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AN EXTENSION METHOD FOR T-NORMS ON SUBINTERVALS TO T-NORMS ON BOUNDED LATTICES

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In this paper, a construction method on a bounded lattice obtained from a given t-norm on a subinterval of the bounded lattice is presented. The supremum distributivity of the constructed t-norm by the mentioned method is investigated under some special conditions. It is shown by an example that the extended t-norm on L from the t-norm on a subinterval of L need not be a supremum-distributive t-norm. Moreover, some relationships between the mentioned construction method and the other construction methods in the literature are presented.

Keywords: T-norm, bounded lattice, construction method, subinterval

Classification: 03E72, 03B52, 03G10, 18B35

1. INTRODUCTION

Triangular norms were first described by Menger as a generalization of triangle inequality in [11]. Then, triangular norms were defined as we know today on $[0,1]$ [13, 14] and have been studied from many different perspectives. Thus, they have been a study topic in itself [2, 4, 8, 9]. Triangular norms, as special aggregation operators, have been proven to be useful in many fields like fuzzy logic, expert systems, neural net-works, aggregation, and fuzzy system modeling [3, 10, 15]. Besides its applicability to computer sciences and engineering, $[0,1]$ is highly preferred by researchers considering the advantages of working on some important mathematical properties (topological structure, continuity on it, etc.)

Due to the presence of incomparable elements and the lack of some important features provided on $[0,1]$, working on bounded lattices is much more complex than working on $[0,1]$. But it is still more attractive since bounded lattice is more general algebraic structure than the unit real interval $[0,1]$. It serves a wide range of applications like coding theory etc. Considering these reasons, to define triangular norms on bounded lattices has been a current area for researchers and is still up to date.

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In [9], there are some construction methods for t-norms via the subset A satisfying $(0,1) \subseteq A \subseteq [0,1]$ or h-additive generators. In literature, some construction methods have been also given on bounded lattices in [2, 4, 12]. In [12], S. Saminger proposed a construction method that do not always produce a t-norm on L and investigated under which assumptions the method produce a t-norm on L. In [4], it is proposed a construction method that extend the t-norm on a interval $[a,1]$ of the bounded lattice L to bounded lattice L. Considering the effective use of fuzzy logic operators in applied sciences, it is important to obtain a new one from a known fuzzy logic operator on a given boundary lattice. Therefore, for triangular norms which are fuzzy logic operators, it is also very important to obtain a triangular norm on a bounded lattice from a triangular norm defined on a subinterval of the bounded lattice. Main aim of this paper is to give a way constructing the t-norm on a bounded lattice from given t-norm on subinterval of the lattice without additional conditions.

The paper is organized as follows: In Section 1, we give some necessary definitions and previous results. Based on the presence of t-norms on $[a,b]$, in Section 2, we propose a construction method extending the t-norm on the subinterval of L to the bounded lattice L being a t-norm again with no additional conditions. In [4], a construction method making the t-norm on a interval $[a,1]$ of L a t-norm on a bounded lattice L is presented. We observe that the extended t-norm obtained by our method is coincident with the method given in [4] when $b = 1$. If we take $a = 0$ for our construction method, our method provide a way to obtain a t-norm on bounded lattice L from the t-norm on a interval $[0,b]$ of bounded lattice L. We also observe that the extended t-norm obtained by our method coincides with the t-norm proposed by Saminger in [12] when a and b are comparable with all elements of L. Therefore, it is clearly understandable that our construction method is much more general than the available methods in the literature. Moreover, our construction method construct a t-norm on bounded lattice L from the t-norm on interval $[a,b]$ (specifically $[0,b]$) of L without any assumptions.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

In this section, we recall some basic notions and results.

Definition 2.1. (Birkhoff [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice. The elements x and y are called comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable. In this situation, the notation $x||y$ is used.

We denote by I_p the set of elements which are incomparable with p, that is, $I_p = \{x \in L \ : \ x||p\}$, where $p \in L$. Similarly, we donote by $I_{p,r}$ the set of incomparable elements to p and r, that is, $I_{p,r} = \{x \in L \ : \ x||p \text{ and } x||r\}$, where $p, r \in L$.

Definition 2.2. (Birkhoff [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The interval $[a,b]$ of L is defined as

$$[a,b] = \{x \in L| \quad a \leq x \leq b\}.$$

Moreover, the intervals $(a,b] = \{x \in L| \quad a < x \leq b\}$, $[a,b) = \{x \in L| \quad a \leq x < b\}$ and $(a,b) = \{x \in L| \quad a < x < b\}$ of L are defined in a similar way.

Definition 2.3. (Klement et al. [9]) An operation T on a bounded lattice L is called a triangular norm if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1.

Let T_1 and T_2 be two t-norms on L. T_1 is called smaller than T_2 if for any elements $x, y \in L$, $T_1(x, y) \leq T_2(x, y)$. Similar relation between two t-conorms can be given. In this sense, the smallest and greatest t-norms on a bounded lattice L are given respectively as follows:

$$T_D(x,y) = \begin{cases} y & x = 1, \\ x & y = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } T_\wedge(x,y) = x \wedge y.$$

Definition 2.4. (Karaçal and Khadjiev [7])

A t-norm T (or t-conorm S) on a bounded lattice L is called $\vee$-distributive if for every $a, b_1, b_2 \in L$

$$T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2) \text{ (or } S(a, b_1 \vee b_2) = S(a, b_1) \vee S(a, b_2))$$

holds.

Similarly, t-norm T (or t-conorm S) on a bounded lattice L is called $\wedge$-distributive if for every $a, b_1, b_2 \in L$

$$T(a, b_1 \wedge b_2) = T(a, b_1) \wedge T(a, b_2) \text{ (or } S(a, b_1 \wedge b_2) = S(a, b_1) \wedge S(a, b_2))$$

holds.

Definition 2.5. (Grabisch et al. [5]) Let T be a t-norm on a bounded lattice L. An element $x \in L \setminus \{0\}$ is called a zero divisor of T if there exists $y \in L \setminus \{0\}$, $x \wedge y \neq 0$ such that $T(x, y) = 0$.

3. CONSTRUCTION OF TRIANGULAR NORMS ON BOUNDED LATTICES

In this section, a construction method is presented to obtain a t-norm on a bounded lattice L from a given t-norm W on a subinterval of L. Additionally, whether the triangular norm obtained by proposed method is a supremum-distributive t-norm is investigated and its relationship with the current methods are investigated.

Theorem 3.1. Let $(L, \leq, 0, 1)$ be bounded lattice and W be a t-norm on a interval $[a, b]$

of L. Then, the function $T : L^2 \to L$ defined as follows

$$T(x,y) = \begin{cases} W(x,y) & (x,y) \in [a,b)^2, \\ W(x \wedge b, y \wedge b) & x \in [a,b),\ (a < y \text{ and } y \in I_b), \\ & (a < x \text{ and } x \in I_b), y \in [a,b), \\ & (a < x \text{ and } x \in I_b), (a < y \text{ and } y \in I_b), \\ x \wedge y \wedge a & (x \in I_a \text{ and } x < b),\ y \in [a,b), \\ & (x \in I_a \text{ and } x < b),\ (y \in I_a \text{ and } y < b), \\ & (x \in I_a \text{ and } x < b),\ (a < y \text{ and } y \in I_b), \\ & (x \in I_a \text{ and } x < b), 1 > y \geq b, \\ & x \in [a,b),\ (y \in I_a \text{ and } y < b), \\ & (a < x \text{ and } x \in I_b), (y \in I_a \text{ and } y < b), \\ & 1 > x \geq b, (y \in I_a \text{ and } y < b), \\ & x \in I_{a,b},\ y \in L \setminus \{1\} \text{ or } y \in I_{a,b},\ x \in L \setminus \{1\} \\ & x < a \text{ or } y < a, \\ x \wedge y \wedge b & (a < x \text{ and } x \in I_b),\ b \leq y < 1, \\ & b \leq x < 1,\ (y > a \text{ and } y \in I_b), \\ x \wedge y & b \leq x, y < 1, \\ & x \in [a,b),\ b \leq y < 1, \\ & b \leq x < 1,\ y \in [a,b) \\ & x = 1 \text{ or } y = 1 \end{cases} \tag{1}$$

is a t-norm on L.

Proof. (i) *Monotonicity*: Let us show that for every elements $x, y \in L$ with $x \leq y$, $T(x,z) \leq T(y,z)$ for all $z \in L$. If $z = 1$, then it is clear that $T(x,z) = x \leq y = T(y,z)$. If there exists one of the cases ($z \in I_a$ and $z < b$), $z < a$ and ($z \in I_{a,b}$), then it is clear that $T(x,z) = x \wedge z \wedge a \leq y \wedge z \wedge a = T(y,z)$. Moreover, if $T(x,z) = x \wedge z \wedge a$, $T(x,z) \leq T(y,z)$ is always satisfied since $T(x,z) = x \wedge z \wedge a \leq y, z, a, W(x,y)$ or $W(x \wedge b, y \wedge b)$. The proof is split into all the other possible cases.

1. Let $x \in [a,b)$. Since $x \leq y$, $y \in I_a$ is not a case.

1.1. $a < y$ and $y \in I_b$,

1.1.1. $z \in [a,b)$,

$$T(x,z) = W(x,z) \leq W(y \wedge b, z) = T(y,z).$$

1.1.2. $a < z$ and $z \in I_b$,

$$T(x,z) = W(x, z \wedge b) \leq W(y \wedge b, z \wedge b) = T(y,z).$$

1.1.3. $b \leq z < 1$,

$$T(x,z) = x \leq y \wedge b = y \wedge z \wedge b = T(y,z).$$

1.2. $b \leq y < 1$,

1.2.1. $z \in [a,b)$,

$$T(x,z) = W(x,z) \leq W(b,z) = z = T(y,z).$$

1.2.2. $a < z$ and $z \in I_b$,

$$T(x,z) = W(x, z \wedge b) \leq y \wedge z \wedge b = T(y,z).$$

1.2.3. $b \leq z < 1$,

$$T(x,z) = x \leq y \wedge z = T(y,z).$$

1.3. $y = 1$. $T(x,z)$ is equal to one of the cases $x \wedge z \wedge a$, $W(x,z)$, x and $W(x, z \wedge b)$. For all the cases, it is clear that $T(x,z) \leq z = T(1,z)$.
2. $a < x$ and $x \in I_b$.
2.1. $b \leq y < 1$.
2.1.1. $z \in [a,b)$,

$$T(x,z) = W(x \wedge b, z) \leq W(b,z) = z = T(y,z).$$

2.1.2. $a < z$ and $z \in I_b$,

$$T(x,z) = W(x \wedge b, z \wedge b) \leq W(b, z \wedge b) = z \wedge b = y \wedge z \wedge b = T(y,z).$$

2.1.3. $b \leq z < 1$,

$$T(x,z) = x \wedge z \wedge b \leq x \wedge z \leq y \wedge z = T(y,z).$$

2.2. If $y = 1$, it is clear that $T(x,z) \leq T(y,z)$.
3. $b \leq x < 1$. Since $x \leq y$, we need to look at the case $y = 1$. In this case, $T(x,z)$ is one of the cases z, $x \wedge z \wedge a$, $x \wedge z \wedge b$ and $x \wedge z$. Since $y = 1$, it is clear that $T(y,z) = z$. Thus, it follows $T(x,z) \leq T(y,z)$ for all possible cases of $T(x,z)$.

(ii) *Associativity*: We present that $T(x, T(y,z)) = T(T(x,y),z)$ for any $x,y,z \in L$. The proof is split into all possible cases by considering the situation of the elements x,y,z. If one of the elements x,y and z is equal to 1, it is clear that the equality is always satisfied. Also, if one of the conditions $y < a$ or ($y \in I_a$ and $y < b$) or ($y \in I_{a,b}$) holds, we have that

$$T(x,T(y,z)) = T(x, y \wedge z \wedge a) = x \wedge y \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z).$$

1. Let $x < a$.
1.1. $y \in [a,b)$,
1.1.1. If $z < a$, then $T(x,T(y,z)) = T(x, y \wedge z \wedge a) = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z)$.
1.1.2. If $z \in [a,b)$, then $T(x,T(y,z)) = T(x, W(y,z)) = x \wedge W(y,z) \wedge a = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z)$.
1.1.3. If $z \in I_a$ and $z < b$, then $T(x,T(y,z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z)$.
1.1.4. If $a < z$ and $z \in I_b$, then $T(x,T(y,z)) = T(x, W(y, z \wedge b)) = x \wedge W(y, z \wedge b) \wedge a = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z)$.
1.1.5. If $b \leq z < 1$, then $T(x,T(y,z)) = T(x,y) = x \wedge y \wedge a = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x,y),z)$.

1.1.6. If $z \in I_a \cap I_b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge z \wedge a = T(x, z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2. $a < y$ and $y \in I_b$,

1.2.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z)) = x \wedge W(y \wedge b, z) \wedge a = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z \wedge a = T(x, z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z \wedge b)) = x \wedge W(y \wedge b, z \wedge b) \wedge a = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = x \wedge (y \wedge b) \wedge a = x \wedge y = x = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.2.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge y \wedge z \wedge a = x \wedge y \wedge z = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3. $b \leq y < 1$,

1.3.1. If $z < a$, $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, z) = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = x \wedge (z \wedge b) \wedge a = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z) = x \wedge (y \wedge z) \wedge a = x \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

1.3.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge z \wedge a = T(x, z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

2. Let $x \in [a, b)$.

2.1. $y \in [a, b)$,

2.1.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = W(x, y) \wedge z \wedge a = T(W(x, y), z) = T(T(x, y), z)$.

2.1.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y, z)) = W(x, W(y, z)) = W(W(x, y), z) = T(W(x, y), z) = T(T(x, y), z)$.

2.1.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x, y) \wedge z \wedge a = T(W(x, y), z) = T(T(x, y), z)$.

2.1.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y, z \wedge b)) = W(x, W(y, z \wedge b)) = W(W(x, y), z \wedge b) = T(W(x, y), z) = T(T(x, y), z)$.

2.1.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y) = W(x, y) = T(W(x, y), z) = T(T(x, y), z)$.

2.1.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) \wedge a = z \wedge a = W(x, y) \wedge z \wedge a = T(W(x, y), z) = T(T(x, y), z)$.

2.2. $a < y$ and $y \in I_b$,

2.2.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = W(x, y \wedge b) \wedge z \wedge a = T(W(x, y \wedge b), z) = T(T(x, y), z)$.

2.2.2. If $z \in [a,b)$, then $T(x,T(y,z)) = T(x,W(y \wedge b,z)) = W(x,W(y \wedge b,z)) = W(W(x,y \wedge b),z) = T(T(x,y),z)$.

2.2.3. If $z \in I_a$ and $z < b$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x,y \wedge b) \wedge z \wedge a = T(W(x,y \wedge b),z) = T(T(x,y),z)$.

2.2.4. If $a < z$ and $z \in I_b$, then $T(x,T(y,z)) = T(x,W(y \wedge b,z \wedge b)) = W(x,W(y \wedge b,z \wedge b)) = W(W(x,y \wedge b),z \wedge b) = T(W(x,y \wedge b),z) = T(T(x,y),z)$.

2.2.5. If $b \leq z < 1$,then $T(x,T(y,z)) = T(x,y \wedge z \wedge b) = T(x,y \wedge b) = W(x,y \wedge b) = T(W(x,y \wedge b),z) = T(T(x,y),z)$.

2.2.6. If $z \in I_{a,b}$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x,y \wedge b) \wedge z \wedge a = T(W(x,y \wedge b),z) = T(T(x,y),z)$.

2.3. $b \leq y < 1$,

2.3.1. If $z < a$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = x \wedge z \wedge a = T(x,z) = T(T(x,y),z)$.

2.3.2. If $z \in [a,b)$, then $T(x,T(y,z)) = T(x,z) = T(T(x,y),z)$.

2.3.3. If $z \in I_a$ and $z < b$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = x \wedge z \wedge a = T(x,z) = T(T(x,y),z)$.

2.3.4. If $a < z$ and $z \in I_b$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge b) = T(x,z \wedge b) = W(x,z \wedge b) = T(x,z) = T(T(x,y),z)$.

2.3.5. If $b \leq z < 1$, then $T(x,T(y,z)) = T(x,y \wedge z) = x = T(x,z) = T(T(x,y),z)$.

2.3.6. If $z \in I_{a,b}$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) \wedge a = z \wedge a = x \wedge z \wedge a = T(x,z) = T(T(x,y),z)$.

3. Let $x \in I_a$ and $x < b$.

3.1. $y \in [a,b)$,

3.1.1. If $z < a$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z) = x \wedge z \wedge a = x \wedge z = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.1.2. If $z \in [a,b)$, then $T(x,T(y,z)) = T(x,W(y,z)) = x \wedge W(y,z) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.1.3. If $z \in I_a$ and $z < b$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.1.4. If $a < z$ and $z \in I_b$, then $T(x,T(y,z)) = T(x,W(y,z \wedge b)) = x \wedge W(y,z \wedge b) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.1.5. If $b \leq z < 1$, then $T(x,T(y,z)) = T(x,y) = x \wedge y \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.1.6. If $z \in I_{a,b}$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z \wedge a) = x \wedge z \wedge a = T(x \wedge a,z) = T(T(x,y),z)$.

3.2. $a < y$ and $y \in I_b$,

3.2.1. If $z < a$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = T(x,z) = x \wedge z \wedge a = x \wedge z = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a,z) = T(T(x,y),z)$.

3.2.2. If $z \in [a,b)$, then $T(x,T(y,z)) = T(x,W(y \wedge b,z)) = x \wedge W(y \wedge b,z) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a,z) = T(T(x,y),z)$.

3.2.3. If $z \in I_a$ and $z < b$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a,z) = T(T(x,y),z)$.

3.2.4. If $a < z$ and $z \in I_b$, then $T(x,T(y,z)) = T(x,W(y \wedge b,z \wedge b)) = x \wedge W(y \wedge b,z \wedge b) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a,z) = T(T(x,y),z)$.

3.2.5. If $b \leq z < 1$, then $T(x,T(y,z)) = T(x,y \wedge z \wedge b) = x \wedge (y \wedge b) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a,z) = T(T(x,y),z)$.

3.2.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) \wedge a = x \wedge z \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a, z) = T(T(x, y), z)$.

3.3. $b \leq y < 1$,

3.3.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = x \wedge z = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

3.3.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, z) = x \wedge z \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

3.3.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = x \wedge z \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

3.3.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = T(x, z \wedge b) = x \wedge (z \wedge b) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

3.3.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z) = x \wedge (y \wedge z) \wedge a = x \wedge a = (x \wedge a) \wedge z \wedge a = T(x \wedge a, z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

3.3.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge y \wedge z \wedge a = x \wedge z \wedge a = T(x \wedge a, z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.

4. $a < x$ and $x \in I_b$,

4.1. $y \in [a, b)$,

4.1.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = W(x \wedge b, y) \wedge z \wedge a = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.1.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y, z)) = W(x \wedge b, W(y, z)) = W(W(x \wedge b, y), z) = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.1.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x \wedge b, y) \wedge z \wedge a = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.1.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y, z \wedge b)) = W(x \wedge b, W(y, z \wedge b)) = W(W(x \wedge b, y), z \wedge b) = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.1.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y) = W(x \wedge b, y) = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.1.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x \wedge b, y) \wedge z \wedge a = T(W(x \wedge b, y), z) = T(T(x, y), z)$.

4.2. $a < y$ and $y \in I_b$,

4.2.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = W(x \wedge b, y \wedge b) \wedge z \wedge a = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.2.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z)) = W(x \wedge b, W(y \wedge b, z)) = W(W(x \wedge b, y \wedge b), z) = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.2.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x \wedge b, y \wedge b) \wedge z \wedge a = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.2.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z \wedge b)) = W(x \wedge b, W(y \wedge b, z \wedge b)) = W(W(x \wedge b, y \wedge b), z \wedge b) = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.2.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = W(x \wedge b, y \wedge b) = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.2.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = W(x \wedge b, z \wedge b) \wedge z \wedge a = T(W(x \wedge b, y \wedge b), z) = T(T(x, y), z)$.

4.3. $b \leq y < 1$,

4.3.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge z \wedge a = z = (x \wedge b) \wedge z \wedge a = T(x \wedge y \wedge b, z) = T(T(x, y), z)$.

4.3.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, z) = W(x \wedge b, z) = T(x \wedge b, z) = T(T(x, y), z)$.

4.3.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = (x \wedge b) \wedge z \wedge a = T(x \wedge b, z) = T(T(x, y), z)$.

4.3.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = T(x, z \wedge b) = W(x \wedge b, z \wedge b) = T(x \wedge b, z) = T(T(x, y), z)$.

4.3.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z) = x \wedge b = T(x \wedge b, z) = T(T(x, y), z)$.

4.3.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) \wedge a = z \wedge a = (x \wedge b) \wedge z \wedge a = T(x \wedge b, z) = T(T(x, y), z)$.

5. $b \leq x < 1$,

5.1. $y \in [a, b)$,

5.1.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = y \wedge z \wedge a = T(y, z) = T(T(x, y), z)$.

5.1.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y, z)) = W(y, z) = T(y, z) = T(T(x, y), z)$.

5.1.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = y \wedge z \wedge a = T(y, z) = T(T(x, y), z)$.

5.1.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y, z \wedge b)) = W(y, z \wedge b) = T(y, z) = T(T(x, y), z)$.

5.1.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$.

5.1.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = y \wedge z \wedge a = T(y, z) = T(T(x, y), z)$.

5.2. $a < y$ and $y \in I_b$,

5.2.1. If $z < a$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = (y \wedge b) \wedge z \wedge a = T(y \wedge b, z) = T(x \wedge y \wedge b, z) = T(T(x, y), z)$.

5.2.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z)) = W(y \wedge b, z) = T(y \wedge b, z) = T(T(x, y), z)$.

5.2.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = (y \wedge b) \wedge z \wedge a = T(y \wedge b, z) = T(T(x, y), z)$.

5.2.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, W(y \wedge b, z \wedge b)) = W(y \wedge b, z \wedge b) = T(y \wedge b, z) = T(T(x, y), z)$.

5.2.5. If $b \leq z < 1$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = T(x, y \wedge b) = y \wedge b = T(y \wedge b, z) = T(T(x, y), z)$.

5.2.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) = z \wedge a = (y \wedge b) \wedge z \wedge a = T(y \wedge b, z) = T(T(x, y), z)$.

5.3. If $b \leq y < 1$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z) = x \wedge z \wedge a = z = (x \wedge y) \wedge z \wedge a = T(x \wedge y, z) = T(T(x, y), z)$.

5.3.2. If $z \in [a, b)$, then $T(x, T(y, z)) = T(x, z) = z = T(x \wedge y, z) = T(T(x, y), z)$.

5.3.3. If $z \in I_a$ and $z < b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = T(x, z \wedge a) = x \wedge (z \wedge a) \wedge a = z \wedge a = T(x \wedge y, z) = T(T(x, y), z)$.

5.3.4. If $a < z$ and $z \in I_b$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge b) = T(x, z \wedge b) = z \wedge b = T(x \wedge y, z) = T(T(x, y), z)$.

5.3.5. If $b \leq z < 1$. then $T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$.

5.3.6. If $z \in I_{a,b}$, then $T(x, T(y, z)) = T(x, y \wedge z \wedge a) = x \wedge (y \wedge z \wedge a) \wedge a = z \wedge a = (x \wedge y) \wedge z \wedge a = T(x \wedge y, z) = T(T(x, y), z)$.

6. $x \in I_a$ and $x \in I_b$. If $T(y, z) = y \wedge z \wedge a$, it is clear that
$T(x, T(y, z)) = x \wedge T(y, z) \wedge a = x \wedge y \wedge z \wedge a = T(x \wedge y \wedge a, z) = T(T(x, y), z)$.
Now, let us look at the cases in which $T(y, z)$ is different from $y \wedge z \wedge a$.

6.1. $y \in [a, b)$ and ($a < z$ and $z \in I_b$). By $T(y, z) = W(y, z \wedge b)$, we have that
$T(x, T(y, z)) = x \wedge W(y, z \wedge b) \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.2. $y \in [a, b)$ and $1 > z \geq b$. Since $T(y, z) = y$, we have that
$T(x, T(y, z)) = x \wedge y \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.3. $a < y$, $y \in I_b$ and $z \in [a, b)$. Since $T(y, z) = W(y \wedge b, z)$, we have that
$T(x, T(y, z)) = x \wedge W(y \wedge b, z) \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.4. $a < y$, $y \in I_b$, $a < z$ and $z \in I_b$. Since $T(y, z) = W(y \wedge b, z \wedge b)$, we have that
$T(x, T(y, z)) = x \wedge W(y \wedge b, z \wedge b) \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.5. $a < y$, $y \in I_b$ and $1 > z \geq b$. Since $T(y, z) = y \wedge z \wedge b$, we have that
$T(x, T(y, z)) = x \wedge (y \wedge z \wedge b) \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.6. $1 > y \geq b$ and $z \in [a, b)$. Since $T(y, z) = y$, we have that
$T(x, T(y, z)) = x \wedge y \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

6.7. $1 > y \geq b$, $z > a$ and $z \in I_b$. Since $T(y, z) = y \wedge z \wedge b$, we have that
$T(x, T(y, z)) = x \wedge (y \wedge z \wedge b) \wedge a = x \wedge a = x \wedge y \wedge z \wedge a = T(T(x, y), z)$.

The commutativity of T and the fact that 1 is a neutral element of T are obvious from the definition of T. Therefore, T is a t-norm on L. □

Remark 3.2. Observe that the restriction of the t-norm to $[0, a]$ obtained by formula 1 in Theorem 3.1 is also a t-norm on $[0, a]$. Similarly, the restriction of the t-norm to $[b, 1]$ obtained by formula 1 in Theorem 3.1 is a t-norm on $[b, 1]$.

Remark 3.3. Note that the t-norm obtained by formula 1 in Theorem 3.1 on bounded lattice L may have zero-divisors even if the t-norm W has no zero-divisors on $[a, b]$. Let us investigate the following example.

Example 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice characterized by the following Hasse diagram, W be the infimum t-norm $T_\wedge$ on $[a, b]$.

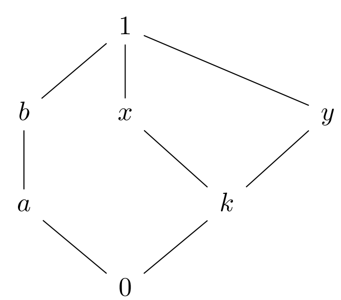

Fig. 1. Lattice diagram of L.

Formula 1 produces the following t-norm T.

T	0	a	b	x	y	k	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	0	a
b	0	a	b	0	0	0	b
x	0	0	0	0	0	0	x
y	0	0	0	0	0	0	y
k	0	0	0	0	0	0	k
1	0	a	b	x	y	k	1

Tab. 1. T-norm T on L.

It is easily shown that T has zero-divisors on L since $T(x, y) = 0$ when $x \wedge y = k \neq 0$.

Remark 3.5. If $I_a = I_b = \emptyset$, the method presented in Theorem 3.1 coincides with the Saminger's construction method [12]. The method proposed by Saminger may not produce a t-norm on any bounded lattice. These two methods do not have to coincide on bounded lattice that provide Saminger's constraints (see following example). Moreover, the method given in Theorem 3.1 coincides with the method given in [4] when $b = 1$.

Example 3.6. Consider the lattice L characterized by Hasse diagram, as shown in Figure 2.

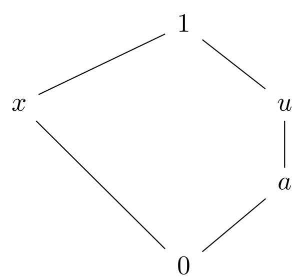

Fig. 2. Lattice diagram of L

Let $T = T_\wedge$ on $[a, 1]$. T_1 and T_2 are the t-norms constructed by the methods given in Saminger [12] and the method in Theorem 3.1, respectively.

T_1	0	x	a	u	1
0	0	0	0	0	0
x	0	x	0	0	x
a	0	0	a	a	a
u	0	0	a	u	u
1	0	x	a	u	1

Tab. 2. T-norm T_1 on L.

and

T_2	0	x	a	u	1
0	0	0	0	0	0
x	0	0	0	0	x
a	0	0	a	a	a
u	0	0	a	u	u
1	0	x	a	u	1

Tab. 3. T-norm T_2 on L.

It is clear that T_1 and T_2 do not coincide.

In the literature, there is no construction method which extends a t-norm on $[0, b]$ to be a t-norm on a bounded lattice L as our best knowledge. We also provided a method for constructing the t-norm on a bounded lattice L using given triangular norms on any interval $[0, b]$ of L.

Corollary 3.7. In Theorem 3.1, if we take $a = 0$, the following t-norm T is obtained. Therefore, Theorem 3.1 provide a way to produce a t-norm on L from the t-norm W on $[0, b]$.

$$T(x,y) = \begin{cases} W(x,y) & (x,y) \in [0,b)^2, \\ W(x, y \wedge b) & x \in [0,b),\ (0 < y \text{ and } y \in I_b), \\ x & x \in [0,b),\ 1 > y \geq b, \\ W(x \wedge b, y) & (0 < x \text{ and } x \in I_b), y \in [0,b), \\ W(x \wedge b, y \wedge b) & (0 < x \text{ and } x \in I_b), (0 < y \text{ and } y \in I_b), \\ x \wedge y \wedge b & (0 < x \text{ and } x \in I_b), 1 > y \geq b, \\ y & 1 > x \geq b,\ y \in [0,b), \\ x \wedge y \wedge b & 1 > x \geq b, (y > 0 \text{ and } y \in I_b), \\ x \wedge y & (x,y) \in [b,1)^2, \\ y & x = 1, \\ x & y = 1. \end{cases}$$

Proposition 3.8. Let $(L, \leq, 0, 1)$ be a distributive bounded lattice and W be a t-norm on a interval $[a, b]$ of L. If W is $\vee$-distributive and $I_a = I_b = \emptyset$, then the t-norm T given in Theorem 3.1 is $\vee$-distributive.

Proof. If $I_a = I_b = \emptyset$, then the t-norm T given in Theorem 3.1 is defined as follows:

$$T(x,y) = \begin{cases} W(x,y) & x, y \in [a,b], \\ x \wedge y & \text{otherwise.} \end{cases}$$

Let us show that T is $\vee$-distributive.

1. $x \in [a, b]$.

1.1. $y \in [a, b]$,

1.1.1. $z \in [a, b]$. In this case, $a \leq y \vee z \leq b$. Thus,

$$T(x, y \vee z) = W(x, y \vee z) = W(x, y) \vee W(x, z) = T(x, y) \vee T(x, z).$$

1.1.2. $z \notin [a, b]$. In this case, either $z < a$ or $z > b$ since $I_a = I_b = \emptyset$.

If $z < a$, it is obtained that $T(x, y \vee z) = T(x, y) = T(x, y) \vee T(x, z)$. Similarly, If $z > b$, it is obtained that $T(x, y \vee z) = T(x, z) = T(x, y) \vee T(x, z)$.

1.2. $y \notin [a, b]$. In this case, $y < a$ or $y > b$.

1.2.1. $y < a$.

1.2.1.1. $z \in [a, b]$. In this case, $y \vee z = z$. In this case, considering $T(x, y) \leq T(x, z)$,

$$T(x, y \vee z) = T(x, z) = T(x, y) \vee T(x, z).$$

1.2.1.2. $z \notin [a, b]$. Then, either $z < a$ or $z > b$. Let $z < a$. Thus, $y \vee z \leq a$. If $y \vee z < a$,

$$T(x, y \vee z) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

If $y \vee z = a$,

$$T(x, y \vee z) = T(x, a) = W(x, a) = a = x \wedge a = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

Let $z > b$. Then, $y \vee z = z$. Thus,

$$T(x, y \vee z) = T(x, z) = x \wedge z = x \wedge (y \wedge z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

1.2.2. $y > b$.

1.2.2.1. $z \in [a, b]$. Then, $y \vee z = y$.

$$T(x, y \vee z) = T(x, y) = x \wedge y = x = x \vee W(x, z) = T(x, y) \vee T(x, z).$$

1.2.2.2. $z \notin [a, b]$. Then, either $z < a$ or $z > b$. Let $z < a$. Then, $y \vee z = y > b$. Thus,

$$T(x, y \vee z) = x \wedge (y \wedge z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

Let $z > b$. Then, $y \vee z > b$. Thus,

$$T(x, y \vee z) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

2. $x \notin [a, b]$. Then,

$$T(x, y \vee z) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = T(x, y) \vee T(x, z).$$

□

Remark 3.9. Even if L is a distributive lattice and W is a $\vee$-distributive t-norm on $[a, b] \subseteq L$, the t-norm T given in Theorem 3.1 need not be $\vee$-distributive. Let us look at the following example.

Example 3.10. Let L be a bounded lattice whose lattice diagram is depicted as follow:

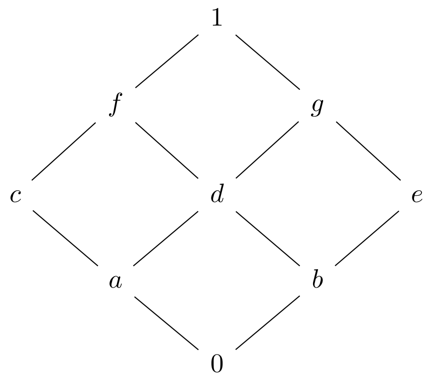

Fig. 3. $(L, \leq)$.

Take the t-norm $W = T_\wedge$ on $[d, 1]$. Then, the t-norm T generated by the method given in Theorem 3.1 is as follow.

T	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a	a	a
b	0	0	b	0	b	b	b	b	b
c	0	a	0	a	a	0	a	a	c
d	0	a	b	a	d	b	d	d	d
e	0	0	b	0	b	b	b	b	e
f	0	a	b	a	d	b	f	d	f
g	0	a	b	a	d	b	d	g	g
1	0	a	b	c	d	e	f	g	1

Tab. 4. The t-norm T

It is clear that, L is a distributive lattice and W is $\vee$-distributive on $[d, 1]$. But, t-norm T is not $\vee$-distributive since

$$T(c, f \vee g) = T(c, 1) = c \neq a = a \vee a = T(c, f) \vee T(c, g).$$

Remark 3.11. Theorem 3.1 provides a way to produce a t-norm different from $\wedge$ on any bounded lattice in general even if $W = \wedge$ in the formula of Theorem 3.1 (Check Example 3.10).

It is known that there may not always be $\vee$-distributive t-norm on any arbitrary bounded lattice [7]. But, there always exists an $\wedge$-distributive t-norm on any bounded lattice (recall $T_W, T_\wedge$). The following poroposition provides to construct an $\wedge$-distributive t-norm different from T_W an on arbitrary bounded lattice in general if the t-norm W on a interval $[a,b]$ of L is $\wedge$-distributive and $I_a = I_b = \emptyset$.

Proposition 3.12. Let $(L, \leq, 0, 1)$ be a bounded lattice and W be a t-norm on a interval $[a,b]$ of L. If W is $\wedge$-distributive and $I_a = I_b = \emptyset$, then the t-norm T given in Theorem 3.1 is $\wedge$-distributive.

Proof. If $I_a = I_b = \emptyset$, then the t-norm T given in Theorem 3.1 is defined as follows:

$$T(x,y) = \begin{cases} W(x,y) & x, y \in [a,b], \\ x \wedge y & \text{otherwise.} \end{cases}$$

Let us show that T is $\wedge$-distributive.

1. $x \in [a,b]$.

1.1. $y \in [a,b]$,

1.1.1. $z \in [a,b]$.

$$\begin{aligned} T(x, y \wedge z) &= W(x, y \wedge z) = W(x,y) \wedge W(x,z) \\ &= T(x,y) \wedge T(x,z). \end{aligned}$$

1.1.2. $z \notin [a,b]$. In this case, either $z < a$ or $z > b$ since $I_a = I_b = \emptyset$.

1.1.2.1. $z < a$ $(z > b)$.

$$T(x, y \wedge z) = T(x,z)\ (T(x,y)) = T(x,y) \wedge T(x,z).$$

1.2. $y \notin [a,b]$. In this case, $y < a$ or $y > b$.

1.2.1. $y < a$.

1.2.1.1. $z \in [a,b]$.

$$T(x, y \wedge z) = T(x,y) = T(x,y) \wedge T(x,z).$$

1.2.1.2. $z \notin [a,b]$. Then, either $z < a$ or $z > b$. Let $z < a$.

$$\begin{aligned} T(x, y \wedge z) &= x \wedge (y \wedge z) \\ &= y \wedge z \\ &= (x \wedge y) \wedge (x \wedge z) \\ &= T(x,y) \wedge T(x,z). \end{aligned}$$

Let $z > b$.

$$T(x, y \wedge z) = T(x,y) = T(x,y) \wedge T(x,z).$$

1.2.2. $y > b$.

1.2.2.1. $z \in [a,b]$.

$$T(x, y \wedge z) = T(x,z) = T(x,y) \wedge T(x,z).$$

1.2.2.2. $z \notin [a,b]$. Then, either $z < a$ or $z > b$. Let $z < a$.

$$\begin{aligned} T(x, y \wedge z) &= T(x,z) = x \wedge z = z \\ &= x \wedge y \wedge z \\ &= (x \wedge y) \wedge (x \wedge z) \\ &= T(x,y) \wedge T(x,z). \end{aligned}$$

Let $z > b$. Then, $y \wedge z = b$ or $y \wedge z > b$. Let $y \wedge z = b$.

$$\begin{aligned} T(x, y \wedge z) &= T(x,b) \\ &= W(x,b) = x \\ &= x \wedge b \\ &= (x \wedge y) \wedge (x \wedge z) \\ &= T(x,y) \wedge T(x,z). \end{aligned}$$

Let $y \wedge z > b$.

$$\begin{aligned} T(x, y \wedge z) &= x \wedge y \wedge z = x \\ &= W(x,b) = x \\ &= x \wedge (y \wedge z) \\ &= (x \wedge y) \wedge (x \wedge z) \\ &= T(x,y) \wedge T(x,z). \end{aligned}$$

2. $x \notin [a,b]$. Then,

$$T(x, y \wedge z) = x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z) = T(x,y) \wedge T(x,z).$$

$\square$

Remark 3.13. Proposition 3.12 produces an $\wedge$-distributive t-norm on any bounded lattice L providing that there exists two elements $a, b \in L$ such that $I_a = I_b = \emptyset$ without checking since there always exists t-norm $T_\wedge$ at least on a interval $[a,b]$ of the bounded lattice L.

4. CONCLUDING REMARKS

Triangular norms are still a current subject matter for many researchers. Therefore, the study of triangular norms on important algebraic structures continues to attract researchers' attentions. In this study, we provide a way to obtain a triangular norm on L from a triangular norm defined on a subinterval of L. It should be noted that unlike existing methods in the literature, this method works without any restrictions and it is a more general construction method. We end this paper with the following some open problems: Is there a construction method extending a given supremum distributive t-norm on interval $[a,b]$ of distributive lattice L to obtain again a supremum distributive t-norm on L even if $I_a \neq \emptyset$ and $I_b \neq \emptyset$? In Remark 3.2, we observe that the restriction of the t-norm obtained by formula 1 in Theorem 3.1 to $[0, a]$ is also a t-norm on $[0, a]$ and similarly the restriction of the t-norm obtained by formula 1 in Theorem 3.1 to $[b, 1]$ is a t-norm on $[b, 1]$. From this point of view, are there another extension methods for triangular norms from the t-norm on interval $[a, b]$ of the bounded lattice L such that the restrictions of the extended t-norm on L to $[0, a]$ and $[b, 1]$ are t-norms on $[0, a]$ and $[b, 1]$, respectively? Can we obtain another extension method of t-norm W of interval $[a, b]$ of L to the bounded lattice L by putting which t-norms on $[0, a]$ and $[b, 1]$?

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