# The asymptotic behavior of the real roots of Fibonacci-like polynomials

FERENC MÁTYÁS*

**Abstract.** The Fibonacci-like polynomials $G_n(x)$ are defined by the recursive formula $G_n(x)=xG_{n-1}(x)+G_{n-2}(x)$ for $n\geq 2$, where $G_0(x)$ and $G_1(x)$ are given seed-polynomials. In this paper the non-zero accumulation points of the set of the real roots of Fibonacci-like polynomials are determined if either both of the seed-polynomials are constants or $G_0(x)=-a$ and $G_1(x)=x\pm a$ ($a\in\mathbf{R}\setminus\{0\}$). The theorems generalize the results of G. A. Moore and H. Prodinger who investigated this problem if $G_0(x)=-1$ and $G_1(x)=x-1$, furthermore we extend a result of Hongquan Yu, Yi Wang and Mingfeng He.

## Introduction

The Fibonacci-like polynomials $G_n(x)$ are defined by the following manner. For $n\geq 2$

$$G_n(x)=xG_{n-1}(x)+G_{n-2}(x), \tag{1}$$

where $G_0(x)$ and $G_1(x)$ are fixed polynomials (so-called seed-polynomials) with real coefficients. If it is necessary to denote the seed-polynomials, then we will use the notation $G_n(x)=G_n\left(G_0(x),G_1(x),x\right)$, too. The polynomials $G_n(0,1,x)$ are the original Fibonacci polynomials and the numbers $G_n(0,1,1)$ are the well-known Fibonacci numbers.

Recently, G. A. Moore [5] investigated the maximal real roots $g_n'$ of the polynomials $G_n(-1,x-1,x)$ and proved that $g_n'$ exists for every $n\geq 1$ and $\lim_{n\to\infty} g_n'=3/2$. (These numbers $g_n'$ are called as "golden numbers". ) H. Prodinger [6] gave the asymptotic formula $g_n'\sim\frac{3}{2}+(-1)^n\frac{25}{12}4^{-n}$. Hongquan Yu, Yi Wang and Mingfeng He [3] investigated the limit of the maximal real roots $g_n'$ of polynomials $G_n(-a,x-a,x)$ if $a\in\mathbf{R}^+$.

For brevity let us introduce the following notations. $B$ denotes the set of the real roots of polynomials $G_n(x)$ ($n=0,1,2,\ldots$) and $A$ denotes the set of the the accumulation points of set $B$. In [4] we investigated these sets.

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Although, the main result of [4] is formulated for seed-polynomials with integer coefficients but it is true for seed-polynomials with real coefficients, too. Since we are going to apply it, therefore we cite it as a lemma.

**Lemma 1.** *Let $G_0(x)$ and $G_1(x)$ be two fixed polynomials with real coefficients, $G_0(0)\cdot G_1(0)\neq 0$ and $x_0\in\mathbf{R}$. $x_0\in A$ if and only if one of the following conditions holds:*

(i) $-\frac{G_1(x_0)}{G_0(x_0)}=\frac{1}{\alpha(x_0)}$ and $x_0>0$;

(ii) $-\frac{G_1(x_0)}{G_0(x_0)}=\frac{1}{\beta(x_0)}$ and $x_0<0$;

(iii) $x_0=0$,

*where*

$$\alpha(x)=\frac{x+\sqrt{x^2+4}}{2}\quad\text{and}\quad\beta(x)=\frac{x-\sqrt{x^2+4}}{2}. \tag{2}$$

The purpose of this paper is to investigate the asymptotic behavior of the elements of the set $B$ in the cases of simple seed-polynomials. In our discussion we are going to use the following explicit formulae for the polynomial $G_n(x)=G_n\left(G_0(x),G_1(x),x\right)$. It is known that

$$G_n(x)=p(x)\alpha^n(x)-q(x)\beta^n(x) \tag{3}$$

for $n\geq 0$, where $\alpha(x)$ and $\beta(x)$ are defined in (2), while

$$p(x)=\frac{G_1(x)-\beta(x)G_0(x)}{\alpha(x)-\beta(x)}\quad\text{and}\quad q(x)=\frac{G_1(x)-\alpha(x)G_0(x)}{\alpha(x)-\beta(x)}.$$

These formulae can be obtained by standard methods or see in [2].

Since we want to investigate the roots of the polynomials $G_n(x)$, therefore it is worth rephasing the expression $G_n(x)=0$ as

$$\frac{p(x)}{q(x)}=\left(\frac{\beta(x)}{\alpha(x)}\right)^n,$$

that is

$$\frac{G_1(x)-\beta(x)G_0(x)}{G_1(x)-\alpha(x)G_0(x)}=\left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^n. \tag{4}$$

Let us consider the polynomial $G_n\left(G_0(x),G_1(x),x\right)$. It is obvious that $G_n(0,0,x)$ is identical to the zero polynomial for every $n\geq 0$. Using (3) the identities $G_n(0, G_1(x), x) = G_1(x) \cdot G_n(0,1,x)$ and $G_n(G_0(x), 0, x) = G_0(x) \cdot G_n(1,0,x)$ yield. But it is known from [2] and can be obtained easily from (4) that neither the Fibonacci polynomials $G_n(0,1,x)$ nor the polynomials $G_n(1,0,x)$ have real root $x'$ except $x' = 0$ if $n$ is even or odd, respectively. Therefore investigating the asymptotic behavior of the roots of polynomials $G_n(G_0(x), G_1(x), x)$ we can assume that the seed-polynomials differ from the zero polynomial and at least one of them is a monic polynomial (since one can simplify the left-hand side of (4) with the leading coefficient of the polynomial $G_1(x)$ or $G_0(x)$).

## Theorems and Proofs

First of all we need the following lemma, which deals with the properties of the functions $\alpha(x)$ and $\beta(x)$ defined in (2).

**Lemma 2.** (a) *On the interval* $[0, \infty)$ *the function* $\frac{1}{\alpha(x)}$ *is continuous and strictly monotonically decreasing, its graph is convex and* $1 \geq \frac{1}{\alpha(x)} > 0$.

(b) *On the interval* $(-\infty, 0]$ *the function* $\frac{1}{\beta(x)}$ *is continuous and strictly monotonically decreasing, its graph is concave and* $0 > \frac{1}{\beta(x)} \geq -1$.

**Proof.** By (2) it is obvious that the functions $\frac{1}{\alpha(x)}$ and $\frac{1}{\beta(x)}$ are continuous on the above mentioned intervals. The rest of the statement can be proved easily using the methods of differential calculus.

Further on we deal with the set $A$ if $G_0(x) = 1$ and $G_1(x) = a$. In this case, using Lemma 1, the set $A$ can be determined in a very simple manner.

**Theorem 1.** *Let* $a \in \mathbf{R} \setminus \{0\}$ *and* $G_n(1, a, x)$ *be Fibonacci-like polinomials. If* $0 < |a| < 1$ *then* $A \setminus \{0\} = \left\{\frac{a^2-1}{a}\right\}$, *while in the case* $|a| \geq 1$ $A \setminus \{0\} = \emptyset$.

**Proof.** According to Lemma 1 to get the elements of the set $A \setminus \{0\}$ we have to solve the equations

$$-a = \frac{2}{x + \sqrt{x^2+4}} \quad \text{for} \quad x > 0 \tag{5}$$

and

$$-a = \frac{2}{x - \sqrt{x^2+4}} \quad \text{for} \quad x < 0. \tag{6}$$

By Lemma 2 the functions $\frac{1}{\alpha(x)} = \frac{2}{x+\sqrt{x^2+4}}$ and $\frac{1}{\beta(x)} = \frac{2}{x-\sqrt{x^2+4}}$ are continuous, $1 > \frac{1}{\alpha(x)} > 0$ for any $x > 0$ and $0 > \frac{1}{\beta(x)} > -1$ for any $x < 0$, therefore $0 < |a| < 1$ is a necessary and sufficient condition for the solvability of (5) and (6). Solving (5) and (6) we get that the single real root $x_0$ is $x_0 = \frac{a^2-1}{a}$, where $x_0 > 0$ if $-1 < a < 0$ and $x_0 < 0$ if $0 < a < 1$. This completes the proof.

In the following theorems we prove asymptotic formulae for those real roots $g_n$ of the polynomials $G_n(-a, x \pm a, x)$ which do not tend to 0 if $n$ tends to infinity.

**Theorem 2.** *Let $G_0(x) = -a$ and $G_1(x) = x - a$, where $a \in \mathbf{R} \setminus \{0\}$. If either $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{\frac{a(a+2)}{a+1}\right\}$, while in the case $-2 \le a < 0$ we have $A \setminus \{0\} = \emptyset$. Furthermore for large $n$*

$$g_n \sim \frac{a(a+2)}{a+1} + (-1)^n \frac{a(a^2+2a+2)^2}{(a+1)^2(a+2)}(a+1)^{-2n}.$$

**Proof.** According to Lemma 1, $x_0 \in A \setminus \{0\}$ if and only if

$$\frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2+4}} \quad \text{and} \quad x_0 > 0 \tag{7}$$

or

$$\frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2+4}} \quad \text{and} \quad x_0 < 0 \tag{8}$$

holds. Using the statements of Lemma 2 one can verify that (7) has a solution for $x_0$ if and only if $a > 0$, while (8) has a solution for $x_0$ if and only if $a < -2$. Solving (7) and (8) we get that

$$x_0 = \frac{a(a+2)}{a+1}.$$

To determine the asymptotic behavior of $g_n$ we apply (4), which in our case has the following form

$$\frac{2(g_n - a) + a\left(g_n - \sqrt{g_n^2+4}\right)}{2(g_n - a) + a\left(g_n + \sqrt{g_n^2+4}\right)} = \left(\frac{g_n - \sqrt{g_n^2+4}}{g_n + \sqrt{g_n^2+4}}\right)^n.$$

This will be much nicer when we substitute

$$g_n = u - \frac{1}{u}. \tag{9}$$

Without loss of generality we can assume that $u > 0$ and we get the equality

$$\frac{(au + u + 1)(u - 1)}{(a + 1 - u)(u + 1)} = -(-u^2)^n. \tag{10}$$

Since $x_0 = u - \frac{1}{u}$ holds for $u = a + 1$ and $u = -\frac{1}{a+1}$ therefore it is plain to see that, for large $n$, (9) can only hold if $u$ is either close to $a + 1$ or $-\frac{1}{a+1}$. In both cases this would mean that $g_n$ is close to $x_0$.

Let us assume that $u$ is close to $a+1$ and so $a > 0$ because of $u > 0$. It is clear from (10) that the cases when $n$ is even or odd have to be distinguished.

We start with $n = 2m$ and rewrite (10) as

$$a + 1 - u = -\frac{(au + u + 1)(u - 1)}{u + 1} \cdot u^{-4m}. \tag{11}$$

We get the asymptotic behavior by a process known as "bootstrapping" which is explained in [1]. First we insert $u = a + 1 + \delta_1$ into the left-hand side of (11) and $u = a + 1$ into the right-hand side of (11). So we get an approximation for $\delta_1$. Then we insert $u = a + 1 + \delta_1 + \delta_2$ into the left-hand side of (11) and $u = a + 1 + \delta_1$ into the rihgt-hand side of (11) and get an approximation for $\delta_2$. This procedure can be repeated to get better and better estiamations for $u$. Now we determine only the number $\delta_1$. From (11) we have

$$\delta_1 \sim \frac{a(a^2 + 2a + 2)}{a + 2}(a + 1)^{-4m}$$

and so

$$u = a + 1 + \delta_1 \sim a + 1 + \frac{a(a^2 + 2a + 2)}{a + 2}(a + 1)^{-4m}.$$

Substituting $u$ into (9) we get that

$$g_{2m} = a+1+\delta_1 - \frac{1}{a + 1 + \delta_1} \sim \frac{a(a + 2)}{a + 1} + \frac{(a(a^2 + 2a + 2)^2}{(a + 1)^2(a + 2)}(a+1)^{-4m}. \tag{12}$$

If $n = 2m + 1$ then (10) can be rewrite as

$$a + 1 - u = \frac{(au + u + 1)(u - 1)}{u^2(u + 1)} u^{-4m}.$$

Using the "bootstrapping" method for $u = a + 1 + \delta_1'$ we get the estimation

$$\delta_1' \sim \frac{a(a^2+2a+2)}{(a+2)(a+1)^2}(a+1)^{-4m},$$

which implies the following form:

$$\begin{aligned} g_{2m+1} &= a + 1 + \delta_1' - \frac{1}{a+1+\delta_1'} \\ &\sim \frac{a(a+2)}{a+1} - \frac{a(a^2+2a+2)^2}{(a+2)(a+1)^4}(a+1)^{-4m}. \end{aligned} \tag{13}$$

Comparing (12) and (13) the desired approximation yields since $a > 0$.

One can verify in the same manner that the estimation for $g_n$ also holds when $a < -2$. This completes the proof.

**Remark.** From our proof one can see that for large $n$ $g_n = g_n'$ if $a > 0$ while $g_n$ is the minimal real root if $a < -2$.

A similar result can be proved for the polynomials $G_n(-a, x+a, x)$.

**Theorem 3.** *Let $G_0(x) = -a$ and $G_1(x) = x + a$ where $a \in \mathbf{R} \setminus \{0\}$. If etiher $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{-\frac{a(a+2)}{a+1}\right\}$, while $A \setminus \{0\} = \emptyset$ if $-2 \le a < 0$. Furthermore for large $n$*

$$g_n \sim -\frac{a(a+2)}{a+1} + (-1)^n \frac{a(a^2+2a+2)^2}{(a+1)^2(a+2)}(a+1)^{-2n},$$

*where $G_n(g_n) = 0$ and $\lim_{n\to\infty} g_n \ne 0$.*

**Proof.** For a real number $x_0$, by our Lemma 2, $x_0 \in A \setminus \{0\}$ if and only if

$$\frac{x_0 + a}{a} = \frac{2}{x_0 + \sqrt{x_0^2+4}} \quad \text{and} \quad x_0 > 0 \tag{14}$$

or

$$\frac{x_0 + a}{a} = \frac{2}{x_0 - \sqrt{x_0^2+4}} \quad \text{and} \quad x_0 < 0 \tag{15}$$

holds. Substituting $-x_0$ for $x_0$ into (14) and (15) we get that

$$\frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2+4}} \quad \text{and} \quad x_0 < 0 \tag{16}$$

and

$$\frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0 \tag{17}$$

Since (16) and (17) are identical to (8) and (7), respectively, therefore all of the statements of our theorem follows from the Theorem 2. Thus the theorem is proved.

## Concluding Remarks

Using our Theorem 2 for $a = 1$ we get that $g_n = g'_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n}$, which matches perfectly with the result of H. Prodinger.

On the other hand it is quite likely that similar results can be obtained for seed-polynomials $G_0(x) = x \pm a$ and $G_0(x) = a$ or for other polynomials. This could be the subject of further research work.

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