

A Necessary and Sufficient Condition for Uniqueness of the Trivial Solution in Semilinear Parabolic Equations

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Abstract

In their (1968) paper Fujita and Watanabe considered the issue of uniqueness of the trivial solution of semilinear parabolic equations with respect to the class of bounded, non-negative solutions. In particular they showed that if the underlying ODE has non-unique solutions (as characterised via an Osgood-type condition) *and* the nonlinearity f satisfies a concavity condition, then the parabolic PDE also inherits the non-uniqueness property. This concavity assumption has remained in place either implicitly or explicitly in all subsequent work in the literature relating to this and other, similar, non-uniqueness phenomena in parabolic equations. In this paper we provide an elementary proof of non-uniqueness for the PDE without any such concavity assumption on f . An important consequence of our result is that uniqueness of the trivial solution of the PDE is equivalent to uniqueness of the trivial solution of the corresponding ODE, which in turn is known to be equivalent to an Osgood-type integral condition on f .

Keywords: Semilinear, parabolic, Osgood, non-uniqueness, uniqueness, lower solution.

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1. Introduction

We consider the issue of uniqueness (with respect to bounded solutions) of the trivial solution of the semilinear parabolic problem

$$(P) \quad \begin{cases} u_t = \mathcal{L}u + q(x)f(u) & \text{in } Q_T := \Omega \times (0, T), \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \times (0, T], \\ u(x, 0) = 0 & \text{in } \bar{\Omega}, \end{cases}$$

where \mathcal{L} is a uniformly elliptic operator, \mathcal{B} is a boundary operator, $q(x) \geq 0$ and $f(0) = 0$. The domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is bounded with boundary $\partial\Omega$ of class $C^{2+\alpha}$, so that classical parabolic regularity and maximum principles apply. Without loss of generality we also assume that Ω contains the origin. As in Fujita and Watanabe [10] we take \mathcal{L} of the form

$$\mathcal{L}u = \sum_{i,j=1}^d a_{ij}(x)u_{x_i x_j} + \sum_{j=1}^d b_j(x)u_{x_j} + c(x)u \quad (1)$$

where a_{ij} is symmetric and satisfies the uniform ellipticity condition

$$k|y|^2 \leq \sum_{i,j=1}^d a_{ij}(x)y_i y_j \leq |y|^2/k, \quad \forall x \in \Omega, \forall y \in \mathbb{R}^d \quad (2)$$

for some $k > 0$. The boundary operator in (P) is given by

$$\mathcal{B}u := \beta(x) \frac{\partial u}{\partial \nu} + (1 - \beta(x))u, \quad (3)$$

where $0 \leq \beta(x) \leq 1$ and $\partial u / \partial \nu$ is the conormal derivative

$$\frac{\partial u}{\partial \nu}(x) = \sum_{i,j=1}^d u_{x_i}(x) a_{ij}(x) n_j(x) \quad (4)$$

with $n(x) = (n_1(x), n_2(x), \dots, n_d(x))$ being the unit outer normal at $x \in \partial\Omega$. The case $\beta \equiv 1$ therefore corresponds to Neumann boundary conditions, whilst $\beta \equiv 0$ corresponds to Dirichlet boundary conditions. Other choices of $\beta(x)$ represent Robin or mixed boundary conditions, which in a certain sense (regarding the ordering of the corresponding heat kernels) is intermediate between the Neumann and Dirichlet cases. We provide precise regularity conditions on the coefficients \mathcal{L} and \mathcal{B} in a later section.

Since $f(0) = 0$ and the initial data in (P) is zero, $u = 0$ is a solution of both the PDE problem (P) and the ODE problem

$$\dot{u} = f(u), \quad u(0) = 0. \quad (5)$$

The nonlinearity f is assumed continuous, non-decreasing and positive for $u > 0$. For such f it is well known ([17]) that $u = 0$ is the unique local solution of (5) if and only if the following Osgood integral condition holds:

$$\int_0^\epsilon \frac{du}{f(u)} = \infty \quad \text{for some } \epsilon > 0. \quad (6)$$

Our main result is that if the integral condition (6) does not hold then problem (P) possesses non-unique, non-negative, bounded solutions; see Theorem 2.5. An important consequence of our result is that uniqueness of the trivial solution of the PDE (P) is equivalent to uniqueness of the trivial solution of the ODE (5). Thus uniqueness of the trivial solution of (P) is equivalent to (6); see Corollary 2.6.

This problem was considered almost half a century ago by Fujita and Watanabe [10] (see also [9]). They proved [10, Theorem 1.4] that the Osgood condition (6) is sufficient for uniqueness of the trivial solution in (P) but did not prove necessity under the same conditions. In order to establish non-uniqueness when condition (6) fails, the authors imposed an additional concavity assumption on the nonlinearity f [10, Theorem 1.5]. We show here that this concavity assumption is not required and thereby obtain a result valid for any increasing function f . Given the many works in the literature which have utilised and extended Fujita and Watanabe's non-uniqueness result it is surprising that their concavity assumption has remained until now. We suspect that this may be due to the enthusiasm for studying the 'model' nonlinearity $f(u) = u^p$, for $0 < p < 1$, which is of course concave.

There have been several papers subsequent to [10] providing non-uniqueness results for parabolic equations of various types, e.g. with unbounded coefficients [14], degenerate p -Laplacian operators [3, 4, 11], and systems [2, 6, 7, 11]. However, all these works either assume explicitly that f is concave near zero [5, 9] or implicitly by working only with nonlinearities of power law type, $f(u) = u^p$ ($0 < p < 1$). To the best of our knowledge non-uniqueness has not been established without assuming concavity of f . We remark that non-uniqueness with respect to *unbounded* solutions can also occur in parabolic equations even when the corresponding ODE has unique solutions (e.g. when

f is locally Lipschitz) [11, 15, 16]. However the non-uniqueness there is due to the superlinear growth of f at infinity.

2. Non-Uniqueness of Bounded Solutions

We state our assumptions on the problem data:

- (H1) The coefficients of \mathcal{L} in (1) satisfy $a_{ij} \in C^{2+\alpha}(\bar{\Omega})$, $b_j \in C^{1+\alpha}(\bar{\Omega})$, $c \in C^\alpha(\bar{\Omega})$ and the uniform ellipticity condition (2).
- (H2) The coefficient of \mathcal{B} in (3) satisfies $0 \leq \beta(x) \leq 1$ and $\beta \in C^{2+\alpha}(\partial\Omega)$.
- (H3) $q \geq 0$, $q \not\equiv 0$ and $q \in C(\bar{\Omega})$,
- (H4) $f : [0, \infty) \rightarrow [0, \infty)$ is continuous, non-decreasing, $f(0) = 0$ and $f > 0$ on $(0, \infty)$.

In all that follows $S_\beta(t) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ ($t \geq 0$) denotes the semigroup generated by $-\mathcal{L}$ with boundary conditions $\mathcal{B}u = 0$. It is well-known (e.g. [1, 8]) that one has the representation formula

$$[S_\beta(t)\psi](x) = \int_{\Omega} K_\beta(x, y; t)\psi(y) dy, \quad \psi \in L^\infty(\Omega), \quad (7)$$

where K_β is the kernel (synonymously known as the fundamental solution or parabolic Green's function) associated with \mathcal{L} with the same boundary conditions. For notational convenience we henceforth write S_D and K_D in the Dirichlet case $\beta \equiv 0$. The open Euclidean ball in \mathbb{R}^d , centred at x , with radius R will be denoted $B_R(x)$ and χ_R denotes the characteristic function on $B_R := B_R(0)$.

Definition 2.1. *We say that u is a bounded generalised solution of (P) on $[0, T]$ if $u \in L^\infty(Q_T)$, $u \geq 0$ and satisfies*

$$u(x, t) = \int_0^t \int_{\Omega} K_\beta(x, y; t - s)[q(y)f(u(y, s))] dy ds, \quad (8)$$

or equivalently,

$$u(t) = \int_0^t S_\beta(t - s)[qf(u(s))] ds. \quad (9)$$

Remark 2.2. *If u is a bounded generalised solution and q and f are Hölder continuous on Ω and $(0, M]$ respectively, where $M = \|u\|_\infty$, then u is a solution of (P) in the classical sense by standard parabolic regularity results of De Giorgi-Nash-Moser type and classical Schauder estimates (cf. [18, Appendix B]).*

Clearly $u \equiv 0$ is a classical solution of (P) on $[0, T]$ for any $T > 0$. We will require the following comparison result from [10], reformulated here for the reader's convenience.

Lemma 2.3. *[10, Lemma 2.7]. If (H1)-(H2) hold then $K_\beta(x, y; t) \geq K_D(x, y; t)$ for all $x, y \in \Omega$, $t > 0$. Consequently the corresponding semigroups satisfy $S_\beta(t)\psi \geq S_D(t)\psi$ for all non-negative $\psi \in L^\infty(\Omega)$, $t > 0$.*

Our goal is to show that if the Osgood condition (6) fails then there exists a non-trivial subsolution of (P). To achieve this we utilise a Gaussian lower bound on the Dirichlet kernel K_D due to Aronson [1]. There the author considered more general linear parabolic operators of the form

$$\mathcal{P}u = u_t - \sum_{i,j=1}^d (A_{ij}(x, t)u_{x_i})_{x_j} - \sum_{j=1}^d (A_j(x, t)u)_{x_j} - \sum_{j=1}^d B_j(x, t)u_{x_j} - C(x, t)u \quad (10)$$

under fairly minimal regularity assumptions on the coefficients. In the special case where

$$A_{ij}(x, t) = a_{ij}(x), \quad A_j(x, t) = 0, \quad B_j(x, t) = b_j(x) - \sum_{i=1}^d (a_{ij}(x))_{x_i} \quad (11)$$

$$C(x, t) = c(x) - \sum_{j=1}^d (b_j(x))_{x_j} - \sum_{i,j=1}^d (a_{ij}(x))_{x_i x_j} \quad (12)$$

then the operator \mathcal{P} reduces to that of $\partial/\partial t - \mathcal{L}$ in problem (P). In particular, if assumption (H1) holds then the coefficients of \mathcal{P} given by (11-12) are all Hölder continuous and thus certainly bounded. This, together with (2), ensure that the results in [1] are applicable to (P) with Dirichlet boundary conditions. For similar estimates in the special case where \mathcal{L} is the Laplacian operator with homogeneous Dirichlet or Neumann boundary conditions see [19, 20] for a more concise treatment.

Lemma 2.4. *Assume (H1) holds and $c(x) \leq 0$. Let $r \in (0, 1)$ be such that $B_{3r} \subset \Omega$ and let $\delta = \text{dist}(B_{2r}, \partial\Omega) > 0$. Then there exists a constant $\kappa = \kappa(d, \mathcal{L}, \delta) > 0$ such that*

$$S_D(t)\chi_r \geq \kappa\chi_r, \quad \forall t \in (0, r^2/8]. \quad (13)$$

Proof. By [1, Theorem 8, Theorem 9 (iii)] (with $\Omega' = B_{2r}$, $T = 1$ and $\tau = 0$ in the notation of that paper) we have the following lower bound on the heat kernel K_D associated with the operator \mathcal{L} with Dirichlet boundary conditions:

$$K_D(x, y; t) \geq c_1 t^{-d/2} e^{-c_2|x-y|^2/t}$$

for all $x, y \in B_{2r}$ and $0 < t \leq \min\{1, \text{dist}^2(y, \partial B_{2r})/8\}$, where c_1 and c_2 are positive constants depending only upon d , δ and \mathcal{L} . In particular, for $y \in B_r$ we have $\text{dist}(y, \partial B_{2r}) \geq r$ and so

$$K_D(x, y; t) \geq c_1 t^{-d/2} e^{-c_2|x-y|^2/t}$$

for all $x, y \in B_r$ and $0 < t \leq \min\{1, r^2/8\} = r^2/8$. Hence for all such t ,

$$[S_D(t)\chi_r](x) = \int_{B_r} K_D(x, y; t) dy \geq c_1 t^{-d/2} \int_{B_r} e^{-c_2|x-y|^2/t} dy.$$

The latter integral is simply a constant multiple of the representation of the solution of a heat equation of the form $u_t = C\Delta u$ on the whole space \mathbb{R}^d with the radially symmetric, non-increasing initial data $\chi_r(x)$. Consequently this integral is also radially symmetric and decreasing with $|x|$ and so for $|x| \leq r$, choosing any unit vector \hat{u} , we can write

$$[S_D(t)\chi_r](x) \geq c_1 t^{-d/2} \int_{B_r(r\hat{u})} e^{-c_2|z|^2/t} dz = c_1 \int_{B_{r/\sqrt{t}}(\frac{r}{\sqrt{t}}\hat{u})} e^{-c_2|w|^2} dw.$$

Observing that for $r/\sqrt{t} \geq 1$ we have

$$B_{r/\sqrt{t}}(\frac{r}{\sqrt{t}}\hat{u}) \supseteq B_1(\hat{u})$$

it follows that

$$[S_D(t)\chi_r](x) \geq c_1 \int_{B_1(\hat{u})} e^{-c_2|w|^2} dw =: \kappa' = \kappa'\chi_r(x)$$

for all $x \in B_r$ and $0 < t \leq \min\{r^2/8, r^2\} = r^2/8$.

Clearly for $x \notin B_r$ we have $[S_D(t)\chi_r](x) \geq 0 = \chi_r(x)$ and so the result follows with $\kappa = \min\{1, \kappa'\}$. \square

Lemma 2.4 is central to our proof of non-uniqueness. Although elementary, similar versions have proved extremely powerful in establishing fundamental non-existence results for semilinear heat equations in Lebesgue spaces. For example, a version was used in [13] to give a complete characterisation of those f for which the local existence property holds, and another in [12] to establish instantaneous blow-up for singular initial data even when all solutions of the corresponding ODE exist globally in time.

We can now prove our main result.

Theorem 2.5. *Assume (H1)-(H4) hold. If f does not satisfy the Osgood condition (6) (i.e., $\int_0^\epsilon du/f(u) < \infty$) then there exists a $T > 0$ and a non-trivial, bounded generalised solution U of (P) on $[0, T]$ satisfying $U(x, t) > 0$ on Q_T . Furthermore, if $q \in C^\alpha(\Omega)$ and $f \in C^\alpha((0, M])$ (where $M = \|U\|_\infty$) for some $\alpha \in (0, 1)$ then U is a classical solution of (P).*

Proof. Suppose initially that $c(x) \leq 0$. By (H3) there exist $\rho > 0$, $\gamma > 0$ and $x_0 \in \Omega$ such that $B_{3\rho}(x_0) \subset \Omega$ and $q(x) \geq \gamma$ for all $x \in B_{3\rho}(x_0)$. Without loss of any generality we may assume that x_0 is the origin. Now choose r as in Lemma 2.4, so that

$$S_D(t)\chi_r \geq \kappa\chi_r, \quad \forall t \leq r^2/8. \quad (14)$$

Setting $R = \min\{r, \rho\}$ we therefore have

$$S_D(t)\chi_R \geq \kappa\chi_R, \quad \forall t \leq R^2/8. \quad (15)$$

Now let $\mu(t)$ denote the unique local solution of the ODE

$$\dot{\mu} = \kappa\gamma f(\mu), \quad \mu(0) = 0$$

which exists and is positive for $t \in (0, T^*]$ for some $T^* > 0$, i.e. $\mu(t) = \int_0^t \kappa\gamma f(\mu(s)) ds$. The existence of such a μ follows by virtue of f failing to satisfy the Osgood condition (6). Setting $v(x, t) = \mu(t)\chi_R(x)$ it is clear that

$v \in L^\infty(Q_{T^*})$. Furthermore, for $t \leq T' := \min\{T^*, R^2/8\}$ we also have that

$$\begin{aligned}
\int_0^t S_\beta(t-s)[qf(v(s))] ds &= \int_0^t S_\beta(t-s)[qf(\mu(s)\chi_R)] ds \\
&= \int_0^t S_\beta(t-s)[q\chi_R f(\mu(s))] ds \quad (f(0) = 0) \\
&= \int_0^t f(\mu(s))S_\beta(t-s)[q\chi_R] ds \quad (S_\beta \text{ linear}) \\
&\geq \int_0^t \gamma f(\mu(s))S_\beta(t-s)[\chi_R] ds \quad (S_\beta \text{ monotone}) \\
&\geq \int_0^t \gamma f(\mu(s))S_D(t-s)[\chi_R] ds \quad (\text{Lemma 2.3}) \\
&\geq \chi_R \int_0^t \gamma \kappa f(\mu(s)) ds \quad (\text{Lemma 2.4}) \\
&= \mu(t)\chi_R = v \quad (\text{by definition of } \mu)
\end{aligned}$$

and so v is a generalised subsolution of (P) on $[0, T']$.

It is easy to see that $w(x, t) = t$ is a classical supersolution of (P) on $[0, \tau]$ for any $\tau > 0$ satisfying $f(\tau)\|q\|_\infty \leq 1$, which is clearly possible by (H4). Furthermore, since $d\mu/dt \rightarrow 0$ as $t \rightarrow 0^+$, τ may also be chosen so that $v \leq w$ on $[0, \tau]$. Standard monotone iteration arguments then guarantee the existence of a bounded generalised solution U of (P) on $[0, T]$ satisfying $v \leq U \leq w$, where $T := \min\{\tau, T'\}$. The positivity of U in Q_T then follows from the integral representation of U and the positivity of K_β . Finally, the regularity of U follows from Remark 2.2 when q and f are Hölder continuous.

For c of indefinite sign let $\sigma = \|c\|_\infty$ and set $\tilde{c}(x) = c(x) - \sigma \leq 0$ and $\tilde{f}(u) = f(u) + \sigma u$. The assumptions (H1) and (H4) are then satisfied with c replaced by \tilde{c} and f replaced by \tilde{f} . Moreover, since $\tilde{f} \geq f$, \tilde{f} fails to satisfy the Osgood condition (6). Hence, arguing as above, there exists a bounded, positive solution \tilde{U} of the problem (P) with \mathcal{L} and f replaced by $\tilde{\mathcal{L}} := \mathcal{L} - \sigma$ and \tilde{f} , respectively. Clearly however, \tilde{U} is a solution of problem (P) with data \mathcal{L} and f if and only if \tilde{U} is a solution of problem (P) with data $\tilde{\mathcal{L}}$ and \tilde{f} , yielding the result. \square

We can now combine Theorem 2.5 and [10, Theorem 1.4] to obtain the following characterisation of uniqueness for (P). We point out that Theorem 2.5 and Corollary 2.6 below remain valid if hypothesis (H4) is replaced by a local one near zero, i.e., for $f : [0, M] \rightarrow [0, \infty)$.

Corollary 2.6. *If (H1)-(H4) hold then the following are equivalent:*

- (i) $u = 0$ is the unique bounded generalised solution of the PDE (P);
- (ii) $u = 0$ is the unique non-negative solution of the ODE (5);
- (iii) $\int_0^\epsilon \frac{du}{f(u)} = \infty$ for some $\epsilon > 0$.

3. Concluding Remarks

We have obtained a simple necessary and sufficient condition on f for uniqueness of the trivial solution in a semilinear parabolic equation with continuous, increasing nonlinearity f . There were several key structural properties required to achieve this: (a) monotonicity of f ; (b) semilinearity of the governing evolution equation; (c) monotonicity of the semigroup S_β (equivalently the kernel K_β) with respect to the boundary data and its action on the underlying phase space $L^\infty(\Omega)$ and (d) a lower bound on the action of the Dirichlet semigroup S_D on characteristic functions (arising from a Gaussian lower bound on the Dirichlet kernel K_D). It seems reasonable to suggest that other evolution problems of the form

$$u' = Au + f(u)$$

having properties (a-d) would be amenable to the method employed here. For example, the fractional Laplacian $Au = -(-\Delta)^s$, $0 < s < 1$, would seem just such a case.

It would also be interesting to see if our method could be extended to continuous but non-monotone f or to quasilinear operators such as the p -Laplacian $A(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ($p > 1$) or the porous medium operator $A(u) = \operatorname{div}(u^m\nabla u)$ ($m > 0$). Whilst no integral equation formulation such as in (8) or (9) exists in the quasilinear case it may still be possible to obtain non-uniqueness results via a weak formulation since monotonicity properties still hold in the weak sense (recalling that our subsolution in the proof of Theorem 2.5 is not a classical subsolution, lacking as it does sufficient regularity).

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