, y_1; t/m) \left\{ \prod_{j=1}^{m-2} K_a(y_j, y_{j+1}; t/m) \right\} K_a(y_{m-1}, y; t/m) \,\mathrm{d}^{m-1}y,$$

[2] Recall that we use the notation $K_a(x, y; t)$ for the one-dimensional heat kernel on $(-a, a)$.

writing $\mathrm{d}^{m-1}y$ for $\mathrm{d}y_1 \cdots \mathrm{d}y_{m-1}$.

Now note that $B(x_j, \epsilon) \subset \Omega$ for every $j = 0, \ldots, m$, and so

$$K_a(x,y;t) \geq \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} K_a(x_j + w_j, x_{j+1} + w_{j+1}; t/m) \, \mathrm{d}^{m-1}w,$$

setting $w_0 = w_m = 0$ and $w_j = y_j - x_j$ for $j = 1, \ldots, m-1$. Using Lemma 2.1 we obtain the lower bound

$$\begin{aligned} K_a &\geq C_{m,t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} G_1(x_j + w_j, x_{j+1} + w_{j+1}; t/m) \, \mathrm{d}^{m-1}w \\ &= C_{m,t} \int_{B(\epsilon)^{m-1}} (4\pi t/m)^{-m/2} \exp\left(-\frac{m \sum_{j=0}^{m-1} |z + w_{j+1} - w_j|^2}{4t} \right) \mathrm{d}^{m-1}w, \end{aligned}$$

where $C_{m,t} = [1 - 2\mathrm{e}^{-m\epsilon^2/t}]^m$.

Elementary algebra gives

$$m \sum_{j=0}^{m-1} |z + w_{j+1} - w_j|^2 = |x-y|^2 + m \sum_{j=0}^{m-1} |w_{j+1} - w_j|^2,$$

and therefore

$$\begin{aligned} K_a &\geq C_{m,t} \mathrm{e}^{-|x-y|^2/4t} \int_{B(\epsilon)^{m-1}} (4\pi t/m)^{-m/2} \exp\left(-\frac{m \sum_{j=0}^{m-1} |w_{j+1} - w_j|^2}{4t} \right) \mathrm{d}^{m-1}w \\ &= C_{m,t} \mathrm{e}^{-|x-y|^2/4t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} G_1(w_j, w_{j+1}; t/m) \, \mathrm{d}^{m-1}w. \end{aligned}$$

Now we can use monotonicity of the heat kernel, $G_1 \geq K_\epsilon$, to obtain

$$\begin{aligned} K_a(x,y;t) &\geq C_{m,t} \mathrm{e}^{-|x-y|^2/4t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} K_\epsilon(w_j, w_{j+1}; t/m) \, \mathrm{d}^{m-1}w \\ &= C_{m,t} \mathrm{e}^{-|x-y|^2/4t} K_\epsilon(0,0,t), \end{aligned}$$

using the semigroup property of K_ϵ and recalling that $w_0 = w_m = 0$. Finally, noting that $C_{m,t} \to 1$ as $m \to \infty$, we obtain (18). □

We now obtain a lower bound on $K_a(0,0;t)$ using the eigenfunction expansion of the kernel.

Lemma 6.1. *For all $t > 0$*

$$K_a(0,0;t) \geq \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\pi^2 t/4a^2}. \tag{19}$$

Proof. The eigenfunctions of $u_{xx} = \lambda u$ with $u(0) = u(2a) = 0$ are $\sin k\pi x/2a$ with corresponding eigenvalues $-k^2\pi^2/4a^2$: the kernel is therefore

$$K_{(0,2a)}(x,y;t) = \frac{1}{a}\sum_{k=1}^{\infty} \mathrm{e}^{-k^2\pi^2 t/4a^2} \sin(k\pi x/2a)\sin(k\pi y/2a).$$

Since $K_a(0,0;t) = K_{(0,2a)}(a,a;t)$ we obtain

$$\begin{aligned} K_a(0,0;t) &= \frac{1}{a}\sum_{k=1}^{\infty} \mathrm{e}^{-k^2\pi^2 t/4a^2} \sin^2(k\pi/2) \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)^2\pi^2 t/4a^2} \geq \frac{1}{a}\mathrm{e}^{-\pi^2 t/4a^2}, \end{aligned}$$

from which (19) follows for $a \leq (4\pi t)^{1/2}$. For $t \leq a^2/4\pi$, we use Lemma 2.1 to give

$$K_a(0,0;t) \geq 1 - 2\mathrm{e}^{-\epsilon^2/t};$$

now simply observe that $\mathrm{e}^{-1/s} < s/4$ and $\mathrm{e}^{-s} < 1 - (s/2)$ for $0 < s < 1/3$, and so certainly $1 - 2\mathrm{e}^{-1/s} \geq \mathrm{e}^{-s}$ for $0 < s \leq 1/(4\pi) < 1/3$, and thus the bound in (19) holds for all $t > 0$ as claimed. □

Combining the results of Proposition 6.1 and Lemma 6.1 finally yields the required lower bound in one dimension.

Corollary 6.1. *If $\Omega = (-a,a)$, $[x,y] \subset \Omega$, and $\epsilon = \mathrm{dist}([x,y],\partial\Omega)$ then*

$$K_a(x,y;t) \geq \mathrm{e}^{-\pi^2 t/4\epsilon^2} G_1(x,y;t) \qquad \textit{for all} \quad t > 0.$$

For $\Omega \subset \mathbb{R}^n$ the result follows using the argument in the proof of Theorem 2.1, in particular the inequality

$$K_\Omega(x,y,t) \geq K_\Pi(x,y,t) \geq K_{\frac{1}{2}|x-y|+\epsilon/\sqrt{n}}(\tilde{x},\tilde{y},t)[K_{\epsilon/\sqrt{n}}(0,0,t)]^{n-1}.$$

Corollary 6.2. *If* $\Omega \subset \mathbb{R}^n$, $[x,y] \subset \Omega$, *and* $\epsilon = \mathrm{dist}([x,y], \partial\Omega)$ *then*

$$K_a(x,y;t) \geq \mathrm{e}^{-n^2\pi^2 t/4\epsilon^2} G_n(x,y;t) \qquad \textit{for all} \quad t > 0.$$

We note that the argument in [8] does not require the line segment $[x,y]$ to be contained in Ω, leading to a lower bound that depends on the curvature of the geodesic joining x and y.

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