Block Transformation of Hybrid Cellular Automata*

Bo Chen[1], Fangyue Chen[1],
Junbiao Guan[1], Genaro J. Martínez[2,3]

December 3, 2016

[1] Department of Mathematics, School of Science, Hangzhou Dianzi University, Hangzhou, Zhejiang, 310018, China.
[2] Escuela Superior de Cómputo, Instituto Politécnico Nacional, México.
[3] Unconventional Computing Lab, University of the West of England, Bristol, BS16 1QY, UK.

Abstract

By introducing the sequence-block transformation and vector-block transformation, a discussion of symbolic dynamics of hybrid cellular automation (HCA) and hybrid cellular automation with memory (HCAM) is presented in this paper. As the local evolution rules of HCA and HCAM are not uniform, the new uniform cellular automata (CAs) with multiple states can be constructed by specific block transformations. It is proved that the new CA rules are topologically conjugate with the originals. Furthermore, the complex dynamics of the HCA and HCAM rules can be investigated via the new CA rules.

Keywords: Hybrid cellular automata, minority memory, block transform, symbolic dynamics, chaos.

1 Introduction

Cellular automata (CAs) are a class of spatially and temporally discrete dynamical systems characterized by local interactions [1]. As a significant renewal of interest in CAs, S. Wolfram introduced spatio-temporal representations of one-dimensional CAs and informally classified them into four classes by using dynamical concepts such as periodicity, stability, chaos and complex [2-5]. He

*This research was supported by the Natural Science Foundation of Zhejiang Province (Grant No. LY13F030014) and the NSFC (Grant Nos. 11301122, 11171084 and 60872093). Paper published in: Journal of Applied Analysis and Computation, Volume 6, Number 4, November 2016, 1164-1179. DOI: 10.11948/2016077.

proposed a scheme of elementary CAs (ECAs) with simple local rules which has drawn a great deal of attentions from various scientific communities. Based on previous work, L. O. Chua et al provided a nonlinear dynamics perspective to Wolfram's empirical observations, and grouped ECAs into six classes hinging on the quantitative analysis of the orbits [6-10]. These six classes are established as period-1, period-2, period-3, Bernoulli-shift, complex Bernoulli-shift and hyper Bernoulli-shift rules. It is worth mentioning that some of their work is consistent with previous related studies.

For an one-dimensional CA, when the evolution of all its cells is only dependent on the unique global function, it is called uniform, otherwise it is called hybrid, i.e. hybrid cellular automata (HCAs) [11,12]. For instance, denoted by HCA(N,M). HCA rule, composed of ECA rule N and ECA rule M, is specified to obey the rule of ECA N at odd sites of the cell array and obey the rule M at even sites of the cell array. A growing number of researches on HCAs have been applied in secure communication, see [13-15] and references therein. Furthermore, the local rule is denoted as HCA$(N_1, N_2, \cdots, N_t)$ when the HCA is composed of t ECA rules. Though HCAs possess simple hybrid rules and act on the same square tile structures, the evolution of HCAs may exhibit rich dynamical behavior through local interactions.

In the 2000s, R. Alonso-Sanz originally proposed ECA with memory whose each output cell is allowed to remember its previous states during a certain fixed period of evolution [16-18]. In this way, memory functions help to "discover" hidden information in dynamical systems from simple functions (or rules), and "transform" simple and chaotic rules to complex rules or vice versa. For instance, under particular majority memory functions, the ECA rule 30 and rule 126 are endowed with gliders phenomena. Their morphological complexity and glider dynamics are analyzed in [19,20]. Meanwhile, a classification of ECA based on memory functions is proposed in [21] as strong, moderate and weak rules.

In this paper, we take into account the particular evolution rule which is composed of the minority memory function and the HCA rule—denoted by HCAM. With respect to the memory function, the number of the cells that perform memory is three; that is, the memory values are determined by the last three states of each instantaneous cell. More specifically, minority memory function implies the ability of recording the values that have the minimum number of the corresponding last three states of each cell. In particular, if all the last three-state values are identical, the recorded value is taken as the minus one. For the instantaneous cells, a line of memory values can be calculated. Then, a row of cell states at the next moment can be obtained via implementing the original HCA rules.

The rest of this article is organized as follows: Section 2 presents the definitions of chaos and topological entropy. By introducing the sequence-block transformation and vector-block transformation, Section 3 and Section 4 carries out the discussion of symbolic dynamics of HCA and HCAM respectively. Finally, Section 4 highlights the main results.

2 The Preliminaries

Let X be a metric space and $\Psi : X \to X$ be a continuous map and distance d is defined on X.

Ψ is chaotic on X in the sense of Li-Yorke if (1) $\lim\limits_{n\to\infty} \ sup\ d(\Psi^n(x), \Psi^n(y)) > 0, \forall x, y \in X,\ x \neq y$; (2) $\lim\limits_{n\to\infty} \ inf\ d(\Psi^n(x), \Psi^n(y)) = 0, \forall x, y \in X$.

$x \in X$ is a n-period point of Ψ if there exists the integer $n > 0$ such that $\Psi^n(x) = x$. Let $P(\Psi)$ stands for the set of all n-period points, that is, $P(\Psi) = \{x \in X|\ \exists n > 0, \Psi^n(x) = x\}$. In particular, if $\Psi(x) = x$ for several $x \in X$, x is fixed point. Then, Ψ is said to be topologically transitive if for any non-empty open subsets U and V of X there exists a natural number n such that $\Psi^n(U) \bigcap V \neq \emptyset$. $P(\Psi)$ is called a dense subset of X if, for any $x \in X$ and any constant $\varepsilon > 0$, there exists a $y \in P(\Psi)$ such that $d(x, y) < \varepsilon$. Ψ is sensitive to initial conditions if there exists a $\delta > 0$ such that, for $x \in X$ and for any neighborhood $B(x)$ of x, there exists a $y \in B(x)$ and a natural number n such that $d(\Psi^n(x), \Psi^n(y)) > \delta$, where d is a distance defined on X.

Ψ is chaotic on X in the sense of Devaney if (1) Ψ is transitive; (2) $P(\Psi)$ is a dense subset of X; (3)Ψ is sensitive to initial conditions.

Let $R \subset X$ is called a (n, ε)-spanning set iff for any $x \in X$ and any constant $n > 0,\ \varepsilon > 0$, there exists a $y \in R$ such that $d(\Psi^i(x), \Psi^i(y)) \leq \varepsilon,\ i = 0, 1, \cdots, n-1$. Thus, $r_n(\varepsilon, X, \Psi)$ stands for the infimum of cardinal number of (n, ε)-spanning set with Ψ. The Bowen's topological entropy is defined as follow: $ent(\Psi) = \underset{\varepsilon\to\infty}{lim} \ \underset{n\to\infty}{lim} \ sup\ \frac{1}{n} log\ r_n(\varepsilon, X, \Psi)$. In addition, Ψ is topologically mixing if there exists a natural number N such that $\Psi^n(U) \bigcap V \neq \emptyset$ for the entire $n \geq N$.

(1)Ψ is both chaos in the sense of Li-Yorke can be deduced from positive topological entropy.

(2)Ψ is both chaos in the sense of Devaney and Li-Yorke can be deduced from topologically mixing.

3 Block transformation in HCA

First and foremost, several terminology and notations are the necessary prerequisite to the rigorous consideration of this subject. The set of bi-infinite configurations is denoted by $S^Z = \cdots S \times S \times S \cdots$ and a metric d on S^Z is defined as $d(x, \overline{x}) = \sum_{i=-\infty}^{+\infty} \frac{\widetilde{d}(x_i, \overline{x}_i)}{2^{|i|}}$, where $S = \{0, 1, \cdots, k-1\}$, $x, \overline{x} \in S^Z$ and $\widetilde{d}(\cdot, \cdot)$ is the metric on S defined as $\widetilde{d}(x_i, \overline{x}_i) = 0$, if $x_i = \overline{x}_i$; otherwise, $\widetilde{d}(x_i, \overline{x}_i) = 1$.

As for S, a word over S is finite sequence $a = \alpha_0, \cdots, \alpha_n$ of elements of S. In S^Z, the cylinder set of a word $a \in S^Z$ is $[a]_k = \{x \in S^Z | x_{[k,k+n]} = a\}$, where $k \in Z$. Such a set is manifestly both open and closed (called clopen). The cylinder sets generate a topology on S^Z and form a countable basis for this topology. Therefore, every open set is a countable union of cylinder sets. In addition, S^Z is a Cantor space.

A set $X \subseteq S^Z$ is F-invariant if $F(X) \subseteq X$ and strongly F-invariant if $F(X) = X$. If X is closed and F-invariant, (X, F) or simply X is called a subsystem of F. For instance, let $\mathcal{A}$ denote a set of some finite words over S, and $\Lambda = \Lambda_{\mathcal{A}}$ is the set which consists of the bi-infinite configurations made up of all the words in $\mathcal{A}$. Subsequently, $\Lambda_{\mathcal{A}}$ is a subsystem of (S^Z, σ), where $\mathcal{A}$ is said to be the determinative block system of Λ. For a closed invariant subset $\Lambda \subseteq S^Z$, the subsystem (Λ, σ) or simply Λ is called a subshift of σ [25,26].

The classical left-shift map $\sigma_L : S^Z \to S^Z$ is defined by $[\sigma_L(x)]_i = x_{i+1}$; the classical right-shift map $\sigma_R : S^Z \to S^Z$ is defined by $[\sigma_R(x)]_i = x_{i-1}$. A map $F : S^Z \to S^Z$ is a CA if and only if it is continuous and commutes with σ, i.e., $\sigma \circ F = F \circ \sigma$, where σ is a left-shift or right-shift. For any CA, there exists a radius $r \geq 0$ and a local rule $N : S^{2r+1} \to S$ such that $[F(x)]_i = N(x_{[i-r,i+r]})$. Moreover, (S^Z, F) is a compact dynamical system. ECA rules in Wolfram's system of identification has captured special attention ever since its publication, and each local rule $F_{ECA\ rule} : S^3 \to S$, $S = \{0, 1\}$ can be represented by a boolean function [22]. For instance, the Boolean function of ECA rule 9 is expressed as $N_9(x_{[i-1,i+1]}) = \overline{x}_{i-1}\overline{x}_i\overline{x}_{i+1} \oplus \overline{x}_{i-1}x_ix_{i+1}, \forall i \in Z$, where $x_i \in S$, "$\cdot$", "$\oplus$" and "$-$" denote "AND", "XOR" and "NOT" logical operations, respectively. Then, the Boolean functions of HCA are represented as

$$[f(x)]_i = \begin{cases} [f_{ECA\ rule\ 1}(x)]_i, & \text{(i mod t)} \equiv 1 \\ [f_{ECA\ rule\ 2}(x)]_i, & \text{(i mod t)} \equiv 2 \\ \qquad \cdots & \\ [f_{ECA\ rule\ t-1}(x)]_i, & \text{(i mod t)} \equiv \text{t-1} \\ [f_{ECA\ rule\ t}(x)]_i, & \text{(i mod t)} \equiv 0 \end{cases}.$$

When the t ECA rules are identical, it is simplified as a special ECA rule.

Whilst we pay close attention on particular subsystems, many of the topological properties are decidable, such as topological entropy, sensitivity and topologically mixing of the compact systems. In particular, under several certain conditions, the compact systems may be only related to subshift σ in the subset $X \subseteq S^Z$. Thus, we could seek out the finite type subshift σ to analyze the asymptotic behavior of the system by the directed graph representation and transition matrix. Because the local rules of HCA are non-uniform, we can not construct graph and matrix of the HCA rules. As the coarse-grained preprocessing, we treat n adjacent cells as a new smallest unit. The HCA can be transformed to a new uniform CA by sequence-block transformation $B_{\langle n \rangle}$, which is defined as

$$y_i = [B_{\langle n \rangle}(x)]_i = \textstyle\sum_{v=1}^{n} x_{n(i-1)+v} \cdot 2^{-v}, x_{n(i-1)+v} \in S$$

Let $\widehat{S} = \{y_i\}$ be a new symbolic set. $\widehat{S}^Z$ is introduced as the space of bi-infinite configurations over $\widehat{S}$ and the metric d^* on $\widehat{S}^Z$ is $d^*(y, \overline{y}) = \sum_{i=-\infty}^{+\infty} \frac{\widehat{d}(y_i, \overline{y}_i)}{2^{|in|}}$, where $y, \overline{y} \in \widehat{S}^Z$ and $\widehat{d}(\cdot, \cdot)$ is the metric on $\widehat{S}$ defined as $\widehat{d}(y_i, \overline{y}_i) = |y_i - \overline{y}_i|$. Obviously, the new uniform CA has 2^n-states and 3-neighbors. Let T stands for the new evolution function. It is easy to prove that the sequence-block transformation B is a homeomorphism and the evolution function T is topologically conjugate with f. Moreover, following the form of Boolean truth table touching upon ECA rules, when the input string is the 3-bit sequences (y_{i-1}, y_i, y_{i+1}) of the whole different values respectively, 2^{3n} evolution results $[T(y)]_i$ can be obtained to identify the particular evolution rule tout court.

Notably, a real CA can be obtained as follow: In bi-infinite symbolic space, let $t \to +\infty$, then $\widehat{S} = [0, 1]$ and each $y_i = [B_{\langle t \rangle}(x)]_i \in \widehat{S}$. There are an infinitely number of states in the real CA, and the state of each cell is a real number in $\widehat{S}$. Roughly speaking, a corresponding binary CA can be constructed for each real CA, and they are mutually topologically conjugate. What is more, the dynamics of real CA can be investigated through the binary CA equivalently. In this article, we try to provide a concise way which is only at an early stage of feasibility exploration for the real CA.

Cite a concrete case, the symbolic dynamics of HCA(45,5,232,138,166,138) is analyzed in the following. Then, ECA rule 232 belongs to the period-1 rules, ECA rule 5 belongs to the period-2 rules, ECA rule 138 belongs to the Bernoulli-shift rules, and ECA rule 45 and rule 166 belong to the hyper Bernoulli-shift rules. The Boolean function of HCA(45,5,232,138,166,138) is induced as

$$[f(x)]_i = \begin{cases} N_{45}(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 1 \\ N_5(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 2 \\ N_{232}(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 3 \\ N_{138}(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 4 \\ N_{166}(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 5 \\ N_{138}(x_{[i-1,i+1]}), & (\text{i mod } 6) \equiv 0 \end{cases}.$$

The sequence-block transformation $B_{\langle 6 \rangle}$ can be defined as

$$y_i = [B_{\langle 6 \rangle}(x)]_i = \sum_{v=1}^{6} x_{6(i-1)+v} \cdot 2^{-v}, x_v \in S.$$

Let $\widehat{S} = \{y_i\}$ be a new symbolic set, and $\widehat{S}^Z$ is the space of bi-infinite configurations over $\widehat{S}$. The new uniform CA have 2^6-states and 3-neighbors. Therefore, 2^{18} evolution results $[T(y)]_i$ can be obtained as the input string (y_{i-1}, y_i, y_{i+1}) of the whole different values respectively. To name only a few, $[T(\frac{1}{2}, \frac{1}{4}, \frac{17}{64})]_i = \frac{1}{4}$, $[T(\frac{27}{32}, \frac{9}{16}, \frac{11}{16})]_i = \frac{33}{64}$, and $[T(\frac{9}{64}, \frac{35}{64}, \frac{3}{64})]_i = \frac{1}{64}$. And, crucially, sequence-block transformation $B_{\langle 6 \rangle}$ is a homeomorphism and T is topologically conjugate with f.

For illustration, a special subset $\sum(I) \subset S^Z$ is introduced to account for the periodic boundary conditions, where $\sum(I) \triangleq \{x \in S^Z | x_{[kI,(1+k)I-1]} = x_{[0,I-1]}, \forall k \in Z\}$. Let $I = 100$, and under random initial string, the spatio-temporal patterns of HCA(45,5,232,138,166,138) and the new CA are illustrated in Fig.1.

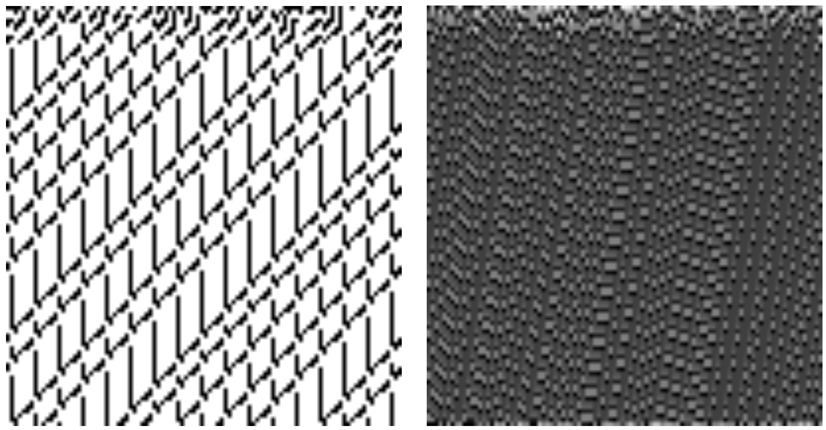

Fig. 1: (a) Spatio-temporal pattern of HCA(45,5,232,138,166,138), where white pixels are cells with state 0, and black pixels are cells with state 1. (b) Spatio-temporal pattern of the new uniform CA, 2^6-states are displayed by different grey levels.

The spatio-temporal patterns presented in Fig.1 implies that two rules in their subsystems, aka attractors, are endowed with Bernoulli-shift dynamical behaviours. In the following, we present an analytical characterization of complex asymptotic dynamics of T [23,24].

For T, there exists a subset $\Lambda_{\mathcal{A}}$ of $\widehat{S}^Z$, such that $T^6(y)|_{\Lambda_{\mathcal{A}}} = \sigma_L(y)|_{\Lambda_{\mathcal{A}}}$, where $\Lambda_{\mathcal{A}} = \{y \in S^Z | y_{[i,i+2]} \in \mathcal{A}, \forall i \in Z\}$ and $\mathcal{A} = \{(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}), (\frac{3}{8}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}, \frac{3}{8}), (\frac{1}{2}, \frac{3}{8}, \frac{1}{4}), (\frac{3}{8}, \frac{1}{4}, \frac{19}{64}), (\frac{1}{4}, \frac{19}{64}, 0), (\frac{19}{64}, 0, \frac{1}{4}), (0, \frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{5}{16}), (\frac{1}{4}, \frac{5}{16}, \frac{17}{64}), (\frac{1}{4}, \frac{1}{8}, \frac{1}{4}), (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{17}{64}), (\frac{1}{4}, \frac{17}{64}, \frac{1}{2}), (\frac{17}{64}, \frac{1}{2}, \frac{1}{8}), (\frac{1}{2}, \frac{1}{8}, \frac{1}{4}), (\frac{1}{8}, \frac{1}{4}, \frac{5}{16}), (\frac{1}{4}, \frac{5}{16}, \frac{1}{4}), (\frac{5}{16}, \frac{1}{4}, \frac{17}{64}), (\frac{17}{64}, \frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{4}, \frac{3}{8}, \frac{19}{64}), (\frac{1}{4}, \frac{1}{4}, \frac{19}{64}), (\frac{19}{64}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{4}, \frac{3}{8}), (0, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{8}), (\frac{1}{4}, \frac{1}{8}, \frac{5}{16}), (\frac{5}{16}, \frac{17}{64}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}), (\frac{5}{16}, \frac{1}{4}, \frac{1}{4}), (\frac{3}{8}, \frac{19}{64}, 0), (\frac{1}{8}, \frac{5}{16}, \frac{1}{4}), (\frac{19}{64}, 0, \frac{17}{64}), (\frac{5}{16}, \frac{1}{4}, \frac{19}{64}), (\frac{3}{8}, \frac{1}{4}, \frac{5}{16}), (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{2}), (0, \frac{17}{64}, \frac{1}{2})\}$. Moreover, $\Lambda_{\mathcal{A}}$ is a subshift of finite type of $(\widehat{S}^Z, \sigma_L)$.

The $y_{[i,i+2]}$ stands for a 3-bit sequence (y_i, y_{i+1}, y_{i+2}) over $\widehat{S}$. Each y_i stands for a 6-bit sequence $(x_{6i}, x_{6i+1}, ..., x_{6i+5})$ over $S = \{0, 1\}$. For instance, (1/4, 3/8, 1/4) refers to the 18-bit sequence (010000, 011000, 010000). $\mathcal{A}$ is called the determinative system of $\Lambda_{\mathcal{A}}$, which is a 3-sequence set in $\widehat{S}^Z$. In addition, we can obtain the determinative system $\mathcal{A}'$ and the subsystem $\Lambda_{\mathcal{A}'}$ of f.

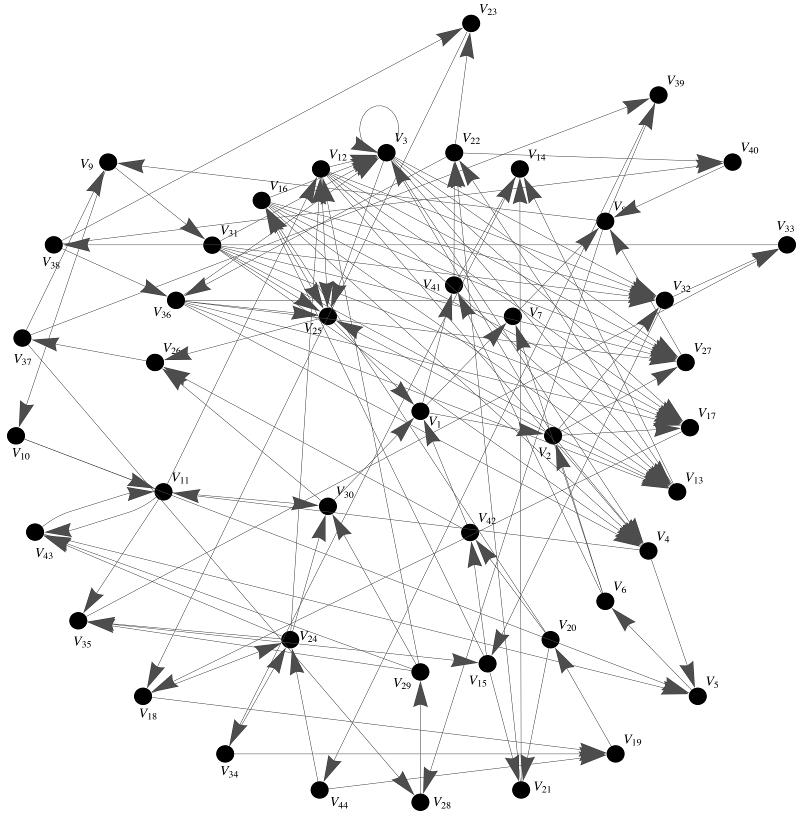

Fig. 2: Graph representation for the subsystem $\Lambda_{\mathcal{A}}$.

In a nutshell, directed graph theory provides a powerful tool for studying the infinite strings. A fundamental method for constructing finite shifts starts with a finite, directed graph and produces the collection of all bi-infinite walks (i.e., strings of edges) on the graph. A graph $G(V, E)$ consists of a finite set V of vertices (or states) together with a finite set E of edges. A finite path $P = V_1 \to V_2 \to \cdots \to V_m$ on a graph $G(V, G)$ is a finite string of vertices V_i from G. The length of P is $|P| = m$. A cycle is a path that starts and terminates at the same vertex. It is addressed that $\Lambda_{\mathcal{A}}$ can be described by a finite directed graph $G_{\mathcal{A}} = G(\mathcal{A}, E)$, where each vertex is a string in $\mathcal{A}$. Each edge $e \in E$ starts at a string denoted by $a = (a_0, a_1, a_2) \in \mathcal{A}$ and terminates at the string $b = (b_0, b_1, b_2) \in \mathcal{A}$ if and only if $a_k = b_{k-1}, k = 1, 2$. One can represent each element of $\Lambda_{\mathcal{A}}$ as a certain path on the graph $G_{\mathcal{A}}$. Figure 2 displays the finite directed graph $G_{\mathcal{A}}$ where each vertex stands for the element of $\mathcal{A}$ by order, i.e., $V_1 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{4})$, $V_2 = (\frac{3}{8}, \frac{1}{4}, \frac{1}{4})$, $V_3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\cdots$, $V_{43} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$,

$V_{44} = (0, \frac{17}{64}, \frac{1}{2})$. The entire bi-infinite walks on the graph constitute the closed invariant subsystem $\Lambda_{\mathcal{A}}$.

$(\cdots, \frac{1}{4}\frac{1}{4}\frac{1}{4}, \cdots)$ a string of period-1 point (fixed point) on $\Lambda_{\mathcal{A}}$. In the $G_{\mathcal{A}}$, the vertex $a = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ has a self-cycle. Then, $T^6(\cdots, \frac{1}{4}\frac{1}{4}\frac{1}{4}, \cdots) = (\cdots, \frac{1}{4}\frac{1}{4}\frac{1}{4}, \cdots)$. However, according to the spatio-temporal patterns, we can gain $T(\cdots, \frac{1}{4}\frac{1}{4}\frac{1}{4}, \cdots) = (\cdots, \frac{1}{4}\frac{1}{4}\frac{1}{4}, \cdots)$.

The diversiform strings of period-$6t$ points are enumerated by the irreducible cycles on $G_{\mathcal{A}}$, where $2 \leq t \leq 27$ and $t = 29$. When one cycle has repeating vertices, it is called the reducible cycle; otherwise, it is called the irreducible cycle. By and large, as any cycle can be compounded by irreducible cycle, we seek out the irreducible cycles in the finite directed graph $G_{\mathcal{A}}$. For instance, $x = (\cdots, \frac{1}{4}\frac{5}{16}\frac{17}{64}\frac{1}{2}\frac{1}{4}\cdots)$ is a string of 30-period point, which is the irreducible closed cycle $(\frac{1}{4}\frac{5}{16}\frac{17}{64}) \rightarrow (\frac{5}{16}\frac{17}{64}\frac{1}{2}) \rightarrow (\frac{17}{64}\frac{1}{2}\frac{1}{4}) \rightarrow (\frac{1}{2}\frac{1}{4}\frac{1}{4}) \rightarrow (\frac{1}{2}\frac{1}{4}\frac{5}{16}) \rightarrow (\frac{1}{4}\frac{5}{16}\frac{17}{64})$ in $G_{\mathcal{A}}$. $x = (\cdots, \frac{19}{64}0\frac{1}{4}\frac{1}{2}\frac{1}{4}\frac{1}{8}\frac{1}{4}\frac{1}{4}\frac{3}{8}, \cdots)$ is one string of 54-period point, which is the irreducible closed cycle $(\frac{19}{64}0\frac{1}{4}) \rightarrow (0\frac{1}{4}\frac{1}{2}) \rightarrow (\frac{1}{4}\frac{1}{2}\frac{1}{4}) \rightarrow (\frac{1}{2}\frac{1}{4}\frac{1}{8}) \rightarrow (\frac{1}{4}\frac{1}{8}\frac{1}{4}) \rightarrow (\frac{1}{8}\frac{1}{4}\frac{1}{4}) \rightarrow (\frac{1}{4}\frac{1}{4}\frac{3}{8}) \rightarrow (\frac{1}{4}\frac{3}{8}\frac{19}{64}) \rightarrow (\frac{3}{8}\frac{19}{64}0) \rightarrow (\frac{19}{64}0\frac{1}{4})$ in $G_{\mathcal{B}}$.

The periodic points set of T is dense on $\Lambda_{\mathcal{A}}$. For any $x \in \Lambda_{\mathcal{A}}$ and $\varepsilon > 0$, there exists a positive integer $M > 1$ such that $\Sigma_{i=M+1}^{\infty}(\frac{1}{2})^i < \frac{\varepsilon}{2}$, and for any $(a_{-M}, \cdots, a_M) \in \mathcal{A}$, it is clear that $(a_{-M}, \cdots a_M) = x_{[-M,M]} \prec x \in \Lambda_{\mathcal{A}}$. As σ_L is topologically transitive on $\Lambda_{\mathcal{A}}$, there exists a closed cycle in the finite directed graph $G_{\mathcal{A}}$: $\overline{c} = (a_{-M}, \cdots, a_M, c_0, c_1, \cdots, c_k, a_{-M}, \cdots, a_M)$, where each $2M+1$-length string in $\overline{c}$ is belong to $\mathcal{A}$. Thus, let $b = (a_{-M}, \cdots, a_M, c_0, c_1, \cdots, c_k)$ and $y = (\cdots, b, b, b, \cdots)$. Obviously, for any $y \in \Lambda_{\mathcal{A}}$, $\sigma_L^{k+2M+1}(y) = y$, where $k+2M+1 = |b|$ is the length of b. $T^{4(k+2M+1)}(y) = \sigma_L^{2(k+2M+1)}(y) = y$ is meaning that y is a periodic point of T and $x_{[-M,M]} = y_{[-M,M]}$, so $d(x,y) \leq 2\Sigma_{i=M+1}^{\infty}(\frac{1}{2})^i < \varepsilon$.

Let $\overline{S} = \{r_0, r_1, \cdots, r_{42}, r_{43}\}$ be a new symbolic set, where $r_i, i = 0, \cdots, 43$, stand for elements of $\mathcal{A}$ respectively. Then, one can construct a new symbolic space $\overline{S}^Z$ on $\overline{S}$. Denote by $\overline{\mathcal{A}} = \{(rr')|r = (b_0b_1b_2), r' = (b'_0b'_1b'_2) \in \overline{S}, \forall 1 \leq j \leq 2 \textit{ such that } b_j = b'_{j-1}\}$. Further, the two-order subshift $\Lambda_{\overline{\mathcal{A}}}$ of σ_L is defined by $\Lambda_{\overline{\mathcal{A}}} = \{r = (\cdots, r_{-1}, r_0^*, r_1, \cdots) \in \overline{S}^Z | r_i \in \overline{S}, (r_i, r_{i+1}) \prec \overline{\mathcal{A}}, \forall i \in Z\}$. Define a map from $\Lambda_{\mathcal{A}}$ to $\Lambda_{\overline{\mathcal{A}}}$ as follows: $\pi : \Lambda_{\mathcal{A}} \rightarrow \Lambda_{\overline{\mathcal{A}}}, x = (..., x_{-1}, x_0^*, x_1, ...) \mapsto (..., r_{-1}, r_0^*, r_1, ...)$, where $r_i = (x_{[i,i+2]}), \forall i \in Z$. Then, it follows from the definition of $\Lambda_{\overline{\mathcal{A}}}$ that for any $x \in \Lambda_{\mathcal{A}}$, one has $\pi(x) \in \Lambda_{\overline{\mathcal{A}}}$; namely, $\pi(\Lambda_{\mathcal{A}}) \subseteq \Lambda_{\overline{\mathcal{A}}}$. One can easily check that π is a homeomorphism and $\pi \circ \sigma_L = \sigma_L \circ \pi$. Therefore, the topologically conjugate relationship between $(\Lambda_{\mathcal{A}}, \sigma_L)$ and a two-order subshift of finite type $(\Lambda_{\overline{\mathcal{A}}}, \sigma_L)$ is established [25,26]. Therefore, it is relatively trivial to calculate the transition matrix $\mathcal{D}$ of the subshift $\Lambda_{\overline{\mathcal{A}}}$, i.e.,

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The matrix $\mathcal{D}$ is positive if all of its entries are non-negative, irreducible if $\forall i, j$, there exist n such that $\mathcal{D}^n_{ij} > 0$, aperiodic if there exists N, such that $\mathcal{D}^n_{ij} > 0, n > N, \forall i, j$. If $\Lambda_{\overline{\mathcal{A}}}$ is a two-order subshift of finite type, then it is topologically mixing if and only if $\mathcal{D}$ is irreducible and aperiodic. Then, the topological dynamics of f on $\Lambda_{\mathcal{A}}$ is largely determined by the properties of $\mathcal{D}$.

T is topologically transitive on $\Lambda_{\mathcal{A}}$. σ_L is topologically transitive on $\Lambda_{\mathcal{A}}$ if the transition matrix $\mathcal{D}$ is irreducible. Further, $\mathcal{D}$ is irreducible if $\mathcal{D} + \mathcal{I}$ is aperiodic, where $\mathcal{I}$ is the 44×44 identity matrix. Meanwhile, it is easy to verify that $(\mathcal{D} + \mathcal{I})^n$ is positive for $n \geq 7$. The matrix is positive if all elements in this matrix is positive. Hence, T is topologically transitive on $\Lambda_{\mathcal{A}}$.

The topological entropy of $T|_{\Lambda_{\mathcal{A}}}$ is $log(\rho(\mathcal{D})) = log(2.55282) = 0.937198$ as $\rho(\mathcal{D})$ is the spectral radius of $\mathcal{D}$. $\rho(\mathcal{D})$ is the maximum positive real root λ^* of characteristic equation in transition matrix. The characteristic equation is $2\lambda^{28} - 6\lambda^{29} + 3\lambda^{31} + 5\lambda^{32} + 7\lambda^{33} - 15\lambda^{34} - 14\lambda^{35} - 31\lambda^{36} - 22\lambda^{37} - 15\lambda^{38} - 8\lambda^{39} - 4\lambda^{40} - 3\lambda^{41} - \lambda^{42} - \lambda^{43} + \lambda^{44} = 0$.

T is topologically mixing on $\Lambda_{\mathcal{A}}$. As a matter of fact, $\mathcal{D}^n_{ij} > 0, n \geq 7$ for $1 \leq i, j \leq 44$, $\mathcal{D}$ is aperiodic accordingly. Thus, the subshift of finite type $(\Lambda_{\mathcal{A}}, \sigma_L)$ is mixing, and $T^6(y)|_{\Lambda_{\mathcal{A}}}$ also is mixing. Then, it is easy to prove $T(y)|_{\Lambda_{\mathcal{A}}}$ also is mixing.

In conclusion, the mathematical analysis presented above provides the rigorous foundation for the following theorem.

T is chaotic in the sense of both Li-Yorke and Devaney on the subsystem $\Lambda_{\mathcal{A}}$. T is topologically mixing on $\Lambda_{\mathcal{A}}$. The topological entropy of $T|_{\Lambda_{\mathcal{A}}}$ is positive. It follows from [25,26] that the chaos in the sense of Li-York can be deduced from positive topological entropy. Suffice it to say that the chaos in the sense of Devaney and Li-Yorke can be deduced from topologically mixing. According to the same way, we can easily get the same dynamical properties of f on its subsystem $\Lambda_{\mathcal{A}'}$. Meanwhile, f is chaotic in the sense of both Li-Yorke and Devaney on its corresponding subsystem $\Lambda_{\mathcal{A}'}$.

For HCA(45,5,232,138,166,138), if we treat $6n(n \in N)$ adjacent cells as a new smallest unit, and the sequence-block transformation $B_{\langle 6n\rangle}$ can be defined as

$$y_i = [B_{\langle 6n\rangle}(x)]_i = \textstyle\sum_{v=1}^{6n} x_{6n(i-1)+v} \cdot 2^{-v}, x_{6n(i-1)+v} \in S.$$

Furthermore, a myriad of new uniform CA of 2^{6n}-states and 3-neighbors can be constructed, which are topologically conjugate with each other. According to the different $B_{\langle 6n\rangle}$, we denote the new evolution function as $T_{\langle 6n\rangle}$ ad infinitum, and the corresponding bi-infinite space as $S^Z_{\langle 6n\rangle}$. In particular, $T_{\langle 6\rangle}$ is remarked as T and $S^Z_{\langle 6\rangle} = \widehat{S}^Z$.

In order to identify the particular evolution rule, 2^{18n} evolution results of $T_{\langle 6n\rangle}$ can be obtained for the input string (y_{i-1}, y_i, y_{i+1}) of the whole different values respectively. The all $T_{\langle 6n\rangle}$ are endowed with Bernoulli-shift dynamics. On their corresponding subsystems Λ, $T^{6n}_{\langle 6n\rangle}(y)|_\Lambda = \sigma_L(y)|_\Lambda$; that is, $T_{\langle 6n\rangle}$ is chaotic in the sense of both Li-Yorke and Devaney. More importantly though, let $n \to \infty$, one can capture a concrete CA with real states, whose dynamics is identical with HCA(45,5,232,138,166,138). For clarity, the following diagram commutes:

$$\begin{tikzcd}
S^Z \arrow[r, "B_{\langle 6n\rangle}"] \arrow[d, "f"'] & S^Z_{\langle 6n\rangle} \arrow[r, "B_{\langle\infty\rangle}"] \arrow[d, "T_{\langle 6n\rangle}"] & S^Z_{\langle\infty\rangle} \arrow[d, "T_{\langle\infty\rangle}"] \\
S^Z \arrow[r, "B_{\langle 6n\rangle}"'] & S^Z_{\langle 6n\rangle} \arrow[r, "B_{\langle\infty\rangle}"] & S^Z_{\langle\infty\rangle}
\end{tikzcd}$$

4 Block transformation in HCAM

According to the description in [21], the memory function ϕ is implemented as $s_i^t = \phi_i(x_i^{t-\tau+1}, \cdots, x_i^{t-1}, x_i^t)^T$, where $t \in Z$ is the instantaneous time step. Here, $1 \le \tau \le t$ determines the degree of memory and ϕ_i denotes the i-th symbol of global memory function ϕ. Thus, $\tau = 1$ means conventional evolution of HCA rules, whereas $\tau = t$ means unlimited trailing memory. Each cell trait $s_i^t \in S$ is a state function of the states of cell i with memory backward up to the value τ. The memory implementation begins to act as soon as t reaches the τ time-step.

Initially, i.e. $t < \tau$, the automata evolves in the conventional way. Furthermore, the original rule is applied on the cell states s to get an evolution with memory as: $f(\cdots, s_{i-1}^t, s_i^t, s_{i+1}^t, \cdots) = x_i^{t+1}$. In particular, the simplified expression of f is $f \circ \phi(x^{t-\tau+1}, \cdots, x^{t-1}, x^t)^T = x^{t+1}$, where $x^{t+k} = (\cdots, x_{i-1}^{t+k}, x_i^{t+k}, x_{i+1}^{t+k}, \cdots)$, k=-τ+1,...,-1,0,1. In this paper, we consider the new evolution rule of HCAM are composed of the memory function and the HCA rule.

Assume that the initial configurations of original stipulation should be applicable, mutatis mutandis, to the mathematical definition of HCAM. The first τ lines of cell array of HCAM rule are all regarded as the random initial configurations; that is, the lines of cell array from second to $\tau - th$ are not regarded as the evolution results according to the original HCA rule. When $t > \tau$, it evolves following the above way. Consequently, the symbolic vector map of HCAM rule F will be conformed to the mathematical definition of the function. Here, we introduce the symbolic vector space and exploit the mathematical definition of HCAM. Firstly, symbolic vector space is introduced as $S_m^Z = \{X = ({x^{(1)}}^T, {x^{(2)}}^T, \cdots, {x^{(m)}}^T)^T | {x^{(j)}}^T \in S^Z, j = 1, 2, \cdots, m\}$, where T refers to the transposed operation. Thus, the metric d^* on S_m^Z is defined as $d^*(X, \overline{X}) = (\sum_{j=1}^n d(x^{(j)},$
$\overline{x}^{(j)}))^{\frac{1}{n}}$. Consequently, the definition of symbolic vector map $F : S_m^Z \to S_m^Z$ is
$F \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \cdots \\ x^{(m)} \end{pmatrix} = \begin{pmatrix} f(x^{(1)}) \\ f(x^{(2)}) \\ \cdots \\ f(x^{(m)}) \end{pmatrix}$, where $f : S^Z \to S^Z$ is the symbolic sequence map.

Then the vector-block transformation $B_{\langle m \times n \rangle}$ can be defined as

$$Y_i = [B_{\langle m \times n \rangle}(X)]_i = \textstyle\sum_{j=1}^m \sum_{v=1}^n x_{n(i-1)+v}^{(j)} \cdot 2^{-(j-1)n-v}.$$

By introducing the extended space $\widetilde{S}^Z$ and distance $\widetilde{d}$, it is demonstrated that the new uniform CA has 2^{mn}-states and 3-neighbors. Let U be the new symbolic sequence map. It could be easily proved that $B_{\langle m \times n \rangle}$ is a homeomorphism and the evolution function U is topologically conjugate with F. Moreover, following the form of Boolean truth table, when the input string is the 3-bit sequences (Y_{i-1}, Y_i, Y_{i+1}) of the whole different values respectively, 2^{3mn} evolution results $[U(Y)]_i$ can be obtained to identify the particular evolution rule.

In this paper, the memory function ϕ is set as the minority memory and $\tau = 3$; that is, $\phi(x_i^{t-2}, x_i^{t-1}, x_i^t) = \overline{(x_i^{t-2} \oplus x_i^{t-1}) \cdot (x_i^{t-1} \oplus x_i^t) \cdot (x_i^t \oplus x_i^{t-2})}$. ECA rule 105 belongs to the complex Bernoulli-shift rules, and ECA rule 60 belongs to the hyper Bernoulli-shift rules. The Boolean function of HCAM(105,60) is induced as

$$f(x_{[i-1,i+1]}) = \begin{cases} N_{105}(x_{[i-1,i+1]}), & \text{(i mod 2)} \equiv 1 \\ N_{60}(x_{[i-1,i+1]}), & \text{(i mod 2)} \equiv 0 \end{cases}.$$

Let $\widetilde{S} = \{Y_i\}$ be a new symbolic set. $\widetilde{S}^Z$ is introduced as the space of bi-infinite configurations over $\widetilde{S}$. Then we define vector-block transformation $B_{\langle 4 \times 2 \rangle}$ as

$$Y_i = [B_{\langle 4\times 2\rangle}(X)]_i = \sum_{j=1}^{4}\sum_{v=1}^{2} x^{(j)}_{2(i-1)+v} \cdot 2^{-(j-1)2-v}.$$

It is demonstrated that the new uniform CA has 2^8-states and 3-neighbors. The 2^{24} evolution results $[U(Y)]_i$ can be obtained as to the input string (Y_{i-1}, Y_i, Y_{i+1}) of the whole different values respectively. For instance, $[U(\frac{11}{32}, \frac{141}{256}, \frac{1}{16})]_i = \frac{5}{16}$, $[U(\frac{51}{128}, \frac{125}{256}, \frac{157}{256})]_i = \frac{93}{128}$, and $[U(\frac{85}{128}, \frac{255}{256}, \frac{5}{8})]_i = \frac{85}{128}$. Vector-block transformation $B_{\langle 4\times 2\rangle}$ is a homeomorphism and the evolution function U of the new uniform CA is topologically conjugate with F. An example of spatio-temporal pattern of HCAM(105,60) and the new CA with random initial configurations is illustrated in Fig.2.

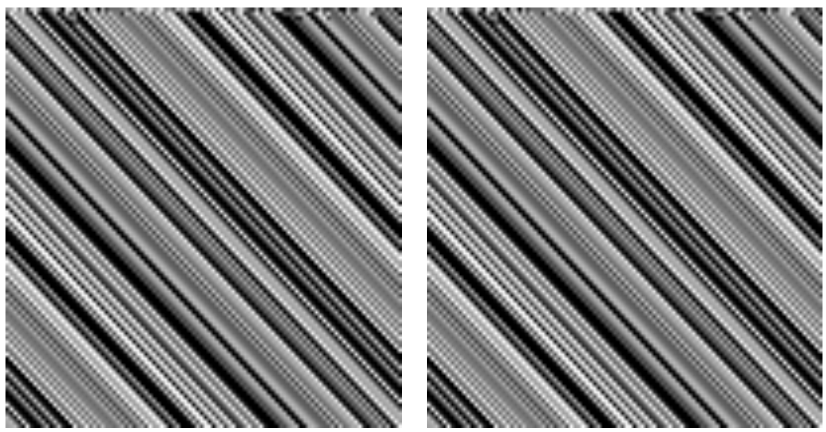

Fig. 3: (a) Spatio-temporal pattern of HCAM(105,60), where white pixels are cells with state 0, and black pixels are cells with state 1. (b) Spatio-temporal pattern of the new uniform CA, 2^8-states are displayed by different grey levels.

For U, there exists a subset $\Lambda_{\mathcal{B}}$ of $\breve{S}^Z$, such that $U(Y)|_{\Lambda_{\mathcal{B}}} = \sigma_R(Y)|_{\Lambda_{\mathcal{B}}}$, where $\Lambda_{\mathcal{B}} = \{Y \in \breve{S}^Z | Y_{[i,i+2]} \in \mathcal{B}, \forall i \in Z\}$ and $\mathcal{B} = \{(\frac{5}{8}, \frac{85}{128}, \frac{175}{256}), (\frac{85}{128}, \frac{175}{256}, \frac{5}{16}), (\frac{175}{256}, \frac{5}{16}, \frac{165}{256}), (\frac{5}{16}, \frac{165}{256}, 0), (\frac{165}{256}, 0, 0), (0, 0, \frac{21}{64}), (0, \frac{21}{64}, \frac{41}{256}), (\frac{21}{64}, \frac{41}{256}, 0), (\frac{41}{256}, 0, 0), (0, 0, \frac{5}{256}), (0, \frac{5}{256}, \frac{125}{128}), (\frac{5}{256}, \frac{125}{128}, \frac{95}{256}), (\frac{125}{128}, \frac{95}{256}, \frac{95}{256}), (\frac{95}{256}, \frac{95}{256}, \frac{15}{256}), (\frac{95}{256}, \frac{15}{256}, \frac{125}{128}), (\frac{15}{256}, \frac{125}{128}, \frac{5}{128}), (\frac{125}{128}, \frac{5}{128}, \frac{5}{16}), (\frac{5}{128}, \frac{5}{16}, \frac{15}{16}), (\frac{5}{16}, \frac{15}{16}, \frac{45}{128}), (\frac{15}{16}, \frac{45}{128}, \frac{15}{16}), (\frac{45}{128}, \frac{15}{16}, \frac{37}{128}), (\frac{15}{16}, \frac{37}{128}, \frac{129}{256}), (\frac{37}{128}, \frac{129}{256}, \frac{127}{128}), (\frac{129}{256}, \frac{127}{128}, \frac{131}{256}), (\frac{127}{128}, \frac{131}{256}, \frac{85}{128}), (\frac{131}{256}, \frac{85}{128}, \frac{125}{128}), (\frac{85}{128}, \frac{125}{128}, \frac{5}{128}), (\frac{125}{128}, \frac{5}{128}, 0), (\frac{5}{128}, 0, 0), (\frac{21}{64}, \frac{41}{256}, \frac{5}{16}), (\frac{41}{256}, \frac{5}{16}, \frac{5}{8}), (\frac{5}{16}, \frac{5}{8}, \frac{117}{128}), (\frac{5}{8}, \frac{117}{128}, \frac{43}{256}), (\frac{117}{128}, \frac{43}{256}, 0), (\frac{43}{256}, 0, \frac{5}{16}), (0, \frac{5}{16}, \frac{15}{16}), (\frac{5}{16}, \frac{15}{16}, \frac{15}{256}), (\frac{15}{16}, \frac{15}{256}, \frac{85}{128}), (\frac{15}{256}, \frac{85}{128}, \frac{85}{128}), (\frac{85}{128}, \frac{85}{128}, \frac{175}{256}), (\frac{85}{128}, \frac{175}{256}, \frac{21}{64}), (\frac{175}{256}, \frac{21}{64}, \frac{61}{256}), (\frac{21}{64}, \frac{61}{256}, \frac{31}{32}), (\frac{61}{256}, \frac{31}{32}, \frac{45}{128}), (\frac{31}{32}, \frac{45}{128}, \frac{5}{8}), (\frac{45}{128}, \frac{5}{8}, \frac{127}{128}), (\frac{5}{8}, \frac{127}{128}, \frac{151}{256}), (\frac{127}{128}, \frac{151}{256}, \frac{1}{128}), (\frac{151}{256}, \frac{1}{128}, \frac{5}{16}), (\frac{1}{128}, \frac{5}{16}, \frac{15}{16}), (\frac{15}{16}, \frac{45}{128}, \frac{5}{8}), (\frac{45}{128}, \frac{5}{8}, \frac{85}{128}), (\frac{5}{8}, \frac{85}{128}, \frac{127}{128}), (\frac{85}{128}, \frac{127}{128}, \frac{131}{256}), (\frac{127}{128}, \frac{131}{256}, \frac{125}{128}), (\frac{131}{256}, \frac{125}{128}, \frac{95}{256}), (\frac{125}{128}, \frac{95}{256}, \frac{5}{128}), (\frac{95}{256}, \frac{5}{128}, \frac{1}{4}), (\frac{5}{128}, \frac{1}{4}, \frac{213}{256}), (\frac{1}{4}, \frac{213}{256}, \frac{151}{256}), (\frac{213}{256}, \frac{151}{256}, \frac{1}{128}), (\frac{151}{256}, \frac{1}{128}, 0), (\frac{1}{128}, 0, \frac{21}{64}), (\frac{41}{256}, 0, \frac{17}{64}), (0, \frac{17}{64}, \frac{55}{64}), (\frac{17}{64}, \frac{55}{64}, \frac{131}{256}), (\frac{55}{64}, \frac{131}{256}, \frac{85}{128}), (\frac{131}{256}, \frac{85}{128}, \frac{85}{128}), (\frac{85}{128}, \frac{85}{128}, \frac{127}{128}), (\frac{85}{128}, \frac{127}{128}, \frac{151}{256}), (\frac{127}{128}, \frac{151}{256}, \frac{35}{128}), (\frac{151}{256}, \frac{35}{128}, \frac{51}{64}), (\frac{35}{128}, \frac{51}{64}), \frac{51}{128}), (\frac{51}{64}), \frac{51}{128}, \frac{55}{64}), (\frac{51}{128}, \frac{55}{64}, \frac{131}{256}), (\frac{55}{64}, \frac{131}{256}, \frac{125}{128}), (\frac{131}{256}, \frac{125}{128}, \frac{15}{256}), (\frac{125}{128}, \frac{15}{256}, \frac{125}{128}), (\frac{15}{256}, \frac{125}{128}, \frac{45}{128}), (\frac{125}{128}, \frac{45}{128}, \frac{245}{256}), (\frac{45}{128}, \frac{245}{256}, \frac{7}{8}), (\frac{245}{256},$

$\frac{7}{8}, \frac{63}{256}), (\frac{7}{8}, \frac{63}{256}, \frac{31}{32}), (\frac{63}{256}, \frac{31}{32}, \frac{5}{128}), (\frac{31}{32}, \frac{5}{128}, \frac{1}{16}), (\frac{5}{128}, \frac{1}{16}, \frac{113}{256}), (\frac{1}{16}, \frac{113}{256}, \frac{45}{128}), (\frac{113}{256}, \frac{45}{128}, \frac{5}{8})\}$. Moreover, $\Lambda_{\mathcal{B}}$ is a subshift of finite type of $(\widetilde{S}^{Z}, \sigma_R)$. Each $Y_{[i,i+2]}$ stands for a 3-bits sequence (Y_i, Y_{i+1}, Y_{i+2}) over $\widetilde{S}$. Each Y_i stands for a 4×2 configuration $\begin{pmatrix} x_{2i-1}^{(1)} & x_{2i}^{(1)} \\ x_{2i+1}^{(2)} & x_{2i}^{(2)} \\ x_{2i-1}^{(3)} & x_{2i}^{(3)} \\ x_{2i-1}^{(4)} & x_{2i}^{(4)} \end{pmatrix}$ over $S = \{0, 1\}$. For instance, (5/8, 85/128, 175/256) refers to the 4×6 configuration

$$\left(\begin{array}{cc:cc:cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

.

$\mathcal{B}$ is called the determinative system of $\Lambda_{\mathcal{B}}$, which is a 4×6 configuration set. Thus, for F, the determinative system $\mathcal{B}'$ and the subsystem $\Lambda_{\mathcal{B}'}$ also can be easily obtained.

Following the similar method presented above, if we calculate the finite directed graph $G_{\mathcal{B}}$ and the transition matrix $\mathcal{E}$, the problem becomes more tractable. In addition, the transition matrices $\mathcal{E}$ is relatively large. Therefore, we only list the indices (i, j) of nonzero elements.
$\mathcal{E} = \{(1,2), (1,41), (2,3), (3,4), (4,5), (5,6), (5,10), (6,7), (7,8), (7,30), (8,9), (8,64), (9,6), (9,10), (10,11), (11,12), (12,13), (12,57), (13,14), (14,15), (15,16), (15,79), (16,17), (16,28), (17,18), (18,19), (18,37), (19,20), (19,51), (20,21), (21,22), (22,23), (23,24), (24,25), (24,55), (25,26), (25,68), (26,27), (27,17), (27,28), (28,29), (29,6), (29,10), (30,31), (31,32), (32,33), (33,34), (34,35), (35,36), (36,19), (36,37), (37,38), (38,39), (39,40), (39,69), (40,2), (40,41), (41,42), (42,43), (43,44), (44,45), (45,46), (45,52), (46,47), (47,48), (47,71), (48,49), (48,62), (49,50), (50,19), (50,37), (51,46), (51,52), (52,1), (52,53), (53,54), (53,70), (54,25), (54,55), (55,56), (55,77), (56,13), (56,57), (57,58), (58,59), (59,60), (60,61), (61,49), (61,62), (62,63), (63,7), (64,65), (65,66), (66,67), (66,76), (67,26), (67,68), (68,40), (68,69), (69,54), (69,70), (70,48), (70,71), (71,72), (72,73), (73,74), (74,75), (75,67), (75,76), (76,56), (76,77), (77,78), (78,16), (78,79), (79,80), (80,81), (81,82), (82,83), (83,84), (84,85), (85,86), (86,87), (87,88), (88,46), (88,52)\}$.

Then, the diversiform strings of period-t points are enumerated by the irreducible cycles on $G_{\mathcal{B}}$, where $7 \leq t \leq 52$ and $t \in \{4, 54, 55, 56, 59\}$. In addition, the periodic points set of U is dense on $\Lambda_{\mathcal{B}}$. As a matter of fact, $(\mathcal{E} + \mathcal{I})_{ij}^{n} > 0, n \geq 24$ for $1 \leq i, j \leq 88$, so $\mathcal{E}$ is irreducible. $U(Y)|_{\Lambda_{\mathcal{B}}}$ is topologically transitive on $\Lambda_{\mathcal{B}}$. And $\mathcal{E}_{ij}^{n} > 0, n \geq 30$ for $1 \leq i, j \leq 88$, so $\mathcal{E}$ is aperiodic. Thus, $U(Y)|_{\Lambda_{\mathcal{B}}}$ also is mixing. Furthermore, the topological entropy $ent(U(Y)|_{\Lambda_{\mathcal{B}}}) = ent(\sigma_R(Y)|_{\Lambda_{\mathcal{B}}})$, and $ent(\sigma_R(Y)|_{\Lambda_{\mathcal{B}}}) = log\ \lambda^* \doteq log(1.42351) = 0.353125$, where λ^* is the maximum positive real root of the characteristic equation of E. In particular, the chaos in the sense of Li-York can be deduced from positive topological entropy. Both the chaos in the sense of Devaney and in the sense of Li-York can be deduced from topologically mixing.

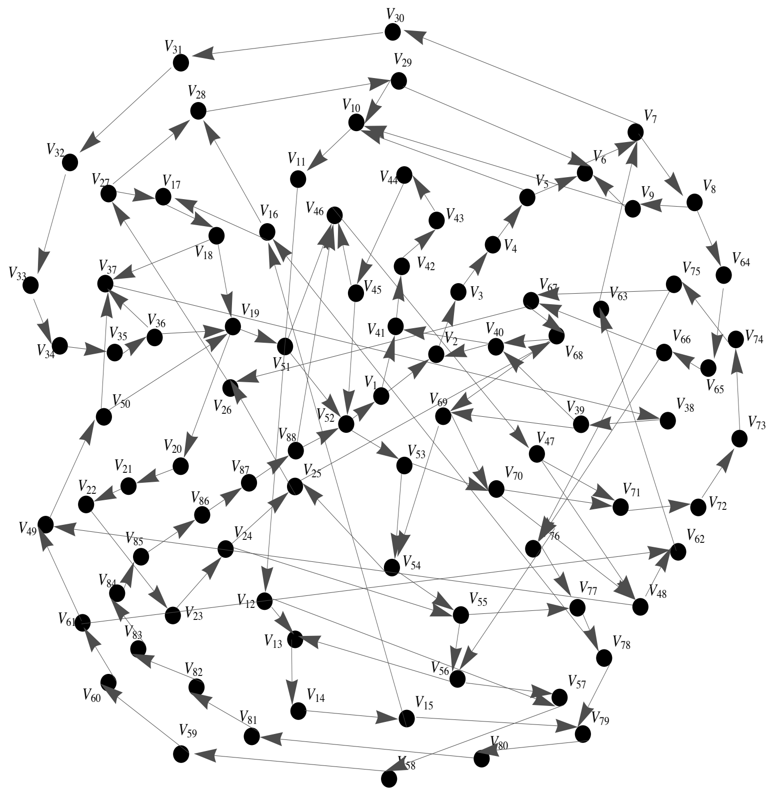

Fig. 4: Graph representation for the subsystem $\Lambda_{\mathcal{B}}$.

U is chaotic in the sense of both Li-Yorke and Devaney on the subsystem $\Lambda_{\mathcal{B}}$. According to the same way, we can easily get the same dynamical properties of F on its subsystem $\Lambda_{\mathcal{B}'}$. Meanwhile, F is chaotic in the sense of both Li-Yorke and Devaney on its corresponding subsystem $\Lambda_{\mathcal{B}'}$.

For HCAM(105,60), we treat $4n\times 2n (n\in N)$ adjacent cells as a new smallest unit, and define vector-block transformation $B_{\langle 4n\times 2n\rangle}$ as

$$Y_i=[B_{\langle 4n\times 2n\rangle}(X)]_i=\textstyle\sum_{j=1}^{4n}\sum_{v=1}^{2n}x_{2n(i-1)+v}^{(j)}\cdot 2^{-(j-1)2n-v}, v\in Z.$$

Then a series of new uniform CA of 2^{8n^2}-states and 3-neighbors can be constructed, which are topologically conjugate with each other. According to the different $B_{\langle 4n\times 2n\rangle}$, we denote the new evolution function as $U_{\langle 4n\times 2n\rangle}$, and the corresponding bi-infinite space as $\widetilde{S}^{Z}_{\langle 4n\times 2n\rangle}$. In this article, $U_{\langle 4\times 2\rangle}$ is remarked as U and $\widetilde{S}^{Z}_{\langle 4\times 2\rangle}$ refers to $\widetilde{S}^{Z}$.

In order to identify the particular evolution rule, 2^{24n^2} evolution results of $U_{\langle 4n\times 2n\rangle}$ can be obtained for the input string (Y_{i-1}, Y_i, Y_{i+1}) assigning different values in order. All $U_{\langle 4n\times 2n\rangle}$ are endowed with Bernoulli shift dynamics. On their corresponding subsystems Λ', $U_{\langle 4n\times 2n\rangle}(Y)|_{\Lambda'} = \sigma_R(Y)|_{\Lambda'}$; that is, $U_{\langle 4n\times 2n\rangle}$ is chaotic in the sense of both Li-Yorke and Devaney. As $n\to\infty$, it is conceivable that a real CA can be obtained, and its dynamics is identical with HCAM(105,60). For clarity, the following diagram commutes:

$$\begin{array}{ccccc}
S^{Z}_{4n} & \xrightarrow{B_{\langle 4n\times 2n\rangle}} & \widetilde{S}^{Z}_{\langle 4n\times 2n\rangle} & \xrightarrow{B_{\langle\infty\rangle}} & \widetilde{S}^{Z}_{\langle\infty\rangle} \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle U_{\langle 4n\times 2n\rangle}} & & \downarrow{\scriptstyle U_{\langle\infty\rangle}} \\
S^{Z}_{4n} & \xrightarrow[B_{\langle 4n\times 2n\rangle}]{} & \widetilde{S}^{Z}_{\langle 4n\times 2n\rangle} & \xrightarrow{B_{\langle\infty\rangle}} & \widetilde{S}^{Z}_{\langle\infty\rangle}
\end{array}$$

5 Conclusion and discussion

In this paper, the chaotic dynamics of HCA and HCAM rules are examined under the framework of symbolic dynamics. By the special block transformations, HCA and HCAM can be transformed to the new uniform and topologically conjugate CAs. Therefore, their dynamical properties on their subsystems can be decided by the directed graph representation and transition matrix of the uniform CAs. As examples, HCA(45,5,232,138,166,138) and HCAM(105,60) here are topologically mixing and possess the positive topological entropy on the concrete subsystems. Therefore, it is concluded that they are chaotic in the sense of both Li-Yorke and Devaney.

The block transforms build the potential bridge between the CAs with real states and the CA with states of 0 and 1 by topological conjugation. It implies that the dynamics of each real CA can be detailedly explored via the corresponding binary CAs. Hence, the investigation of the relationship between real CAs and binary CAs is of great interest in the future work.

References

[1] J. von-Neumann, *Theory of Self-reproducing Automata*, University of Illinois Press, Urbana, 1966.

[2] S. Wolfram, *Statistical mechanics of cellular automata*, Rev. Mod. Phys., 55(1983)(3), 601-644.

[3] S. Wolfram, *Universality and complexity in cellular automata*, Phys. D, 10(1984)(1-2), 1-35.

[4] S. Wolfram, *Theory and Applications of Cellular Automata*, World Scientifc, Singapore, 1986.

[5] S. Wolfram, *A New Kind of Science*, Wolfram Media, Champaign, 2002.

[6] L. O. Chua, V. I. Sbitnev and S. Yoon, *A nonlinear dynamics perspective of Wolfram's new kind of science. Part I: threshold of complexity*, Int. J. Bifur. Chaos, 12(2002)(12), 2655-2766.

[7] L. O. Chua, G. E. Pazienza and J. Shin, *A nonlinear dynamics perspective of Wolfram's new kind of science. Part X: period-1 rules*, Int. J. Bifur. Chaos, 19(2009)(5), 1425-1654.

[8] L. O. Chua, G. E. Pazienza and J. Shin, *A nonlinear dynamics perspective of Wolfram's new kind of science. Part XI: period-2 rules*, Int. J. Bifur. Chaos, 19(2009)(6), 1751-1930.

[9] L. O. Chua and G. E. Pazienza, *A nonlinear dynamics perspective of Wolfram's new kind of science. Part XII: period-3, period-6, and permutive rules*, Int. J. Bifur. Chaos, 19(2009)(12), 3887-4038.

[10] L. O. Chua and G. E. Pazienza, *A nonlinear dynamics perspective of Wolfram's new kind of science. Part XIII: Bernoulli σ_τ-shift rules*, Int. J. Bifur. Chaos, 20(2010)(7), 1859-2003.

[11] K. Cattell and J. C. Muzio, *Synthesis of one-dimensional linear hybrid cellular automata*, IEEE Trans. Computers, 15(1996)(3), 325-335.

[12] J. Binghama and B. Binghamb, *Hybrid one-dimensional reversible cellular automata are regular*, Discrete Appl. Math., 155(2007)(18), 2555-2566.

[13] C. F. Rubio, L. H. Encinas, S. H. White, A. M. Rey and G. R. Sánchez, *The use of linear hybrid cellular automata as pseudorandom bit generators in cryptography*, Neural Parallel Sci. Comput., 12(2004)(2), 175-192.

[14] D. Radu, *Hybrid cellular automata as pseudo-random number generators with binary synchronization property*, in Proceedings of the International Symposium on Signals Circuits and Systems, (2009), 389-392.

[15] D. Radu and D. Ioana, *Efficient and cryptographically secure pseudorandom number generators based on chains of hybrid cellular automata maps*, in Proceedings of COMM 2014 10th International Conference on Communications, (2014).

[16] R. Alonso-Sanz and M. Martin, *Elementary cellular automata with memory*, Complex Systems, 14(2003)(2), 99-126.

[17] R. Alonso-Sanz and M. Martin, *Three-state one-dimensional cellular automata with memory*, Chaos, Solitons and Fract., 21(2004)(4), 809-834.

[18] R. Alonso-Sanz, *Elementary cellular automata with memory of delay type*, 19th International Workshop, AUTOMATA 2013, 8155(2013), 67-83.

[19] G. J. Martínez, A. Adamatzky, R. Alonso-Sanz and J. C. Seck-Tuoh-Mora, *Complex dynamic emerging in rule 30 with majority memory*, Complex Systems, 18(2010)3, 345-365.

[20] G. J. Martínez, A. Adamatzky, J. C. Seck-Tuoh-Mora and R. Alonso-Sanz, *How to make dull cellular automata complex by adding memory: rule 126 case study*, Complexity, 15(2010)(6), 34-49.

[21] G. J. Martínez, A. Adamatzky and R. Alonso-Sanz, *Designing complex dynamics in cellular automata with memory*, Int. J. Bifur. Chaos, 23(2013)(10), 1330035.

[22] J. B. Guan, S. W. Shen, C. B. Tang and F. Y. Chen, *Extending Chua's global equivalence theorem on Wolfram's new kind of science*, Int. J. Bifur. Chaos, 17(2007)(12), 4245-4259.

[23] F. Y. Chen, W. F. Jin ,G. R. Chen, F. F. Chen and L. Chen, *Chaos of elementary cellular automata rule 42 of wolfram's class II*, Chaos, 19(2009)(1), 013140.

[24] W. F. Jin, F. Y. Chen, G. R. Chen, L. Chen, and F. F. Chen, *Extending the symbolic dynamics of Chua's Bernoulli-shift rule 56*, J. Cell. Automata, 5(2010)(1-2), 121-138.

[25] Z. L. Zhou, *Symbolic Dynamics*, Shanghai Scientific and technological Education Publishing House, Shanghai, 1997.

[26] B. Kitchens, *Symbolic Dynamics: One-Sided, Two-Sided and Countable State Markov Shifts*, Springer-Verlag, Berlin, 1998.