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TO REALISATION OF CHROMATIC POLYNOMIAL CALCULATION ALGORITHM

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Abstract

We calculate chromatic polynomial of an undirected graph using the fundamental reduction theorem and reducing to complete graphs. We also find the chromatic number using the chromatic polynomial. The C++ program was created, the result is obtained in the form of falling factorials and afterwards by the powers of x, the applications of chromatic polynomial are given.

Keywords: graph, chromatic polynomial, chromatic number, fundamental reduction theorem, deletion-contraction recurrence, reduction to complete graphs

Introduction

This work is related to analysis of different graphs by calculating such characteristics as chromatic polynomial and chromatic number.

Chromatic polynomial was proposed by George David Birkhoff to find the solution for the four color problem. It counts the number of graph colorings as a function of the number of colors.

The chromatic number of a graph is the smallest number of colors needed to color the vertices of this graph so that any two adjacent vertices have different colors.

In this work we use the fundamental reduction theorem and reducing to complete graphs. Generally, any reasonably-sized, undirected graph without multiple edges and loops can be analyzed by the program, that uses this method.

1. Preliminaries

Let us briefly recall the method, proposed in [1] (see also [2, 3]). The idea is to calculate chromatic polynomials of two modified graphs, instead of calculating chromatic polynomial of the original graph itself. Beneath is a theorem that this method is based on:

Let G be the input graph, u and v – two different, not adjacent vertices of graph G. Graph G_1 is obtained from original graph G by adding an edge $e = uv$ between vertices u and v. Graph G_2 is obtained from graph G by merging vertices u and v together (multiple edges achieved on this step are understood as single ones).

Then the chromatic polynomial $P(G, x)$ has a property:

$$P(G, x) = P(G_1, x) + P(G_2, x),$$

where $P(G_1, x)$ and $P(G_2, x)$ are chromatic polynomials of graphs G_1 and G_2 respectively.

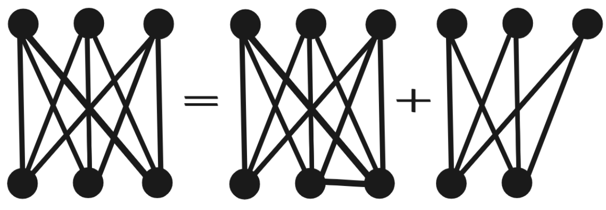

Fig. 1. The utility graph $K_{3,3}$

We illustrate the described procedure for the utility graph, for example (see Fig. 1).

Let us further describe the way this method works. We repeat this procedure recursively until we obtain complete graphs only. At this point the main algorithm stops, because it can't add an edge to the complete graph. However, chromatic polynomials of complete n-sized graphs K_n are easily calculated using falling factorial:

$$P(K_n, x) = x(x-1) \cdot \ldots \cdot (x-n+1) = x^{(n)}.$$

As previously mentioned, this method is called reduction to complete graphs. However it is also possible to use reduction to empty graphs according to the fact that the chromatic polynomial of empty n-sized graph O_n can also be easily calculated using next formula:

$$P(O_n, x) = x^n.$$

2. Main results

We obtain the chromatic polynomial for the original graph G in the following form:

$$P(G, x) = \alpha_k \cdot x^{(k)} + \alpha_{k+1} \cdot x^{(k+1)} + \ldots + \alpha_n \cdot x^{(n)}.$$

Here n denotes the number of vertices of the original graph, k is the size of the smallest complete graph, obtained by algorithm, α_i, $i = k, \ldots, n$ are quantities

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of complete graphs, obtained by algorithm. Chromatic number is equal to the size of the smallest complete graph k, i.e. the least falling factorial. Chromatic polynomial by the powers of x is further calculated by summing up chromatic polynomials of complete graphs $x^{(k)}, \ldots, x^{(n)}$ multiplied by their quantities $\alpha_k, \ldots, \alpha_n$.

Application was created using C++ programming language in Visual Studio development environment without using any specific libraries.

Effectiveness of the algorithm is mainly dependent on amount of the edges in input graph (the fewer edges it has – the more calculations are made).

As already mentioned, after all the calculations are made the program returns all the coefficients of the chromatic polynomial and chromatic number. For instance, for graph $K_{3,3}$ result is:

$$P(K_{3,3}, x) = x^{(6)} + 6x^{(5)} + 11x^{(4)} + 6x^{(3)} + x^{(2)} = \\ = x^6 - 9x^5 + 36x^4 - 75x^3 + 78x^2 - 31x,$$

which matches with the chromatic polynomial for $K_{3,3}$ given in [2]. This formula implies that the least falling factorial is $x^{(2)}$, so the chromatic number of $K_{3,3}$ is 2 (it is obvious that $K_{3,3}$ is bipartite).

3. Applications

It is well-known that the chromatic number can be found in other ways by both exact and approximate algorithms, but chromatic polynomial contains more useful information such as

1) number of vertices – the degree of $P(G, x)$;
2) number of edges – the absolute value of the second coefficient of $P(G, x)$;
3) number of graph components – the least power of x;
4) the chromatic number – the power of the least falling factorial;
5) number of graph acyclic orientations – $(-1)^n P(G, -1)$, etc.

Also the chromatic polynomial is connected with the Stirling numbers of the first kind.

The expressions for the first four coefficients of the chromatic polynomial can be found in [4]. For the graph G with n vertices and e edges the chromatic polynomial is equal to

$$P(G, x) = x^n - ex^{n-1} + \left(\binom{e}{2} - A \right) x^{n-2} + \\ + \left(-\binom{e}{3} + (e-2)A + B - 2C \right) x^{n-3} + \ldots,$$

where A, B and C are the numbers of subgraphs of G, which are triangles, pure quadrilaterals and complete 4-sized graphs, respectively.

For instance, let us apply the following results to $K_{3,3}$. This graph contains 9 edges, so the second coefficient is equal to -9, the number of it's subgraphs, which are triangles, is equal to zero, according to the fact that it is a bipartite graph, the third coefficient is equal to $\binom{9}{2} - 0 = 36$. This matches with the formula stated above. Also the number of acyclic orientations of $K_{3,3}$ is equal to $(-1)^6 P(K_{3,3}, -1) = 230$.

Also the chromatic polynomial of the bipartite graph $K_{n,m}$ is calculated in [5] in the following form, where $s(l,k)$ and $S(n,i)$ are the Stirling numbers of the first kind and the second kind respectively:

$$P(K_{n,m}, x) = \sum_k x^k \sum_l s(l,k) \sum_i S(n,i) S(m, l-i) = \\ = \sum_k S(m,k) x(x-1) \cdot \ldots \cdot (x-k+1)(x-k)^n.$$

4. Conclusion

We have developed stable and effective app that makes it easy to calculate chromatic number and chromatic polynomial, which are, as previously desribed, useful in many different ways. Such methods have already been developed in solutions like Maple or Wolfram. However, it is not clear, whether they use exactly this method or not. Therefore, this work is, in a way, unique, as it is open and might help to develop more complex software solutions.

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