Arcs, Caps and Codes

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Abstract

Apart from being an interesting and exciting area in combinatorics with beautiful results, finite projective spaces or Galois geometries have many applications to coding theory, algebraic geometry, design theory, graph theory, cryptology and group theory. As an example, the theory of linear maximum distance separable codes (MDS codes) is equivalent to the theory of arcs in $PG(n,q)$; so all results of Section 2 can be expressed in terms of linear MDS codes. In this paper we survey interesting results from the theory of arcs and caps in Galois geometries.

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1 Introduction

A non-singular conic of the projective plane $PG(2,q)$ over the finite field $\mathbb{F}_q$ consists of $q+1$ points no three of which are collinear. It is natural to ask if this non-collinearity condition for $q+1$ points is sufficient for them to be a conic. In other words, does this combinatorial property characterize non-singular conics? For q odd, this question was affirmatively answered by Segre [47, 48]. Generalizing, Segre considers sets of k points in $PG(2,q)$, $k \geq 3$, no three of which are collinear, and also sets of k points in the n-dimensional projective space $PG(n,q)$ over $\mathbb{F}_q$, $k \geq n+1$, no $n+1$ of which lie in a hyperplane; these objects are k-arcs. There is a close relationship between k-arcs and certain algebraic curves and hypersurfaces of $PG(n,q)$. Later on, it appeared that arcs and linear maximum distance separable (MDS) codes of dimension at least 3 are equivalent objects, yielding many new results about these codes.

The concept of a k-arc in $PG(2,q)$ was generalized to that of a k-cap in $PG(n,q)$; a k-cap of $PG(n,q)$, $n \geq 3$, is a set of k points no three of which are collinear. An elliptic quadric of $PG(3,q)$ is a cap of size q^2+1. In 1955, Barlotti [3] and Panella [44] independently showed that, for q odd, the converse is true. Also, q^2+1 is the maximum size of a k-cap in $PG(3,q)$ for $q \neq 2$. This leads to the definition of an ovoid of $PG(3,q)$ as a cap of size q^2+1 for $q>2$ and, for $q=2$, a cap of size 5 with no 4 points in a plane. Ovoids of particular interest were discovered by Tits [61]. Ovoids are important objects in the theories of circle geometries, projective planes, designs, generalized polygons and finite simple groups.

Most of this paper is taken from "Open problems in finite projective spaces" by J.W.P. Hirschfeld and J.A. Thas [38].

2 k-arcs

2.1 Definitions

A <u>k-arc</u> in $PG(n,q)$ is a set of k points, with $k \geq n+1 \geq 3$, such that no $n+1$ of its points lie in a hyperplane. An arc K is <u>complete</u> if it is not properly contained in a larger arc. Otherwise, if $K \cup \{P\}$ is an arc for some point P of $PG(n,q)$, the point P <u>extends</u> K.

A <u>normal rational curve</u> (NRC) of $PG(n,q)$, $n \geq 2$, is any set of points of $PG(n,q)$ which is projectively equivalent to

$$\left\{(t^n, t^{n-1}, \dots, t, 1) | t \in \mathbb{F}_q\right\} \cup \left\{(1,0,\dots,0)\right\}.$$

A NRC contains $q+1$ points. A NRC is a $(q+1)$-arc. For $n=2$, it is a non-singular conic. For $n=3$, it is a <u>twisted cubic</u>. Any $(n+3)$-arc in $PG(n,q)$ is contained in a unique NRC of this space; see [30, Chapter 21].

2.2 k-arcs and linear MDS codes

Let C be a <u>linear code</u> over $\mathbb{F}_q$ of <u>length</u> k and <u>dimension</u> m, that is, C is an m-dimensional subspace of the k-dimensional vector space $V(k,q)$ over $\mathbb{F}_q$. If $\overline{x} = (x_1, x_2, \dots, x_k)$ and $\overline{y} = (y_1, y_2, \dots, y_k)$ are distinct <u>code words</u> in C, that is, distinct elements of C, then the <u>distance</u> $d(\overline{x}, \overline{y})$ between $\overline{x}$ and $\overline{y}$ is the number of indices i for which $x_i \neq y_i$, $i = 1, 2, \dots, k$. The minimum distance $d = d(C)$ of C

is the minimum of the distances $d(\overline{x},\overline{y})$, with $\overline{x},\overline{y} \in C$ and $\overline{x} \neq \overline{y}$. It can be shown that $d \leq k-m+1$; see, for example, [27, Chapter 2]. If $d = k-m+1$, then C is maximum distance separable (MDS).

Let G be an $m \times k$ generator matrix for C, that is, the rows of G form a basis for C. It can be shown that C is MDS if and only if any m columns of G are linearly independent; this property is preserved under multiplication of the columns by non-zero scalars. So consider the columns of G as points $P_1, P_2, \ldots, P_k$ of $PG(m-1,q)$. It follows that, for $m \geq 3$, C is MDS if and only if $\left\{P_1, P_2, \ldots, P_k\right\}$ is a k-arc of $PG(m-1,q)$. Hence, for $m \geq 3$, linear MDS codes and arcs are equivalent objects; see, for example, [32, Chapter 13].

2.3 Segre's three problems

I. For given n and q, what is the maximum value of k such that a k-arc exists in $PG(n,q)$?

II. For what values of n and q, with $q > n+1$, is every $(q+1)$-arc of $PG(n,q)$ a normal rational curve?

III. For given n and q, with $q > n+1$, what are the values of k such that each k-arc of $PG(n,q)$ is contained in a $(q+1)$-arc of $PG(n,q)$?

For a survey of solutions to problems I, II, III, see [34], [32, Chapter 13], [1], [2]. Many of the bounds obtained come from relating k-arcs to both algebraic curves and algebraic hypersurfaces; see [27], [36, 37], [54], [9].

2.4 k-arcs in $PG(2,q)$

Theorem 1. *(See[27, Chapter 8]) Let K be a k-arc of* $PG(2,q)$. *Then*

1. $k \leq q+2$;
2. *for q odd,* $k \leq q+1$;
3. *any non-singular conic is a* $(q+1)$*-arc;*
4. *for q even, a* $(q+1)$*-arc extends to a* $(q+2)$*-arc.*

$(q+1)$-arcs of $PG(2,q)$ are called ovals; $(q+2)$-arcs of $PG(2,q)$, q even, are called complete ovals or hyperovals.

Theorem 2. *(See [47, 48], [27, Chapter 8]) In* $PG(2,q)$, *q odd, every oval is a non-singular conic.*

Remark 3. For q even many ovals are known which are not conics.

Theorem 4. *(See [51], [57], [27, Chapter 8])*

1. *For q even, every k-arc with* $k > q - \sqrt{q} + 1$ *extends to a hyperoval.*
2. *For q odd, every k-arc with* $k > q - \frac{1}{4}\sqrt{q} + \frac{25}{16}$ *extends to a conic.*

Remark 5. 1. For many particular values of q, the bounds in Theorem 4 can be considerably improved; see [32, Chapter 13].

2. For q square and $q > 4$, there exist complete $(q-\sqrt{q}+1)$-arcs in $PG(2,q)$, see [27, Chapter 10]. In $PG(2,9)$ there exists a complete 8-arc; see [27, Chapter 10].

In $PG(2,q)$, the size of the largest complete arc is denoted by $m(2,q)$, the second largest by $m'(2,q)$. For $m'(2,q)$ see [27, Chapter 10], [29, 31], [32, Chapter 13].

Progress on the value of $m'(2,q)$ is as follows :

$m'(2,q)$, with $q=p^h$, p prime		
$\leq q-1$	$q \geq 7$	Segre 1955, Tallini 1957
$\leq q-\frac{1}{4}\sqrt{q}+\frac{7}{4}$	q odd	Segre 1967
$= q-\sqrt{q}+1$	q even	Segre 1967, Fisher et al. 1986
$< q-\frac{1}{4}\sqrt{q}+\frac{25}{16}$	q odd	Thas 1987
$\leq \frac{44}{45}q+\frac{8}{9}$	q prime	Voloch 1990
$\leq q-\sqrt{2q}+2$	$q=2^{2e+1}$	Voloch 1991
$\leq q-\frac{1}{4}\sqrt{pq}+\frac{29}{16}+1$	$q=p^{2e+1}$	Voloch 1991
$\leq q-\frac{1}{2}\sqrt{q}+5$	$p \geq 5$	Hirschfeld-Korchmáros 1996
$\leq q-\frac{1}{2}\sqrt{q}+3$	$q \geq 23^2, p \geq 3, q \neq 3^6, 5^5$	Hirschfeld-Korchmáros 1998

2.5 k-arcs in $PG(3,q)$

Theorem 6. *(See [47, 48], [11])*

1. *For any k-arc of $PG(3,q)$, q odd and $q>3$, we have $k \leq q+1$; any k-arc of $PG(3,3)$ has at most 5 points.*
2. *For any k-arc of $PG(3,q)$, q even and $q>2$, we have $k \leq q+1$; any k-arc of $PG(3,2)$ has at most 5 points.*

Theorem 7. *(See [49], [12], [25, 26, 30])*

1. *Any $(q+1)$-arc of $PG(3,q)$, q odd, is a twisted cubic; that is, is projectively equivalent to*
$$\left\{(t^3,t^2,t,1) | t \in \mathbb{F}_q\right\} \cup \left\{(1,0,0,0)\right\}.$$
2. *Every $(q+1)$-arc of $PG(3,q)$, $q=2^h$, is projectively equivalent to*
$$\left\{(t^{e+1},t^e,t,1) | t \in \mathbb{F}_q\right\} \cup \left\{(1,0,0,0)\right\},$$
where $e=2^m$ and $(m,h)=1$.

2.6 k-arcs in $PG(4,q)$ and $PG(5,q)$

Theorem 8. *(See [49], [11], [13], [39], [21])*

1. *For q odd and $q \geq 5$, a k-arc in $PG(4,q)$ has $k \leq q+1$; a k-arc in $PG(4,3)$ has at most 6 points.*
2. *In $PG(4,9)$, up to projective equivalence there are precisely two 10-arcs, the normal rational curve and the Glynn 10-arc*
$$\left\{(t^4,t^3,t^2+\sigma t^6,t,1) | t \in \mathbb{F}_9\right\} \cup \left\{(1,0,0,0,0)\right\},$$
where σ is a primitive element of $\mathbb{F}_9$ with $\sigma^2=\sigma+1$.
3. *For q even and $q>4$, a k-arc in $PG(4,q)$ has $k \leq q+1$; a k-arc in $PG(4,2)$ or $PG(4,4)$ has at most 6 points.*
4. *For q even, a $(q+1)$-arc K in $PG(4,q)$ is a normal rational curve.*
5. *For q even and $q \geq 8$, a k-arc in $PG(5,q)$ has $k \leq q+1$.*

2.7 k-arcs in $PG(n,q)$, $n \geq 3$

Theorem 9. *(See [54, 57], [39]) For q odd and $n \geq 3$, let K be a k-arc in $PG(n,q)$.*

1. *If $k > q - \frac{1}{4}\sqrt{q} + n - \frac{7}{16}$, then K lies on a unique normal rational curve of $PG(n,q)$.*
2. *If $k = q+1$ and $q > \left(4n - \frac{23}{4}\right)^2$, then K is a normal rational curve of $PG(n,q)$.*
3. *If $q > \left(4n - \frac{39}{4}\right)^2$, then $k \leq q+1$ for any k-arc in $PG(n,q)$.*

Theorem 10. *(See [9], [6], [52])*

1. *For q even, $q \neq 2$, $n \geq 3$, if K is a k-arc in $PG(n,q)$ with $k > q - \frac{1}{2}\sqrt{q} + n - \frac{3}{4}$, then K lies on a unique $(q+1)$-arc of $PG(n,q)$.*
2. *A $(q+1)$-arc in $PG(n,q)$, q even and $n \geq 4$, with $q > \left(2n - \frac{7}{2}\right)^2$, is a normal rational curve.*
3. *If K is a k-arc in $PG(n,q)$, q even and $n \geq 4$, with $q > \left(2n - \frac{11}{2}\right)^2$, then $k \leq q+1$.*

Several interesting theorems were proved for particular values of q. In particular, the following result holds for q a prime.

Theorem 11. *(See[1]) For a k-arc in $PG(n,q)$, $q > n+1$ and q a prime, the cardinality k satisfies $k \leq q+1$.*

Theorem 12. *(See[55]) A k-arc exists in $PG(n,q)$, $n \geq 2$, if and only if a k-arc exists in $PG(k-n-2,q)$.*

2.8 Conjectures

Conjecture 13. *(I) If K is a k-arc in $PG(n,q)$, q odd and $q > n+1$, then $k \leq q+1$.*

(II) For any k-arc K of $PG(n,q)$, q even, $q > n+1$ and $n \notin \{2, q-2\}$, we have $k \leq q+1$.

Remark 14. From Theorem 1 and Theorem 12 it follows that for any q even, $q \geq 4$, there exists a $(q+2)$-arc in $PG(q-2,q)$.

2.9 Open problems

1. Classify all ovals and hyperovals of $PG(2,q)$, q even.
2. Is every k-arc of $PG(2,q)$, q odd, $q > 9$ and $k > q - \sqrt{q} + 1$ extendable?
3. What is the size of the second largest complete k-arc in $PG(2,q)$ for q odd and for q an even non-square?
4. Is every 6-arc of $PG(3,q)$, $q = 2^h$, $h > 2$, contained in exactly one $(q+1)$-arc projectively equivalent to

$$\left\{(t^{e+1}, t^e, t, 1) | t \in \mathbb{F}_q\right\} \cup \left\{(1,0,0,0)\right\},$$

where $e = 2^m$ and $(m,h) = 1$?

5. In $PG(2,q)$ a $(q-1)$-arc is incomplete for $q > 13$, except possibly for the 14 values of q consisting of 49, 81 and the twelve primes 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83; see [32, Chapter 13]. For $q \in \{4,5,8\}$ a $(q-1)$-arc is incomplete, but for $q \in \{7,9,11,13\}$ there exists a complete $(q-1)$-arc in $PG(2,q)$; see [27, Chapter 13]. What about the fourteen remaining cases? Note that every q-arc of $PG(2,q)$ is incomplete; see Theorem 4 and [27, Chapter 10].
6. Are conjectures (I) and (II) true?
7. Find more solutions for Segre's problems I, II and III.
8. In $PG(n,q)$, q odd and $q \geq n$, are there $(q+1)$-arcs other than the 10-arc of Glynn which are not normal rational curves?
9. Is a rational curve of $PG(n,q)$, $2 < n < q-2$, always complete?

3 k-caps

3.1 Definitions

A k-cap in $PG(n,q)$, $n \geq 3$, is a set of k points no three of which are collinear. A k-cap is complete if it is not contained in a $(k+1)$-cap. A line of $PG(n,q)$ is a secant, tangent or external line as it meets K in 2, 1 or 0 points.

The maximum size of a k-cap in $PG(n,q)$ is denoted $m_2(n,q)$.

3.2 k-caps in $PG(3,q)$

Theorem 15. *(See [30], [7], [46])*

1. *For a k-cap in $PG(3,q)$ with $q \neq 2$, $k \leq q^2+1$.*
2. *Each elliptic quadric of $PG(3,q)$ is a (q^2+1)-cap.*
3. *$m_2(3,2) = 8$ and any 8-cap of $PG(3,2)$ is the complement of a plane.*

(q^2+1)-caps of $PG(3,q)$, $q \neq 2$, are called ovoids; the ovoids of $PG(3,2)$ are its elliptic quadrics.

Remark 16. Tits [62] defines an ovoid of $PG(3,\mathbb{K})$, for any field $\mathbb{K}$, as a cap such that, for any of its points P, the lines containing P and intersecting the cap just in P form a plane. For $\mathbb{K}$ finite, the two definitions are equivalent, but not for $\mathbb{K}$ infinite.

Theorem 17. *(See [30]) In $PG(3,q)$,*

1. *an elliptic quadric is an ovoid;*
2. *at each point P of an ovoid O, there is a unique tangent plane π such that $\pi \cap O = \{P\}$;*
3. *every non-tangent plane meets O in a $(q+1)$-arc;*
4. *for q even, the $(q^2+1)(q+1)$ tangent lines of O are the self-polar lines of a sympletic polarity η, that is, the lines l for which $l^\eta = l$.*

Remark 18. A correlation of $PG(3,q)$ is a bijection η of the point set of $PG(3,q)$ onto the plane set of $PG(3,q)$, such that the $q+1$ points of any line l are mapped onto the $q+1$ planes containing a line l'. The line l' is the image of l under η, and denoted by l^η. The point set of any plane π of $PG(3,q)$ is mapped onto the set of all planes containing a point P. The point P is the image of π under η, denoted π^η. If $\eta^2 = 1$, then η is a polarity. If l is a line for which $l^\eta = l$, with η a polarity, then l is self-polar. A symplectic polarity is a polarity such that $P \in P^\eta$ for each point P of $PG(3,q)$.

Theorem 19. *(See [30], [3], [44]) In $PG(3,q)$, q odd, every ovoid is an elliptic quadric.*

Theorem 20. *(See [8]) In $PG(3,q)$, q even, an ovoid containing at least one conic is an elliptic quadric.*

Let $W(q)$ be the point-line geometry formed by all points of $PG(3,q)$ and all self-polar lines with respect to a symplectic polarity η of $PG(3,q)$. The number of points of $PG(3,q)$ is $(q^2+1)(q+1)$ and so is the number of self-polar lines of η. Let $\mathbb{P}$ be the point set of $PG(3,q)$ and let $\mathbb{B}$ be the set of all self-polar lines. A polarity of $W(q)$ is a bijection α of $\mathbb{P} \cup \mathbb{B}$ onto itself such that $\mathbb{P}^\alpha = \mathbb{B}$, $\mathbb{B}^\alpha = \mathbb{P}$, $P \in l$ if and only if $l^\alpha \in P^\alpha$ for $P \in \mathbb{P}$ and $l \in \mathbb{B}$, and $\alpha^2 = 1$.

Theorem 21. *(See [61]) The geometry $W(q)$ admits a polarity α if and only if $q = 2^{2e+1}$. In such a case the absolute points of α, that are the points which lie on their images for α, form an ovoid O of $PG(3,q)$. These ovoids are called Tits ovoids. A Tits ovoid is an elliptic quadric if and only if $q = 2$.*

Theorem 22. *(See [61])*

1. *With $q = 2^{2e+1}$, the canonical form of a Tits ovoid O is the following :*

$$O = \Big\{(1,z,y,x) | z = xy + x^{\sigma+2} + y^\sigma\Big\} \cup \Big\{(0,1,0,0)\Big\},$$

where σ is the automorphism $t \mapsto t^{2^{e+1}}$ of $\mathbb{F}_q$.

2. *The group of all projectivities of $PG(3,q)$ fixing O is the Suzuki group $Sz(q)$, which acts doubly transitively on O.*

Remark 23. 1. For q even, no other ovoids than the elliptic quadrics and the Tits ovoids are known.

2. For $q = 8$, the Tits ovoid was first discovered by Segre; see [50], [20].

Remark 24. 1. Each ovoid of $PG(3,4)$ is an elliptic quadric; see [3], [30].

2. For $q = 8$, the elliptic quadric and the Tits ovoid are the only ovoids; see [20].
3. For $q = 16$, every ovoid is an elliptic quadric; see [41, 42].
4. For $q = 32$, the elliptic quadric and the Tits ovoid are the only ovoids; see [43].

3.3 Ovoids and inversive planes

Definition 25. 1. An inversive plane of order n, $n \geq 2$, is a set $\mathbb{P}$ of size n^2+1 together with a set $\mathbb{B}$ of subsets of size $n+1$ of $\mathbb{P}$, such that any three distinct elements of $\mathbb{P}$ are contained in just one element of $\mathbb{B}$. The elements of $\mathbb{P}$ are called points and the elements of $\mathbb{B}$ are circles.

2. If O is an ovoid of $PG(3,q)$, then O together with the intersections $O \cap \pi$, where π is a non-tangent plane of O, is an inversive plane of order q. Such an inversive plane is called egglike and is denoted $I(O)$.

3. If O is an elliptic quadric, then the inversive plane $I(O)$ is called classical or Miquelian.

Remark 26. By Theorem 19, each egglike inversive plane of odd order is Miquelian.

Theorem 27. *(See [16, Chapter 6]) Every inversive plane of even order is egglike.*

Consider an inversive plane I of order n with point set $\mathbb{P}$ and circle set $\mathbb{B}$. Let P be any point of $\mathbb{P}$, let $\mathbb{P}' = \mathbb{P} \setminus \{P\}$, and let $\mathbb{B}'$ be the set of all circles containing P with P removed. Then $|\mathbb{P}'| = n^2$, any element of $\mathbb{B}'$ contains n elements of $\mathbb{P}'$, and any two elements of $\mathbb{P}'$ are contained in exactly one element of $\mathbb{B}'$. Hence $\mathbb{P}'$ together with $\mathbb{B}'$ is an affine plane of order n. This plane is denoted I_P and is called the internal or derived plane of I at P.

Remark 28. For any egglike inversive plane of order q, each internal plane is Desarguesian, that is, the affine plane $AG(2,q)$ over $\mathbb{F}_q$.

Theorem 29. *(See [58]) Let I be an inversive plane of odd order n. If for at least one point P of I, the internal plane I_P is Desarguesian, then I is Miquelian.*

Remark 30. Up to isomorphism, there is a unique inversive plane of order n for $n = 2,3,4,5,7$; see [15], [17, 18], [63]. As a corollary to Theorem 29 and the uniqueness of the projective planes of orders 3, 5, 7, the uniqueness of the inversive planes of these orders follow.

3.4 k-caps in $PG(n,q)$, $n \geq 3$

The maximum size of a k-cap in $PG(n,q)$ is denoted by $m_2(n,q)$.

Theorem 31. *(See [36, Chapter 27], [37, Chapter 6], [34], [63], [7], [22, 23, 24], [45], [19])*

1. $m_2(n,2) = 2^n$; *a* 2^n*-cap of* $PG(n,2)$ *is the complement of a hyperplane.*
2. $m_2(4,3) = 20$; *there are nine projectively distinct* 20*-caps in* $PG(4,3)$.
3. $m_2(5,3) = 56$; *the* 56*-cap in* $PG(5,3)$ *is projectively unique.*
4. $m_2(4,4) = 41$; *there exist exactly two projectively distinct* 41*-caps in* $PG(4,4)$.

Remark 32. No other values $m_2(n,q)$, $n > 3$, are known.

Several bounds were obtained for the number k for which there exist complete k-caps in $PG(3,q)$ which are not ovoids; these bounds are used to determine bounds for $m_2(n,q)$, with $n > 3$. See [36, Chapter 27], [37, Chapter 6], [28], [40], [35], [53], [14], [59], [38]. Here we mention just three bounds.

Theorem 33. *(See [36, Chapter 27], [37, Chapter 6], [28]) In* $PG(3,q)$, *q odd and* $q \geq 67$, *if K is a complete k-cap which is not an elliptic quadric, then*

$$k < q^2 - \frac{1}{4}q^{\frac{3}{2}} + 2q.$$

Theorem 34. *(See [40]) For $n \geq 4$, $q = p^h$ and p an odd prime,*

$$m_2(n,q) \leq \frac{nh+1}{(nh)^2} q^n + m_2(n-1,q).$$

Theorem 35. *(See [59]) In $PG(3,q)$, q even and $q \geq 8$, if K is a complete k-cap which is not an ovoid, then*

$$k < q^2 - (\sqrt{5}-1)q + 5.$$

Remark 36. The bound of Theorem 35, which is not yet published, is better than the bound $k \leq q^2 - q + 5$ (q even and $q \geq 8$) of Chao [14]. In 2014, Cao and Ou [10] published the bound $k \leq q^2 - 2q + 8$ (q even and $q \geq 128$), which is better than the one in Theorem 35. I did not understand some reasoning in their proof, so I sent two mails to one of the authors explaining why I did not follow part of the proof. I never received an answer.

3.5 Open problems

1. Improve the bounds of Theorems 33-35.
2. Classify all ovoids of $PG(3,q)$ for q even.
3. Is every inversive plane of odd order Miquelian?
4. Determine $m_2(n,q)$ for $n \geq 4$, $q \neq 2$.

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