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RECURSIVE FORMULAE FOR SPECIAL CONTINUED FRACTION CONVERGENTS

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Abstract: Let α and β be the zeros of the polynomial $x^2 - Ax - B$, where $A \in Z \setminus \{0\}, B \in \{1, -1\}, D = A^2 + 4B > 0, |\alpha| > |\beta|$ and D is not a square number. In this paper some recursive formulae are given for the continued fraction convergents to α.

1. Introduction

Let the sequence $R = \{R_n\}_{n=0}^{\infty}$ be defined for $n \geq 0$ by the recursion

(1) $$R_{n+2} = AR_{n+1} + BR_n,$$

where $A, B \in \mathbb{Z}\setminus\{0\}, R_0 = 0, R_1 = 1, D = A^2 + 4B > 0$ and D is not a perfect square. If $R_0 = R_1 = 1$ then the terms of sequence R are denoted by $R_n^\star$, while if $A = B = R_1 = 1$ and $R_0 = 0$ then the terms of sequence R are the Fibonacci numbers, which are denoted by F_n.

The polynomial $f(x) = x^2 - Ax - B$ is called to be the characteristic polynomial of the sequence R, and the zeros of $f(x)$ are denoted by α and β. By our condition, the zeros α and β are irrationals and we suppose that $|\alpha| > |\beta|$. It is known that for $n \geq 0$

(2) $$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad R_n^\star = \frac{(1-\beta)\alpha^n - (1-\alpha)\beta^n}{\alpha - \beta},$$

from which

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \lim_{n\to\infty} \frac{R_{n+1}^\star}{R_n^\star} = \alpha$$

immediately follows.

P. Kiss [2] proved that

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{\sqrt{D}R_n^2}$$

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holds for infinitely many positive integer n if and only if $|B| = 1$, and in this case all of the rational solutions p/q of the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{D}q^2}$$

have the form $p/q = R_{n+1}/R_n$. Because of this fact, in this paper we deal with only the case $|B| = 1$.

The connection between diophantine approximation and continued fraction convergents is well-known (see [1]). The simple periodic continued fraction expansion of α is denoted by $[a_0, a_1, \dots, \overline{a_k, \dots, a_m}]$, where $[\overline{a_k, \dots, a_m}]$ denotes the minimal periodic part, while the n^{th} convergent to α by $r_n(\alpha)$.

G. J. Rieger [4] has created a special function having the zero $\frac{\sqrt{5}-1}{2}$ and he has proved that the Newton approximants x_n to this zero satisfy the recursive formula

$$x_{n+1} = \frac{x_n + 1}{x_n + 2} \qquad (x_0 = 0, n \geq 0),$$

and $x_n = r_{2n}\left(\frac{\sqrt{5}-1}{2}\right)$. Since $r_{2n}\left(\frac{\sqrt{5}-1}{2}\right) = \frac{F_{2n}}{F_{2n+1}}$, thus G. J. Rieger obtained a recursive formula for the (even) continued fraction convergents to $\frac{\sqrt{5}-1}{2}$, which is in close relation with the Fibonacci numbers. On the other hand, $\frac{\sqrt{5}-1}{2}$ is a zero of the characteristic polynomial $x^2 - Ax - B$ of sequence R defined in (1) if $A = -1$ and $B = 1$.

The aim of this paper is to generalize the result of G. J. Rieger for $A \in \mathbb{Z}\backslash\{0\}$ and $|B| = 1$. We give some recursive formulae for the continued fraction convergents $r_n(\alpha)$ to α.

2. Results

It is known that $r_0(\alpha) < r_2(\alpha) < r_4(\alpha) < \ldots < \alpha < \ldots < r_5(\alpha) < r_3(\alpha) < r_1(\alpha)$ (see [1]), therefore we are looking for recursive formulae for the odd and for the even convergents to α.

Theorem 1. *Let $B = 1$ in (1) and let the approximant x_{n+1} be defined by the recursive formula*

$$x_{n+1} = \frac{(A^2+1)x_n + A}{Ax_n + 1} \qquad (n \geq 0),$$

let α and β denote the zeros of the polynomial $x^2 - Ax - 1$, where $|\alpha| > |\beta|$.

(1.) Let $A \geq 1$.

(a) If $x_0 = \frac{A^2+1}{A}$, then $x_n = r_{2n+1}(\alpha) \qquad (n \geq 0)$.

(b) If $x_0 = A$*, then* $x_n = r_{2n}(\alpha) \qquad (n \geq 0)$.

(2.) Let $A < -1$.

(a) If $x_0 = A$*, then* $x_n = r_{2n+1}(\alpha) \qquad (n \geq 0)$.

(b) If $x_0 = \frac{A^2+1}{A}$*, then* $x_n = r_{2n+2}(\alpha) \qquad (n \geq 0)$.

Remark. If $A = -1$ and $B = 1$ then $\alpha = \frac{-\sqrt{5}-1}{2}$ and $\beta = \frac{\sqrt{5}-1}{2}$. The cited paper of G. J. Rieger [4] investigated exactly the even convergents to this β.

Theorem 2. *Let* $B = -1$ *in (1) and let the approximant* x_{n+1} *be defined by the recursive formula*

$$x_{n+1} = \frac{Ax_n - 1}{x_n} \quad (n \geq 0),$$

let α *and* β *denote the zeros of the polynomial* $x^2 - Ax + 1$*, where* $|\alpha| > |\beta|$.

(1.) Let $A \geq 3$.

(a) If $x_0 = A$*, then* $x_n = r_{2n+1}(\alpha) \qquad (n \geq 0)$.

(b) If $x_0 = A - 1$*, then* $x_n = r_{2n}(\alpha) \qquad (n \geq 0)$.

(2.) Let $A \leq -3$.

(a) If $x_0 = A - 1$*, then* $x_n = r_{2n+1}(\alpha) \qquad (n \geq 0)$.

(b) If $x_0 = A$*, then* $x_n = r_{2n}(\alpha) \qquad (n \geq 0)$.

Further on, using the known Newton approximation to approximate the root α of the equation $x^2 - Ax - B = 0 \quad (B = \pm 1)$, we will investigate the connection between the convergents to α and the Newton approximants.

Theorem 3. *Let* $f(x) = x^2 - Ax - B \quad (B = \pm 1, A^2 + 4B > 0)$ *and let* α *and* β *denote the root of* $f(x) = 0$ *with the condition* $|\alpha| > |\beta|$*. Then the Newton iteration gives the formula*

$$x_{n+1} = \frac{x_n^2 + B}{2x_n - A}.$$

(1.) Let $B = 1$ *and* $x_0 = \frac{A^2+1}{A}$.

(a) If $A \geq 1$*, then* $x_n = r_{2^{n+1}-1}(\alpha) \qquad (n \geq 0)$.

(b) If $A < -1$*, then* $x_n = r_{2^{n+1}}(\alpha) \qquad (n \geq 0)$.

(2.) Let $B = -1$ *and* $x_0 = A$.

(a) If $A \geq 3$*, then* $x_n = r_{2^{n+1}-1}(\alpha) \qquad (n \geq 0)$.

(b) If $A \leq -3$*, then* $x_n = r_{2^{n+1}-2}(\alpha) \qquad (n \geq 0)$.

3. Proofs

Before the proofs of our theorems, we need the following lemma.

Lemma. *Let the sequence R be defined by (1), where* $B = \pm 1$. *Then for all* $k > 0$

$$\frac{\left(\frac{R_{k+1}}{R_k}\right)^2 + B}{2\frac{R_{k+1}}{R_k} - A} = \frac{R_{2k+1}}{R_{2k}}.$$

Proof. We are going to show the proof only in the case $B = 1$, because the proof would be very similar if $B = -1$. Using (1), (2) and $\alpha\beta = -1$, one can verify the following:

$$\frac{\left(\frac{R_{k+1}}{R_k}\right)^2 + 1}{2\frac{R_{k+1}}{R_k} - A} = \frac{R_{k+1}^2 + R_k^2}{2R_{k+1}R_k - AR_k^2} =$$

$$\frac{R_{k+1}^2 + R_k^2}{R_k\left(R_{k+1} - AR_k\right) + R_kR_{k+1}} = \frac{R_{k+1}^2 + R_k^2}{R_k\left(R_{k-1} + R_{k+1}\right)} =$$

$$\frac{\alpha^{2k+2} + \beta^{2k+2} - 2\alpha^{k+1}\beta^{k+1} + \alpha^{2k} + \beta^{2k} - 2\alpha^k\beta^k}{\alpha^{2k-1} + (-1)^k(\alpha+\beta) + \alpha^{2k+1} + (-1)^{k+1}(\alpha+\beta) + \beta^{2k-1} + \beta^{2k+1}} =$$

$$\frac{\alpha^{2k+1}(\alpha + \frac{1}{\alpha}) + \beta^{2k+1}(\beta + \frac{1}{\beta})}{\alpha^{2k}(\frac{1}{\alpha} + \alpha) + \beta^{2k}(\frac{1}{\beta} + \beta)} = \frac{(\alpha-\beta)(\alpha^{2k+1} - \beta^{2k+1})}{(\alpha-\beta)(\alpha^{2k} - \beta^{2k})} = \frac{R_{2k+1}}{R_{2k}}.$$

This completes the proof.

In the proofs of our theorems we omit the numerical calculation of the continued fraction expansions and the convergents. For the calculations we used the general algorithms that can be found in [1].

Proof of Theorem 1. First we deal with the case (1.) Now $\alpha = \frac{A+\sqrt{A^2+4}}{2} = [\overline{A}]$ and so the n^{th} convergent to α is

$$r_n(\alpha) = \frac{R_{n+2}}{R_{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{3}$$

that is, the odd and the even convergents are

$$r_{2n+1}(\alpha) = \frac{R_{2n+3}}{R_{2n+2}} \quad \text{and} \quad r_{2n}(\alpha) = \frac{R_{2n+2}}{R_{2n+1}}. \tag{4}$$

(These can be easily verified or see [2].)
The cases (a) and (b) will be proved by induction on n. By (3) and (1), in (a)

$r_1(\alpha) = \frac{R_3}{R_2} = \frac{A^2+1}{A} = x_0$, while in (b) $r_0(\alpha) = \frac{R_2}{R_1} = \frac{A}{1} = x_0$. Let us suppose that (a) and (b) hold for some $n \geq 0$. Then in (a), by (4),

$$x_{n+1} = \frac{(A^2+1)r_{2n+1}(\alpha)+A}{Ar_{2n+1}(\alpha)+1} = \frac{(A^2+1)\frac{R_{2n+3}}{R_{2n+2}}+A}{A\frac{R_{2n+3}}{R_{2n+2}}+1} =$$

$$\frac{A(AR_{2n+3}+R_{2n+2})+R_{2n+3}}{AR_{2n+3}+R_{2n+2}} = \frac{R_{2n+5}}{R_{2n+4}} = r_{2(n+1)+1}(\alpha),$$

and in (b)

$$x_{n+1} = \frac{(A^2+1)r_{2n}(\alpha)+A}{Ar_{2n}(\alpha)+1} = \ldots = \frac{R_{2n+4}}{R_{2n+3}} = r_{2(n+1)}(\alpha).$$

Now, let us see the case (2.), where $\alpha = \frac{A-\sqrt{A^2+4}}{2} = [A-1, 1, -A-1, \overline{-A}]$ and so the n^{th} convergents to α is

$$(5) \qquad r_n(\alpha) = \frac{R_{n+1}}{R_n} \qquad (n = 1, 2, \ldots).$$

By (5) and (1), in (a) $r_1(\alpha) = \frac{R_2}{R_1} = \frac{A}{1} = x_0$, while in (b) $r_2(\alpha) = \frac{R_3}{R_2} = \frac{A^2+1}{A} = x_0$. But for $n \geq 0$

$$x_{n+1} = \frac{(A^2+1)r_{2n+1}(\alpha)+A}{Ar_{2n+1}(\alpha)+1} = \ldots = \frac{R_{2n+4}}{R_{2n+3}} = r_{2(n+1)+1}(\alpha)$$

and

$$x_{n+1} = \frac{(A^2+1)r_{2n+2}(\alpha)+A}{Ar_{2n+2}(\alpha)+1} = \ldots = \frac{R_{2n+5}}{R_{2n+4}} = r_{2(n+1)+2}(\alpha)$$

in the case (a) and (b), respectively, which proves the theorem by induction.

Proof of Theorem 2. Similarly, as we have done it in the previous proof, first let us deal with the case (1.). Then $\alpha = \frac{A+\sqrt{A^2-4}}{2} = [A-1, \overline{1, A-2}]$ and so the odd and the even convergents to α are

$$(6) \qquad r_{2n+1}(\alpha) = \frac{R_{n+2}}{R_{n+1}} \quad \text{and} \quad r_{2n}(\alpha) = \frac{R^\star_{n+2}}{R^\star_{n+1}} \qquad (n = 0, 1, 2, \ldots).$$

In the case (a) and (b), by (6) and (1), $r_1(\alpha) = \frac{R_2}{R_1} = \frac{A}{1} = x_0$ and $r_0(\alpha) = \frac{R^\star_2}{R^\star_1} = \frac{A-1}{1} = x_0$, respectively. By induction on n, by (6) and (1), we get that in (a)

$$x_{n+1} = \frac{Ax_n - 1}{x_n} = \frac{Ar_{2n+1}(\alpha)-1}{r_{2n+1}(\alpha)} = \frac{A\frac{R_{n+2}}{R_{n+1}}-1}{\frac{R_{n+2}}{R_{n+1}}} = \frac{R_{n+3}}{R_{n+2}} = r_{2(n+1)+1}(\alpha),$$

while in (b)

$$x_{n+1} = \frac{Ar_{2n}(\alpha) - 1}{r_{2n}(\alpha)} = \ldots = \frac{R^\star_{n+3}}{R^\star_{n+2}} = r_{2(n+1)}(\alpha).$$

In the case (2.) $\alpha = \frac{A-\sqrt{A^2-4}}{2} = [A, -A-1, \overline{1, -A-2}]$ and so the odd and the even convergents to α are

$$(7) \qquad r_{2n+1}(\alpha) = \frac{R^\star_{n+2}}{R^\star_{n+1}} \quad \text{and} \quad r_{2n}(\alpha) = \frac{R_{n+2}}{R_{n+1}} \qquad (n = 0, 1, 2, \ldots).$$

Using (7), by induction on n, the proof can be terminated in this case, too.

Proof of Theorem 3. It is known that, under some conditions, the Newton approximants for the zero of the function $f(x)$ can be derived from the equality

$$(8) \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(see [3]). Now, $f(x_0) > 0$ and $f''(x) = 2 > 0$, that is, the approximants converge to α (see [3]), and from (8) we get the following iterative formula

$$(9) \qquad x_{n+1} = \frac{x_n^2 + B}{2x_n - A}.$$

First, let us see the case (1.), when $A \geq 1$. Then, by (1) and (4), $x_0 = r_1(\alpha) = r_{2^{0+1}-1}(\alpha)$. Supposing that (a) holds for some $n \geq 0$, by (9) and (4),

$$x_{n+1} = \frac{x_n^2 + 1}{2x_n - A} = \frac{r_{2^{n+1}-1}(\alpha)^2 + 1}{2r_{2^{n+1}-1}(\alpha) - A} = \frac{\left(\frac{R_{2^{n+1}+1}}{R_{2^{n+1}}}\right)^2 + 1}{2\frac{R_{2^{n+1}+1}}{R_{2^{n+1}}} - A}.$$

From this, applying the Lemma and (4), we get

$$x_{n+1} = \frac{R_{2^{n+2}+1}}{R_{2^{n+2}}} = r_{2^{(n+1)+1}-1}(\alpha).$$

If $A < -1$ then, by (1) and (5), $x_0 = r_2(\alpha) = r_{2^{0+1}}(\alpha)$. By (5) and the Lemma, using induction on n we obtain

$$x_{n+1} = \frac{x_n^2 + 1}{2x_n - A} = \frac{r_{2^{n+1}}(\alpha)^2 + 1}{2r_{2^{n+1}}(\alpha) - A} = \frac{\left(\frac{R_{2^{n+1}+1}}{R_{2^{n+1}}}\right)^2 + 1}{2\frac{R_{2^{n+1}+1}}{R_{2^{n+1}}} - A} =$$

$$\frac{R_{2^{n+2}+1}}{R_{2^{n+2}}} = r_{2(n+1)+1}(\alpha).$$

The part (2.) of the theorem can be proved similarly, therefore we omit it.

References

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