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FERMAT k-FIBONACCI AND k-LUCAS NUMBERS

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Abstract. Using the lower bound of linear forms in logarithms of Matveev and the theory of continued fractions by means of a variation of a result of Dujella and Pethő, we find all k-Fibonacci and k-Lucas numbers which are Fermat numbers. Some more general results are given.

Keywords: generalized Fibonacci number; Fermat number, linear form in logarithms; reduction method

MSC 2010: 11B39, 11J86

1. Introduction and preliminary results

For an integer $k \geqslant 2$ we consider the linear recurrence sequence $G^{(k)} := (G_n^{(k)})_{n\geqslant 2-k}$ of order k, defined as

$$G_n^{(k)} = G_{n-1}^{(k)} + G_{n-2}^{(k)} + \ldots + G_{n-k}^{(k)} \quad \forall\, n \geqslant 2,$$

with the initial conditions $G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \ldots = G_{-1}^{(k)} = 0$, $G_0^{(k)} = a$ and $G_1^{(k)} = b$, where a and b are both integers.

If $a = 0$ and $b = 1$, then $G^{(k)}$ is known as the *k-Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n\geqslant 2-k}$. We shall refer to $F_n^{(k)}$ as the *nth k-Fibonacci number*. We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Fibonacci numbers are obtained for $k = 2$. For small values of k, these sequences are called Tribonacci ($k = 3$), Tetranacci ($k = 4$), Pentanacci ($k = 5$), Hexanacci ($k = 6$), Heptanacci ($k = 7$) and Octanacci ($k = 8$). In a similar way, if $a = 2$ and $b = 1$, then $G^{(k)}$ is known as the *k-Lucas sequence* $L^{(k)} := (L_n^{(k)})_{n\geqslant 2-k}$, which extends the usual Lucas sequence $L^{(2)}$. Other generalization for Lucas numbers can be found in [14].

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An interesting fact about the k-Fibonacci sequence is that the first $k+1$ nonzero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2}, \quad 2 \leqslant n \leqslant k+1, \tag{1}$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, the inequality

$$F_n^{(k)} < 2^{n-2} \quad \text{holds for all } n \geqslant k+2 \tag{2}$$

(see [3]). Similarly, the k-Lucas sequence $L^{(k)}$ has the remarkable property that the first few terms are given by

$$L_n^{(k)} = 3 \cdot 2^{n-2}, \quad 2 \leqslant n \leqslant k.$$

Below we present the values of these numbers for the first few values of k and n.

k	Name	First nonzero terms ($n \geqslant 1$)
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...
9	Nonanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ...
10	Decanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ...

Table 1. First nonzero k-Fibonacci numbers

k	Name	First nonzero terms ($n \geqslant 0$)
2	Lucas	2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364,...
3	3-Lucas	2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, ...
4	4-Lucas	2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, ...
5	5-Lucas	2, 1, 3, 6, 12, 24, 46, 91, 179, 352, 692, 1360, 2674, 5257, 10335, ...
6	6-Lucas	2, 1, 3, 6, 12, 24, 48, 94, 187, 371, 736, 1460, 2896, 5744, 11394, ...
7	7-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 190, 379, 755, 1504, 2996, 5968, 11888, ...
8	8-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 382, 763, 1523, 3040, 6068, 12112, ...
9	9-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 766, 1531, 3059, 6112, 12212, ...
10	10-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1534, 3067, 6131, 12256, ...

Table 2. First nonzero k-Lucas numbers

Several authors have worked on problems involving generalized Fibonacci sequences. For instance, Luca in [11] and Marques in [12] proved that 55 and 44 are the largest repdigits in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, Marques conjectured that there are no repdigits with at least two digits belonging to $F^{(k)}$ for $k > 3$. This conjecture was confirmed in [4]. In addition, the Diophantine equation $F_n^{(k)} = 2^m$ was studied in [3]. Similar equations have been considered for $L^{(k)}$ (see, for example, [1] and [5]).

When $k = 2$, Finkelstein found that the only Fibonacci and Lucas numbers of the form $y^2 + 1$, $y \in \mathbb{Z}$, $y \geqslant 0$ are $F_1 = F_2 = 1$, $F_3 = 2$, $F_5 = 5$, $L_0 = 2$ and $L_1 = 1$ (see [8], [9]). In 2006, Bugeaud et al. generalized the problem discussed above and proved that the only nonnegative integer solutions (n, y, m) of equations $F_n \pm 1 = y^m$ with $m \geqslant 2$ are

$$\begin{aligned} &F_0 + 1 = 0 + 1 = 1, && F_1 - 1 = F_2 - 1 = 1 - 1 = 0,\\ &F_4 + 1 = 3 + 1 = 2^2, && F_3 - 1 = 2 - 1 = 1,\\ &F_6 + 1 = 8 + 1 = 3^2, && F_5 - 1 = 5 - 1 = 2^2. \end{aligned}$$

As a consequence of the above, the only nonnegative integer solutions (n, m) of equation

$$F_n = 2^m + 1 \tag{3}$$

are $(n, m) \in \{(3, 0), (4, 1), (5, 2)\}$.

In the present paper we aim to generalize the above equation (3) for generalized Fibonacci sequences, i.e. we consider the more general Diophantine equations

$$F_n^{(k)} = 2^m + 1, \tag{4}$$

$$L_n^{(k)} = 2^m + 1 \tag{5}$$

in nonnegative integers n, k, m with $k \geqslant 2$. As a particular case of the above equations (4) and (5), we determine all k-Fibonacci and k-Lucas numbers which are Fermat numbers. Recall that a *Fermat number* is a number of the form $\mathcal{F}_m = 2^{2^m} + 1$, where m is a nonnegative integer. The first six Fermat numbers are

$$\mathcal{F}_0 = 3,\ \mathcal{F}_1 = 5,\ \mathcal{F}_2 = 17,\ \mathcal{F}_3 = 257,\ \mathcal{F}_4 = 65537 \text{ and } \mathcal{F}_5 = 4294967297.$$

It is important to mention that equation (3) can also be solved by using the well known factorization $F_n - 1 = F_{(n-\delta)/2} L_{(n+\delta)/2}$, where $\delta \in \{-2, 1, 2, -1\}$ depends on the class of n modulo 4. In this case, the resulting equation can be easily solved by using prime factorization. However, similar divisibility properties for $F^{(k)}$ when $k \geqslant 3$ are not known and therefore it is necessary to attack the problem differently.

We begin our analysis of equations (4) and (5) by noting that $F_3^{(k)} = 2$, $L_0^{(k)} = 2$ and $L_2^{(k)} = 3$ are valid for all $k \geqslant 2$; thus, the triples

$$(n, k, m) = (3, k, 0) \quad \text{are the solutions of (4) for all } k \geqslant 2,$$

and

$$(n, k, m) \in \{(0, k, 0), (2, k, 1)\} \quad \text{are the solutions of (5) for all } k \geqslant 2.$$

The above solutions will be called *trivial solutions.* In this paper, we prove the following theorems.

Theorem 1. *The only nontrivial solutions of the Diophantine equation* (4) *in nonnegative integers n, k, m with $k \geqslant 2$ are $(n, k, m) \in \{(4, 2, 1), (5, 2, 2)\}$.*

Theorem 2. *The Diophantine equation* (5) *has no nontrivial solutions in nonnegative integers n, k, m with $k \geqslant 2$.*

As an immediate consequence of Theorem 1 and Theorem 2 we have the following corollaries.

Corollary 1. *The only Fermat numbers in the k-Fibonacci family of sequences are $F_4 = 3$ and $F_5 = 5$.*

Corollary 2. *The only Fermat number in the k-Lucas family of sequences is $L_2^{(k)} = 3$, which holds for all $k \geqslant 2$.*

To prove our main results we use lower bounds for linear forms in logarithms (Baker's theory) to bound n and m polynomially in terms of k. When k is small, we use the theory of continued fractions by means of a variation of a result of Dujella and Pethő to lower such bounds to cases that allow us to treat our problem computationally. For large values of k, Bravo, Gómez and Luca in [2], [3], [5] developed some ideas for dealing with Diophantine equations involving k-Fibonacci and k-Lucas numbers.

Before proceeding further, it may be mentioned that the characteristic polynomial of $G^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \ldots - x - 1,$$

is irreducible in $\mathbb{Q}[x]$ and has just one zero root outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single zero. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number of degree k. Moreover, it is also known that

$\alpha(k)$ is located between $2(1-2^{-k})$ and 2, see [10], Lemma 2.3 or [15], Lemma 3.6. To simplify the notation, we shall omit the dependence on k of α.

We now consider the function $f_k(x) = (x-1)/(2+(k+1)(x-2))$ for an integer $k \geqslant 2$ and $x > 2(1-2^{-k})$. It is easy to see that the inequalities

$$(6) \qquad \frac{1}{2} < f_k(\alpha) < \frac{3}{4} \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \leqslant i \leqslant k$$

hold, where $\alpha := \alpha^{(1)}, \dots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, by computing norms from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$, for example, we see that the number $f_k(\alpha)$ is not an algebraic integer. Proofs for this fact and for (6) can be found in [2].

With the above notation, Dresden and Du showed in [6] that

$$(7) \qquad F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \quad \text{and} \quad |F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < \frac{1}{2}$$

hold for all $n \geqslant 1$ and $k \geqslant 2$.

In addition to this, Bravo and Luca proved in [4] that

$$(8) \qquad \alpha^{n-2} \leqslant F_n^{(k)} \leqslant \alpha^{n-1} \quad \text{holds for all } n \geqslant 1 \text{ and } k \geqslant 2.$$

The observations in expressions (7) and (8) lead us to call α the *dominant zero* of $G^{(k)}$.

Note that sequences $G^{(k)}$ and $F^{(k)}$ have the same recurrence relation. This makes us think that there is some relationship between them. In this sense, Bravo and Luca in [5] proved that $G_n^{(k)} = aF_{n+1}^{(k)} + (b-a)F_n^{(k)}$. In particular,

$$(9) \qquad L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)}.$$

The above result supports the following lemma (see the proof in [5]).

Lemma 1. *Let $k \geqslant 2$ be an integer. Then*

(a) $\alpha^{n-1} \leqslant L_n^{(k)} \leqslant 2\alpha^n$ *for all* $n \geqslant 1$,

(b) $L^{(k)}$ *satisfies the following "Binet-like" formula*

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i)\alpha_i^{n-1},$$

where $\alpha = \alpha_1, \dots, \alpha_n$ *are the zeros of* $\Psi_k(x) = x^k - x^{k-1} - \ldots - x - 1$,

(c) $|L_n^{(k)} - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{3}{2}$ *for all* $n \geqslant 2 - k$,

(d) *If* $2 \leqslant n \leqslant k$, *then* $L_n^{(k)} = 3 \cdot 2^{n-2}$.

2. Linear forms in logarithms

In order to prove our main result, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers, and such a bound, which plays an important role in this paper, was given by Matveev (see [13]). We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\right)$$

is called the *logarithmic height* of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$.

The following properties of the logarithmic height, which will be used in next sections without special reference, are also known:

▷ $h(\eta \pm \gamma) \leqslant h(\eta) + h(\gamma) + \log 2$.
▷ $h(\eta\gamma^{\pm 1}) \leqslant h(\eta) + h(\gamma)$.
▷ $h(\eta^s) = |s|h(\eta)$.

Matveev in [13] proved the following deep theorem.

Theorem 3 (Matveev's theorem)**.** *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1 \quad \text{and} \quad B \geqslant \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geqslant \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t).$$

To conclude this section, we give estimates for the logarithmic heights of some algebraic numbers. Let $\mathbb{K} = \mathbb{Q}(\alpha)$. Knowing that $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$ and that $|f_k(\alpha^{(i)})| \leqslant 1$ for all $i = 1, \ldots, k$ and $k \geqslant 2$, we obtain that $h(\alpha) = (\log \alpha)/k$

and $h(f_k(\alpha)) = (\log a_0)/k$, where a_0 is the leading coefficient of minimal primitive polynomial over the integers of $f_k(\alpha)$. Put

$$g_k(x) = \prod_{i=1}^{k}(x - f_k(\alpha^{(i)})) \in \mathbb{Q}[x] \quad \text{and} \quad \mathcal{N} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(2 + (k+1)(\alpha - 2)) \in \mathbb{Z}.$$

We conclude that $\mathcal{N} g_k(x) \in \mathbb{Z}[x]$ vanishes at $f_k(\alpha)$. Thus, a_0 divides $|\mathcal{N}|$. But for $k \geqslant 2$,

$$\begin{aligned}
|\mathcal{N}| = \left|\prod_{i=1}^{k}(2 + (k+1)(\alpha^{(i)} - 2))\right| &= (k+1)^k \left|\prod_{i=1}^{k}\Big(2 - \frac{2}{k+1} - \alpha^{(i)}\Big)\right| \\
&= (k+1)^k \left|\Psi_k\Big(2 - \frac{2}{k+1}\Big)\right| \\
&= \frac{2^{k+1}k^k - (k+1)^{k+1}}{k-1} < 2^k k^k.
\end{aligned}$$

Hence, we will use the following inequalities:

$$h(\alpha) < \frac{7}{10k} \quad \text{and} \quad h(f_k(\alpha)) < 2\log k, \quad k \geqslant 2. \tag{10}$$

Additionally, Bravo and Luca in [5] proved that $h(2\alpha - 1) < \log 3$ for all $k \geqslant 2$. So,

$$h((2\alpha - 1)f_k(\alpha)) < \log 3 + 2\log k < 4\log k, \quad k \geqslant 2. \tag{11}$$

3. Proof of Theorem 1

Assume first that we have a nontrivial solution (n, k, m) of equation (4). If $n = 1$, then $1 = 2^m + 1$, which is impossible because $m \geqslant 0$. Now, if $2 \leqslant n \leqslant k+1$, then we obtain from (1) that $2^{n-2} = 2^m + 1$. From this, we get only the trivial solutions $(n, k, m) = (3, k, 0)$ for all $k \geqslant 2$. So, from now on, we assume that $n \geqslant k+2$ and therefore $n \geqslant 4$. In fact, after a quick inspection of the first table presented above, we can assume that $n \geqslant 6$ since the only solutions for the values $n = 4, 5$ are given by $F_4 = 3$ and $F_5 = 5$. By inequalities (2) and (4), we have

$$2^m < 2^m + 1 = F_n^{(k)} < 2^{n-2}$$

obtaining

$$m \leqslant n - 3. \tag{12}$$

We shall have some use for it later. Using now (4) once again and (7) we get that

$$|f_k(\alpha)\alpha^{n-1} - 2^m| < \frac{1}{2} + 1 = \frac{3}{2},$$

giving

$$\left|1 - \frac{2^m}{\alpha^{n-1}} \frac{1}{f_k(\alpha)}\right| < \frac{3}{\alpha^{n-1}}, \tag{13}$$

where we used the fact that $f_k(\alpha) > \frac{1}{2}$ as has already been mentioned (see (6)). In order to use the result of Matveev theorem 3, we take $t := 3$ and

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := f_k(\alpha).$$

We also take $b_1 := m$, $b_2 := -(n-1)$ and $b_3 := -1$. We begin by noticing that the three numbers γ_1, γ_2, γ_3 are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. The left-hand side of (13) is not zero. Indeed, if this were zero, we would then get that $f_k(\alpha) = 2^m \cdot \alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, contradicting something previously mentioned. Note that α^{-1} is an algebraic integer, because it is a root of the monic polynomial $x^k\Psi_k(1/x) \in \mathbb{Z}[x]$, and recall that the set of algebraic integers form a ring.

Since $h(\gamma_1) = \log 2$, it follows that we can take $A_1 := k \log 2$. Further, in view of (10), we can take $A_2 = \frac{7}{10}$ and $A_3 := 2k \log k$. Finally, by recalling that $m \leqslant n-3$, we can take $B := n-1$. Then Matveev's theorem together with a straightforward calculation gives

$$|1 - 2^m\alpha^{-(n-1)}(f_k(\alpha))^{-1}| > \exp(-8.34 \times 10^{11} k^4 \log^2 k \log(n-1)), \tag{14}$$

where we used that $1 + \log k \leqslant 3 \log k$ for all $k \geqslant 2$ and $1 + \log(n-1) \leqslant 2\log(n-1)$ for all $n \geqslant 4$. Comparing (13) and (14), taking logarithms and then performing the respective calculations, we get that

$$\frac{n-1}{\log(n-1)} < 1.76 \times 10^{12} k^4 \log^2 k. \tag{15}$$

We next use the fact that the inequality $x/\log x < A$ implies $x < 2A \log A$ whenever $A \geqslant 3$ in order to get an upper bound for n depending on k. Indeed, taking $x := n-1$ and $A := 1.76 \times 10^{12} k^4 \log^2 k$, and performing the respective calculations, inequality (15) yields $n < 1.7 \times 10^{14} k^4 \log^3 k$. We record what we have proved so far as a lemma.

Lemma 2. *If (n, m, k) is a nontrivial solution in positive integers of equation* (4), *then $n \geqslant k+2$ and*

$$m + 3 \leqslant n < 1.7 \times 10^{14} k^4 \log^3 k.$$

3.1. The case $k > 170$. In this case the following inequalities hold:

$$m+3 \leqslant n < 1.7 \times 10^{14} k^4 \log^3 k < 2^{k/2}.$$

We recall the following result due to Bravo, Gómez and Luca (see [2]).

Lemma 3. *If $r < 2^k$, then the following estimate holds:*

$$F_r^{(k)} = 2^{r-2}\Big(1 + \frac{k-r}{2^{k+1}} + \zeta(k,r)\Big),$$

where $\zeta = \zeta(k,r)$ is a real number such that $|\zeta| < 4r^2/2^{2k+2}$.

So, from (4) and Lemma 3 applied to $r := n < 2^{k/2}$, we get

$$|2^{n-2} - 2^m| = \Big|(F_n^{(k)} - 2^m) - 2^{n-2}\Big(\frac{k-n}{2^{k+1}} + \zeta\Big)\Big| < 1 + 2^{n-2}\Big(\frac{n-k}{2^{k+1}} + \frac{4n^2}{2^{2k+2}}\Big).$$

Factoring 2^{n-2} on the right-hand side of the above inequality and taking into account that $1/2^{n-2} < 1/2^{k/2}$ (because $n \geqslant k+2$ by Lemma 2), $(n-k)/2^{k+1} < 1/2^{k/2}$ and $4n^2/2^{2k+2} < 1/2^{k/2}$, which are all valid for $k > 170$, we conclude that

$$|1 - 2^{m-n+2}| < \frac{3}{2^{k/2}}. \tag{16}$$

By recalling that $m \leqslant n-3$ (see (12)), we have that $m-n+2 \leqslant -1$. So, from (16) and the previous result we have

$$\frac{1}{2} \leqslant 1 - 2^{m-n+2} < \frac{3}{2^{k/2}}$$

giving $2^{k/2} < 6$, which contradicts the fact that $k > 170$. Consequently, equation (4) has no solutions for $k > 170$.

3.2. The case $2 \leqslant k \leqslant 170$. For these values of k, we will use the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [7], and will be the key tool used in this paper to reduce the upper bounds on the variables of the Diophantine equation (4).

Lemma 4. *Let A, B, γ, μ be positive real numbers and M a positive integer. Suppose that p/q is a convergent of the continued fraction expansion of the irrational γ such that $q > 6M$. Put $\varepsilon = \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance* *from the nearest integer. If* $\varepsilon > 0$*, then there is no positive integer solution* (u, v, w) *to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

subject to the restrictions that

$$u \leqslant M \quad \textit{and} \quad w \geqslant \frac{\log A + \log q - \log \varepsilon}{\log B}.$$

In order to apply this result, we let $z := m \log 2 - (n-1) \log \alpha - \log f_k(\alpha)$ and we observe that (13) can be rewritten as

$$|\mathrm{e}^z - 1| < \frac{3}{\alpha^{n-1}}. \tag{17}$$

Note that $z \neq 0$; thus, we distinguish the following cases. If $z > 0$, then $\mathrm{e}^z - 1 > 0$, so from (17) we obtain

$$0 < z < \frac{3}{\alpha^{n-1}}.$$

Suppose now that $z < 0$. Since the dominant zeros of $F^{(k)}$ are strictly increasing as k increases, we deduce that $3/\alpha^{n-1} \leqslant 3/(\alpha(2))^{n-1} < \frac{1}{2}$ for all $n \geqslant 5$. Here, $\alpha(2)$ denotes the golden section as mentioned before. Then from (17) we have that $|\mathrm{e}^z - 1| < \frac{1}{2}$ and therefore $\mathrm{e}^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \leqslant \mathrm{e}^{|z|} - 1 = \mathrm{e}^{|z|}|\mathrm{e}^z - 1| < \frac{6}{\alpha^{n-1}}.$$

In any case, we have that the inequality

$$0 < |z| < \frac{6}{\alpha^{n-1}}$$

holds for all $k \geqslant 2$ and $n \geqslant 5$. Replacing z in the above inequality by its formula and dividing it across by $\log \alpha$, we conclude that

$$0 < \left| m \frac{\log 2}{\log \alpha} - n + \left(1 - \frac{\log f_k(\alpha)}{\log \alpha}\right) \right| < \frac{13}{\alpha^{(n-1)}}, \tag{18}$$

where we have used the fact that $1/\log \alpha < 2.1$. We put

$$\widehat{\gamma} := \widehat{\gamma}(k) = \frac{\log 2}{\log \alpha}, \quad \widehat{\mu} := \widehat{\mu}(k) = 1 - \frac{\log f_k(\alpha)}{\log \alpha}, \quad A := 13 \quad \text{and} \quad B := \alpha.$$

We also put $M_k := \lfloor 1.7 \times 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on m by Lemma 2. The fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$, ensures that $\widehat{\gamma}$ is an irrational number. Even more, $\widehat{\gamma}$ is transcendental by the Gelfond-Schneider Theorem. Then, the above inequality (18) yields

$$0 < |m\widehat{\gamma} - n + \widehat{\mu}| < AB^{-(n-1)}. \tag{19}$$

It then follows from Lemma 4, applied to inequality (19), that

$$n - 1 < \frac{\log A + \log q - \log \varepsilon}{\log B},$$

where $q = q(k) > 6M_k$ is a denominator of a convergent of the continued fraction of $\widehat{\gamma}$ such that $\varepsilon = \varepsilon(k) = \|\widehat{\mu}q\| - M_k\|\widehat{\gamma}q\| > 0$. A computer search with *Mathematica* revealed that if $k \in [2, 170]$, then the maximum value of $(\log A + \log q - \log \varepsilon)/\log B$ is < 360. Hence, we deduce that the possible solutions (n, k, m) of equation (4) for which k is in the range $[2, 170]$ all have $n < 360$.

Finally, a brute force search with *Mathematica* in the range

$$2 \leqslant k \leqslant 170 \quad \text{and} \quad k + 2 \leqslant n < 360$$

allows us to conclude that the only nontrivial solutions of (4) are

$$(n, k, m) \in \{(4, 2, 1), (5, 2, 2)\}.$$

This completes the analysis in the case $k \in [2, 170]$ and therefore the proof of Theorem 1. $\square$

4. Proof of Theorem 2

Assume first that we have a nontrivial solution (n, k, m) of equation (5). Thus, $n \neq 0$ and $n \neq 2$. Note that if $3 \leqslant n \leqslant k$, then by (5) and Lemma 1 (d) we get $3 \cdot 2^{n-2} = 2^m + 1$, which is not possible. Hence, from now on, we can assume that $m \geqslant 2$ and $n \geqslant k + 1$.

On the other hand, by Lemma 1 (a) and (5) we get

$$2^m < 2^m + 1 = L_n^{(k)} \leqslant 2\alpha^n < 2^{n+1}$$

implying that $m \leqslant n$. However, using (2) and (9), and taking into account that $n \geqslant k + 1$, we have that

$$F_n^{(k)} + 2^m + 1 = 2F_{n+1}^{(k)} < 2^n.$$

From the expression above we see that $m = n$ cannot be. Hence $m < n$. Using now (5) and Lemma 1 (c), we get that

$$|2^m - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{5}{2}. \tag{20}$$

Dividing both sides of the above inequality by the second term of the left-hand side (which is positive), we obtain

$$\Big|\frac{2^m\alpha^{-(n-1)}}{(2\alpha-1)f_k(\alpha)} - 1\Big| < \frac{3}{\alpha^{n-1}}, \tag{21}$$

where we used the facts $1/f_k(\alpha) < 2$ and $1/(2\alpha - 1) < \frac{1}{2}$. The left-hand size of (21) is not zero. Indeed, if this were zero, we would then get that

$$2^m = (2\alpha - 1)f_k(\alpha)\alpha^{n-1}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geqslant 2$ we have

$$4 \leqslant 2^m = |(2\alpha_i - 1)f_k(\alpha_i)\alpha_i^{n-1}| < 3,$$

which is a contradiction.

In order to use Theorem 3, we take $t := 3$,

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := (2\alpha - 1)f_k(\alpha)$$

and

$$b_1 := m, \quad b_2 := -(n-1), \quad b_3 := -1.$$

For this choice we have $D = k$ (because $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$) and $B = n - 1$. Thus, we can take $A_1 := k \log 2$, $A_2 := \frac{7}{10}$ (see (10)) and $A_3 := 4k \log k$ (see (11)).

By Matveev's theorem and proceeding as in the proof of Lemma 2 we obtain the following lemma.

Lemma 5. *If (n, m, k) is a nontrivial solution in positive integers of equation (5), then $n \geqslant k + 1$ and*

$$m < n < 1.68 \times 10^{14} k^4 \log^3 k.$$

4.1. The case $k > 170$. For these values of k, from Lemma 5 we deduce that $n < 2^{k/2}$. Bravo and Luca in [5] established that if $r > 1$ is an integer satisfying $r - 1 < 2^{k/2}$, then

$$(2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2} + 3 \cdot 2^{r-1}\eta + \frac{\delta}{2} + \eta\delta, \tag{22}$$

where δ and η are real numbers such that $|\delta| < 2^{r+2}/2^{k/2}$ and $|\eta| < 2k/2^k$. Consequently, from (22) (with $r := n$) and (20) we obtain

$$\begin{aligned} |3 \cdot 2^{n-2} - 2^m| &\leqslant |(2\alpha - 1)f_k(\alpha)\alpha^{n-1} - 2^m| + 3|\eta|2^{n-1} + \frac{|\delta|}{2} + |\eta\delta| \\ &< 3 \cdot 2^{n-2}\Big(\frac{5}{3 \cdot 2^{n-1}} + \frac{4k}{2^k} + \frac{8}{3 \cdot 2^{k/2}} + \frac{32k}{3 \cdot 2^{3k/2}}\Big). \end{aligned}$$

Dividing the above inequality across by 2^{n-2} we conclude that

$$|3 - 2^{m-n+2}| < 3\Big(\frac{1}{2^{k/2}} + \frac{4k}{2^k} + \frac{8}{3 \cdot 2^{k/2}} + \frac{32k}{3 \cdot 2^{3k/2}}\Big) < \frac{18}{2^{k/2}}. \tag{23}$$

In the last inequality we have used that $5/(3 \cdot 2^{n-1}) < 1/2^{k/2}$ (because $n \geqslant k + 1$), $4k/2^k < 1/2^{k/2}$, $8/(3 \cdot 2^{k/2}) < 3/2^{k/2}$ and $32k/(3 \cdot 2^{3k/2}) < 1/2^{k/2}$, which are all valid for $k > 170$. By recalling that $m < n$, we have $m - n + 2 \leqslant 1$ and so, from (23), we get

$$1 \leqslant 3 - 2^{m-n+2} < \frac{18}{2^{k/2}}.$$

That is, $2^{k/2} < 18$ which is impossible since $k > 170$. Then (5) has no solutions for $k > 170$.

4.2. The case $2 \leqslant k \leqslant 170$. If we take $z = m \log 2 - (n - 1) \log \alpha - \log \mu$, where $\mu = (2\alpha - 1)f_k(\alpha)$, and proceeding as in Section 3.2, we deduce that the possible solutions (n, k, m) of equation (5) for which k is in the range $[2, 170]$ all have $n < 340$.

Finally, we conclude by a brute force search in *Mathematica* that equation (5) has no solutions in the range

$$2 \leqslant k \leqslant 170 \quad \text{and} \quad k + 1 \leqslant n < 340.$$

This proves Theorem 2. □

Finally, Corollary 1 and Corollary 2 are immediate consequences of Theorem 1 and Theorem 2, respectively.

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References

[1] *E. F. Bravo, J. J. Bravo, F. Luca*: Coincidences in generalized Lucas sequences. Fibonacci Q. *52* (2014), 296–306. zbl MR

[2] *J. J. Bravo, C. A. Gómez , F. Luca*: Powers of two as sums of two k-Fibonacci numbers. Miskolc Math. Notes *17* (2016), 85–100. zbl MR doi

[3] *J. J. Bravo, F. Luca*: Powers of two in generalized Fibonacci sequences. Rev. Colomb. Mat. *46* (2012), 67–79. zbl MR

[4] *J. J. Bravo, F. Luca*: On a conjecture about repdigits in k-generalized Fibonacci sequences. Publ. Math. *82* (2013), 623–639. zbl MR doi

[5] *J. J. Bravo, F. Luca*: Repdigits in k-Lucas sequences. Proc. Indian Acad. Sci., Math. Sci. *124* (2014), 141–154. zbl MR doi

[6] *G. P. Dresden, Z. Du*: A simplified Binet formula for k-generalized Fibonacci numbers. J. Integer Seq. *17* (2014), Article No. 14.4.7, 9 pages. zbl MR

[7] *A. Dujella, A. Pethő*: A generalization of a theorem of Baker and Davenport. Quart. J. Math., Oxf. II. Ser. *49* (1998), 291–306. zbl MR doi

[8] *R. Finkelstein*: On Fibonacci numbers which are more than a square. J. Reine Angew. Math. *262/263* (1973), 171–178. zbl MR doi

[9] *R. Finkelstein*: On Lucas numbers which are one more than a square. Fibonacci Q. *136* (1975), 340–342. zbl MR

[10] *L. K. Hua, Y. Wang*: Applications of Number Theory to Numerical Analysis. Springer, Berlin; Science Press, Beijing, 1981. zbl MR

[11] *F. Luca*: Fibonacci and Lucas numbers with only one distinct digit. Port. Math. *57* (2000), 243–254. zbl MR

[12] *D. Marques*: On k-generalized Fibonacci numbers with only one distinct digit. Util. Math. *98* (2015), 23–31. zbl MR

[13] *E. M. Matveev*: An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II. Izv. Math. *64* (2000), 1217–1269; translated from Izv. Ross. Akad. Nauk Ser. Mat. *64* (2000), 125–180. zbl MR doi

[14] *T. D. Noe, J. Vos Post*: Primes in Fibonacci n-step and Lucas n-step sequences. J. Integer Seq. *8* (2005), Article No. 05.4.4, 12 pages. zbl MR

[15] *D. A. Wolfram*: Solving generalized Fibonacci recurrences. Fibonacci Q. *36* (1998), 129–145. zbl MR

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