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Robust Model Predictive Control of Constrained Supply Networks via Invariant Ellipsoids Technique

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Abstract: The problem of robust control strategy synthesis for distributed supply network under demand uncertainty, time delays and state and control constraints is considered. An invariant ellipsoids approach is used for robust control problem solving, since the uncertain demands are regarded as an external disturbance. On the base Model Predictive Control approach, the designed control law implements in the form of linear feedback signal based on mismatch between the current state and safety stock level and provides external disturbances effect suppression with simultaneous robust stabilization of closed-loop system. Via invariant ellipsoids technique the considered problem was presented in the terms of Linear Matrix Inequalities and a solution of corresponding semi-definite optimization problem was also obtained. As an example, the three-tier supply network with five nodes robust control problem is considered.

Keywords: Supply Network, Inventory Control, Constrained Control, Model Predictive Control, Invariant Ellipsoid, Linear Matrix Inequality, Semi-Definite Programming.

1. INTRODUCTION

Supply network is a complex system consisting of a set of interrelated objects carrying raw materials extraction, production, storage, transportation and distribution of products in order to meet customer demand (see Bartmann and Beckmann (1992)). Typically, supply network is represented as a directed graph whose vertices correspond to network nodes and determine the types and amounts of controlled inventory levels, and the arcs represent the controlled and uncontrolled flows in the network. Controlled flows represent the processes of recycling and redistribution of resources between network nodes and the processes of supplying of raw materials from outside of the system. Uncontrolled flows in turn represent the demand for resources, which is produced by the external customers.

Thus under the influence of production line operation as well as external customers demand, the levels of the reserve resources at the supply network nodes change over the time. The main goal of Supply Network Control (SNC) is complete and prompt external customers demand satisfaction with simultaneous operation cost minimization. In such case, the control strategy should describe a set of rules for determination of moments and amount of orders to stock replenishment. Therefore such a problem is closely related with multivariable inventory control problem under uncertainty.

Model Predictive Control (MPC) is widely used for the synthesis of inventory control strategy with a known demand model (see Bemporad and Morari (1999), Mayne et al. (2000)). Such approach takes into account state and control constraints and can be implemented as a nonlinear state feedback control law. In most cases, solution can be reduced to the *on-line* solution of the sequence of Quadratic Programming (QP) problems.

In the modern robust control theory the concept of invariant sets is actively used to solve the problems of uncertain disturbances suppression (see Blanchini and Miani (2008)). Among the various forms of invariant sets ellipsoids take special place due to their simple structure and a direct relation with quadratic Lyapunov functions. In Kothare et al. (1996) the problem of closed-loop system stabilization under constraints is solved using the MPC approach and the technique of Linear Matrix Inequalities (LMI). Evaluation of a reachable set is performed using the Semi-Definite Programming (SDP) method. In Poznyak and Alazki (2011) solution of the inventory control problem based on ellipsoidal technique for two-dimensional case without a delays is considered. However, obtained control strategy which ensures the closed-loop system phase trajectories to the origin is applicable only for traditional formulation of the constrained control problem, when $|u| \leq u^+$. In the inventory control problems input actions are meaningful volumes ordered resources. So they can take only non-negative values and must meet the asymmetric constraints.

In practice the design of predictive control is faced with the lack of the necessary information to describe the demand in terms of random process model with known structure. In such case the uncertain external demand may be regarded as unknown disturbance. Therefore the SNC technique should be based on disturbance suppression methods under uncertainty.

The concept of "unknown-but-bounded" (UBB) disturbances is proposed in Blanchini et al. (1997) to solve the problem of inventory control under demand uncertainty. The corresponding UBB demand model is characterized by interval uncertainty, signifying that each component of the vector-function describing the demand belongs to some interval whose boundaries are determined based on the apriori information.

Another source of uncertainty in SNC is the presence of delays between the moment when the supply order is issued and when ordered resources are delivered. Discrete delay model is used to describe the influence of the lead-times on a supply network. In order to use this model it is assumed that the values of the lead-times are known and fixed. However, during the operation of the network, those parameters may change and significantly differ from their nominal values. As a result, it is necessary to ensure the robustness of the SNC system with respect to possible parameters variations.

The purpose of this paper is the synthesis of robust control strategy for supply network in the presence of uncertain external demand and delays meanwhile taking into account asymmetric constraints on the state and control variables. The control law is proposed in the form of linear feedback signal based on mismatch between the current state and safety stock level and provides external disturbances suppression with simultaneous robust stabilization of closed-loop system.

2. PROBLEM STATEMENT

2.1 Discrete-time state space model formulation

It is assumed that network state can be obtained and control actions can be formed at discrete time points with a given sampling period $\Delta \mathrm{t}$. Hence a discrete state-space model is used for the mathematical description of the considered supply network. It is assumed that all values of time delay intervals, which characterize the functioning of the supply network, can be described as multiple of $\Delta \mathrm{t}$ and are the integer positive numbers. State variables would represent the inventory levels of resources that already processed and placed in the storage nodes. Control variables would represent the supply orders levels, and disturbance variables would represent external demand for the finished products. Additional limitations on a state and control variables are taken into account to describe the capacity limits and limits on supply order levels.

Consider the mathematical model of the supply network which describes the changes in inventory levels of each type of resource. It is assumed that the structure of the supply network is known and states are accessible to direct measurement. Then the supply network may be represented as a linear discrete model with delays:

$$x(k+1) = x(k) + \sum_{t=0}^{\Lambda} \mathrm{B}_t u(k-t) + \mathrm{E}d(k),\ k = 0, 1, \dots \quad (1)$$

where $x(k) \in \mathbb{R}^n$ - state vector; $u(k) \in \mathbb{R}^m$ - control inputs; $d(k) \in \mathbb{R}^q$ - is external disturbances; the structure of the system is determined by control matrixes $\mathrm{B}_t \in \mathbb{R}^{n \times m}$ $t = \overline{0, \Lambda}$ and matrix that describes the impact of external disturbances $\mathrm{E} \in \mathbb{R}^{n \times q}$; Λ - maximum delay value material flows between nodes.

During operation the control system is reguired to ensure the following constraints fulfillment:

$$x(k) \in X = \{x \in \mathbb{R}^n :\ 0 \le x \le x^+\}, \quad (2a)$$

$$u(k) \in U = \{u \in \mathbb{R}^m :\ 0 \le u \le u^+\}, \quad (2b)$$

where x^+ and u^+ are the known vectors which describe the upper bound limits for storage capacity and control respectively. We assume that the external disturbances vectors satisfy the constraints:

$$d(k) \in D = \{d \in \mathbb{R}^q :\ d^- \le d \le d^+\}, \quad (3)$$

where components of vectors d^- and d^+ are known and describe the lower and upper bounds of external disturbances.

Then sets X, U, D are bounded polyhedra, which are determined by the intersection of finite number of closed half-spaces, i.e. are compact convex sets with $0 \notin \mathrm{int}(X)$, $0 \notin \mathrm{int}(U)$, $0 \notin \mathrm{int}(D)$.

For the system (1) we consider the problem of SNC strategy synthesis, which for any initial conditions $x(0) \in X$ and uncertain demand $d(k) \in D$, $k \ge 0$ and under the constraints on state and control (2) provides the minimization of a quadratic cost function, which describes the total losses from the the current inventory levels deviations from the given safety stock level as well as ensure the asymptotic stability of the closed-loop system.

2.2 Supply network model transformation

The first step of control problem solving is model (1) transformation to the standard form with no delays based on the state vector expending (see Blanchini et al. (1997)) by including the volumes of previously ordered amounts of resources that are currently in processing or transportation:

$$\xi(k) = \left(x(k)', u(k-1)', u(k-2)', ..., u(k-\Lambda)'\right)',$$

where "′" denotes the transpose of a vector.

Then the equations of the augmented supply network model will take the form:

$$\begin{aligned} \xi(k+1) &= \mathrm{A}\xi(k) + \mathrm{B}u(k) + \mathrm{G}d(k), \\ x(k) &= \mathrm{C}\xi(k), \end{aligned} \quad (4)$$

where model matrices have a block structure:

$$\mathrm{A} = \begin{vmatrix} \mathrm{I}_{n\times n} & \mathrm{B}_1 & \cdots & \mathrm{B}_{\Lambda-1} & \mathrm{B}_\Lambda \\ 0_{m\times n} & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times n} & \mathrm{I}_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m\times n} & 0_{m\times m} & \cdots & \mathrm{I}_{m\times m} & 0_{m\times m} \end{vmatrix}, \mathrm{B} = \begin{vmatrix} \mathrm{B}_0 \\ \mathrm{I}_{m\times m} \\ 0_{m\times m} \\ \vdots \\ 0_{m\times m} \end{vmatrix},$$

$$\mathrm{G}' = \left|\mathrm{E}\ 0_{m\times q}\ \cdots\ 0_{m\times q}\right|, \mathrm{C} = \left|\mathrm{I}_{n\times n}\ 0_{m\times n}\ \cdots\ 0_{m\times n}\right|.$$

During operation, supply network parameters may differ from their nominal values. Typically delay value of the material flow associated with a node i increases. Depending on which value Λ_i would take element (i, i) in one of the control matrices B_t, $t = \overline{0, \Lambda}$ would be equal to 1: $[\mathrm{B}_{\Lambda_i}]_{ii} = 1$. Then the dynamic matrix of the augmented network model can be represented as the sum:

$$\mathrm{A}(k) = \mathrm{A}_0 + \sum_{i=1}^{\Lambda} \mathrm{A}_i(\lambda_i) = \begin{vmatrix} \mathrm{I} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{I} & 0 \end{vmatrix}$$

$$+\begin{vmatrix}0 & B_1(\lambda_1) & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{vmatrix} + \cdots + \begin{vmatrix}0 & 0 & \cdots & 0 & B_\Lambda(\lambda_\Lambda)\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{vmatrix},$$

which is equivalent to

$$A(k) \in \{\sum_{i=1}^{\Lambda} \lambda_i(k)[A_i] : \lambda_i(k) \geq 0, \sum_{i=1}^{\Lambda} \lambda_i(k) = 1\}.$$

Assuming that delays of all controlled flows may change in the similar manner, the set of possible values of the dynamic matrix can be represented as

$$A(k) \in \Omega = \{\sum_{j=1}^{n}\sum_{i=1}^{\Lambda} \lambda_i^j(k)[A_i^j] : \lambda_i^j(k) \geq 0, \sum_{i=1}^{\Lambda} \lambda_i^j(k) = 1\}.$$

Thus, augmented supply network model with varying lead-times in control flows may be represented as a model with uncertainty in the form of a convex polyhedron, which is described by a list of vertices (see Blanchini and Miani (2008)) or, in other words, in the form of Linear Parameter-Varying (LPV) system:

$$\begin{aligned}&\xi(k+1) = A(k)\xi(k) + Bu(k) + Gd(k),\\ &x(k) = C\xi(k),\\ &A(k) \in \Omega = Co\{A^{(1)}, \ ..., \ A^{(L)}\},\end{aligned} \qquad (5)$$

where Co{.} - the convex hull, L - the number of vertices of the set Ω, and equal to the maximum cardinality of the set of possible values of the lead-time material flows between network nodes.

3. BASIC MPC STRATEGY SYNTHESIS

The basic approach for the synthesis of MPC control strategy is based on the prediction of external demand during prediction horizon N_p (see Bemporad and Morari (1999)). In Blanchini et al. (1997) it is shown that if all pairs of matrixes $\left(A^{(i)}, B\right)$, $i = \overline{1, L}$ for the network model (6) are controllable, then the control strategy that provides the condition $x(k) \in X$ $k \geq 0$ for any initial state $x(0) \in X$ under the disturbances $d(k) \in D$ exists only if a convex polyhedron, which describes the influence of external demand, is strictly inside convex polyhedron, which describes the constraints on control actions:

$$ED \subset -BU. \qquad (6)$$

Verification of condition (6) could be done by solving $K = 2^{n+m}$ number of Linear Programming (LP) problems.

Therefore the problem of MPC control strategy synthesis may be stated as follows:

$$\min_{u(k),...,u(k+N_c-1)} J(k), \qquad (7)$$

where N_c - control horizon. A cost function that takes into account losses from current inventory level deviations from safety stock level as well as control costs, is chosen as:

$$\begin{aligned}J(k) = &\sum_{l=1}^{N_p}\left(\xi(k+l|k) - \xi^*\right)' R_\xi\left(\xi(k+l|k) - \xi^*\right)\\ &+ \sum_{l=1}^{N_c} u(k+l-1|k)' R_u u(k+l-1|k),\end{aligned} \qquad (8)$$

where $\xi(k+l|k)$, $u(k+l|k)$ - are the state and control vectors predictions for l steps forward respectively; ξ^* - safety stock level; $R_\xi \succ 0$, $R_u \succ 0$ - positive definite diagonal weighting matrices of appropriate dimensions. The prediction horizon should not be less than the maximum delay in the network $N_p \geq \Lambda$, and the control horizon is chosen to be less or equals to the prediction horizon $N_c \leq N_p$.

When the problem (7) is solved only first element of obtained control sequence $u(k) = u(k|k)$ is used as a control action at time k. After that new state vector $\xi(k+1)$ is measured and procedure is repeated again using the principle of receding horizon.

Since the constraints (2) take place, it is impossible to find general solution of (7) in analytic form, and corresponding QP-problem is solved numerically in *on-line* mode.

It is well known that basic MPC control strategy do not guarantee the stability of the closed-loop system. Consequently, there is a need for the synthesis of a stabilizing control strategy.

4. ROBUST MPC STRATEGY SYNTHESIS

According to the theorem on the approximation of arbitrary convex sets not symmetric around the origin (see Chernousko (1994)), external demand can be approximated by an ellipsoid:

$$\begin{aligned}&E(d^*, q^2P_d) = \{d \in \mathbb{R}^q : (d(k) - d^*)'(q^2P_d)^{-1}\\ &\quad\cdot(d(k) - d^*) \leq 1\}, \qquad d^* = 1/2(d^- + d^+),\\ &P_d = \mathrm{diag}\big(1/4(d_1^+ - d_1^-)^2, \ ..., \ 1/4(d_q^+ - d_q^-)^2\big).\end{aligned} \qquad (9)$$

Control strategy should ensure the suppression of the demand effect by minimizing the functional constructed for the invariant set of the system states. Hence this set is represented by the invariant ellipsoid, trace of this ellipsoid can be selected as a corresponding functional. Choose the control law in the form of static linear feedback signal based on mismatch between the current state and safety stock level:

$$u(k) = K(\xi(k) - \xi^*), \ k \geq 0. \qquad (10)$$

Vector of safety stock levels is calculated based on the average value of the external demand by applying the theorem on the approximation of arbitrary convex sets and equivalent model Leontief:

$$\xi^* = \left(x^{*'}, ..., x^{*'}\right)', \ x^* = \begin{cases}\frac{n}{2}(d_i^- + d_i^+), & i = \overline{1, q},\\ \left(I - \Pi\right)^{-1} d_{mean}, & i = \overline{q+1, n},\end{cases} \qquad (11)$$

where $d_{mean} = \begin{cases}\frac{1}{2}(d_i^- + d_i^+), & i = \overline{1, q},\\ 0, & i = \overline{q+1, n},\end{cases}$ $\Pi \in \mathbb{R}^{n \times n}$ - technological matrix, which element (i, j) equals to the units of resource amount of type i, required to produce one resource unit of type j.

Therefore augmented closed-loop system model with control (10) may be represented as:

$$\begin{aligned}&\xi(k+1) = A_f(k)(\xi(k) - \xi^*) + (A(k) - I)\xi^*\\ &\qquad + G(d(k) - d^*) + Gd^*,\\ &x(k) = C\xi(k),\\ &A_f(k) = A(k) + BK, \ A(k) \in \Omega.\end{aligned} \qquad (12)$$

Note that the output vector $x(k)$ of the model (12) actually is a state vector of the supply network model (1) under consideration.

Control strategy synthesis problem can be solved by obtaining the feedback gain matrix K in a way that closed-loop system (12) would be asymptotically stable.

Introduce the "shift" quadratic Lyapunov function on the system (12) solutions:

$$V\big(\xi(k)-\xi^*\big) = \big(\xi(k)-\xi^*\big)^{'}\mathrm{P}\big(\xi(k)-\xi^*\big), \ \mathrm{P}=\mathrm{P}^{'} \succ 0. \quad (13)$$

Asymptotic stability of the closed-loop system is guaranteed if the Lyapunov function (13) is non-increasing for $\forall k \geq 0$. Following Kothare et al. (1996), using the fact that function (13) determines the upper bound of cost function in the infinite horizon case:

$$V\big(\xi(k)-\xi^*\big) \geq \max_{\mathrm{A}(k)\in\Omega} J_\infty(k)$$
$$= \sum_{k=0}^{\infty}\Big(\big(\xi(k)-\xi^*\big)^{'}\mathrm{R}_\xi\big(\xi(k)-\xi^*\big) + u(k)^{'}\mathrm{R}_u u(k)\Big).$$

As a result the problem of control strategy synthesis is reduced to the solution of the minimax problem

$$u(k) = \arg\min_{u(k)\in U}\Big(\max_{\mathrm{A}(k)\in\Omega} J_\infty(k)\Big)$$

or to the solution of equivalent problem

$$u(k) = \arg\min_{u(k)\in U} V\big(\xi(k)-\xi^*\big).$$

So our goal is to find the smallest scalar $\gamma > 0$ such that $\big(\xi(k)-\xi^*\big)^{'}\mathrm{P}\big(\xi(k)-\xi^*\big) \leq \gamma, \ \forall k \geq 0$.

According to Boyd et al. (1994) introduce notation:

$$\mathrm{Q} = \gamma\mathrm{P}^{-1}. \quad (14)$$

Then appropriate inequality will have the form $\big(\xi(k)-\xi^*\big)^{'}\mathrm{Q}^{-1}\big(\xi(k)-\xi^*\big) \leq 1, \ \forall k \geq 0$.

According to the invariant ellipsoids method (see Nazin et al. (2007)) define invariant ellipsoid for evaluation of reachable set for system (12)

$$\mathrm{E}(x^*,\mathrm{R}) = \{x \in \mathbb{R}^n : \ (x-x^*)^{'}\mathrm{R}^{-1}(x-x^*) \leq 1\} \quad (15)$$

with center at vector $x^* = \mathrm{C}\xi^*$ and matrix $\mathrm{R} \in \mathbb{R}^{n\times n}$.

Desired control law (10) may be found by minimizing the linear function $f = \gamma + \mathrm{tr}(\mathrm{R})$ under constraints (2).

Corresponding result is giving by the following theorem.

Theorem 1. Let the matrix $\mathrm{K} = \mathrm{YQ}^{-1}$, where $\mathrm{Q} = \mathrm{Q}^{'} \in \mathbb{R}^{(n+m\Lambda)\times(n+m\Lambda)}$, $\mathrm{Y} \in \mathbb{R}^{m\times(n+m\Lambda)}$ obtained by solving a one-dimensional convex optimization problem to the scalar parameter α and the semi-definite programming of the form:

$$\min_{\alpha,\gamma,\mathrm{Q},\mathrm{Y}} \gamma + \mathrm{tr}(\mathrm{R}) \quad (16a)$$

$$\begin{vmatrix} \mathrm{Q} & \mathrm{G} & 0 & 0 & \big(\mathrm{A}^{(i)}\mathrm{Q}+\mathrm{BY}\big)^{'} & \mathrm{QR}_\xi^{1/2} & \mathrm{Y}^{'}\mathrm{R}_u^{1/2} \\ * & \alpha\big(q^2\mathrm{P}_d\big)^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & \big(\mathrm{A}^{(i)}-\mathrm{I}\big)^{'} & 0 & 0 \\ * & * & * & 0 & \mathrm{G}^{'} & 0 & 0 \\ * & * & * & * & \mathrm{Q} & 0 & 0 \\ * & * & * & * & * & \gamma\mathrm{I} & 0 \\ * & * & * & * & * & * & \gamma\mathrm{I} \end{vmatrix} \succeq 0,$$
$$i = \overline{1,L} \quad (16b)$$

(inequality matrix has symmetric structure),

$$\begin{vmatrix} 1 & \big(\xi(k)-\xi^*\big)^{'} \\ * & \mathrm{Q} \end{vmatrix} \succeq 0, \quad (16c)$$

$$\begin{vmatrix} \mathrm{R} & \mathrm{C} \\ * & \mathrm{Q} \end{vmatrix} \succeq 0, \quad (16d)$$

$$\begin{vmatrix} \gamma\big(u_j^+\big)^2 & \mathrm{Y}_j \\ * & \mathrm{Q} \end{vmatrix} \succeq 0, \ j = \overline{1,m}, \quad (16e)$$

where Y_j denotes j-th raw of matrix Y,

$$\begin{vmatrix} \gamma\big(x_l^+\big)^2 & \mathrm{C}_l \\ * & \mathrm{Q} \end{vmatrix} \succeq 0, \ l = \overline{1,n}. \quad (16f)$$

If problem (16) is feasible, then closed-loop system (12) with initial state $x(0) \in X$ with linear feedback control law (10) under disturbances $d(k) \in \mathrm{E}(d^*, q^2\mathrm{P}_d)$ $\forall k \geq 0$ is asymptotically stable and satisfies the constraints (2).

Proof. Lyapunov function (13) minimization is equivalent to solving the optimization problem:

$$\min_{\gamma,\mathrm{P}} \gamma$$
$$\big(\xi(k)-\xi^*\big)^{'}\mathrm{P}\big(\xi(k)-\xi^*\big) \leq \gamma.$$

Using invariant ellipsoid Lemma (Kothare et al. (1996)) and substitution (14) obtain the equivalent problem:

$$\min_{\gamma,\mathrm{Q}} \gamma$$
$$\begin{vmatrix} 1 & \big(\xi(k)-\xi^*\big)^{'} \\ * & \mathrm{Q} \end{vmatrix} \succeq 0.$$

Introduce vector and matrices:

$s(k) = \Big(\big(\xi(k)-\xi^*\big)^{'}, \xi^{*'}, d^{*'}, \big(d(k)-d^*\big)^{'}\Big)^{'}$, $\mathrm{M}_0 =$

$$\begin{vmatrix} \mathrm{A}_f(k)^{'}\mathrm{PA}_f(k) - \mathrm{P} + \mathrm{R}_\xi + \mathrm{K}^{'}\mathrm{R}_u\mathrm{K} & \mathrm{A}_f(k)^{'}\mathrm{P}\big(\mathrm{A}(k)-\mathrm{I}\big) & \mathrm{A}_f(k)^{'}\mathrm{PG} & \mathrm{A}_f(k)^{'}\mathrm{PG} \\ \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\mathrm{PA}_f(k) & \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\mathrm{P}\big(\mathrm{A}(k)-\mathrm{I}\big) & \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\mathrm{PG} & \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\mathrm{PG} \\ \mathrm{G}^{'}\mathrm{PA}_f(k) & \mathrm{G}^{'}\mathrm{P}\big(\mathrm{A}(k)-\mathrm{I}\big) & \mathrm{G}^{'}\mathrm{PG} & \mathrm{G}^{'}\mathrm{PG} \\ \mathrm{G}^{'}\mathrm{PA}_f(k) & \mathrm{G}^{'}\mathrm{P}\big(\mathrm{A}(k)-\mathrm{I}\big) & \mathrm{G}^{'}\mathrm{PG} & \mathrm{G}^{'}\mathrm{PG} \end{vmatrix}, \ \mathrm{M}_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \big(q^2\mathrm{P}_d\big)^{-1} \end{vmatrix},$$

$f_i(s) = s^{'}\mathrm{M}_i s, \ i = 0,1.$

Then inequality:

$$V\big(\xi(k+1)-\xi^*\big) - V\big(\xi(k)-\xi^*\big)$$
$$\leq -\Big(\big(\xi(k)-\xi^*\big)^{'}\mathrm{R}_\xi\big(\xi(k)-\xi^*\big) + u(k)^{'}\mathrm{R}_u u(k)\Big)$$

and inequality (9) may be rewritten as:

$f_0(s) \leq 0 \ \ \forall s: \ f_1(s) \leq 1.$

According to S-theorem with one constraint (see Polyak and Shcherbakov (2002)) last statement is equal to LMI $\mathrm{M}_0 \leq \alpha\mathrm{M}_1$ for some $\alpha \geq 0$.

Applying Schur complement and matrices inversion Lemma (see Golub et al. (1996)), get:

$$\begin{vmatrix} -\mathrm{P} + \mathrm{R}_\xi + \mathrm{K}^{'}\mathrm{R}_u\mathrm{K} + \mathrm{A}_f(k)^{'}\Psi\mathrm{A}_f(k) & \mathrm{A}_f(k)^{'}\Psi\big(\mathrm{A}(k)-\mathrm{I}\big) & \mathrm{A}_f(k)^{'}\Psi\mathrm{G} \\ \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\Psi\mathrm{A}_f(k) & \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\Psi\big(\mathrm{A}(k)-\mathrm{I}\big) & \big(\mathrm{A}(k)-\mathrm{I}\big)^{'}\Psi\mathrm{G} \\ \mathrm{G}^{'}\Psi\mathrm{A}_f(k) & \mathrm{G}^{'}\Psi\big(\mathrm{A}(k)-\mathrm{I}\big) & \mathrm{G}^{'}\Psi\mathrm{G} \end{vmatrix} \preceq 0,$$

where $\Psi = \big(\mathrm{P}^{-1} - \frac{q^2}{\alpha}\mathrm{GP}_d\mathrm{G}^{'}\big)^{-1}$.

According to Boyd et al. (1994) introduce new matrix variable $\mathrm{Y} = \mathrm{KQ}$. Using substitution (14) and pre- and post-multiplying last matrix inequality by $\begin{vmatrix} \mathrm{Q} & 0 \\ 0 & \mathrm{I} \end{vmatrix}$:

$$\begin{vmatrix} -\gamma \mathrm{Q} + \mathrm{Q}\mathrm{R}_\xi \mathrm{Q} + \mathrm{Y}^{'}\mathrm{R}_u \mathrm{Y} + \big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big)^{'}\Psi\big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big) & \big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big)^{'}\Psi\big(\mathrm{A}(k) - \mathrm{I}\big) & \big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big)^{'}\Psi \mathrm{G} \\ \big(\mathrm{A}(k) - \mathrm{I}\big)^{'}\Psi\big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big) & \big(\mathrm{A}(k) - \mathrm{I}\big)^{'}\Psi\big(\mathrm{A}(k) - \mathrm{I}\big) & \big(\mathrm{A}(k) - \mathrm{I}\big)^{'}\Psi \mathrm{G} \\ \mathrm{G}^{'}\Psi\big(\mathrm{A}(k)\mathrm{Q} + \mathrm{BY}\big) & \mathrm{G}^{'}\Psi\big(\mathrm{A}(k) - \mathrm{I}\big) & \mathrm{G}^{'}\Psi \mathrm{G} \end{vmatrix} \preceq 0\ .$$

In Bertsekas (2005) it is shown, that for convex cost function and model uncertainties represented by convex set, to guarantee the stability of the considered system only vertices of convex set has to be checked. It is worth noting that vertices of uncertainties set not unnecessary maps to the vertices of states set of a system. However, the opposite statement is true.

As a result, for supply networks stability check for matrices $\mathrm{A}(k) \in \Omega$ is equivalent to the stability check for all vertices of set Ω. Considering this, and using Schur complement, last inequality can be represented as (16b).

Since R is a matrix of invariant ellipsoid (15) must hold inequality $\mathrm{CPC}^{'} \leq \gamma \mathrm{R}$. Using substitution (14) and Schur complement this inequality may be represented as LMI (16d).

Consider control constraint (2a). According to the Theorem about minimum of a linear function in ellipsoid (see Lofberg (2003)) and considering (10):

$$\begin{aligned} &\max_{\mathrm{E}(\xi^*,\mathrm{P})} u_j(k) = \max_{\mathrm{E}(\xi^*,\mathrm{P})} \mathrm{K}_j\big(\xi(k) - \xi^*\big) \\ &= \sqrt{\mathrm{K}_j \mathrm{P}^{-1}\mathrm{K}_j^{'}} = \sqrt{\frac{1}{\gamma}\mathrm{K}_j \mathrm{Q}\mathrm{K}_j^{'}},\ j = \overline{1,m}, \end{aligned} \tag{17}$$

where K_j denotes j-th raw of matrix K.

Then constraint (2a) by applying (17) may be rewritten as:

$$\mathrm{K}_j \mathrm{Q}\mathrm{K}_j^{'} \leq \gamma (u_j^+)^2,\ j = \overline{1,m}. \tag{18}$$

Using Schur complement (18) may be represented as LMI:

$$\begin{vmatrix} \gamma(u_j^+)^2 & \mathrm{K}_j \\ * & \mathrm{Q}^{-1} \end{vmatrix} \succeq 0. \tag{19}$$

Pre- and post-multiplying (19) by $\begin{vmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{Q} \end{vmatrix}$ and using $\mathrm{Y} = \mathrm{KQ}$ rewrite constraint (2a) as (16e).

Considering constraint (2b). The same as (17),(18) get:

$$\big(\mathrm{C}_l \mathrm{P}\mathrm{C}_l^{'}\big)^{-1} \leq \big(x_l^+\big)^2,\ l = \overline{1,n}, \tag{20}$$

Using Schur complement (20) can be rewritten as (16f).

As a result, we come to the problem minimization of linear function $f = \gamma + \mathrm{tr}\big(\mathrm{R}\big)$ with constraints (16b)-(16f).

For fixed α problem (16) is a SDP problem. After solving problem (16) and calculating feedback gain matrix K found control input $u(k) = \mathrm{K}\big(\xi(k) - \xi^*\big)$ used for supply network management at the time k. The next time a new value of the state vector $\xi(k+1)$ is measured and, in accordance with the receding horizon principle, one-dimensional convex optimization and SDP problems solving in *on-line* mode to calculate the new value of the feedback gain matrix K.

5. NUMERICAL EXAMPLE

As an example, consider the supply network, which was studied in Hennet (2003). Supply network model can be represented as a graph $G = \big(V = \{1,2,3,4,5\}, E = \{(5,1),(5,2),(5,3),(4,3),(3,1),(3,2)\}\big)$. Supply network consists of $n = 5$ nodes, which are divided into three levels. Amount of time required for resource processing in nodes: $T_1 = T_3 = T_5 = 1$, $T_2 = T_4 = 2$, and for transportation of resources between nodes: $T_{5,1} = T_{5,2} = T_{5,3} = T_{4,3} = T_{3,1} = T_{3,2} = 1$ are known. Using $\Lambda_i = \max_j\{T_{j,i} + T_i,\ j = \overline{1,5}\},\ i = \overline{1,5}$ determine the delays in material flows in the supply network, and as a result calculate $\Lambda = \max_i\{\Lambda_i\} = 3$.

Let us represent control flows in a form of hyper-arc and add two additional flows which represent the supply of resources from outside of the system. Number them as shown on Fig. 1. Arcs d_1 and d_2 are shown by the dashed lines and represent the external demand. Time intervals $T_{j,i}$ and values of technological matrix elements $\pi_{j,i}$ are represented next to each arc in round and square brackets, respectively. Next to each node time required for resourse processing T_i is shown.

Storage capacities $x^+ = \big(80\ 75\ 300\ 850\ 850\big)^{'}$, control volume limits $u^+ = \big(35\ 35\ 200\ 500\ 500\big)^{'}$, and demand bounds $d^- = \big(10\ 5\big)^{'}$, $d^+ = \big(20\ 10\big)^{'}$ are known. The initial conditions are: $x(0) = \big(100\ 50\ 150\ 400\ 250\big)^{'}$.

Let us assume that transportation time between nodes 5 and 1 may increase by one period during the operation of the supply network, i.e. $T_{5,1} \in \{1,2\}$. Then the lead-time between nodes 5 and 1 can take any value from set $\Lambda_1 \in \{2,3\}$. As a result $\mathrm{A}(k) \in \Omega = \mathrm{Co}\{\mathrm{A}^{(1)},\ \mathrm{A}^{(2)}\}$, where

$$\mathrm{A}^{(1)} = \begin{vmatrix} \mathrm{I}_{5\times5} & \mathrm{B}_1 & \mathrm{B}_2^{(1)} = \begin{vmatrix} \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} & \mathrm{B}_3^{(1)} = \begin{vmatrix} \underline{0} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \\ 0_{5\times5} & 0_{5\times5} & 0_{5\times5} & 0_{5\times5} \\ 0_{5\times5} & \mathrm{I}_{5\times5} & 0_{5\times5} & 0_{5\times5} \\ 0_{5\times5} & 0_{5\times5} & \mathrm{I}_{5\times5} & 0_{5\times5} \end{vmatrix},$$

$$\mathrm{A}^{(2)} = \begin{vmatrix} \mathrm{I}_{5\times5} & \mathrm{B}_1 & \mathrm{B}_2^{(2)} = \begin{vmatrix} \underline{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} & \mathrm{B}_3^{(2)} = \begin{vmatrix} \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \\ 0_{5\times5} & 0_{5\times5} & 0_{5\times5} & 0_{5\times5} \\ 0_{5\times5} & \mathrm{I}_{5\times5} & 0_{5\times5} & 0_{5\times5} \\ 0_{5\times5} & 0_{5\times5} & \mathrm{I}_{5\times5} & 0_{5\times5} \end{vmatrix},$$

(the uncertain varying parameters are underlined).

It can be checked that pairs of matrixes $\big(\mathrm{A}^{(1)}, \mathrm{B}\big)$, $\big(\mathrm{A}^{(2)}, \mathrm{B}\big)$ are controllable, and condition of existence of control strategy (6) holds. By (11) we compute the safety stock level $x^* = \big(75\ \ 38\ \ 45\ \ 135\ \ 120\big)^{'}$. The values of weighting matrices are chosen as $\mathrm{R}_\xi = \mathrm{diag}\big(300,\ ...,\ 300\big)$, $\mathrm{R}_u = \mathrm{diag}\big(0.1,\ ...,\ 0.1\big)$.

The numerical solution of the SDP problem (16) was obtained by freely distributed software package `CVX` for MATLAB (see Grant and Boyd (2008)) with solver `SeDuMi` (see Sturm (1999)).

Simulation results for 15 steps with $\alpha = 0.99$ and step-changing demand volumes are represented on Fig. 2 and Fig. 3.

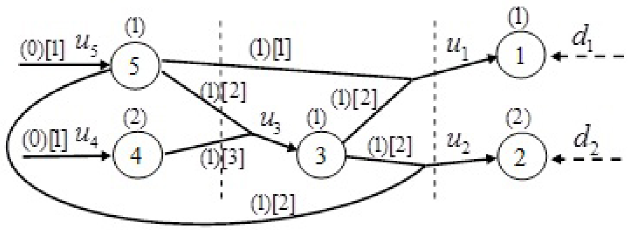

Fig. 1. Graphical representation of the supply network model

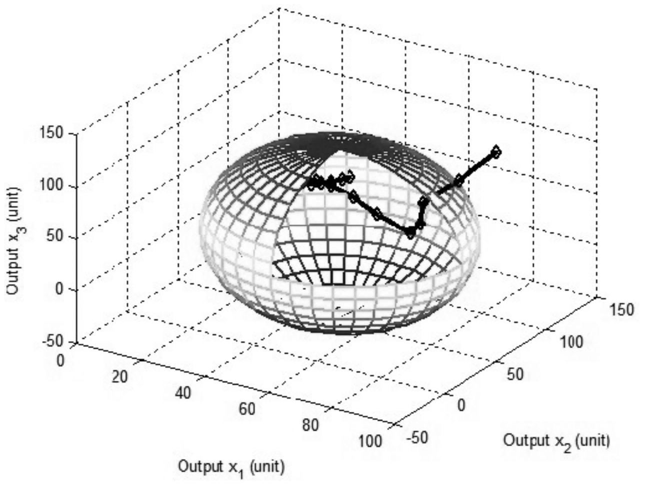

Fig. 2. Phase trajectory and invariant ellipsoid for nodes 1, 2 and 3 of last sample period

6. CONCLUSION

The approach is suggested for synthesis of robust predictive control for supply networks under uncertain demand in the presence of asymmetric constraints on state and control variables. To suppress disturbances, caused by the effect of uncertain demand, while ensuring the stability of the closed-loop system, the Invariant Ellipsoids Technique is used. This allows formulating the problem in terms of Linear Matrix Inequalities, and reduces the synthesis of control to a problem of Semi-Definite Programming and one-dimensional convex optimization.

Obtained control depends on the inventory safety-stock levels, which have a significant impact on the control actions and the operation quality of the entire system. In case when the demand is known, safety-stock levels may be assigned to zero and system would operate on zero-reserves. However, in practice, when uncertain demand is changing, usage of safety stock levels compensate fluctuations in demand and reduce the risk of bumping into system constraints. Suggested approach allows choosing optimal safety stock levels, because the proposed solution in fact describes algorithmic relationship between the safety stock levels and optimal value of the cost function.

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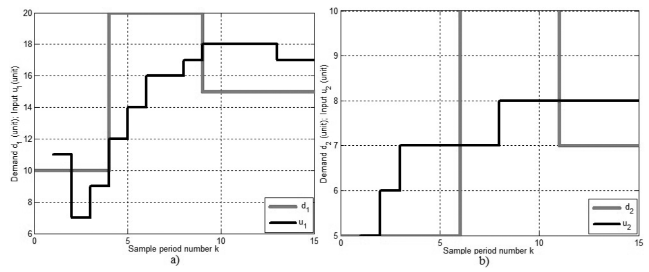

Fig. 3. External demands and control inputs: a) node 1; b) node 2

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