An approximation problem concerning linear recurrences

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Abstract. Let $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ $(n=0,1,2,\ldots)$ be sequences of integers defined by $R_n=AR_{n-1}-BR_{n-2}$ and $V_n=AV_{n-1}-BV_{n-2}$, where A and B are fixed non-zero integers. We prove that the distence from the points $P_n(R_n,R_{n+1},V_n)$ to the line L, L is defined by $x=t,y=\alpha t,z=\sqrt{D}t$, tends to zero in some case. Moreover, we show that there is no lattice points (x,y,z) nearer to L than $P_n(R_n,R_{n+1},V_n)$ if and only if $|B|=1$.

Let $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ be second order linear recurring sequences of integers defined by

$$R_n = AR_{n-1} - BR_{n-2} \qquad (n > 1),$$
$$V_n = AV_{n-1} - BV_{n-2} \qquad (n > 1),$$

where $A > 0$ and B are fixed non-zero integers and the initial terms of the sequences are $R_0 = 0, R_1 = 1, V_0 = 2$ and $V_1 = A$. Let α and β be the roots of the characteristic polynomial $x^2 - Ax + B$ of these sequences and denote by D its discriminant. Then we have

$$(1) \qquad \sqrt{D} = \sqrt{A^2 - 4B} = \alpha - \beta, \quad A = \alpha + \beta, \quad B = \alpha\beta.$$

Throughout the paper we suppose that $D > 0$ and D is not a perfect square. In this case, α and β are two irrational real numbers and $|\alpha| \neq |\beta|$, so we can suppose that $|\alpha| > |\beta|$.
Furthermore, as it is well known, the terms of the sequences R and V are given by

$$(2) \qquad R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

Some results are known about points whose coordinate are terms of linear recurrences from a geometric points of view. G. E. Bergum [1] and A. F.

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Horadam [2] showed that the points $P_n = (R_n, R_{n+1})$ lie on the conic section $Bx^2 - Axy + y^2 + eB^n = 0$, where $e = AR_0R_1 - BR_0^2 - R_1^2$ and the intial terms R_0 and R_1 are not necessarily 0 and 1. For the Fibonacci sequence, when $A = 1$ and $B = -1$, C. Kimberling [6] characterized conics satisfied by infinetely many Fibonacci lattice points $(x, y) = (F_m, F_n)$. J. P. Jones and P. Kiss [4] considered the distance of points $P_n = (R_n, R_{n+1})$ and the line $y = \alpha x$. They proved that this distance tends to zero if and only if $|\beta| < 1$. Moreover, they showed that in the case $|B| = 1$ there is not such a lattice point (x, y) which is nearer to the mentioned line than P_n, if $|x| < |R_n|$. They proved similar arguments in three-dimensional case, too.

In this paper we investigate the geometric properties of the lattice points $P_n = (R_n, R_{n+1}, V_n)$. We shall use the following result of P. Kiss [5]: if $|B| = 1$ and p/q is a rational number such that $(p, q) = 1$, then the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{D}q^2}$$

implies that $p/q = R_{n+1}/R_n$ for some $n \geq 1$.

It is known, that

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \alpha \tag{3}$$

and

$$\lim_{n\to\infty} \frac{V_n}{R_n} = \sqrt{D} \tag{4}$$

(see. e.g. [3], [7]).

Let us consider the vectors (R_n, R_{n+1}, V_n). Since by (3) and (4) and using the equality

$$(R_n, R_{n+1}, V_n) = R_n \left(1, \frac{R_{n+1}}{R_n}, \frac{V_n}{R_n}\right)$$

we get that the direction of vectors (R_n, R_{n+1}, V_n) tends to the direction of the vector $\left(1, \alpha, \sqrt{D}\right)$. However, the sequence of the lattice points $P_n = (R_n, R_{n+1}, V_n)$ does not always tend to the line passing through the origin and parallel to the vector $\left(1, \alpha, \sqrt{D}\right)$, we give a condition when it is hold.

Theorem 1. Let L be the line defined by $x = t$, $y = \alpha t$, $z = \sqrt{D}t$, $t \in \mathbf{R}$. Futhermore, let d_n be the distance from the point (R_n, R_{n+1}, V_n) $(n = 0, 1, 2, \ldots)$ to the line L. Then $\lim_{n\to\infty} d_n = 0$ if and only if $|\beta| < 1$.

Proof. It is known that the distance from the point (x_0, y_0, x_0) to the line L is

$$(5) \qquad d_{x_0,y_0,z_0} = \sqrt{\frac{(\sqrt{D}x_0 - z_0)^2 + (\alpha x_0 - y_0)^2 (\sqrt{D}y_0 - \alpha z_0)^2}{1 + \alpha^2 + D}}.$$

By (1), (2) and (5), we have

$$(6) \qquad \sqrt{\frac{4\beta^{2n} + \left(\frac{-\alpha\beta^n + \beta^{n+1}}{\alpha - \beta}\right)^2 + (-\beta^{n+1} - \alpha\beta^n)^2}{1+\alpha^2+D}}$$
$$= \sqrt{\frac{4\beta^{2n} + \beta^{2n}\left(\frac{-\alpha+\beta}{\alpha-\beta}\right)^2 + \beta^{2n}(-\beta-\alpha)^2}{1+\alpha^2+D}} = \sqrt{\frac{\beta^{2n}(5+A^2)}{1+\alpha^2+D}} = |\beta|^n \sqrt{\frac{5+A^2}{1+\alpha^2+D}}.$$

From this the theorem follows.

It is easy to see that points P_n are on a plane. We investigate whether there is a lattice point $P = (x, y, z)$ in the plane such that $|x| < |R_n|$ and P is nearer to the line L than P_n. We use the previous denotations.

Theorem 2. The points $P_n = (R_n, R_{n+1}, V_n)$ are in a plane. Furthermore if n is sufficiently large, than there is no lattice pont (x, y, z) in this plane such that $d_{x,y,z} \leq d_n$ and $|x| < |R_n|$ if and only if $|B| = 1$.

Proof. First suppose $|B| = 1$. In this case, obviously, $|\beta| < 1$ and α is irrational, as it was supposed.

Using (2), we have

$$R_{n+1} = \alpha\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha\beta^n - \beta^{n+1}}{\alpha - \beta} = \alpha R_n + \beta^n$$

and similarly

$$R_{n+1} = \beta R_n + \alpha^n.$$

Adding these equation, we get

$$(7) \qquad 2R_{n+1} = (\alpha + \beta)R_n + V_n.$$

Consequently, the points P_n are on the plane which is defined by the equation $Ax - 2y + z = 0$. It is easy to prove that L is also on this plane. Assume that for some n there is lattice point (x, y, z) on this plane such that

$$(8) \qquad d_{x,y,z} \leq d_n$$

and $|x| < |R_n|$. Using the equation of the plane

$$(\alpha+\beta)x - 2y + z = 0 \tag{9}$$

we get the following equalities

$$\left|\sqrt{D}x - z\right| = |(\alpha-\beta)x - z|$$
$$= |(\alpha x - (\beta x + z)| = |\alpha x - (2y - \alpha x)| = 2\,|\alpha x - y|$$

and

$$\left|\sqrt{D}y - \alpha z\right| = |(\alpha-\beta)y - (2\alpha y - \alpha(\alpha+\beta)x)| = |\alpha+\beta|\,|\alpha x - y|\,. \tag{11}$$

Thus, from (1), (5), (6), (8), (10) and (11) we obtain the inequality

$$d_{x,y,z} = \sqrt{\frac{A^2+5}{1+\alpha^2+D}}\,|\alpha x - y| \le |\beta|^n \sqrt{\frac{A^2+5}{1+\alpha^2+D}},$$

and so using $|x| < |R_n|$ and (1), we get

$$\left|\alpha - \frac{y}{x}\right| \le \frac{|\beta|^n}{|x|} = \frac{1}{|\alpha|^n\,|x|} = \frac{1-(\beta/\alpha)^n}{|R_n|\sqrt{D}\,|x|} < \frac{1-(\beta/\alpha)^n}{\sqrt{D}|x|^2}.$$

From this, using the mentioned theorem of P. Kiss and its proof, we obtain $x = R_i$, $y = R_{i+1}$ and by (9) $z = 2y - (\alpha+\beta)x = V_n$, for some i, if n is sufficiently large. Thus $d_{x,y,z} \le d_n$. But by (6), $d_k < d_n$, only if $k > n$, so $i \ge n$. It can be seen that $|R_t|, |R_{t+1}|, \ldots$ is an increasing sequence if t is sufficiently large, so $|x| = |R_i| \ge |R_n|$, which contradicts the assumption $|x| < |R_n|$.

To complete the proof, we have to show that in the case $|B| > 1$ there are lattice points (x,y,z) for which $d_{x,y,z} \le d_n$ and $|x| < |R_n|$ for some n.

Suppose $|B| > 1$. If $|\beta| > 1$, then by (6), $d_n \to \infty$ as $n \to \infty$, so there are such lattice points for any sufficiently large n.

If $|\beta| = 1$ the d_n is a constant and there are infinetely many n and points (x,y,z) which fulfill the assumptions.

Suppose $|\beta| < 1$. Let y/x be a convergent of the simple continued fraction expansion of the irrational α. Then, by the elementary properties of continued fraction expansions of irrational numbers and by (10), (11), we have the inequalities

$$|\alpha x - y| < \frac{1}{x},$$
$$\left|\sqrt{D}x - z\right| = 2\,|\alpha x - y| < \frac{2}{x},$$
$$\left|\sqrt{D}y - \alpha z\right| = |\alpha+\beta|\,|\alpha x - y| < |\alpha+\beta|\,\frac{1}{x}.$$

Using by (5) we obtain

$$d_{x,y,z} < \frac{1}{|x|}\sqrt{\frac{A^2+5}{1+\alpha^2+D}}. \tag{12}$$

Let the index n be defined by $|R_{n-1}| \leq |x| < |R_n|$. For this n, by (1), (2), (6) and (12), we have

$$\begin{aligned} d_n &= |\beta|^n\sqrt{\frac{A^2+5}{1+\alpha^2+D}} = \frac{|B|^n}{|\alpha|^n}\sqrt{\frac{A^2+5}{1+\alpha^2+D}} \\ &= \frac{|B|^n}{|\alpha|^{n-1}}\cdot\frac{1}{|\alpha|}\sqrt{\frac{A^2+5}{1+\alpha^2+D}} = \frac{\left(1-(\beta/\alpha)^{n-1}\right)}{|\alpha|\sqrt{D}\,|R_{n-1}|}|B|^n\sqrt{\frac{A^2+5}{1+\alpha^2+D}} \\ &> \sqrt{\frac{A^2+5}{1+\alpha^2+D}}\cdot\frac{1}{x} > d_{x,y,z} \end{aligned}$$

if n is sufficiently large, since $|B| > 1$.

This shows that, for any lattice point (x, y, z) defined as above, there is an n such that $d_{x,y,z} < d_n$ and $|x| < |R_n|$. This completes the proof.

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