Norm convergence of Fejér means of certain functions with respect to UDMD product systems

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Abstract. In this paper we investigate the norm convergence of Feje'r means of functions belonging to Lipschitz classes in that case when the orthonormal system is the unitary dyadic martingale difference system (UDMD system). We give an estimation of the order of norm convergence. The result of the paper shows a sharp constrast between the corresponding known statement which relatives to the ordinary Walsh-Paley system.

Introduction

Let $\mathbf{N}$ denote the set of natural numbers, $\mathbf{P}$ denote the set of positiv intergers, and $\mathbf{A} = \{0,1\}$. For each $m \in \mathbf{N}$ let $(m^{(j)}, j \in \mathbf{N})$ represent the binary coefficient of m, that is,

$$m = \sum_{j=0}^{\infty} m^{(j)} 2^j \quad (m^{(j)} \in \mathbf{A}, j \in \mathbf{N}).$$

Let (Ω, λ) be a measure space with $\lambda(\Omega) = 1$ and $\Phi := (\phi_j, j \in \mathbf{N})$ be a sequence of λ-measurable functions on Ω which are a.e. $[\lambda]$ bounded by 1. The product system generated by Φ is the sequence of functions $\Psi := (\psi_m, m \in \mathbf{N})$ defined by

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m^{(j)}}$$

for $m \in \mathbf{N}$. Each ψ_m is a finite product of ϕ_n's, $\psi_0(x) = 1$ for $x \in \Omega$, $|\psi_m| \leq 1$ for $m \in \mathbf{N}$, and $\phi_n = \psi_{2^n}$ for $n \in \mathbf{N}$.

Let $\Psi := (\psi_m, m \in \mathbf{N})$ be any orthonormal system on Ω. The Fourier coefficients of a function $f \in L^1(\Omega, \lambda)$ are defined by

$$\{f, \psi_m\} := \int_{\Omega} f \overline{\psi_m} d\lambda \quad (m \in \mathbf{N}).$$

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The Fourier series of f with respect to the system Ψ is the series

$$\sum_{m=0}^{\infty} \{f, \psi_m\}\psi_m.$$

The partial sums of order n of the Fourier series of f are defined by

$$S_n^{\Psi} f := \sum_{m=0}^{n-1} \{f, \psi_m\}\psi_m$$

for $n \in \mathbf{P}$.

The Cesaro means of order n of the Fourier series of an f are defined by

$$\sigma_n^{\Psi} f := \frac{1}{n} \sum_{m=1}^{n} S_m^{\Psi} \quad (n \in \mathbf{P}).$$

Denote the dyadic, or 2-series, group by $(G, +)$. Thus G consists of sequences $x := (x^{(j)}, j \in \mathbf{N})$ with $x^{(j)} = 0$, or 1 and addition $+$ is coordinatewise, modulo 2.

Let $\Omega = [0,1)$ or G. By the additive digits $(x^{(j)}, j \in \mathbf{N})$ of an $x \in \Omega$ we shall mean the coordinates of $x = (x^{(0)}, x^{(1)}, \dots)$ if $x \in G$ and the binary coefficients of $x = \sum_{j=0}^{\infty} x^{(j)} 2^{-j-1}$ if $x \in [0,1)$, where the finite binary expansion of x is used when x is a dyadic rational.

Let λ be the Lebesgue measure when $\Omega = [0,1)$ and Haar measure when $\Omega = G$. Denote the corresponding Lebesgue spaces by $L^p(\Omega)$ for $0 < p \leq \infty$. By a dyadic interval of rank n in $\Omega = [0,1)$ we mean an interval of the form $\left[\frac{p}{2^n}, \frac{(p+1)}{2^n}\right)$ where $0 \leq p < 2^n$ and $n \in \mathbf{N}$. Given $a \in [0,1)$ and $n \in \mathbf{N}$, there is one and only one interval of rank n which contains a. Let it be denoted by $I_n(a)$. By a dyadic interval of rank n centered at $a \in \Omega = G$ we mean a set of the form

$$I_n(a) = \{x \in G : x^{(k)} = a^{(k)}, k \leq n\}.$$

Denote the σ-algebra generated by the intervals $I_n(a)$ $(a \in \Omega)$ by $\mathcal{A}_n$. The intervals $I_n(a)$ $(a \in \Omega)$ are called the atoms of $\mathcal{A}_n$. Each element of $\mathcal{A}_n$ is a finite disjoint union of atoms.

A function f defined on Ω is said to be $\mathcal{A}_n$-measurable if f is constant on the atoms of $\mathcal{A}_n$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on Ω. Each $f \in L(\mathcal{A}_n)$ is integrable.

Let the Rademacher system on Ω be denoted by $\{r_n : n \in \mathbf{N}\}$, that is

$$r_n(x) = (-1)^{x^{(n)}} \quad (n \in \mathbf{N})$$

where $\{x^{(j)} : j \in \mathbf{N}\}$ represents the additive digits of x. Let $\alpha_n : \Omega \to \mathbf{C}$ be a function such that each α_n is $\mathcal{A}_n$-measurable for any $n \in \mathbf{N}$. A sequence $\{\phi_n : n \in \mathbf{N}\}$ is said to be a UDMD system if

$$\phi_n := r_n \alpha_n \quad (n \in \mathbf{N})$$

for some $\mathcal{A}_n$-measurable functions α_n and if $|\phi_n(x)| = 1$ $(n \in \mathbf{N}, x \in \Omega)$ (see [3]). The simplest UDMD system is the Rademacher system. The product system generated by the UDMD system is said to be UDMD product system.

If $\{\psi_m : m \in \mathbf{N}\}$ is a UDMD product system then it is orthonormal on $L^2(\Omega)$.

The Dirichlet kernels of the product system $\Psi := \{\psi_n : n \in \mathbf{N}\}$ are defined as follows $D_0^{\Psi}(x,t) := 0$ and

$$D_n^{\Psi}(x,t) := \sum_{j=0}^{n-1} \psi_j(x)\overline{\psi_j}(t) \quad (x, t \in \Omega).$$

for $n \in \mathbf{P}$. The partial sums $S_n^{\Psi} f$ can be expressed using the Dirichlet kernels D_n^{Ψ} :

$$(S_n^{\Psi} f)(x) = \int_{\Omega} f(t) D_n^{\Psi}(x,t) d\lambda(t).$$

The subsequence $D_{2^n}^{\Psi}$ has a closed form. For every $n \in \mathbf{N}$

$$(1) \qquad D_{2^n}^{\Psi}(x,t) = \prod_{j=0}^{n-1} (1 + \phi_j(x)\overline{\phi_j}(t)) = \begin{cases} 2^n, & \text{if } t \in I_n(x) \\ 0, & \text{if } t \notin I_n(x). \end{cases}$$

The Fejér kernels of the product system Ψ are defined by $K_0^{\Psi}(x,t) := 0$ and

$$K_n^{\Psi}(x,t) := \frac{1}{n} \sum_{m=0}^{n} D_m^{\Psi}(x,t) \quad (x, t \in \Omega)$$

for $n \in \mathbf{N}$. The Cesaro means of a Fourier series $S_j^{\Psi} f$ can be expressed using the Fejér kernels:

$$(\sigma_n^{\Psi} f)(x) = \int_{\Omega} f(t) K_n^{\Psi}(x,t) d\lambda(t).$$

Introduce the following notation:

$$(\phi_k \otimes \overline{\phi_k})(x,t) := \phi_k(x)\overline{\phi_k}(t) \quad (x, t \in \Omega, k \in \mathbf{N}).$$

For every $n \in \mathbf{N}$ and $x \in \Omega$ (see [4]):

$$(2) \qquad K_{2^n}^{\Psi} = 2^{-n} D_{2^n}^{\Psi} + \sum_{j=0}^{n-1} 2^{j-n} \prod_{\substack{k=0 \\ j \neq k}}^{n-1} (1 + \phi_k \otimes \overline{\phi_k}),$$

$$\int_{\Omega} |K_n^{\Psi}(x,t)| d\lambda(t) \le 8. \tag{3}$$

Let X be a Banach space with norm $\| \ \|$. The space X is called a homogeneous Banach space if $P \subseteq X \subseteq L^1(\Omega)$ where P is the set of Walsh polynomials, $\tau_x f(y) := f(y+x)$ and if the following three properties hold (see [1]):

$$\|f\|_1 \le \|f\| \quad (f \in X), \tag{i}$$

$$\tau_x f \in X, \quad \|\tau_x f\| = \|f\| \quad (x \in \Omega, f \in X), \tag{ii}$$

and, for a given $f \in X$ there is a sequence of Walsh polynomials $(P_n, n \in \mathbf{N})$ such that

$$\lim_{n \to \infty} \|P_n - f\| = 0. \tag{iii}$$

Define the modulus of continuity in X of an $f \in X$ by

$$\omega^X(f,\delta) := \sup_{|y|<\delta} \|f - \tau_y f\| \quad (\delta > 0).$$

For each $\alpha > 0$, Lipschitz classes in X can be defined by

$$\mathrm{Lip}(\alpha, X) := \{f \in X : \omega^X(f,\delta) = O(\delta^\alpha) \ \text{ as } \ \delta \to \infty\}.$$

Results

THEOREM. Suppose $f \in \mathrm{Lip}(\alpha, X)$ and $\alpha > 0$. Then

$$\|\sigma_n f - f\| = \begin{cases} O(n^{-\alpha}) & 0 < \alpha < \frac{1}{2} \\ O(\frac{\log n}{\sqrt{n}}) & \alpha = \frac{1}{2} \\ O(\frac{1}{\sqrt{n}}) & \alpha > \frac{1}{2}, \end{cases}$$

as $n \to \infty$.

PROOF. Let $n \in \mathbf{P}$ and choose $s \in \mathbf{N}$ such that $2^s \le n < 2^{s+1}$. Let

$$\Delta(s) := 2^{-s\alpha} + 2^{-\frac{s}{2}} \sum_{k=0}^{s-1} 2^{k(\frac{1}{2}-\alpha)}.$$

First of all we will show that

$$\|\sigma_{2^s} f - f\| = O(\Delta(s)), \quad \text{as } s \to \infty.$$

Since

$$\int_\Omega K_{2^s}^\Psi(x,t)d\lambda(t) = 1$$

we have

$$\sigma_{2^s}^\Psi f(x) - f(x) = \int_\Omega f(t)K_{2^s}^\Psi(x,t)d\lambda(t) - f(x) = \int_\Omega (f(t)-f(x))K_{2^s}^\Psi(x,t)d\lambda(t)$$

for any $f \in L^1(\Omega)$ and any $x \in \Omega$. For any $t \in I_s(x)$ we have

$$|K_{2^s}^\Psi(x,t)| \leq 2^{s-1}.$$

A disjoint decomposition of Ω is

$$\Omega = I_s(x) \bigcup \left(\bigcup_{k=0}^{s-1} I_k(x) \backslash I_{k+1}(x) \right)$$

for any $x \in \Omega$. Let $I_k(x)\backslash I_{k+1}(x)$ be denoted by L_k^x. The following inequality holds for any $x \in \Omega$ and any $f \in \mathrm{Lip}(\alpha, X)$:

$$|\sigma_{2^s}^\Psi f(x) - f(x)| \leq \int_\Omega |f(t) - f(x)||K_{2^s}^\Psi(x,t)|d\lambda(t) \leq$$

$$\int_{I_s(x)} |f(t) - f(x)||K_{2^s}^\Psi(x,t)|d\lambda(t) +$$

$$+ \sum_{k=0}^{s-1} \int_{L_k^x} |f(t) - f(x)||K_{2^s}^\Psi(x,t)|d\lambda(t) \leq$$

$$2^{s-1} \int_{I_s(x)} |f(t) - f(x)|d\lambda(t) +$$

$$\sum_{k=0}^{s-1} 2^{-s} \int_{L_k^x} |f(t) - f(x)| D_{2^s}^\Psi(x,t)d\lambda(t) +$$

$$\sum_{k=0}^{s-1} \sum_{j=0}^{s-1} 2^{j-s} \int_{L_k^x} |f(t) - f(x)| \prod_{\substack{l=0 \\ l \neq j}}^{s-1} |1 + \phi_l(x)\overline{\phi_l}(t)|d\lambda(t) \leq$$

$$2^{s-1}2^s\omega^X(f,2^{-s})+\sum_{k=0}^{s-1}2^{k-s}\int_{L_k^x}|f(t)-f(x)|\prod_{\substack{l=0\\ l\neq k}}^{s-1}|1+$$

$$\phi_l(x)\overline{\phi_l}(t)|d\lambda(t)\leq\frac{1}{2}\omega^X(f,2^{-s})+\sum_{k=0}^{s-1}2^{k-s}\omega^X(f,2^{-k})$$

$$\int_{L_k^x}\prod_{\substack{l=0\\ l\neq k}}^{s-1}|1+\phi_l(x)\overline{\phi_l}(t)|d\lambda(t)\leq\omega^X(f,2^{-s})+\sum_{k=0}^{s-1}2^{\frac{k}{2}-\frac{s}{2}}\omega^X(f,2^{-k}).$$

From this inequality we have

$$\|\sigma_{2^s}^{\Psi}f-f\|\leq\omega^X(f,2^{-s})+\sum_{k=0}^{s-1}2^{\frac{k}{2}-\frac{s}{2}}\omega^X(f,2^{-k})=O(\Delta(s))\quad\text{as } s\to\infty.$$

for any $f\in\mathrm{Lip}(\alpha,X)$.
We have used the following result:

$$\int_{L_k^x}\prod_{\substack{l=0\\ l\neq k}}^{s-1}|1+\phi_l(x)\overline{\phi_l}(t)|d\lambda(t)\leq 2^{\frac{s}{2}-\frac{k}{2}}.$$

To prove this, let $K'_{s-1}(x,t):=\prod_{\substack{l=0\\ l\neq k}}^{s-1}(1+\phi_l(x)\overline{\phi_l}(t))$ and suppose for a moment that

$$\sqrt{\int_{L_k^x}|K'_{s-1}(x,t)|^2d\lambda(t)}\leq\sqrt{2^s}.\tag{4}$$

Using the Cauchy–Buniakovski inequalities we have

$$\int_{L_k^x}|K'_{s-1}(x,t)|d\lambda(t)\leq\sqrt{\lambda(L_k^x)}\sqrt{\int_{L_k^x}|K'_{s-1}(x,t)|^2d\lambda(t)}$$
$$\leq\sqrt{\frac{1}{2^{k+1}}}\sqrt{2^s}\leq\sqrt{2^{s-k}}.$$

Now we will prove (4) (see [4]):

$$\int_{L_k^x}|K'_{s-1}(x,t)|^2d\lambda(t)=\int_{L_k^x}\prod_{\substack{l=0\\ l\neq k}}^{s-1}(1+\phi_l(x)\overline{\phi_l}(t))\prod_{\substack{j=0\\ j\neq k}}^{s-1}(1+\overline{\phi_j}(x)\phi_j(t))d\lambda(t)=$$

$$\int_{L_k^x} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_{s-1}\in\{0,1\}\\ \tilde\varepsilon_1,\ldots,\tilde\varepsilon_{s-1}\in\{0,1\}\\ \varepsilon_{s-1}=\varepsilon_{s-1}=0}} \phi_1^{\varepsilon_1}(x)\overline{\phi}_1^{\tilde\varepsilon_1}(x)\overline{\phi}_1^{\varepsilon_1}(t)\phi_1^{\tilde\varepsilon_1}(t)\ldots$$

$$\phi_{s-1}^{\varepsilon_{s-1}}(x)\overline{\phi}_{s-1}^{\tilde\varepsilon_{s-1}}(x)\overline{\phi}_{s-1}^{\varepsilon_{s-1}}(t)\phi_{s-1}^{\tilde\varepsilon_{s-1}}(t)d\lambda(t):=$$

$$:=\sum_{\varepsilon,\tilde\varepsilon\in\{0,1\}^{s-2}\times\{0\}}\int_{L_k^x}\chi_\varepsilon(x)\overline{\chi_{\tilde\varepsilon}}(x)\overline{\chi_\varepsilon}(t)\chi_{\tilde\varepsilon}(t)d\lambda(t).$$

$\int_{L_k^x}\chi_\varepsilon(x)\overline{\chi_{\tilde\varepsilon}}(x)\overline{\chi_\varepsilon}(t)\chi_{\tilde\varepsilon}(t)d\lambda(t)\neq 0$ if $\varepsilon_{k+1}=\tilde\varepsilon_{k+1},\ldots,\varepsilon_{s-1}=\tilde\varepsilon_{s-1}$. (see [3], [4]). From this fact and $|\phi_j(x)|=1$ $(j\in\mathbf{N}, x\in\Omega)$ we get

$$\int_{L_k^x}|K_{s-1}'(x,t)|^2d\lambda(t)\le\sum_{\substack{\varepsilon,\tilde\varepsilon\in\{0,1\}^{s-2}\times\{0\}\\ \varepsilon_{k+1}=\tilde\varepsilon_{k+1},\ldots,\varepsilon_{s-1}=\tilde\varepsilon_{s-1}}}\int_{L_k^x}1\,d\lambda(t)\le\frac{1}{2^k}2^k2^s=2^s.$$

Let $P:=S_{2^s}^\Psi f$ and observe that

$$\sigma_n^\Psi f-f=\sigma_n^\Psi(f-P)+(P-f)+(\sigma_n^\Psi P-P).$$

Using the fact $(S_iS_jf)(x)=(S_{\min(i,j)}f)(x)$ we can show

$$\sigma_n^\Psi P-P=\frac{1}{n}\sum_{i=1}^{2^s}(S_if-S_{2^s}f)=\frac{2^s}{n}(\sigma_{2^s}^\Psi P-P).$$

From the inequality

$$|f(t)-P(t)|\le\omega^X(f,2^{-s})$$

we have

$$\|f-P\|\le\omega^X(f,2^{-s})$$

and

$$|\sigma_n^\Psi(f-P)(x)|\le\int_\Omega|f(t)-P(t)||K_n^\Psi(x,t)|d\lambda(t)$$

$$\le\omega^X(f,2^{-s})\int_\Omega|K_n^\Psi(x,t)|d\lambda(t)\le 8\omega^X(f,2^{-s}),$$

that is

$$\|\sigma_n^\Psi(f-P)\|\le 8\omega^X(f,2^{-s}).$$

$$\|\sigma_n^{\Psi} f - f\| \le \|\sigma_n^{\Psi}(f-P)\| + \|P - f\| + \|\sigma_n^{\Psi} P - P\|$$
$$\le 9\omega^X(f, 2^{-s}) + \|\sigma_{2^s}^{\Psi} P - P\|.$$

Since

$$\|\sigma_{2^s}^{\Psi} P - P\| = \|S_{2^s}^{\Psi}(\sigma_{2^s}^{\Psi} f - f\| \le \|\sigma_{2^s}^{\Psi} f - f\|$$

$\|\sigma_n^{\Psi} f - f\|$ can be estimated by $\Delta(s)$.
For $0 < \alpha < \frac{1}{2}$ we have

$$\Delta(s) = O\left(2^{-s\alpha} + 2^{-\frac{s}{2}} 2^{s(\frac{1}{2}-\alpha)}\right) = O(2^{-s\alpha}) = O(n^{-\alpha}) \quad \text{as } n \to \infty$$

For $\alpha > \frac{1}{2}$ we have

$$\Delta(s) = 2^{-\frac{s}{2}\alpha} + 2^{-s} \sum_{k=0}^{s-1} 2^{k(\frac{1}{2}-\alpha)} = O\left(n^{-\alpha} + \frac{1}{\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right) \quad as\ n \to \infty.$$

For $\alpha = \frac{1}{2}$ we have

$$\Delta(s) = 2^{-\frac{s}{2}} + s 2^{-\frac{s}{2}} = O\left(\frac{s}{\sqrt{n}}\right) = O\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{as } n \to \infty.$$

This completes the proof of the theorem.

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