Two problems related to the Bernoulli numbers

FERENC MÁTYÁS*

Abstract. In this paper we deal with two similar problems. First we look for those polynomials $f_k(n)$ with rational coefficients for which the equality $S_k(n)=1^k+2^k+\cdots+n^k=(f_k(n))^m$ holds for every positive integer n with some positive integer k and $m(\geq 2)$. In our first theorem we prove for $m\geq 2$ that $S_k(n)=(f_k(n))^m$ holds for every positive integer n if and only if $m=2$, $k=3$ and $f_3(n)=\frac{1}{2}n^2+\frac{1}{2}n$. In the second part of this paper we look for those polynomials $f(n)$ with complex coefficients for which the equality

$$P_k(n,c)=\sum_{j=k}^{2n-2} n^{\frac{j-c}{2n-j}}\binom{2n-1}{j}B_{2n-j}=(f(n))^m$$

holds for every integer $n\geq k$ with some integer $m\geq 2$, where $k\in\{2,3,4\}$, B_j is the j^{th} Bernoulli number and c is a complex parameter. In our second theorem we prove for $m\geq 2$ that $P_2(n,c)=(f(n))^m$ holds for every integer $n\geq 2$ if and only if $m=2$, $c=1\pm i2\sqrt{2}$ and $f(n)=n+p$ where $p=-1\pm i\frac{\sqrt{2}}{2}$; while in the cases of $k=3$ or 4 and $m\geq 2$ the equality $P_k(n,c)=(f(n))^m$ doesn't hold for any polynomial $f(n)$.

Let us introduce the following notations: $\binom{n}{k}$ is the usual binomial coefficient; B_j is the j^{th} Bernoulli number defined by the recursion

$$\sum_{j=0}^{k-1}\binom{k}{j}B_j=0 \quad (k\geq 2) \tag{1}$$

with $B_0 = 1$. $S_k(n) = 1^k + 2^k + \cdots + n^k$ ($n \geq 1$, $k \geq 1$ are integers); $P_k(n,c) = \sum_{j=k}^{2n-2} n^{\frac{j-c}{2n-j}}\binom{2n-1}{j}B_{2n-j}$, where $n\geq k\geq 2$ are integers and c is a complex parameter; $f_k(n)$ and $f(n)$ are polynomials of n with rational and complex coefficients, respectively.

The problem of looking for those polynomials $f_k(n)$ and integers $m\geq 2$ for which $S_k(n) = \left(f_k(n)\right)^m$ for every positive integer n was proposed and

* Research supported by the Hungarian OTKA foundation, N° T 020295.

solved in [1] and the author of this paper also was among the solvers. In our Theorem 1., using the Bernoulli numbers, we give a new proof for this problem.

Theorem 1. If $m \geq 2$ is an integer then there exists a polynomial $f_k(n)$ such that $S_k(n) = \left(f_k(n)\right)^m$ for every positive integer n if and only if $m = 2$, $k = 3$ and $f_3(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

Proof. It is known that $S_k(n)$ can be expressed by the Bernoulli numbers and the binomial coefficients, that is

$$\begin{aligned} S_k(n) = \frac{1}{k+1}\Bigg(\binom{k+1}{0}B_0 n^{k+1} + \binom{k+1}{1}B_1 n^k \\ + \cdots + \binom{k+1}{k-1}B_{k-1}n^2 + \binom{k+1}{k}B_k n\Bigg) \end{aligned} \tag{2}$$

moreover

$$\begin{aligned} &B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \dots \quad \text{and} \\ &B_j = 0 \quad \text{if and only if } j \geq 3 \text{ and } j \text{ is odd.} \end{aligned} \tag{3}$$

Let $f_k(n) = a_j n^j + \cdots + a_1 n + a_0$ be a polynomial of n over the rationals and $a_j \neq 0$. If $S_k(n) = \left(f_k(n)\right)^m$ for some m, then by (2) $a_0 = 0$ follows and since $m \geq 2$ so the degree of the polynomial $\left(f_k(n)\right)^m$ is at least two, that is $B_k = 0$ in (2). But by (3) $B_k = 0$ implies that $k \geq 3$, k is an odd integer and $B_{k-1} \neq 0$.

From the equality $S_k(n) = \left(f_k(n)\right)^m$ it follows that

$$\frac{1}{k+1}\left(\binom{k+1}{0}B_0 n^{k+1} + \cdots + \binom{k+1}{k-1}B_{k-1}n^2\right) = a_j^m n^{mj} + \cdots + a_1^m n^m$$

and from this we get $m = 2$ and $a_1 \neq 0$.

So we have to investigate the equality

$$\frac{1}{k+1}\left(n^{k+1} + \cdots + \binom{k+1}{k-1}B_{k-1}n^2\right) = \left(a_j n^j + \cdots + a_1 n\right)^2 \tag{4}$$

from which we obtain that $k + 1 = 2j$ and $\frac{1}{k+1} = a_j^2$. Moreover a_j is a rational number, therefore $k + 1 = 4f$ and $j = 2f$. (4) can be written in the following form:

$$\begin{aligned} \frac{1}{k+1}\left(n^{k+1} + \cdots + \binom{k+1}{k-1}B_{k-1}n^2\right) \\ = \left(a_{2f}n^{2f} + \cdots + a_2 n^2 + a_1 n\right)^2 \end{aligned} \tag{5}$$

and from (5) by $a_1 \neq 0$ and (3) $a_2 = a_4 = \ldots + a_{2f-2} = 0$ follows. But $2a_{2f}a_1 = \frac{1}{k+1}\binom{k+1}{k-2f}B_{k-2f} \neq 0$, that is $B_{k-2f} = B_{4f-1-2f} = B_{2f-1} \neq 0$. It implies that $2f - 1 = 1$, and so $f = 1$, $j = 2$ and $k = 3$. Thus we have got the only solution $m = 2$, $k = 3$, $f_3(n) = \frac{1}{2}n^2 + \frac{1}{2}n$ and $S_3(n) = \left(\frac{1}{2}n^2 + \frac{1}{2}n\right)^2$

In [2], using the definition $P_k(n,c)$, one can find the proof of the equality $\frac{1}{n}P_2(n,3) = n - 3 + \frac{3}{2n}$. In our Theorem 2. we generalize this result and the proof will be similar to that proof which was sent for the original problem by the author of this paper.

Theorem 2. a) If $m \geq 2$ is an integer then there exists a polynomial $f(n)$ such that $P_2(n,c) = (f(n))^m$ for every integer $n \geq 2$ if and only if $m = 2$, $c = 1 \pm i2\sqrt{2}$ and $f(n) = n - 1 \pm i\frac{\sqrt{2}}{2}$.

b) If $m \geq 2$ and $k = 3$ or 4 then the equality $P_k(n,c) = (f(n))^m$ can not be solved by any polynomial $f(n)$ and parameter c.

Proof. One can easily verify the following equality:

$$(6) \qquad \frac{j-c}{2n-j}\binom{2n-1}{j} = \binom{2n-1}{2n-j} - \frac{c}{2n}\binom{2n}{2n-j}.$$

Using (6), we have

$$(7) \qquad P_k(n,c) = n\sum_{j=k}^{2n-2}\binom{2n-1}{2n-j}B_{2n-j} - \frac{c}{2}\sum_{j=k}^{2n-2}\binom{2n}{2n-j}B_{2n-j}.$$

By the recursive definition (1) of the Bernoulli number we can write that

$$(8) \qquad \begin{aligned}\sum_{j=k}^{2n-2}\binom{2n-1}{2n-j}B_{2n-j} &= -\binom{2n-1}{k-1}B_{k-1} \\ &-\binom{2n-1}{k-2}B_{k-2} - \cdots - \binom{2n-1}{1}B_1 - \binom{2n-1}{0}B_0\end{aligned}$$

and

$$(9) \qquad \begin{aligned}\sum_{j=k}^{2n-2}\binom{2n}{2n-j}B_{2n-j} &= -\binom{2n}{2n-1}B_{2n-1} - \binom{2n}{k-1}B_{k-1} \\ &-\binom{2n}{k-2}B_{k-2} - \cdots - \binom{2n}{1}B_1 - \binom{2n}{0}B_0.\end{aligned}$$

First let us deal with the case a) of the Theorem 2. If $k = 2$ then by (7), (8), (9) and (3)

$$P_2(n,c) = n\left(-\binom{2n-1}{1}B_1 - \binom{2n-1}{0}B_0\right)$$
$$-\frac{c}{2}\left(-\binom{2n}{2n-1}B_{2n-1} - \binom{2n}{1}B_1 - \binom{2n}{0}B_0\right) = n^2 - \frac{3+c}{2}n + \frac{c}{2}$$

follows. From this, investigating the polinomial equality $P_2(n,c) = (f(n))^m$ in the case $m \geq 2$, we can see that $m = 2$ and $f(n) = (n+p)^2$, where $p = -1 \pm i\frac{\sqrt{2}}{2}$ and $c = 1 \pm i2\sqrt{2}$.

Now let us consider the case b) of the Theorem 2.

If $k = 3$ then by (7), (8), (9), and (3)

$$P_3(n,c) = n\left(-\binom{2n-1}{2}B_2 - \binom{2n-1}{1}B_1 - \binom{2n-1}{0}B_0\right)$$
$$-\frac{c}{2}\left(-\binom{2n}{2}B_2 - \binom{2n}{1}B_1 - \binom{2n}{0}B_0 - \binom{2n}{2n-1}B_{2n-1}\right)$$
$$= \cdots = -\frac{n^3}{3} + \frac{9+c}{6}n^2 - \frac{7c+20}{12}n + \frac{c}{2}\,.$$

If $P_3(k,n) = (f(n))^m$ and $m \geq 2$ then $m = 3$ and $f(n)$ should have the form $f(n) = -\frac{1}{\sqrt[3]{3}}n + \sqrt[3]{\frac{c}{2}}$. But it is easy to verify that such complex numbers c don't exist.

If $k = 4$ then B_3 appears on the right side of (8) and (9). But $B_3 = 0$ and so $P_3(n,c) = P_4(n,c)$. Therefore $P_4(n,c) = (f(n))^m$ $(m \geq 2)$ is also unsolvable.

Remark. The statement of the Theorem 2 can also be extended for $k \geq 5$ too, but it seems, that there is no polynomial $f(n)$ such that $P_k(n,c) = (f(n))^m$ where $m \geq 2$.

References

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