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PELL AND PELL-LUCAS NUMBERS OF THE FORM $-2^a - 3^b + 5^c$

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Abstract. In this paper, we find all Pell and Pell-Lucas numbers written in the form $-2^a - 3^b + 5^c$, in nonnegative integers a, b, c, with $0 \leqslant \max\{a, b\} \leqslant c$.

Keywords: Pell number; Pell-Lucas number; linear form in logarithms; continued fraction; reduction method

MSC 2010: 11B39, 11J86, 11D61

1. Introduction

Let $\{F_n\}_{n\geqslant 0}$ be the Fibonacci sequence defined as $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$ and its companion Lucas sequence $\{L_n\}_{n\geqslant 0}$ follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$. The Pell sequence $\{P_n\}_{n\geqslant 0}$ is the binary recurrent sequence given by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geqslant 0$ and its companion Pell-Lucas sequence $\{Q_n\}_{n\geqslant 0}$ follows the same recursive pattern as the Pell numbers, but with initial values $Q_0 = 2$ and $Q_1 = 2$. These numbers are well-known for possessing amazing properties (consult [5]). The problem of finding binary recurrent sequence of a particular form has a very rich history. Bugeaud, Mignotte and Siksek in [3] concluded that 0, 1, 8, 144 and 1, 4 are the only perfect power in Fibonacci and Lucas numbers, respectively. Other related papers searched for Fibonacci numbers of the particular forms such as px^2+1, px^3+1 (see [11]), k^2+k+2 (see [6]), $p^a \pm p^b + 1$ (see [7]). There are also many papers that searched for Pell numbers and Pell-Lucas numbers of a particular form. For example, in 1996, McDaniel found that 1 is the

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only triangular number in the Pell sequence (see [9]). In 1991, Pethő in [10] found all perfect powers in the Pell sequence. In 2015, Bravo, Das, Guzmán and Laishram in [2] found the powers in products of terms of Pell and Pell-Lucas sequences.

In this paper, we are interested in Pell and Pell-Lucas numbers which are sums of three perfect powers of some prescribed distinct bases. More precisely, our results are as follows.

Theorem 1.1. *The only solutions of the Diophantine equation*

$$P_n = -2^a - 3^b + 5^c \tag{1.1}$$

in nonnegative integers n, a, b, c with $0 \leqslant \max\{a,b\} \leqslant c$ are

$$(n,a,b,c) \in \{(0,1,1,1),(1,0,1,1),(2,1,0,1),(4,2,2,2)\}.$$

Theorem 1.2. *The only solutions of the Diophantine equation*

$$Q_n = -2^a - 3^b + 5^c \tag{1.2}$$

in nonnegative integers n, a, b, c with $0 \leqslant \max\{a,b\} \leqslant c$ are

$$(n,a,b,c) \in \{(0,1,0,1),(1,1,0,1),(3,1,2,2)\}.$$

2. Preliminaries

Before proceeding further, we shall recall some facts and lemmas which will be used later. First, we recall Binet's formulae for Pell and Pell-Lucas sequences:

$$P_n = \frac{\gamma^n - \mu^n}{\gamma - \mu}$$

and

$$Q_n = \gamma^n + \mu^n,$$

where $\gamma = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of P_n. The inequalities

$$\gamma^{n-2} \leqslant P_n \leqslant \gamma^{n-1}, \quad \gamma^{n-1} \leqslant Q_n \leqslant 2\gamma^n \tag{2.1}$$

hold for all positive integers n.

In order to prove our theorem, one result is a Baker type lower bound for a linear form in logarithms of algebraic numbers and such a bound was shown by the following result of Matveev (see [8]).

Lemma 2.1. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_t$ *be real algebraic numbers and let* $b_1, \ldots, b_t$ *be nonzero rational integers. Let* D *be the degree of the number field* $\mathbb{Q}(\gamma_1, \gamma_2, \ldots, \gamma_t)$ *over* $\mathbb{Q}$ *and let* A_j *be a real number satisfying*

$$A_j \geqslant \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}$$

for $j = 1, \ldots, t$. *Assume that*

$$B \geqslant \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \ldots \gamma_t^{b_t} \neq 1$, *then*

$$|\gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1| \geqslant \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t).$$

In the above statement, the logarithmic height of an s-degree algebraic number γ is given by

$$h(\gamma) = \frac{1}{s}\left(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\gamma^{(j)}|\}\right),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $\gamma^{(j)}$, $1 \leqslant j \leqslant s$ are the conjugates of γ (over $\mathbb{Q}$).

After finding an upper bound on n which is in general too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő (see [4]). For a real number x, we denote the distance from x to the nearest integer by $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$.

Lemma 2.2 (see [1]). *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational number* α *such that* $q > 6M$, *and let* A, B, τ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\varepsilon = \|\tau q\| - M\|\alpha q\|$, *where* $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\varepsilon > 0$, *then there exists no solution to the inequality*

$$0 < |u\alpha - v + \tau| < AB^{-\omega}$$

in positive integers u, v, *and* ω *with* $u \leqslant M$ *and*

$$w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

Now, we are ready to deal with the proofs of our theorems.

3. Proof of Theorem 1.1

3.1. Bounding n. Combining the Binet formula together with (1.1) we get

$$\left|\frac{\gamma^n}{2\sqrt{2}} - 5^c\right| = \left|-2^a - 3^b + \frac{\mu^n}{2\sqrt{2}}\right| \leqslant 2^a + 3^b + \frac{|\mu|^n}{2\sqrt{2}}, \tag{3.1}$$

which yields

$$\left|\frac{\gamma^n 5^{-c}}{2\sqrt{2}} - 1\right| < \frac{3}{5^{0.3c}}. \tag{3.2}$$

We claim that $\gamma^n 5^{-c}/2\sqrt{2} - 1 \neq 0$. In fact, if $\gamma^n 5^{-c}/2\sqrt{2} - 1 = 0$, then $\gamma^n = 5^c\sqrt{8}$, hence $\overline{\gamma}^n = -5^c\sqrt{8}$, where $\overline{\gamma}$ is the conjugate of γ. Thus, we can get $\gamma^n\overline{\gamma}^n = -5^{2c}8$, hence $(-1)^{n-1} = 5^{2c}8$, which is an absurdity. So, we have

$$0 < \left|\frac{\gamma^n 5^{-c}}{2\sqrt{2}} - 1\right| < \frac{3}{5^{0.3c}}. \tag{3.3}$$

If $n = 0$, from (1.1), we get $5^c = 2^a + 3^b \leqslant 2^c + 3^c$, this implies that $c = 0, 1$. If $n > 0$, from the first inequality of (2.1), we obtain the estimate $\gamma^{n-2} < 5^c$; this yields $n < 1.83c + 2$. If $c \leqslant 10$, then $n \leqslant 20$. A brute force search with Mathematica in the range $0 \leqslant c \leqslant 10$ and $0 \leqslant n \leqslant 20$ turned up that the only solutions of (1.1) are $(n, a, b, c) \in \{(0,1,1,1), (1,0,1,1), (2,1,0,1), (4,2,2,2)\}$. Thus, we assume that $c > 10$. From the first inequality of (2.1), we get

$$\gamma^{n-1} \geqslant 5^c - 2^a - 3^b > 5^c - 5^{0.44c} - 5^{0.69c} = 5^c\Big(1 - \frac{1}{5^{0.56c}} - \frac{1}{5^{0.31c}}\Big) > 0.9 \times 5^c,$$

which implies that $1.82c + 0.8 < n$, and this also yields $c < n$.

We apply Matveev's result Lemma 2.1 to the left-hand side of (3.2). According to (3.3) we have proved that the expression on the left-hand side of (3.2) is nonzero. We take $t := 3$, $\gamma_1 := \gamma$, $\gamma_2 := 5$, $\gamma_3 := \sqrt{8}$ and $b_1 := n$, $b_2 := -c$, $b_3 := -1$. For this choice, we have $D = [\mathbb{Q}(\sqrt{2})\colon \mathbb{Q}] = 2$. Note that $h(\gamma_1) = \frac{1}{2}\log\gamma$, $h(\gamma_2) = \log 5$ and $h(\gamma_3) = \log\sqrt{8}$. Thus, we can take $A_1 := 0.89$, $A_2 := 3.22$ and $A_3 := 2.1$. Note that $B = \max\{|b_1|, |b_2|, |b_3|\} = \max\{n, c, 1\} = n$. According to Matveev's result Lemma 2.1 together with a straightforward calculation, we get

$$\left|\frac{\gamma^n 5^{-c}}{2\sqrt{2}} - 1\right| > \exp(-C(1 + \log n)), \tag{3.4}$$

where $C = 5.84 \times 10^{12}$. Thus from (3.3), (3.4) and $c > (n-2)/1.83$, taking logarithms in inequalities (3.3), (3.4) and comparing the resulting inequalities, we get that

$$0.263n - 1.63 < 5.84 \times 10^{12} \times (1 + \log n),$$

giving $n < 7.84 \times 10^{14}$. We obtain the conclusion of this section as follows.

Lemma 3.1. *If* (n, a, b, c) *is a solution in positive integers of equation* (1.1) *with* $0 \leqslant \max\{a, b\} \leqslant c$, *then*

$$c < n < 7.84 \times 10^{14}.$$

3.2. Reducing the bound on n. We use several times Lemma 2.2 to reduce the bound for n. We return to (3.3). Put

$$\Lambda_P := n \log \gamma - c \log 5 - \log \sqrt{8}.$$

Note that $\Lambda_P \neq 0$, thus, we distinguish the following cases. If $\Lambda_P > 0$, then $\mathrm{e}^{\Lambda_P} > 1$, so from (3.3) we obtain $0 < \Lambda_P < \mathrm{e}^{\Lambda_P} - 1 < 3/5^{0.3c}$. Suppose now that $\Lambda_P < 0$. It is easy to check that $3/5^{0.3c} < \frac{1}{2}$ for all $c > 10$. Then, from (3.3), we have that $|\mathrm{e}^{\Lambda_P} - 1| < \frac{1}{2}$ and therefore $\mathrm{e}^{|\Lambda_P|} < 2$. Since $\Lambda_P < 0$, we have

$$0 < |\Lambda_P| < \mathrm{e}^{|\Lambda_P|} - 1 \leqslant \mathrm{e}^{|\Lambda_P|}|\mathrm{e}^{\Lambda_P} - 1| < \frac{6}{5^{0.3c}}.$$

In any case, we have that the inequality $0 < |\Lambda_P| < 6/5^{0.3c}$ holds for all $c > 10$. Replacing Λ_P in the above inequality by its formula and dividing by $\log 5$, we conclude that

$$0 < \left| n \frac{\log \gamma}{\log 5} - c - \frac{\log \sqrt{8}}{\log 5} \right| < \frac{6}{\log 5 \times 5^{0.3c}} < 3.73 \times 1.6^{-c}. \tag{3.5}$$

We are now ready to apply Lemma 2.2 with the obvious parameters

$$\alpha := \frac{\log \gamma}{\log 5}, \quad \tau := -\frac{\log \sqrt{8}}{\log 5}, \quad A := 3.73, \quad B := 1.6.$$

It is easy to prove that α is irrational and we omit this step here. Let p_k/q_k be its continued fraction's kth convergent. We can take $M := 7.84 \times 10^{14}$. Applying Lemma 2.2 and performing the calculations with $q_{28} > 6M$ and $\varepsilon = \|\tau q_{28}\| - M\|\alpha q_{28}\| = 0.02427\ldots$, we get that if (n, a, b, c) is a solution in positive integers of equation (1.1), then $c < 88$, which implies that

$$n < 1.83 \times 88 + 2 = 163.04 < 164.$$

Then we can take $M := 164$. Applying Lemma 2.2 again and performing the calculations with $q_7 > 6M$ and $\varepsilon = \|\tau q_7\| - M\|\alpha q_7\| = 0.10378\ldots$, we get that if (n, a, b, c) is a solution in positive integers of equation (1.1), then $c < 25$, which implies that

$$n < 1.83 \times 25 + 2 = 47.75 < 48.$$

Finally, we use a program written in Mathematica to find the solutions of (1.1) in the range $0 \leqslant \max\{a,b\} \leqslant c < 25$ and $n < 48$. Quickly, the program returns the solutions $(n,a,b,c) \in \{(0,1,1,1),(1,0,1,1),(2,1,0,1),(4,2,2,2)\}$. This completes the proof. $\square$

4. Proof of Theorem 1.2

4.1. Bounding n. Combining the Binet formula together with (1.2), we get

$$|\gamma^n 5^{-c} - 1| = \Big|-\frac{2^a}{5^c} - \frac{3^b}{5^c} - \mu^n 5^{-c}\Big| < \frac{3}{5^{0.3c}}. \tag{4.1}$$

From the second inequality of (2.1) and (1.2), we obtain the estimate $\gamma^{n-1} < 5^c$, this yields $n < 1.83c + 1$. If $c \leqslant 20$, then $n < 38$. A brute force search with Mathematica in the range $0 \leqslant c \leqslant 20$ and $0 \leqslant n < 38$ turned up that the only solutions of (1.2) are $(n,a,b,c) \in \{(0,1,0,1),(1,1,0,1),(3,1,2,2)\}$. Thus, we assume that $c > 20$. From the second inequality of (2.1), we get

$$2\gamma^n \geqslant 5^c - 2^a - 3^b > 5^c - 5^{0.44c} - 5^{0.69c} = 5^c\Big(1 - \frac{1}{5^{0.56c}} - \frac{1}{5^{0.31c}}\Big) > 0.9 \times 5^c,$$

which implies that $1.82c - 0.91 < n$, and this also yields $c < n$.

We also apply Matveev's result Lemma 2.1 to the left-hand side of (4.1). The expression on the left-hand side of (4.1) is nonzero, since this expression being zero means that $\gamma^n = 5^c \in \mathbb{Z}$, so $\gamma^n \in \mathbb{Z}$ for some positive integer n, which is false. We take $t := 2$, $\gamma_1 := \gamma$, $\gamma_2 := 5$ and $b_1 := n$, $b_2 := -c$. For this choice, we have $D = [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Note that $h(\gamma_1) = \frac{1}{2}\log\gamma$, $h(\gamma_2) = \log 5$. Thus, we can take $A_1 := 0.89$, $A_2 := 3.22$. Note that $B = \max\{|b_1|,|b_2|\} = \max\{n,c\} = n$. According to Matveev's result Lemma 2.1 together with a straightforward calculation, we get

$$|\gamma^n 5^{-c} - 1| > \exp(-C(1 + \log n)), \tag{4.2}$$

where $C = 1.5 \times 10^{10}$. Thus from (4.1), (4.2) and $c > (n-1)/1.83$, taking logarithms in inequalities (4.1), (4.2) and comparing the resulting inequalities, we get that

$$0.263n - 1.37 < 1.5 \times 10^{10} \times (1 + \log n),$$

giving $n < 1.662 \times 10^{12}$. We obtain the conclusion of this section as follows.

Lemma 4.1. *If* (n, a, b, c) *is a solution in positive integers of equation* (1.2), *with* $0 \leqslant \max\{a, b\} \leqslant c$, *then*

$$c < n < 1.662 \times 10^{12}.$$

4.2. Reducing the bound on n. We reduce the bound for n by using the extremality property of continued fractions. We return to (4.1). Put

$$\Lambda_Q := n \log \gamma - c \log 5.$$

Note that $\Lambda_Q \neq 0$, thus, we distinguish the following cases. If $\Lambda_Q > 0$, then $e^{\Lambda_Q} > 1$, so from (4.1) we obtain $0 < \Lambda_Q < e^{\Lambda_Q} - 1 < 3/5^{0.3c}$. Suppose now that $\Lambda_Q < 0$. It is easy to check that $3/5^{0.3c} < \frac{1}{2}$ for all $c > 20$. Then, from (4.1), we have that $|e^{\Lambda_Q} - 1| < \frac{1}{2}$ and therefore $e^{|\Lambda_Q|} < 2$. Since $\Lambda_Q < 0$, we have

$$0 < |\Lambda_Q| < e^{|\Lambda_Q|} - 1 \leqslant e^{|\Lambda_Q|}|e^{\Lambda_Q} - 1| < \frac{6}{5^{0.3c}}.$$

In any case, we have that the inequality $0 < |\Lambda_Q| < 6/5^{0.3c}$ holds for all $c > 20$. Replacing Λ_Q in the above inequality by its formula and dividing by $\log 5$, we conclude that

$$0 < \left| n \frac{\log \gamma}{\log 5} - c \right| < \frac{6}{\log 5 \times 5^{0.3c}}. \tag{4.3}$$

Let $[a_0, a_1, a_2, a_3, a_4, \ldots,] = [0, 1, 1, 4, 1, 2, \ldots]$ be the continued fraction of the ratio $\log \gamma / \log 5$, and let p_k/q_k be its kth convergent. Recall that $n < 1.662 \times 10^{12}$ by Lemma 4.1. A quick inspection using Mathematica reveals that $q_{19} < 1.662 \times 10^{12} < q_{20}$. Furthermore, $a_M := \max\{a_i \colon i = 0, 1, \ldots, 20\} = a_{17} = 163$. So, from the extremality property of continued fractions we obtain that

$$\left| n \frac{\log \gamma}{\log 5} - c \right| > \frac{1}{(a_M + 2)n} = \frac{1}{165n}. \tag{4.4}$$

Comparing estimates (4.3) and (4.4), we get that

$$\frac{1}{165n} < \frac{6}{\log 5 \times 5^{0.3c}}.$$

Since $c > (n-1)/1.83$, leading to

$$0.263n < 6.69 + \log n,$$

which implies that $0 \leqslant n < 40$, this yields $c < (n + 0.91)/1.82 < 23$. Finally, we use a program written in Mathematica to find the solutions of (1.2) in the range $0 \leqslant \max\{a, b\} \leqslant c < 23$ and $n < 40$. Quickly, the program returns the only solutions of (1.2) are $(n, a, b, c) \in \{(0, 1, 0, 1), (1, 1, 0, 1), (3, 1, 2, 2)\}$. This completes the proof. $\square$

5. Conclusion

In this paper, we solve the Diophantine equation (1.1) by using Matveev's result Lemma 2.1 and Lemma 2.2 from Diophantine approximation to reduce the upper bounds on the variables of the equation. For the Diophantine equation (1.2), we find its all solutions by using Matveev's result Lemma 2.1 and the properties of continued fractions to reduce the upper bounds on the variables of the equation.

6. Final comments

We remark that we can use our approach to conclude that there are only finitely many solutions which are effectively computable for the Diophantine equation $P_n = \pm 2^a \mp 3^b + 5^c$, $Q_n = \pm 2^a \mp 3^b + 5^c$ in nonnegative integers n, a, b, c with $0 \leqslant \max\{a,b\} \leqslant c$. We leave this as a problem for other readers.

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