Unitary subgroup of the Sylow p-subgroup of the group of normalized units in an infinite commutative group ring

ATTILA SZAKÁCS

Abstract. Let G be an abelian group, K a ring of prime characteristic p and let V(KG) denote the group of normalized units of the group ring KG. An element $u = \sum_{g\in G} \alpha_g g \in V(KG)$ is called unitary if u^{-1} coincides with the element $u^* = \sum_{g\in G} \alpha_g g^{-1}$. The set of all unitary elements of the group $V(KG)$ forms a subgroup $V_*(KG)$.

S. P. Novikov had raised the problem of determining the invariants of the group $V_*(KG)$ when G has a p-power order and K is a finite field of characteristic p. This was solved by A. Bovdi and the author. Here we give the invariants of the unitary subgroup of the Sylow p-subgroup of $V(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring of odd prime characteristic p without nilpotent elements.

1. Introduction

Let G be an abelian group, K a ring of prime characteristic p and let V(KG) denote the group of normalized units (i.e. of augmentation 1) of the group ring KG. We say that for $x = \sum_{g\in G} \alpha_g g \in KG$ the element $x^* = \sum_{g\in G} \alpha_g g^{-1}$ is conjugate to x, and if $x^* = x$, then x is selfconjugate. It can be seen that the map $x \to x^*$ is an anti-isomorphism (involution) of the ring KG. An element $u \in V(KG)$ is called unitary if $u^{-1} = u^*$. The set of all unitary elements of the group $V(KG)$ obviously forms a subgroup, which we therefore call the unitary subgroup of $V(KG)$, and we denote it by $V_*(KG)$.

Let G^p denote the subgroup $\{g^p \mid g \in G\}$ of p-th powers elements of G and ω an arbitrary ordinal. The subgroup G^{p^ω} of the group G is defined by transfinite induction in following way: $G^{p^0} = G$, for non-limited ordinals

Research supported (partially) by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T014279.

(that is if $\omega = \nu + 1$): $G^{p^\omega} = \left(G^{p^\nu}\right)^p$, and if ω is a limited ordinal, then $G^{p^\omega} = \bigcap_{\nu<\omega} G^{p^\nu}$.

The subring K^{p^ω} of the ring K is defined similarly. The ring K is called p-divisible if $K^p = K$.

Let $G[p]$ denote the subgroup $\{g \in G \mid g^p = 1\}$ of G. The factorgroup $G^{p^\omega}[p]/G^{p^{\omega+1}}[p]$ can be considered as a vector space over $GF(p)$ the field of p elements, and the cardinality of a basis of this vector space is called the ω-th Ulm–Kaplansky invariant $f_\omega(G)$ of the group G concerning p.

S. P. Novikov had raised the problem of determining the invariants of the group $V_*(KG)$ when G has a p-power order and K is a finite field of characteristic p. This was solved by A. Bovdi and the author in [1]. Here we continue this work by giving the Ulm–Kaplansky invariants of the unitary subgroup $W(KG)$ of the Sylow p-subgroup $V_p(KG)$ of $V(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring of odd prime characteristic p without nilpotent elements.

Theorem. *Let ω be an arbitrary ordinal, K a commutative ring of odd prime characteristic p without nilpotent elements, P the maximal divisible subgroup of the Sylow p-subgroup S of an abelian group G, $G_\omega = G^{p^\omega}$, $S_\omega = S^{p^\omega}$ and $K_\omega = K^{p^\omega}$. Let, further on, $V_p = V_p(KG)$ denote the Sylow p-subgroup of the group $V = V(KG)$ of normalized units in the group ring KG and $W = W(KG)$ the unitary subgroup of $V_p(KG)$. In case $P \neq 1$ we assume that the ring K is p-divisible. If $G_\omega \neq G_{\omega+1}$, $S_\omega \neq 1$ and at least one of the ordinals $|K_\omega|$ or $|G_\omega|$ is infinite, then the ω-th Ulm–Kaplansky invariant $f_\omega(W)$ of the group W concerning p equals*

$$f_\omega(W) = f_\omega(V_p) = \max\{|G_\omega|, |K_\omega|\}.$$

PROOF. Note that if $G_\omega = G_{\omega+1}$ or $S_\omega = 1$ then, according to [2], $f_\omega(V_p) = 0$ and hence $f_\omega(W) = 0$.

Let $C(KG)$ denote the subgroup of selfconjugate elements of the group $V_p(KG)$. Then the following statements are true:

$$V_p(KG) = C(KG) \times W(KG)$$

and

$$W(KG) = \{x^{-1}x^* \mid x \in V_p(KG)\}.$$

Really, if $x \in C(KG) \bigcap W(KG)$, then $x = x^*$ and $xx^* = 1$. Hence $x^2 = 1$ and since $p > 2$, it follows that $x = 1$. Therefore, $C(KG) \times W(KG)$ is a subgroup of $V_p(KG)$. Let H be a finite subgroup of the group G and ψ

the map, defined in following way: $\psi(x) = x^{-1}x^* \, (x \in V_p(KH))$. Then ψ is an endomorphism of the group $V_p(KH)$, $\psi(V_p(KH)) \subseteq W(KH)$ and the kernel of ψ coincides with the subgroup $C(KH)$. Hence the index of the group $V_p(KH)$ by the subgroup $C(KH)$ not greater than the order of the group $W(KH)$. Since $C(KH) \bigcap W(KH) = 1$, it follows that this index coincides with the order of the group $W(KH)$, and so $V_p(KH) = C(KH) \times W(KH)$. The statements are proved.

It is easy to prove the following statements (see [2]):

1) $|K^p| = |K|$;

2) if n is a nonnegative integer and $J(G^{p^n}[p])$ is the ideal of the ring $(KG)^{p^n}$, generated by the elements of the form $g - 1 \; (g \in G^{p^n}[p])$, then $V^{p^n}(KG)[p] = V(K_nG_n)[p] = 1 + J(G^{p^n}[p])$.

First we shall prove the theorem for a finite ordinal $\omega = n$. Suppose that n is a nonnegative integer, the Sylow p-subgroup S_n of the group G_n is not singular, $G_n \neq G_{n+1}$ and at least one of the ordinals $|K_n|$ or $|G_n|$ is infinite. Since

$$W^{p^{n+1}}[p] \subseteq W^{p^n}[p] \subseteq V^{p^n} = V(K_nG_n),$$

it follows that

$$f_n(W) \leq |V^{p^n}| \leq \max\{|K_n|, |G_n|\} = \beta.$$

In the proof of the equation $f_n(W) = \beta$ we shall consider the following cases:

A) $|K_n| \geq |G_n|$,

B) $|G_n| > |K_n|$ and $S_n \neq S_{n+1}$,

C) $|G_n| > |K_n|$ and $S_n = S_{n+1}$,

and in each of this cases we shall construct a set $M \subseteq W^{p^n}(KG)[p]$ of cardinality $\beta = \max\{|K_n|, |G_n|\}$ elements of which belong to different cosets of the group $V^{p^n}(KG)[p]$ by the subgroup $V^{p^{n+1}}(KG)[p]$. This will be sufficient for the proof of the theorem, because the elements of a set M constructed in this way can be considered as the representatives of the cosets of the group $W^{p^n}(KG)[p]$ by the subgroup $W^{p^{n+1}}(KG)[p]$. Note that we will choose the elements of the set M in form $y^{-1}y^* \; (y \in V^{p^n}(KG))$.

Suppose A) holds, i.e. $|K_n| \geq |G_n|$.

It is easy to prove that in this case the Sylow p-subgroup S_n of the group G_n has an element g of order p and there exists an $a \in G_n$ such that one of the following conditions holds:

A_1) $G_n \neq \langle g \rangle$, $a \notin \langle g \rangle$ and $a^2 \notin \langle g \rangle$,

A_2) $G_n \neq \langle g \rangle$, $a \notin \langle g \rangle$ and $a^2 \in \langle g \rangle$,

$A_3)$ $G_n = \langle g \rangle$,
and in cases $A_1)$ and $A_2)$ one of the elements a or g does not belong to the subgroup G_{n+1}. Indeed, if $g \in G_{n+1}$ then, by the condition $G_n \neq G_{n+1}$, the set $G_n \setminus G_{n+1}$ has the required element a.

Suppose $A_1)$ holds. Let α be a nonzero element of the ring K_n and $y_\alpha = 1 - \alpha a(1 + g + \dots + g^{p-1})$. We shall prove that the set

$$M = \{x_\alpha = {y_\alpha}^{-1}{y_\alpha}^* = 1 + \alpha(a - a^{-1})(1 + g + \dots + g^{p-1}) \mid 0 \neq \alpha \in K_n\}$$

has the above declared property. Indeed, since $a^2 \notin \langle g \rangle$, it follows that the elements a and a^{-1} belong to different cosets of the group G_n by the subgroup $\langle g \rangle$. Hence $x_\alpha \neq 1$. It is easy to see that

$$ {x_\alpha}^* = 1 - \alpha(a - a^{-1})(1 + g + \dots + g^{p-1}) = {x_\alpha}^{-1}$$

and ${x_\alpha}^p = 1$. Therefore x_α is a unitary element of order p of the group $V(K_nG_n)$. If $x_\alpha \in V^{p^{n+1}}$ then, from the condition $a^2 \notin \langle g \rangle$, it follows that $ag^i \in G_{n+1}$ for every $i = 0, 1, \dots, p-1$. Hence the elements a and ag belong to the group G_{n+1}, but this contradicts the choice of elements a and g. Therefore $x_\alpha \in W^{p^n}[p] \setminus W^{p^{n+1}}[p]$.

Suppose that the cosets $x_\alpha V^{p^{n+1}}[p]$ and $x_\nu V^{p^{n+1}}[p]$ coincide for some different α and ν from K_n. Then $x_\alpha = x_\nu z$ for a suitable $z \in V^{p^{n+1}}$. Since ${x_\nu}^* = {x_\nu}^{-1}$, it follows that

$$z = x_\alpha {x_\nu}^* = 1 + (\alpha - \nu)(a - a^{-1})(1 + g + \dots + g^{p-1}) = x_{\alpha - \nu}$$

and $x_{\alpha-\nu}$ belongs to the subgroup $V^{p^{n+1}}$, which contradicts the proved above. Obviously $|M| = |K_n|$. Therefore, the constructed set M has the above declared property.

Suppose now that $A_2)$ holds.

Then $y_\alpha = 1 + \alpha a(g - 1)$ is not a selfconjugate element in the group $V^{p^n}[p] \setminus V^{p^{n+1}}[p]$ and the set M can be choosen in the following way:

$$M = \{x_\alpha = {y_\alpha}^{-1}{y_\alpha}^* \mid 0 \neq \alpha \in K_n\}.$$

It is easy to prove that ${x_\alpha}^* = {x_\alpha}^{-1}$, so, from the assumption $x_\alpha = x_\nu z$ $(\alpha \neq \nu,\ z \in V^{p^{n+1}})$ the equation

$$(1) \qquad \begin{aligned}(1 + \nu a(g-1))\left(1 + \alpha a^{-1}(g^{-1} - 1)\right) &= \\ = (1 + \alpha a(g-1))&\left(1 + \nu a^{-1}(g^{-1} - 1)\right) z\end{aligned}$$

follows. If we multiply (1) by $(g-1)^{p-1} = 1+g+\cdots+g^{p-1}$, then we get the equation $1+g+\cdots+g^{p-1} = (1+g+\cdots+g^{p-1})z$. Hence

$$(1+g+\cdots+g^{p-1})(z-1) = 0. \tag{2}$$

Suppose that $g \notin G_{n+1}$. Since the support of the element $z-1$ belongs to the subgroup G_{n+1} and the elements of the group G_{n+1} belong to the cosets of the group G_n by the subgroup $\langle g \rangle$, it follows that $z=1$. According to the statement

$$1 = 1+(g-1)^p = (1+g-1)(1-(g-1)+(g-1)^2-\cdots+(g-1)^{p-1})$$

it is easy to prove that

$$g^{-1}-1 = -(g-1)+(g-1)^2-\cdots+(g-1)^{p-1}. \tag{3}$$

If in the equation (1) the element $g^{-1}-1$ is substituted by the right side of (3) and the obtained equation is multiplied by $(g-1)^{p-2}$, then we get

$$(\nu-\alpha)a(1+g+\cdots+g^{p-1}) = (\alpha-\nu)a(1+g+\cdots+g^{p-1}).$$

Hence $(\nu-\alpha) = -(\nu-\alpha)$ and this is impossible in a ring of characteristic $p>2$ whenever $\alpha \neq \nu$.

Now let $g \in G_{n+1}$. The element $y = z-1$ can be presented in the form

$$y = z-1 = z_1u_1+\cdots+z_su_s$$

where $z_i \in K_n\langle g \rangle$ and $u_i \quad (i=1,\dots,s)$ are the representatives of the cosets of the group G_{n+1} by the subgroup $\langle g \rangle$. Then, according to (2), every $z_i \quad (i=1,\dots,s)$ belongs to the fundamental ideal of the ring $K_n\langle g \rangle$ and hence it can be written in the form $z_i = \alpha_1(g-1)+\cdots+\alpha_{p-1}(g-1)^{p-1}$. Therefore

$$z = 1+y_1(g-1)+\cdots+y_{p-1}(g-1)^{p-1} \tag{4}$$

where the support of the elements $y_i \quad (i=1,\dots,p-1)$ consists of the representatives of the cosets of the group G_{n+1} by the subgroup $\langle g \rangle$. If in the equation (1) the elements $g^{-1}-1$ and z are substituted by the expressions shown in (3) and (4), and the obtained equation is multiplied by $(g-1)^{p-2}$, then we get

$$2(\nu-\alpha)a(1+g+\cdots+g^{p-1}) = y_1(1+g+\cdots+g^{p-1}).$$

Hence the element a from the support of the left side of this equation coincides with some element from the support of the right side. But this contradicts the condition $a \notin G_{n+1}$, since the support of the right side belongs to the group G_{n+1}. Therefore, the case A_2) is completed.

Suppose A_3) holds, i.e. $G_n = \langle g \rangle$. Then $G_{n+1} = 1$. As in the previous case it is easy to prove that the set

$$M = \left\{x_\alpha = (1 + \alpha(g-1))^{-1} \left(1 + \alpha(g^{-1} - 1)\right) \mid 0 \neq \alpha \in K_n\right\}$$

has the needed property.

Therefore, the proof is complete whenever A) holds.

Suppose now that B) holds, i.e. $|G_n| > |K_n|$ and the Sylow p-subgroup S_n of the group G_n does not coincide with the Sylow p-subgroup S_{n+1} of the group G_{n+1}. Then the set $S_n \setminus S_{n+1}$ has an element g of order $q = p^r$. Let $\pi = \pi(G_n/\langle g \rangle)$ denote the full set of representatives of the cosets of the group G_n by the subgroup $\langle g \rangle$. Consider two disjunct subsets

$$\pi_1 = \{a \in \pi \mid a^2 \notin \langle g \rangle\} \quad \text{and} \quad \pi_2 = \{a \in \pi \mid a^2 \in \langle g \rangle\}$$

of the set π. It is easy to see that $|G_n| = |\pi| = \max\{|\pi_1|, |\pi_2|\}$.

Let us suppose first that $|G_n| = |\pi_1|$. Without loss of generality we can assume that the representative of the set $a^{-1}\langle g \rangle$ is the element a^{-1}. Let E denote a set which has a unique representative in every subset of the form $\{a, a^{-1} \mid a \in \pi_1\}$ and $y_a = 1 - a^{-1}(1 + g + \cdots + g^{q-1})$. Then $|G_n| = |E|$ and the elements of the set

$$M = \{x_a = {y_a}^{-1}{y_a}^* = 1 + (a^{-1} - a)(1 + g + \cdots + g^{q-1}) \mid a \in E\}$$

belong to different cosets of the group $V^{p^n}[p]$ by the subgroup $V^{p^{n+1}}[p]$. Indeed, it is easy to see that $x_a \in W^{p^n}[p] \setminus W^{p^{n+1}}[p]$. Suppose that a and c are distinct elements of the set E. If $x_a = x_c z$ for some $z \in V^{p^{n+1}}$, then

$$z = x_a {x_c}^* = 1 + (a^{-1} - a - c^{-1} + c)(1 + g + \cdots + g^{q-1}).$$

According to the choice of the elements of the set E we have that the elements a, a^{-1}, c, c^{-1} belong to distinct cosets of the group G_n by the subgroup $\langle g \rangle$. Hence from the condition $z \in V^{p^{n+1}}$ it follows that $ag^i \in G_{n+1}$ $(i = 0, 1, \cdots, q-1)$, which contradicts the choice of the element $g \in S_n \setminus S_{n+1}$.

Let be $|G_n| = |\pi_2|$. Then the set M can be choosen in the following way:

$$M = \{x_a = (1 + a(g-1))^{-1} \left(1 + a^{-1}(g^{-1} - 1)\right) \mid a \in \pi_2\}.$$

Indeed, from the supposition $x_a = x_c z$ $(z \in V^{p^{n+1}}, a \neq c)$ the equation

$$(5)\qquad (1 + c(g-1))\left(1 + a^{-1}(g^{-1}-1)\right) = \\ = (1 + a(g-1))\left(1 + c^{-1}(g^{-1}-1)\right) z$$

follows. Multiplying the equation (5) by $(g-1)^{q-1}$ we get the statement $(1 + g + \cdots + g^{q-1}) = (1 + g + \cdots + g^{q-1})z$. As in above, we can prove that from this equation and the condition $g \notin G_{n+1}$ the statement $z = 1$ follows. Substituting the element $g-1$ in (5) by the right side of (3) and the obtained equation is multiplied by $(g-1)^{q-2}$ we have that $2(c-a)(1+g+\cdots+g^{q-1}) = 0$. But it contradicts the fact that a and c belong to distinct cosets of the group G_n by the subgroup $\langle g \rangle$. So the case B) is fully considered.

Suppose C) holds , i. e. $|G_n| > |K_n|$ and the Sylow p-subgroup S_n of the group G_n is p-divisible.

Let us fix an element $g \in S_n[p]$ and choose $v \in G_n \setminus G_{n+1}$ such that p does not divide the order of v. Since $|S_n| = [S_n : \langle g \rangle] \geq |\langle v \rangle|$ and $v \notin S_n$, it follows that the cardinality of the set $\pi' = \pi(G_n / \langle g, v \rangle)$ coincides with $|G_n|$. Obviously the set π decomposes to two disjoint subsets $\pi_1 = \{a \in \pi \mid a^2 \notin \langle v, g \rangle\}$ and $\pi_2 = \{a \in \pi \mid a^2 \in \langle v, g \rangle\}$.

Let $|G_n| = |\pi_1|$,

$$\widetilde{v} = \begin{cases} 1 + v + v^{-1}, & \text{if } v^2 \neq 1, \\ 1 + v, & \text{if } v^2 = 1, \end{cases}$$

E be a set which has a unique representative in every subset of the form $\{a, a^{-1} \mid a \in \pi_1\}$ and $y_a = 1 - a\widetilde{v}(1 + g + \cdots + g^{p-1})$. Then the set M can be choosen in the following way:

$$M = \{x_a = {y_a}^{-1} {y_a}^* = 1 + (a - a^{-1})\widetilde{v}(1 + g + \cdots + g^{p-1}) \mid a \in E\}.$$

Indeed, from the equation $x_a = x_c z$ $(z \in V^{p^{n+1}}, a \neq c)$ follows that

$$z = 1 + (a - a^{-1} - c + c^{-1})\widetilde{v}(1 + g + \cdots + g^{p-1}) \in V^{p^{n+1}}.$$

Hence, according to the construction of the set E, the elements a and av belong to the group G_{n+1}, but this contradicts the condition $v \notin G_{n+1}$.

Assume $|G_n| = |\pi_2|$. The elements of

$$M = \{x_a = (1 + a(1+v)(g-1))^{-1}\left(1 + a^{-1}(1+v^{-1})(g^{-1})\right) | a \in \pi_2\}$$

belong to distinct cosets of the group $V^{p^n}[p]$ by the subgroup $V^{p^{n+1}}[p]$. Indeed, suppose that $x_a = x_c z$ for distinct elements a and c from the set π_2 and for some $z \in V^{p^{n+1}}[p]$. Then

$$(6) \qquad \begin{aligned} &\left(1 + c(1+v)(g-1)\right)\left(1 + a^{-1}(1+v^{-1})(g^{-1}-1)\right) = \\ &= \left(1 + a(1+v)(g-1)\right)^{-1}\left(1 + c^{-1}(1+v^{-1})(g^{-1}-1)\right) z. \end{aligned}$$

If we multiply the equation (6) by $(g-1)^{p-1}$, then it follows that $(1 + g + \cdots + g^{p-1})(z-1) = 0$ and we can write z in the form (4). Let us now multiply the equation (6) by $(g-1)^{p-2}$. Then, by useing (3) and (4), we get

$$(c-a)(2 + v + v^{-1})(1 + g + \cdots + g^{p-1}) = y_1(1 + g + \cdots + g^{p-1}).$$

Since a and c are from distinct cosets of the group G_n by the subgroup $\langle g, v\rangle$, it follows that c and cv belong to the support of the left side of this equation. Hence they coincide with some elements from the support of the right side, which belongs to the subgroup G_{n+1}. Therefore $c \in G_{n+1}$ and $cv \in G_{n+1}$, but this contradicts the choice of $v \notin G_{n+1}$.

Therefore, the case C) is fully considered and the theorem is proved for a finite ordinal $\omega = n$.

Let us consider the case of infinite ordinal ω.

Let ω be an arbitrary infinite ordinal $R = K_\omega, H = G_\omega$, $G_\omega \neq G_{\omega+1}$ and the Sylow p-subgroup S_ω of the group G_ω is not singular. Then

$$W(KG)^{p^\omega} \subseteq W(RH) \subseteq V_p(RH)$$

and by transfinite induction it is easy to prove

$$(7) \qquad (V_p(KG))^{p^\omega} = V_p(RH).$$

For the group $V_p(RH)$ we can construct the set M as in the above shown cases A), B) and C). Since in each of these cases the set M consists of elements of form $x = y^{-1}y^*$ and, by (7), y belongs to the group $V_p(RH) = (V_p(KG))^{p^\omega}$, it follows that the elements x are the representatives of the cosets of the group $W^{p^\omega}(KG)[p]$ by the subgroup $W^{p^{\omega+1}}(KG)[p]$.

Therefore, for an arbitrary infinite ordinal ω the Ulm–Kaplansky invariants of the group $W(KG)$ can be calculated in the above shown way for the case $\omega = n$.

References

[1] A. A. Bovdi and A. A. Szakács, The unitary subgroup of the group of units in a modular group algebra of a finite abelian p-group, *Math. Zametki.* *6* **45** (1989), 23–29 (in Russian). (see English translation in *Math. Notes*, 5–6 **45** (1989), 445–450.)

[2] A. A. Bovdi and Z. F. Patay, The structure of the centre of the multiplicative group of the group ring of a p-group over a ring of characteristic p. *Vesci Akad. Nauk. Bssr. Ser. Fiz. Math. Nauk.* (1978) No. 1, 5–11.