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A SEMILATTICE OF VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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Abstract. Completely regular semigroups are unions of their (maximal) subgroups with the unary operation within their maximal subgroups. As such they form a variety whose lattice of subvarieties is denoted by $\mathcal{L}(\mathcal{CR})$.

We construct a 60-element $\cap$-subsemilattice and a 38-element sublattice of $\mathcal{L}(\mathcal{CR})$. The bulk of the paper consists in establishing the necessary joins for which it uses Polák's theorem.

Keywords: completely regular semigroup; lattice; variety; $\cap$-subsemilattice

MSC 2010: 20M07

1. Introduction and summary

Completely regular semigroups endowed with the operation of inversion within their maximal subgroups form a variety. It is denoted by $\mathcal{CR}$ and its lattice of subvarieties by $\mathcal{L}(\mathcal{CR})$.

In the study of $\mathcal{L}(\mathcal{CR})$, there emerged a great number of subvarieties creating a copious choice of further varieties. Author's article [6] was an attempt to collect them, classify them according to their bases of identities, and present them as a subset of $\mathcal{L}(\mathcal{CR})$ ordered by inclusion. This appeared in 1982. Since that time, the collection of known varieties grew considerably. The diagram in that paper showed large lacunae. With the present article, we will close some of them, but also create new ones.

Gradually, progress was made in identifying new varieties and determining the relationship between the old and new ones. Besides the enlarged and more complete diagram, we identify a large sublattice of $\mathcal{L}(\mathcal{CR})$. In fact, the bulk of this paper consists in proving that the stated set of varieties indeed represents a sublattice of

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$\mathcal{L}(\mathcal{CR})$. All this is illustrated by a diagram depicting an $\cap$-subsemilattice and a sublattice of $\mathcal{L}(\mathcal{CR})$.

By sections, the paper runs as follows. Section 2 contains most of the terminology and notation used in the paper. This is supplemented in Section 3 by citations from the literature and lemmas. Section 4 comprises proof of the single theorem.

2. Terminology and notation

For the terminology, notation and results we depend heavily on text [13], and will not repeat here what can be found in that book. In particular, varieties are denoted by their acronyms as in [13]. In them, in a meet $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \ldots \cap \mathcal{V}_n$, we often omit writing the symbol $\cap$ of meet, and write them by juxtaposition. In order to facilitate the reading and avoiding ambiguity, we place brackets in strategic positions. For example, $(\mathcal{BG})CH\mathcal{A}$ stands for the variety of cryptic (bands of groups) completely regular semigroups all of whose (maximal) subgroups of its core (subsemigroup generated by idempotents) are abelian.

All relations on $\mathcal{L}(\mathcal{CR})$ we allude to (possibly implicitly) have all their classes intervals. Hence, for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and relation A we write the A-class of $\mathcal{V}$ as $\mathcal{V}A = [\mathcal{V}_A, \mathcal{V}^A]$. Conforming to the custom, and for typographical reasons, we write $\mathcal{V}^A$ also as $A\mathcal{V}$.

We generally omit stating the dual statements except in very few cases for the sake of clarity.

3. Citations and lemmas

First we state the results from the literature needed in the paper but not contained in book [13]. In some cases, the needed results are not available in the literature in the form we need them. We will use them without reference for the sake of uninterrupted train of thought.

For the kernel and trace relations we recommend paper [11], and for local and core relations, paper [12]. General results we need follow.

Fact 3.1.

(i) *If $\mathcal{V} \in \mathcal{L}(L\mathcal{O})$ then*

$$\mathcal{V}_K = \begin{cases} \mathcal{V} \cap \mathcal{G} & \text{if } \mathcal{V} \subseteq \mathcal{O}, \\ \mathcal{V} \cap \mathcal{CS} & \text{otherwise.} \end{cases}$$

(ii) *The intervals $[\mathcal{T}, \mathcal{B}]$, $[\mathcal{G}, \mathcal{O}]$, $[\mathcal{CS}, L\mathcal{O}]$ are K-classes.*

P r o o f. (i) See [15], Theorem 2 and [11], Theorem 5.8.

(ii) This can be deduced from [15], Theorem 2. □

Fact 3.2.

(i) *The mappings* $\mathcal{V} \to \mathcal{V}^P$, $P \in \{T_l, T, T_r\}$ *are* $\cap$*-endomorphisms of* $\mathcal{L}(\mathcal{CR})$.

(ii) *The mappings* $\mathcal{V} \to \mathcal{V}^K$, $\mathcal{V} \to \mathcal{V}_{T_l}$, $\mathcal{V} \to \mathcal{V}_{T_r}$ *are endomorphisms of* $\mathcal{L}(\mathcal{CR})$.

P r o o f. (i) For T, see [11], Proposition 7.10; for T_l, see [11], remark after Proposition 8.4; T_r follows by duality.

(ii) For K, this follows from [15], Theorem 1(3) via [4], Theorem 14; for T_l, see [11], Theorem 8.2 and Proposition 8.4; T_r follows by duality. □

Next we consider some relations among operators.

Fact 3.3. *Let* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$.

(i) $\mathcal{V}^K = (\mathcal{V}^K)_{T_l} = (\mathcal{V}^K)_{T_r}$.

(ii) $\mathcal{V}^T = \mathcal{V}^{T_l} \cap \mathcal{V}^{T_r}$.

(iii) $\mathcal{B}^{T_l} = \mathcal{LBG}$, $\mathcal{B}^{T_r} = \mathcal{R}t\mathcal{BG}$.

P r o o f. (i) See [15], Theorem 2.4(4).

(ii) This forms part of the varietal version of [13], Corollary VII.4.2.

(iii) For the first formula, see [8], Lemma 5.3; the second is its dual. □

We will also need the following lemmas.

Lemma 3.4. *We have* $\mathcal{O}^T = \mathcal{BG}^C$.

P r o o f. Using [10], Lemma 5.5, we obtain $\mathcal{O}^T = \mathcal{B}^{CT} = \mathcal{B}^{TC} = \mathcal{BG}^C$. □

Lemma 3.5. *For* $\mathcal{V} \in \{H\mathcal{A}, CH\mathcal{A}, \mathcal{C}\}$ *we have* $\mathcal{V}^K = \mathcal{V}$.

P r o o f. The case $(H\mathcal{A})^K = H\mathcal{A}$ is the content of [9], Lemma 5.2. Using this and [10], Lemma 5.3, we obtain

$$(CH\mathcal{A})^K = (H\mathcal{A})^{CK} = (H\mathcal{A})^{KC} = (H\mathcal{A})^C = CH\mathcal{A}.$$

Next let $S \in \mathcal{C}^K$, $a \in S$ and $e, f \in E(S)$ satisfy $ef \; \mathcal{H} \; a$. Then $(ef)\tau \; \mathcal{H} \; a\tau$, so $(e\tau)(f\tau) \; \mathcal{H} \; a\tau$. By [2], Proposition 7.2(ii), we obtain $(e\tau)(f\tau)(a\tau) = (a\tau)(e\tau)(f\tau)$, whence $efa \; \tau \; aef$, which by [13], Lemma II.3.4 yields

$$xefay \in E(S) \Leftrightarrow xaefy \in E(S), \quad x, y \in S^1. \tag{3.1}$$

Let $x = (efa)^{-1}$ and $y = 1$, so $(efa)^{-1}aef \in E(S)$ by (3.1). The hypothesis that $ef \; \mathcal{H} \; a$ implies that $aef \; \mathcal{H} \; efa$ and thus $aef = efa$. Therefore $S \in \mathcal{C}$, which proves that $\mathcal{C}^K \subseteq \mathcal{C}$. The reverse inclusion is trivial. □

Lemma 3.6.

(i) $\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$, $\mathcal{V} \in [\mathcal{T}, \mathcal{O}(\mathcal{BG})]$,

(ii) $\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{CS})$, $\mathcal{V} \in [\mathcal{RB}, (L\mathcal{O})\mathcal{BG}]$,

(iii) $\mathcal{V} = (\mathcal{V} \cap \mathcal{O}) \vee (\mathcal{V} \cap \mathcal{CS})$, $\mathcal{V} \in [\mathcal{R}e\mathcal{G}, (L\mathcal{O})T\mathcal{O}]$.

P r o o f. (i) See [5], Lemma 1.

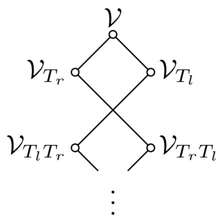
(ii) See the proof of [1], Corollary 5.7.

(iii) See [16], Theorem 4.9. □

We will use the following consequence of Polák's theorem, see [14], [15] concerning the computation of a basis of a join of varieties. To this end, we need the following construction.

Let $\mathbb{N}_3 = \{\mathcal{LNB}, \mathcal{S}, \mathcal{RNB}\}$ and by Θ denote the set of all finite sequences of alternating letters T_l and T_r. For each $\theta \in \Theta$ denote by $\bar{\theta}$ the mirror image of θ.

For any $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$ we call



the *network* of $\mathcal{V}$ and denote it by net $\mathcal{V}$. Further, we call

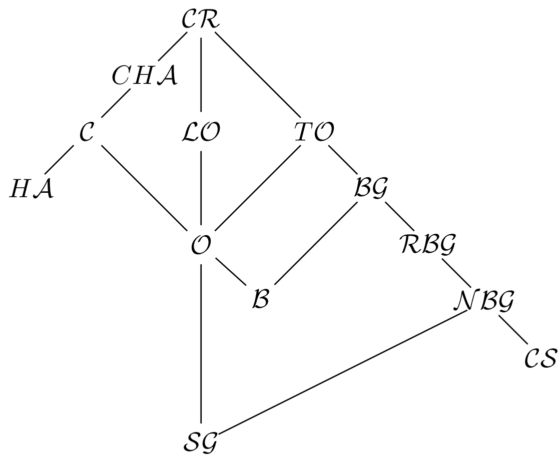
$$\mathcal{V}^K \cap \Big(\bigcap\{(\mathcal{V}_\theta)^{K\bar{\theta}} \colon \theta \in \Theta, \mathcal{V}_\theta \notin \mathbb{N}_3\}\Big) \cap \Big(\bigcap\{(\mathcal{V}_\theta)^{\bar{\theta}} \colon \theta \in \Theta, \mathcal{V}_\theta \in \mathbb{N}_3\}\Big)$$

the *evaluation* of net $\mathcal{V}$, and denote it by evalnet $\mathcal{V}$.

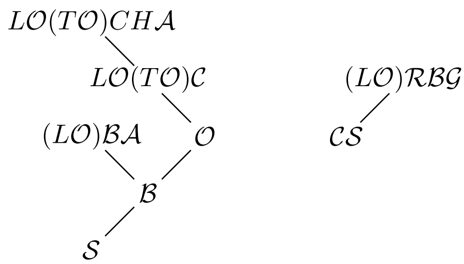
Fact 3.7. *Let $\{\mathcal{V}_\alpha\}_{\alpha \in A}$ be a family in $[\mathcal{S}, \mathcal{CR}]$. We define the join $\bigvee_{\alpha \in A} \operatorname{net} \mathcal{V}_\alpha$ componentwise. Then*

$$\bigvee_{\alpha \in A} \mathcal{V}_\alpha = \operatorname{eval}\Big(\bigvee_{\alpha \in A} \operatorname{net} \mathcal{V}_\alpha\Big).$$

We will need the following subsets of $\mathcal{CR}$.



Sketch 1.



Sketch 2.

Lemma 4.1. *Both Sketch 1 and Sketch 2 represent inclusion ordered subsets of $\mathcal{CR}$.*

P r o o f. These inclusions are obvious except $\mathcal{C} \subseteq CH\mathcal{A}$, which follows from [13], Theorem II.6.5, and $\mathcal{BG} \subseteq T\mathcal{O}$, which follows from Lemma 3.4. $\square$

We are finally ready for the single theorem of the paper. It contains a complete $\cap$-subsemilattice and a complete sublattice of $\mathcal{L}(\mathcal{CR})$, both finite. For them we provide the number of elements, the relevant lattice properties and the set of generators.

Theorem 4.2.

(i) *Diagram 1 represents a 60-element $\cap$-subsemilattice Γ of $\mathcal{L}(\mathcal{CR})$ generated by the set of varieties in Sketch 1.*

(ii) *The set*

$$\Delta = [\mathcal{T}, (L\mathcal{O})T\mathcal{O}] \setminus \{\mathcal{O}(H\mathcal{A}), L\mathcal{O}(T\mathcal{O})H\mathcal{A}\}$$

in Diagram 1 represents a 38-element sublattice of $\mathcal{L}(\mathcal{CR})$ generated by the varieties in Sketch 2.

(iii) *For $\mathcal{V} \in \{\mathcal{BG}, (L\mathcal{O})\mathcal{BG}, \mathcal{RBG}, (L\mathcal{O})\mathcal{RBG}, \mathcal{NBG}, \mathcal{CS}\}$ we have $\mathcal{V}(H\mathcal{A}) \vee \mathcal{G} = \mathcal{VC}$.*

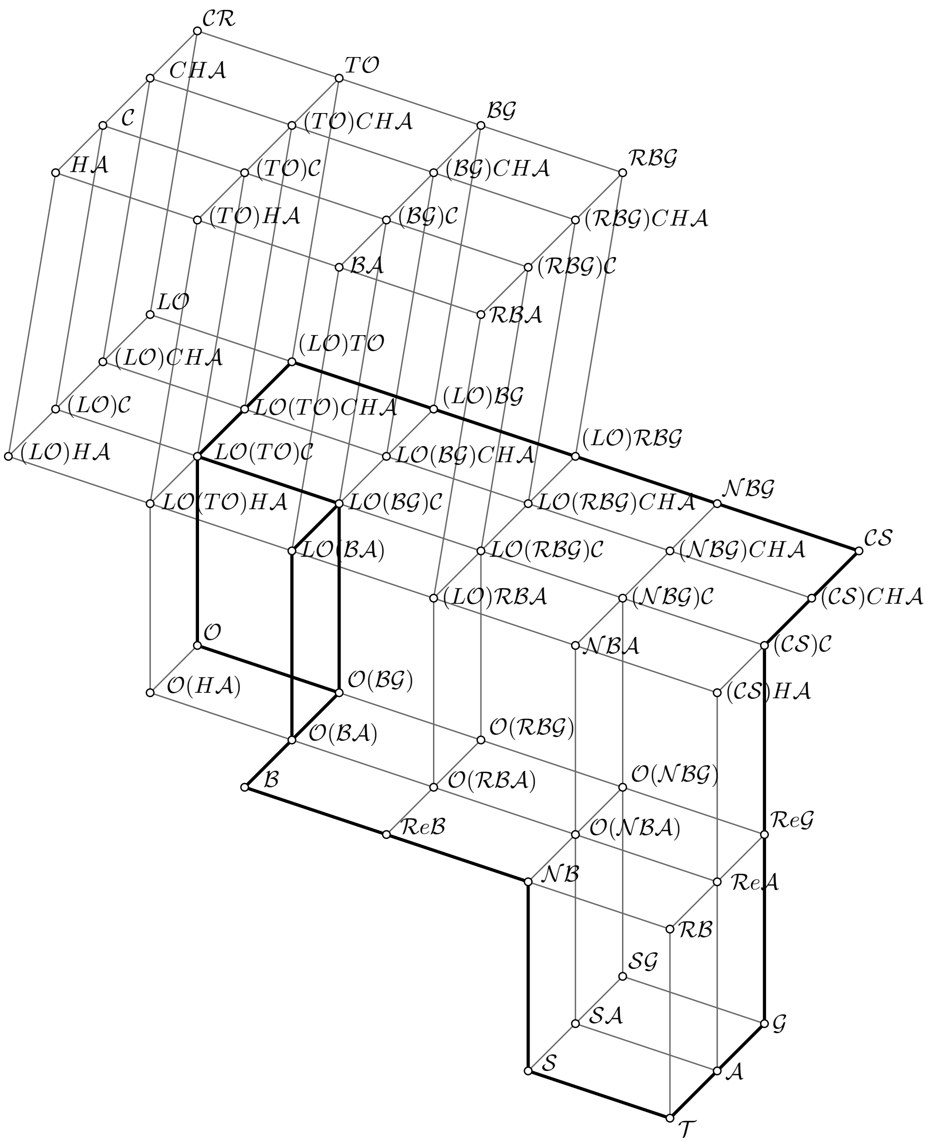


Diagram 1. Subsemilattice Γ and sublattice Δ (within boldface lines) of $\mathcal{L}(\mathcal{CR})$.

P r o o f. (i) One can simply count the number of elements. Alternatively, one may do this by "blocks", which is more instructive, for it induces a useful decomposition of the set into blocks which will be essential in most of the coming discussion. In fact,

$$4 \times 4 + 4 \times 6 + 2 \times 5 + 4 + 2 \times 3 = 60.$$

The fact that Γ is an $\cap$-subsemilattice of $\mathcal{L}(\mathcal{CR})$ follows directly from the notation of the labels of its vertices. We can thus pass to the generation by varieties in Sketch 1.

For any $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{Z}$, let

$$[\mathcal{W}, \mathcal{X}] \wedge [\mathcal{Y}, \mathcal{Z}] = \{\mathcal{P} \cap \mathcal{Q} | \mathcal{P} \in [\mathcal{W}, \mathcal{X}], \mathcal{Q} \in [\mathcal{Y}, \mathcal{Z}]\}.$$

If $\mathcal{W} = \mathcal{X}$, we write simply $\mathcal{W}$ for $[\mathcal{W}, \mathcal{X}]$. Next we consider numerous cases.

▷ $[H\mathcal{A}, \mathcal{CR}] \wedge [\mathcal{RBG}, \mathcal{CR}] = [\mathcal{RBA}, \mathcal{CR}]$,
▷ $L\mathcal{O} \wedge [\mathcal{RBG}, \mathcal{CR}] = [(L\mathcal{O})\mathcal{RBG}, L\mathcal{O}]$,
▷ $\{\mathcal{CS}, \mathcal{NBG}\}$ together with the preceding case gives $[\mathcal{CS}, L\mathcal{O}]$,
▷ $L\mathcal{O} \wedge [H\mathcal{A}, \mathcal{CR}] = [(L\mathcal{O})H\mathcal{A}, L\mathcal{O}]$,

from the preceding two cases, or directly, we get

▷ $[\mathcal{CS}, L\mathcal{O}] \wedge [(L\mathcal{O})H\mathcal{A}, L\mathcal{O}] = [(\mathcal{CS})H\mathcal{A},\ (L\mathcal{O})H\mathcal{A}]$,
▷ $\mathcal{O} \wedge [\mathcal{CS}, (L\mathcal{O})\mathcal{BG}] = [\mathcal{R}e\mathcal{G}, \mathcal{O}(\mathcal{BG})]$,
▷ $\mathcal{O} \wedge H\mathcal{A} = \mathcal{O}(H\mathcal{A})$,
▷ $\mathcal{O}(H\mathcal{A}) \wedge [\mathcal{CS}, (L\mathcal{O})\mathcal{BG}] = [\mathcal{R}e\mathcal{A}, \mathcal{O}(\mathcal{BA})]$,
▷ $\mathcal{B} \wedge [\mathcal{CS},\ (L\mathcal{O})\mathcal{RBG}] = [\mathcal{RB}, \mathcal{R}e\mathcal{B}]$,
▷ $\mathcal{SG} \wedge (\mathcal{CS})H\mathcal{A} = \mathcal{A}$,
▷ $\mathcal{SG} \wedge \mathcal{NB} = \mathcal{S}$,
▷ $\mathcal{SG} \wedge \mathcal{R}e\mathcal{G} = \mathcal{G}$,
▷ $\mathcal{S} \wedge \mathcal{G} = \mathcal{T}$.

By direct inspection of Diagram 1, we can see that with this type of meet we have covered all varieties in Γ, proving the assertion of generation. Note that for varieties $\mathcal{U}, \mathcal{V} \in \Gamma$ we have $\mathcal{U} \wedge \mathcal{V} = \mathcal{U} \cap \mathcal{V}$.

Recall that the semilattice Γ being an $\cap$-subsemilattice of $\mathcal{L}(\mathcal{CR})$ means that the label of the meet is the meet (that is, the intersection) of labels of factors in $\mathcal{L}(\mathcal{CR})$. We will have a similar situation for varieties in Δ but relating both to meet and join.

(ii) Part (i) takes care of the meets. Guided by Lemma 3.6, we set

$$A = [\mathcal{T}, \mathcal{O}(\mathcal{BG})], \quad B = [\mathcal{RB}, (L\mathcal{O})\mathcal{BG}], \quad C = [\mathcal{R}e\mathcal{G}, (L\mathcal{O})T\mathcal{O}].$$

The sets A, B, C are depicted in Diagram 2. First, the three equalities in Lemma 3.6 guarantee that the operations within A, B and C coincide with those of their labels

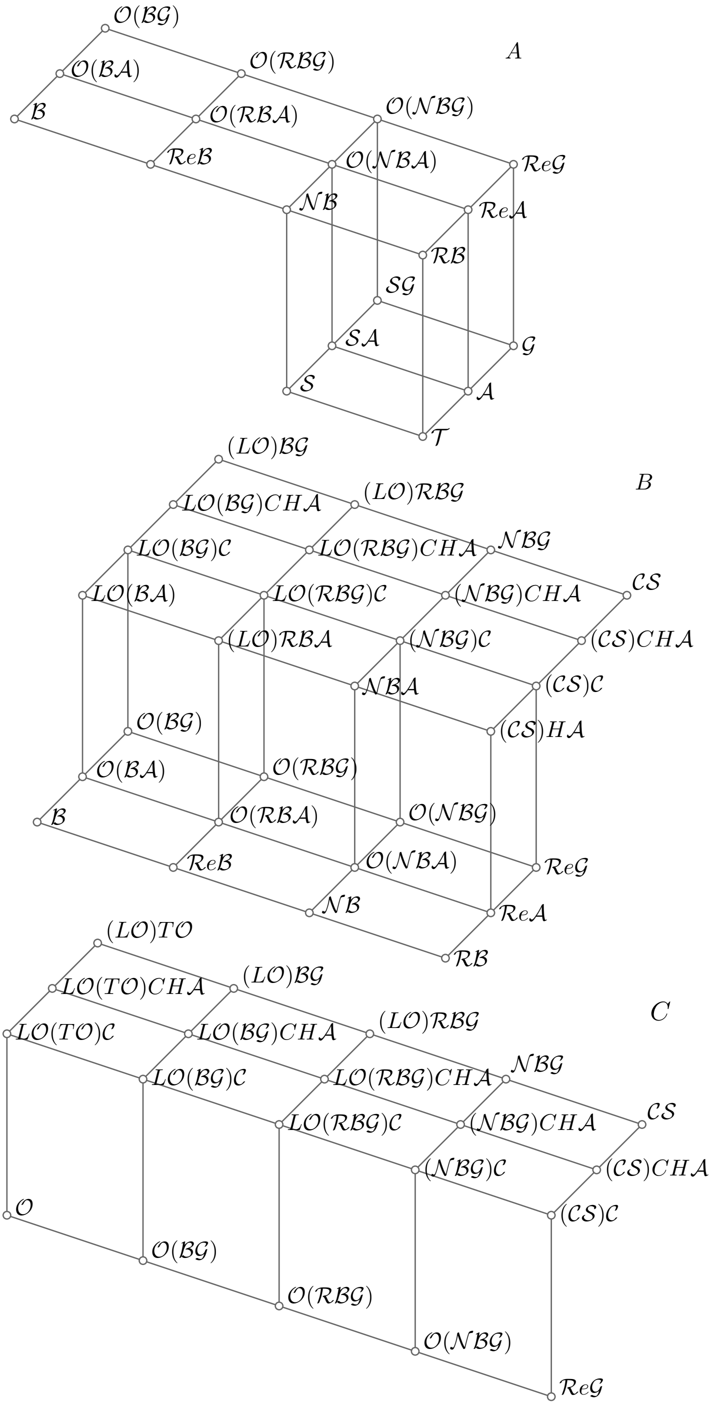


Diagram 2. The sets A, B, C

in $\mathcal{L}(\mathcal{CR})$. It thus remains to check the join $\mathcal{U} \vee \mathcal{V}$ in each of the following cases. Consult Diagram 2 for the following information.

(α) $A \setminus B = [\mathcal{T}, \mathcal{SG}]$, $B \setminus A = [(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BG}]$,
(β) $B \setminus C = [\mathcal{RB}, (L\mathcal{O})\mathcal{BA}]$, $C \setminus B = [\mathcal{O}, (L\mathcal{O})T\mathcal{O}]$,
(γ) $A \setminus C = [\mathcal{T}, \mathcal{O}(\mathcal{BA})]$, $C \setminus A = [(\mathcal{CS})\mathcal{C}, (L\mathcal{O})T\mathcal{O}] \cup \{\mathcal{O}\}$.

The pattern of the proof in each of these cases is the same. Let $X, Y \in \{A, B, C\}$ be any choice. To each member $\mathcal{U}$ of $X \setminus Y$ we associate the set of all varieties $\mathcal{V}$ in $Y \setminus X$ noncomparable to $\mathcal{U}$, and perform the join $\mathcal{U} \vee \mathcal{V}$. We then verify whether the join of their labels in $\mathcal{L}(\mathcal{CR})$ equals the label of $\mathcal{U} \vee \mathcal{V}$ in Γ.

For the sake of the economy of exposition, we will refer to the intervals in Diagram 2, and will not state their members explicitly. Hence, Diagram 2 will be referred to every time we speak of an interval. In particular, if the interval in question is linearly ordered, we shall refer to its members, as first, second, etc., counting from below.

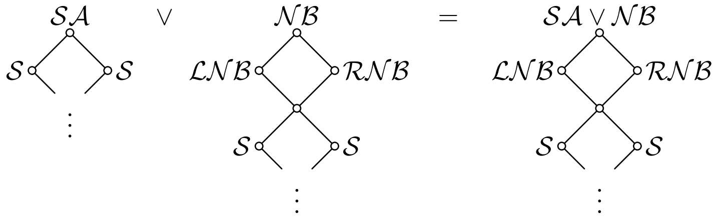
We can now pass to the three cases (α), (β), (γ) indicated above. For the networks we will freely use the results in Section 3.

Case (α)**.** To each $\mathcal{U} \in A \setminus B = [\mathcal{T}, \mathcal{SG}]$ we associate the set of all varieties $\mathcal{V}$ in $B \setminus A = [(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BG}]$ noncomparable to $\mathcal{U}$ as follows:

$\mathcal{A}$: $[\mathcal{RB}, \mathcal{B}]$,
$\mathcal{G}$: $[\mathcal{RB}, \mathcal{B}]$, $[\mathcal{R}e\mathcal{A}, \mathcal{O}(\mathcal{BA})]$, $[(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BG}]$,
$\mathcal{S}$: $[\mathcal{RB}, \mathcal{CS}]$,
$\mathcal{SA}$: $[\mathcal{RB}, \mathcal{B}]$, $[\mathcal{RB}, \mathcal{CS}]$,
$\mathcal{SG}$: $[\mathcal{RB}, \mathcal{B}]$, $[\mathcal{R}e\mathcal{A}, \mathcal{O}(\mathcal{BA})]$, $[(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BA}]$, $[\mathcal{RB}, \mathcal{CS}]$.

Subcase $\mathcal{A}$. The fact that $\mathcal{A} \vee \mathcal{RB} = \mathcal{R}e\mathcal{A}$ is a simple consequence of the material in [13], Section VIII.1. For the remaining varieties in the interval $[\mathcal{RB}, \mathcal{B}]$ we state the argument only for $\mathcal{A} \vee \mathcal{NB}$; the remaining two cases, that is, $\mathcal{A} \vee \mathcal{R}e\mathcal{B}$ and $\mathcal{A} \vee \mathcal{B}$ are very similar. We use Fact 3.7.

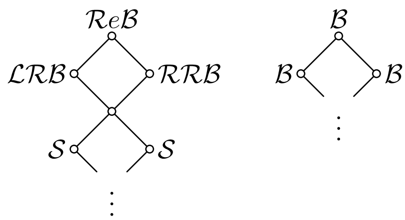
The networks are



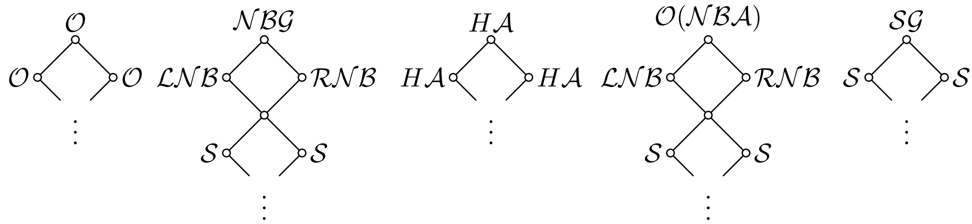
and the evaluation

$$\begin{aligned}\mathcal{A} \vee \mathcal{NB} &= \mathcal{SA} \vee \mathcal{NB} = (\mathcal{SA} \vee \mathcal{NB})^K \cap \mathcal{LNB}^{T_r} \cap \mathcal{RNB}^{T_l} \\ &= \mathcal{A}^K \cap \mathcal{LNB}^{T_r} \cap \mathcal{RNB}^{T_l} = \mathcal{O}(H\mathcal{A}) \cap \overline{\mathcal{H}}_3 \cap \mathcal{H}_3 \\ &= \mathcal{O}(H\mathcal{A}) \cap \mathcal{NBG} = \mathcal{O}(\mathcal{NBA}),\end{aligned}$$

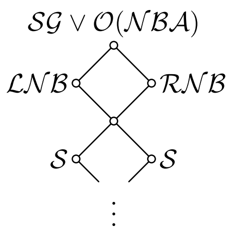
where $\mathcal{H}_3$ and $\overline{\mathcal{H}}_3$ can be found in [7] and in particular the equality $\mathcal{H}_3 \cap \overline{\mathcal{H}}_3 = \mathcal{NBG}$ in [7], Theorem 5.1(iii). The networks of the remaining varieties, $\mathcal{R}e\mathcal{B}$ and $\mathcal{B}$, are:



Subcase $\mathcal{G}$. The instances in the interval $[\mathcal{RB}, \mathcal{B}]$ are treated in a manner very similar to the preceding subcase using $\mathcal{SG}$ instead of $\mathcal{SA}$. For the second interval $[\mathcal{R}e\mathcal{A}, \mathcal{O}(\mathcal{BA})]$ we first have $\mathcal{G} \vee \mathcal{R}e\mathcal{A} = \mathcal{R}e\mathcal{G}$. For the second instance $\mathcal{O}(\mathcal{NBA})$, the networks are



which gives the network

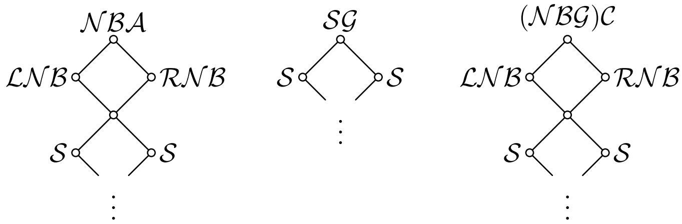


and its evaluation

$$\begin{aligned}\mathcal{SG} \vee \mathcal{O}(\mathcal{NBA}) &= (\mathcal{SG} \vee \mathcal{O}(\mathcal{NBA}))^K \cap \mathcal{LNB}^{T_r} \cap \mathcal{RNB}^{T_l} \\ &= \mathcal{O} \cap \overline{\mathcal{H}}_3 \cap \mathcal{H}_3 = \mathcal{O}(\mathcal{NBG})\end{aligned}$$

as in Subcase $\mathcal{A}$. The remaining two instances in this interval require the same type of argument.

For the variety in the third interval we have $\mathcal{G} \vee (\mathcal{CS})H\mathcal{A} = (\mathcal{CS})\mathcal{C}$ by [13], Corollary VIII.8.3. For the remaining varieties in this interval we need the following networks:



and the evaluation, using Lemma 3.5,

$$\begin{aligned}\mathcal{G} \vee \mathcal{NBA} &= ((\mathcal{NBG})\mathcal{C})^K \cap \mathcal{LNB}^{T_r} \cap \mathcal{RNB}^{T_l} \\ &= \mathcal{NBG}^K \cap \mathcal{C}^K \cap \mathcal{H}_3 \cap \overline{\mathcal{H}}_3 = (L\mathcal{O})\mathcal{C}(\mathcal{NBG}) = (\mathcal{NBG})\mathcal{C}.\end{aligned}$$

The remaining two instances in this interval are treated in a similar way.

Subcase $\mathcal{S}$. This follows directly from [13], Corollary IV.1.1.

Subcase $\mathcal{SA}$. First

$$\mathcal{SA} \vee \mathcal{RB} = \mathcal{S} \vee \mathcal{A} \vee \mathcal{RB} = \mathcal{S} \vee \mathcal{R}e\mathcal{A} = \mathcal{O}(\mathcal{NBA})$$

and for $\mathcal{V} \in [\mathcal{R}e\mathcal{G}, \mathcal{CS}]$ we have $\mathcal{SA} \vee \mathcal{V} = \mathcal{S} \vee \mathcal{A} \vee \mathcal{V} = \mathcal{S} \vee \mathcal{V}$.

Subcase $\mathcal{SG}$. Similarly,

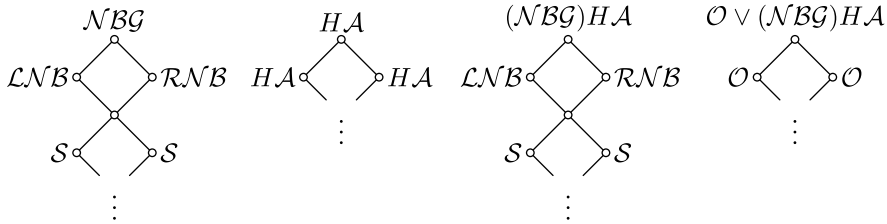
$$\begin{aligned}\mathcal{SG} \vee \mathcal{RB} &= \mathcal{S} \vee \mathcal{G} \vee \mathcal{RB} = \mathcal{S} \vee \mathcal{R}e\mathcal{G} = \mathcal{O}(\mathcal{NBG}), \\ \mathcal{SG} \vee \mathcal{R}e\mathcal{A} &= \mathcal{S} \vee \mathcal{G} \vee \mathcal{RB} \vee \mathcal{A} = \mathcal{S} \vee \mathcal{R}e\mathcal{G} = \mathcal{O}(\mathcal{NBG}),\end{aligned}$$

and for $\mathcal{V} \in [(\mathcal{CS})\mathcal{C}, \mathcal{CS}]$ we have $\mathcal{SG} \vee \mathcal{V} = \mathcal{S} \vee \mathcal{G} \vee \mathcal{V} = \mathcal{S} \vee \mathcal{V}$.

Case (β)**.** To each $\mathcal{U} \in C \setminus B = [\mathcal{O}, (L\mathcal{O})T\mathcal{O}]$ we associate the set of all varieties $\mathcal{V} \in B \setminus C = [\mathcal{RB}, (L\mathcal{O})\mathcal{BA}]$ noncomparable to $\mathcal{U}$ as follows. This amounts to the single subcase.

Subcase $\mathcal{O}$*:* $[(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BG}]$. Since the remaining three varieties in $C \setminus B$ are all greater than the varieties in $B \setminus C$, we now consider subsubcases of this subcase.

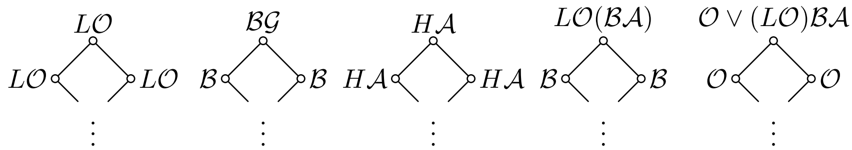
Subsubcase $(\mathcal{CS})H\mathcal{A}$. The networks are



where we have used Lemma 3.5 for $H\mathcal{A}$, and will now use it for $\mathcal{C}$ together with [13], Corollary VIII.8.3:

$$\begin{aligned}\mathcal{O} \vee (\mathcal{CS})H\mathcal{A} &= (\mathcal{O} \vee (\mathcal{CS})H\mathcal{A})^K \cap \mathcal{O}^{T_r} \cap \mathcal{O}^{T_l} = (\mathcal{G} \vee (\mathcal{CS})H\mathcal{A})^K \cap \mathcal{O}^T \\ &= ((\mathcal{CS})\mathcal{C})^K \cap \mathcal{O}^T = \mathcal{CS}^K \cap \mathcal{C}^K \cap \mathcal{O}^T = L\mathcal{O}(T\mathcal{O})\mathcal{C}.\end{aligned}$$

Subsubcases $\{\mathcal{NBA}, (L\mathcal{O})\mathcal{RBA}, (L\mathcal{O})\mathcal{BA}\}$. The networks are



and the evaluation, with references as above,

$$\begin{aligned}\mathcal{O} \vee (L\mathcal{O})\mathcal{BA} &= (\mathcal{O} \vee (L\mathcal{O})\mathcal{BA})^K \cap \mathcal{O}^{T_r} \cap \mathcal{O}^{T_l} = (\mathcal{G} \vee (\mathcal{CS})H\mathcal{A})^K \cap \mathcal{O}^T \\ &= ((\mathcal{CS})\mathcal{C})^K \cap \mathcal{O}^T = L\mathcal{O}(T\mathcal{O})\mathcal{C}.\end{aligned}$$

Since

$$(\mathcal{CS})H\mathcal{A} \subseteq \mathcal{NBA} \subseteq (L\mathcal{O})\mathcal{RBA} \subseteq (L\mathcal{O})\mathcal{BA},$$

it follows from the above that

$$\mathcal{O} \vee \mathcal{NBG} = \mathcal{O} \cap (L\mathcal{O})\mathcal{RBA} = L\mathcal{O}(T\mathcal{O})\mathcal{C}.$$

Case (γ)**.** To each $\mathcal{U} \in C \setminus A = [(\mathcal{CS})\mathcal{C}, (L\mathcal{O})T\mathcal{O}] \cup \{\mathcal{O}\}$ we associate the set of all varieties $\mathcal{V} \in A \setminus C = [\mathcal{T}, \mathcal{O}(\mathcal{BA})]$ noncomparable to $\mathcal{U}$.

All pairs of noncomparable varieties in A:

$$\mathcal{O}(\mathcal{BA}),\ \mathcal{B},\ \mathcal{O}(\mathcal{RBA}),\ \mathcal{R}e\mathcal{G},\ \mathcal{O}(\mathcal{NBA}),\ \mathcal{NB},$$

and those in C:

$$\mathcal{CS},\ (\mathcal{CS})CH\mathcal{A},\ (\mathcal{CS})\mathcal{C},$$
$$\mathcal{NBG},\ (\mathcal{NBG})CH\mathcal{A},\ (\mathcal{NBG})\mathcal{C},$$
$$(L\mathcal{O})\mathcal{RBG},\ L\mathcal{O}(\mathcal{RBG})CH\mathcal{A},\ L\mathcal{O}(\mathcal{RBG})\mathcal{C}$$

are contained in B, and therefore have been accounted for by the statement at the very start of the proof, see Diagram 2.

This establishes the coincidence of the lattice properties. We can now pass to the assertion of generation of the lattice Δ.

Similarly as in the case of Γ, but now faster with fewer details because of it, we proceed as follows.

$$\mathcal{O} \vee \mathcal{CS} = L\mathcal{O}(T\mathcal{O}), \quad \mathcal{B} \vee \mathcal{CS} = (L\mathcal{O})\mathcal{BG}, \quad \mathcal{S} \vee \mathcal{CS} = \mathcal{NBG}$$

yield $[\mathcal{CS}, (L\mathcal{O})T\mathcal{O}]$. By hypothesis, we have $[\mathcal{O}, (L\mathcal{O})T\mathcal{O}]$. Performing meets, we get (recall the meaning of $\wedge$)

$$(L\mathcal{O})\mathcal{BA} \wedge [\mathcal{CS}, (L\mathcal{O})\mathcal{BG}] = [(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BG}],$$
$$\mathcal{O} \wedge [(\mathcal{CS})H\mathcal{A}, (L\mathcal{O})\mathcal{BA}] = [\mathcal{R}e\mathcal{A}, \mathcal{O}(\mathcal{BA})],$$
$$\mathcal{B} \wedge [\mathcal{CS}, (L\mathcal{O})\mathcal{RBG}] = [\mathcal{RB}, \mathcal{R}e\mathcal{B}],$$
$$\mathcal{S} \wedge \mathcal{RB} = \mathcal{T}, \quad \mathcal{S} \wedge \mathcal{NBA} = \mathcal{SA}, \quad \mathcal{S} \wedge (\mathcal{CS})H\mathcal{A} = \mathcal{A}, \quad \mathcal{S} \wedge \mathcal{R}e\mathcal{G} = \mathcal{G}.$$

This concludes the proof of part (ii).

(iii) By [3], Theorem 2.7, we have $\mathcal{BA} \vee \mathcal{O}(\mathcal{BA}) = (\mathcal{BG})\mathcal{C}$, and from Corollary in [5], it follows that $\mathcal{O}(\mathcal{BG}) = \mathcal{B} \vee \mathcal{G}$. Therefore by [13], Corollary VIII.8.3, we obtain

$$\mathcal{BA} \vee \mathcal{G} = \mathcal{BA} \vee \mathcal{B} \vee \mathcal{G} = \mathcal{BA} \vee \mathcal{O}(\mathcal{BG}) = (\mathcal{BG})\mathcal{C}. \tag{4.1}$$

According to [17], Corollary 2.9, $\mathcal{G}$ is neutral in $\mathcal{L}(\mathcal{CR})$ which together with (4.1) and $\mathcal{G} \subseteq \mathcal{V} \subseteq \mathcal{BG}$ yields

$$\begin{aligned}\mathcal{V}(H\mathcal{A}) \vee \mathcal{G} &= (\mathcal{V} \cap \mathcal{BA}) \vee \mathcal{G} = (\mathcal{V} \vee \mathcal{G}) \cap (\mathcal{BA} \vee \mathcal{G}) \\ &= \mathcal{V} \vee (\mathcal{BG})\mathcal{C} = \mathcal{V} \cap \mathcal{BG} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C} = \mathcal{VC}.\end{aligned}$$

□

Where do we go from here? One way would be to enlarge the lattice Δ by adjoining some of the varieties in $\Gamma \backslash \Delta$. Theorem 4.2(iii) might be of some help, but many more joins, or even meets, would have to be computed. This may be termed extension outward.

As a contrast, extension inward would be to adjoin new subvarieties of $(L\mathcal{O})T\mathcal{O}$. This would entail computing meets and joins with varieties already in Δ. For example, the lattice $\mathcal{L}(\mathcal{B})$ is well known and understood, and the short sequence of band varieties in Δ can be integrated into $\mathcal{L}(\mathcal{B})$. In a different direction, the sequence $\mathcal{T}$, $\mathcal{A}$, $\mathcal{G}$ can be augmented by any group varieties. Similarly for $\mathcal{L}(\mathcal{CS})$, where a number of varieties has already been identified.

This is the first instalment of a trilogy. The second paper will contain at least one basis for each of the varieties in Γ, while the third paper will deal with classes of kernel, trace, local and core relations of all varieties in Γ.

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